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Symmetry, Compact Closure and Dagger Compactness for Categories of Convex Operational Models

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Abstract In the categorical approach to the foundations of quantum theory, one begins with a symmetric monoidal category, the objects of which represent physical systems, and the morphisms of which represent physical processes. Usually, this category is taken to be at least compact closed, and more often, dagger compact, enforcing a certain self-duality, whereby preparation processes (roughly, states) are interconvertible with processes of registration (roughly, measurement outcomes). This is in contrast to the more concrete "operational" approach, in which the states and measurement outcomes associated with a physical system are represented in terms of what we here call a *convex operational model*: a certain dual pair of ordered linear spaces–generally, *not* isomorphic to one another. On the other hand, state spaces for which there *is* such an isomorphism, which we term *weakly self-dual*, play an important role in reconstructions of various quantum-information theoretic protocols, including teleportation and ensemble steering. In this paper, we characterize compact closure of symmetric monoidal categories of convex operational models in

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two ways: as a statement about the existence of teleportation protocols, and as the principle that every process allowed by that theory can be realized as an instance of a remote evaluation protocol—hence, as a form of classical probabilistic conditioning. In a large class of cases, which includes both the classical and quantum cases, the relevant compact closed categories are degenerate, in the weak sense that every object is its own dual. We characterize the dagger-compactness of such a category (with respect to the natural adjoint) in terms of the existence, for each system, of a *symmetric* bipartite state, the associated conditioning map of which is an isomorphism.

Keywords Quantum foundations · Convex operational theories · Compact closed category · Dagger-compact category

1 Categorical Semantics and Quantum Foundations

One natural way to formalize a physical theory is as some kind of *category*, $\mathcal{C}$, the objects of which are the systems, and the morphisms of which are the *processes*, contemplated by that theory. In order to provide some apparatus for representing compound systems, it is natural to assume further that $\mathcal{C}$ is a *symmetric monoidal* category. In the categorical semantics for quantum theory pioneered by Abramsky and Coecke [1], Selinger [38, 39], and others (e.g., [2, 3, 7, 17]), it is further assumed that $\mathcal{C}$ is at least compact closed, and more usually, dagger compact. This last condition enforces a certain self-duality, in that there is a bijection between the states of a system $A \in \mathcal{C}$, represented by elements of $\mathcal{C}(I, A)$, and and the measurement-outcomes associated with that system, represented by elements of $\mathcal{C}(A, I)$. The motivating example here is the category **FDHilb** of finite-dimensional complex Hilbert spaces and unitary mappings—that is, the category of finite-dimensional "closed" quantum systems and unitary processes. Many of the information-processing features of finite-dimensional quantum systems occur in any dagger-compact category, notably, conclusive (that is, post-selected) teleportation and entanglement-swapping protocols. On the other hand, if our interest in a categorical reformulation of quantum theory is mainly foundational, rather than strictly one of systematization, these strong structural assumptions need further justification, or at any rate, further motivation.

There is an older tradition, stemming from Mackey's work on the foundations of quantum mechanics [35], in which an individual physical (or, more generally, probabilistic) system is represented by a set of states, a set of observables or measurements, and an assignment of probabilities to measurement outcomes, conditional upon the state. From this basic idea, one is led to a representation of systems by pairs of ordered real vector spaces—the *convex operational models* of our title—and of physical processes, by certain positive linear mappings between such spaces. The motivating example is the category of Hermitian parts of C^* algebras and completely positive mappings.

This "convex operational" approach, in contrast to the categorical one, is conservative of classical probabilistic concepts, but liberal as to how systems may be combined and transformed, so long as this probabilistic content is respected. In particular, there is no standing assumption of monoidality; rather, systems are combined using any of a variety of "non-signaling" products. Nor is there, in general, any hint of the kind of self-duality mentioned above—indeed, the natural dual object for a convex operational model is not itself an operational model. Nevertheless, here again various familiar "quantum" phenomena—such as no-cloning and no-broadcasting theorems, information-disturbance tradeoffs, teleportation and entanglement swapping protocols, and ensemble steering—emerge naturally and in some generality [8, 9, 12, 15]. A key idea here is that of a *remote evaluation protocol* [9] (of which teleportation is a special case), which reduces certain kinds of dynamical processes to purely classical *conditioning*.

It is obviously of interest to see how far such convex operational theories can be treated formally, that is, as categories, and more especially, as symmetric monoidal categories; equally, one would like to know how much of the special structure assumed in the categorical approach can be given an operational motivation. Some first steps toward addressing these issues are taken in [13, 14]. Here, we aim to make further progress, albeit along a somewhat narrower front. We focus on *symmetric monoidal* categories of convex operational models—what we propose to call *probabilistic theories*. We show that such a theory admits a compact closed structure if and only if every system allowed by the theory can be teleported (conclusively, though not necessarily with probability 1) through a copy of itself—or, equivalently, if and only if every process contemplated by the theory can be represented as a remote evaluation protocol. We then specialize further, to consider *weakly self-dual* theories, in which for every system A there is a bipartite state γ_A on $A \otimes A$ corresponding to an isomorphism between A and its dual, and an effect corresponding to its inverse. (Such state spaces figure heavily in earlier treatments of teleportation protocols [9] and ensemble steering [12] in general probabilistic theories.) We show that if the state implementing weak self-duality can be chosen to be symmetric for every A, then a weakly self-dual monoidal probabilistic theory is not merely compact closed, but dagger compact.

Organization and Notation Sections 2 and 3 provide quick reviews of the category-theoretic and the convex frameworks, respectively, mainly following [1] for the former and [8, 9, 12–14] for the latter. Section 4 makes precise what we mean by a monoidal probabilistic theory, as a symmetric monoidal category of convex operational models, and establishes that all such theories have the property of allowing *remote evaluation* [9]; when the state spaces involved are weakly self-dual, teleportation arises as a special case. Section 5 contains the results on categories of weakly self-dual state spaces described above. Section 6 discusses some of the further ramifications of these results.

We assume that the reader is familiar with basic category-theoretic ideas and notation, as well as with the probabilistic machinery of quantum theory. We write $\mathcal{C}, \mathcal{D}$ etc. for categories, $A \in \mathcal{C}$, to indicate that A is an object of $\mathcal{C}$, and $\mathcal{C}(A, B)$ for the set of morphisms between objects $A, B \in \mathcal{C}$. Except as noted, all vector spaces considered here will be finite-dimensional and *real*. We write $\mathbf{Vec}_{\mathbb{R}}$ for the category of finite-dimensional real vector spaces and linear maps. The dual space of a vector space A is denoted by A^*. An *ordered vector space* is a real vector space V equipped with a *regular*—that is, closed, convex, pointed, generating—cone V_+, and ordered by the relation $x \leq y \Leftrightarrow y - x \in A_+$. A linear mapping $\phi : V \to W$ between ordered linear spaces V and W is *positive* if $\phi(V_+) \subseteq W_+$. We write $\mathcal{L}_+(A, B)$ for the cone of positive linear mappings from A to B. The special case in which $B = \mathbb{R}$, the positive linear functionals on A, is the *dual cone* of A_+, denoted A^*_+. The category of ordered linear spaces and positive linear maps we denote by **Ordlin**. Finally, we make the standing assumption that, except where otherwise indicated, *all vector spaces considered here are finite dimensional*.

2 The Category-Theoretic Perspective

A *monoidal category* [36] is a category $\mathcal{C}$ equipped with a bifunctor[1] $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a distinguished unit object I, and natural associativity and left and right unit deletion isomorphisms,

$$\alpha_{A,B,C} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C,$$
$$\lambda_A : I \otimes A \cong A, \qquad \rho_A : A \otimes I \cong A,$$

subject to some coherence conditions (for which, see [36]). If these isomorphisms are identities, the category is called *strict monoidal*. Every monoidal category is equivalent to a strict one, so setting $A \otimes I = A$ etc, is harmless and we will do this throughout.

A *symmetric monoidal category* (SMC) is a monoidal category further equipped with a natural family of symmetry isomorphisms,

$$\sigma_{A,B} : A \otimes B \cong B \otimes A,$$

again, subject to some coherence conditions (for which, again, see [36]). Unlike the other isomorphisms, these symmetry isomorphisms cannot generally be made strict.

Examples of SMCs include commutative monoids (as one-object categories), the category of sets and mappings (with $A \otimes B = A \times B$), and—of particular relevance for us—the category of (say, finite-dimensional) vector spaces over a field K and K-linear maps, with $A \otimes B$ the usual tensor product. Another source of examples comes from logic: one can regard the set of sentences of a logical calculus as a category, with *proofs*, composed by concatenation, as morphisms. In this context, one can take conjunction, $\wedge$, as a monoidal product.

Much more broadly, if somewhat less precisely, if one views the objects of a category $\mathcal{C}$ as "systems" (of whatever sort), and morphisms as "processes" between systems, then a natural interpretation of the product in a symmetric monoidal category is as a kind of accretive composition: $A \otimes B$ is the system that consists of the two systems A and B sitting, as it were, side by side, without any special interaction; $f \otimes g$ represents the processes $f : A \to X$ and $g : A \to Y$ acting

[1] Bifunctoriality means that: (i) $1_{A \otimes B} = 1_A \otimes 1_B$; and (ii) given morphisms $f : A \to X$ and $g : B \to Y$ in $\mathcal{C}$, there is a canonical product morphism $f \otimes g : A \otimes B \to X \otimes Y$, such that $(f \otimes g) \circ (f' \otimes g') = (f \circ f') \otimes (g \circ g')$.

in parallel. Taking this point of view, it is helpful to regard processes of the form $I \to A$, where I is the monoidal unit in $\mathcal{C}$, as *states*, associated with ways of preparing the system A. Similarly, we regard processes of the form $a : A \to I$ as "effects", or measurement-outcomes. We shall henceforth adhere to the convention of denoting states by lower-case Greek letters α, β, ... and effects, lower-case Roman letters $a, b, \ldots$.

In any monoidal category, one can regard endomorphisms $s \in \mathcal{C}(I, I)$ as "scalars" acting on elements of $\mathcal{C}(A, B)$ by $sx = s \otimes x$. In every monoidal category, $\mathcal{C}(I, I)$ is a commutative monoid, even if $\mathcal{C}$ is not symmetric. When $\mathcal{C}(I, I)$ is isomorphic to a particular monoid S, we shall say that $\mathcal{C}$ is a symmetric monoidal category *over S*.

2.1 Compact Closed Categories

A *dual* for an object A of a symmetric monoidal category $\mathcal{C}$ is an object B and two morphisms, the *unit*, $\eta : I \to B \otimes A$ (not to be confused with the *tensor unit I*) and the *co-unit*, $\epsilon : A \otimes B \to I$, such that

$$\begin{array}{l} A \xrightarrow{1_A \otimes \eta} A \otimes B \otimes A \xrightarrow{\epsilon \otimes 1_A} A \;= 1_A \\ B \xrightarrow{\eta \otimes 1_B} B \otimes A \otimes B \xrightarrow{1_B \otimes \epsilon} B \;= 1_B. \end{array} \tag{1}$$

Duals are unique up to a canonical isomorphism. Indeed, if $(B_1, \eta_1, \epsilon_1)$ and $(B_2, \eta_2, \epsilon_2)$ are duals for A, then $\phi := (1_{B_2} \otimes \epsilon_1) \circ (\eta_2 \otimes 1_{B_1}) : B_1 \to B_2$ has inverse $\phi^{-1} = (1_{B_1} \otimes \epsilon_2) \circ (\eta_1 \otimes 1_{B_2})$; moreover, $\eta_2 = (\phi \otimes 1_A) \circ \eta_1$. Some, for example the authors of [17], use the term *compact structure* to refer to what we are calling a dual, i.e., a particular choice of (A', η_A, ϵ_A) for a given object $A \in \mathcal{C}$ or, when applied to a category, a particular choice of dual for each object.

A symmetric monoidal category $\mathcal{C}$ is *compact closed*[2] if for every object A in the category, there is a dual, (A', η_A, ϵ_A), where A' is an object of the category.[3] As thus defined, compact closedness is a *property* of the SMC $\mathcal{C}$, not an additional structure: it requires the existence of at least one dual for each object, but not the explicit specification of a distinguished one. The alternative definition of compact closed category, which differs only in requiring a choice of duals be specified [32], is perhaps more common. Owing to the uniqueness up to isomorphism mentioned above, the various possible choices of duals are largely—but not entirely—equivalent. A compact structure is said to be *degenerate* iff $A' = A$; a compact closed category $\mathcal{C}$ *with a distinguished compact structure* is said to be degenerate if every object's compact structure is degenerate. This *does* depend on an explicit choice of duals, and thus imposes some non-trivial structure[4] beyond compact closure. This is the setting that will most interest us below.

[2] Sometimes just *compact*.

[3] We use the notation A', rather than the more standard A^*, for the designated dual of an object in a compact closed category, because we wish to reserve the latter to denote, specifically, the dual space of a vector space.

[4] The self-dual setting requires some additional coherence conditions; see [40]

Remark 1 If (A', η_A, ϵ_A) and (B', η_B, ϵ_B) are duals for objects $A, B \in \mathcal{C}$, then we can construct a canonical dual $(A' \otimes B', \eta_{AB}, \epsilon_{AB})$ for $A \otimes B$ by setting $\eta_{AB} = \tau \circ (\eta_A \otimes \eta_B)$ and $\epsilon_{AB} = (\eta_A \otimes \eta_B) \circ \tau^{-1}$, where

$$\tau = 1_{A'} \otimes \sigma_{AB'} \otimes 1_B : (A' \otimes A) \otimes (B' \otimes B) \simeq (A' \otimes B') \otimes (A \otimes B).$$

Since all duals are isomorphic we are free to assume that $A \otimes B$ has this particular dual.

In any compact closed category, an assignment $A \mapsto A'$ extends to a contravariant functor $(-)' : \mathcal{C}^{op} \to \mathcal{C}$, called the *adjoint*, taking morphisms $\phi : A \to B$ to $\phi' : B' \to A'$ defined by:

$$\begin{array}{ccc} B' & \xrightarrow{\eta_A \otimes 1_{B'}} & A' \otimes A \otimes B' \\ \Big\downarrow{\scriptstyle \phi'} & & \Big\downarrow{\scriptstyle 1_{A'} \otimes \phi \otimes 1_{B'}} \\ A' & \xleftarrow[1_{A'} \otimes \epsilon_B]{} & A' \otimes B \otimes B'. \end{array} \quad (2)$$

The functor $'$ is *nearly* involutive, in that there are natural isomorphisms $w_A : A'' \to A$. To say that $'$ is involutive is just to say that $A'' = A$ and $\phi = \phi''$; note that this does not imply that $w_A = 1_A$.

Remark 2 In the classic treatment of coherence for compact closed categories in [32], one has that $\sigma \circ \eta_A = (1_A \otimes w_A) \circ \eta_{A'}$; a similar condition holds for ϵ ([32], eq. (6.4)ff.). In the case of a degenerate category, this implies that

$$\sigma \circ \eta_A = (1_A \otimes w_A) \circ \eta_A. \quad (3)$$

It is easy to show that if the units—or, equivalently, co-units—are symmetric, in the sense that, for every object $A \in \mathcal{C}$, $\eta_A = \sigma_{A,A} \circ \eta_A$ or, equivalently, $\epsilon_A = \epsilon_A \circ \sigma_{A,A}$, then the the functor $'$ is involutive. However the converse does not necessarily hold, unless $w_A = 1_A$. In general, it is not clear what coherence requirements are appropriate for the degenerate categories we consider, nor whether the functors involved are always strict. Therefore, in Section 5 we will establish explicitly that the involutiveness of the adjoint is equivalent to the symmetry of the unit and co-unit for the compact closed categories of convex operational models considered in this paper.

Remark 3 For an arbitrary degenerate compact closed category, there is no guarantee that η_A will be symmetric. (We thank Peter Selinger (P. Selinger 2010, personal communication. See also [40]) for supplying a nice example involving a category of plane tangles). Thus, it is a non-trivial constraint on such a category that the canonical adjoint be an involution. This will be important below.

2.2 Daggers

A *dagger category* [1, 39] is a category $\mathcal{C}$ together with an involutive functor $(-)^\dagger$: $\mathcal{C}^{op} \to \mathcal{C}$ that acts as the identity on objects. That is, $A^\dagger = A$ for all $A \in \mathcal{C}$, and, if $f \in \mathcal{C}(A, B)$, then $f^\dagger \in \mathcal{C}(B, A)$, with

$$(g \circ f)^\dagger = f^\dagger \circ g^\dagger \quad \text{and} \quad f^{\dagger\dagger} = f$$

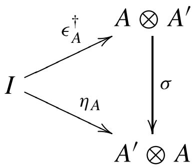
for all $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$.[5] We say that f is *unitary* iff $f^\dagger = f^{-1}$. A *dagger-monoidal* category is a symmetric monoidal category with a dagger such that (i) all the canonical isomorphisms defining the symmetric monoidal structure are unitary, and (ii)

$$(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$$

for all morphisms f and g in $\mathcal{C}$. Finally, a dagger-monoidal category $\mathcal{C}$ is *dagger compact* if it is compact closed and

$$\eta_A = \sigma_{A,A'} \circ \epsilon_A^\dagger$$

for every A, i.e.:



commutes. In the case of a degenerate compact closed category, the canonical adjoint $'$ functions as a dagger *if* it is involutive. However, as remarked above, this is a nontrivial condition.

In the work of Abramsky and Coecke [1], a symmetric monoidal category is interpreted as a physical theory, in which a morphism $\alpha : I \to A$ is interpreted as a *state* on the system A, and a morphism $b = \beta^\dagger : A \to I$ is understood as the registration of an effect (e.g., a measurement outcome) on A. The scalar $\beta^\dagger \circ \alpha : I \to I$ is understood, somewhat figuratively in the abstract setting, as the "probability" that the given effect will occur when the given state obtains.[6] This raises the obvious question of how to implement the compelling idea that probabilities should be identified with real numbers in the interval [0, 1] without passing through Hilbert space. One way to do this is simply to posit a mapping $p : \mathcal{C}(I, I) \to [0, 1]$, whereby the scalars of $\mathcal{C}$ can be interpreted probabilistically. Another is to examine the symmetric monoidal possibilities in cases in which the category consists, *ab initio*, of concretely described probabilistic models of a reasonably simple and general sort. In this paper, we concentrate on this second strategy. As a first step, in the next section we describe the kinds of concrete probabilistic models we have in mind.

[5]Those new to categories should note that a functor from $\mathcal{C}^{op}$ to $\mathcal{C}$ is sometimes called a *contravariant* functor from $\mathcal{C} \to \mathcal{C}$; the description we have just given (minus the involutiveness condition) defines this notion without reference to $\mathcal{C}^{op}$.

[6]We can be more precise here: given a dagger-compact category of states and processes $\mathcal{C}$, any dagger-monoidal functor from $\mathcal{C}$ to **FDHilb**, the category of finite dimensional Hilbert spaces, will send the scalar $\beta^\dagger \circ \alpha$ to the inner product $\langle \beta \mid \alpha \rangle$.

3 Convex Operational Models and their Duals

The more traditional approach to modeling probabilistic physical theories [16, 35] begins by associating to each individual physical (or other probabilistic) system a triple (X, Σ, p)—sometimes called a *Mackey triple*—where Σ is a set of possible *states*, X is a set of possible measurement-outcomes, and $p : X \times \Sigma \to [0, 1]$ assigns to each pair (x, s) the probability, $p(x, s)$, that x will occur, if measured, when the system's state is s.

This minimal apparatus can be "linearized" in a natural way. The probability function p gives us a mapping $\Sigma \to [0, 1]^X$, namely $s \mapsto p(\cdot, s)$. We can plausibly identify each state $s \in \Sigma$ with its image under this mapping (thus identifying states if they cannot be distinguished statistically by the outcomes in X). Having done so, let Ω denote the point-wise closed, hence compact, convex hull of $\Sigma \subseteq [0, 1]^X$, representing the set of possible probabilistic *mixtures* of states in Σ. Every measurement outcome $x \in X$ can be represented by the affine functional $a_x : \Omega \to [0, 1]$, given by $a_x(\alpha) = \alpha(x)$ for all $\alpha \in \Omega$. More broadly, we can regard *any* affine functional $a : \Omega \to [0, 1]$ as representing a mathematically possible measurement outcome, having probability $a(\alpha)$ in state $\alpha \in \Omega$. Such functionals are called *effects* in the literature.

In general, the (mixed) state space Ω we have just constructed will have infinite affine dimension. Accordingly, for the next few paragraphs, we suspend our standing finite-dimensionality assumption. Now, any compact convex set Ω can be embedded, in a canonical way, as a *base* for the positive cone $V_+(\Omega)$ of a regularly ordered linear $V(\Omega)$ [4]. This means that every $\rho \in V_+(\Omega)$ has the form $\rho = t\alpha$ for a unique scalar $t \geq 0$ and a unique vector $\alpha \in \Omega$ (hence, Ω spans $V(\Omega)$). This space $V(\Omega)$ is complete in a natural norm, the *base norm*, the unit ball of which is given by the closed convex hull of $\Omega \cup -\Omega$. Moreover, $V(\Omega)$ has the following universal property: every bounded affine mapping $L : \Omega \to \mathbf{M}$, where $\mathbf{M}$ is any real Banach space, extends uniquely to a bounded linear mapping $L : V(\Omega) \to \mathbf{M}$.

In particular, every affine functional on Ω—in particular, every effect—extends uniquely to a linear functional in $V(\Omega)^*$. In particular, there is a unique *unit functional* $u_\Omega \in V(\Omega)^*$ such that $u_\Omega(\alpha) = 1$ for $\alpha \in \Omega$, and $\Omega = u_\Omega^{-1}(1) \cap V_+(\Omega)$. Thus, effects correspond to positive functionals $a \in V(\Omega)^*$ with $0 \leq a \leq u_\Omega$.

One often regards *any* effect $a \in V(\Omega)^*$ as a *bona fide* measurement outcome. This is the point of view, e.g., of [8, 9]. However, we may sometimes wish to privilege certain effects as "physically accessible". This suggests the following more general formulation:

Definition 4 A *convex operational model* (COM) is a triple $(A, A^\#, u_A)$ where

(i) A is a complete base-normed space with (strictly positive) unit functional u_A, and
(ii) $A^\#$ is a weak-$*$ dense subspace of A^*, ordered by a chosen regular cone $A^\#_+ \subseteq A^*_+$ containing u_A.

An *effect* on A is a functional $a \in A^\#_+$ with $a \leq u_A$.

Henceforth, where no ambiguity seems likely, we write A for the triple $(A, A^{\#}, u_A)$. Also, we now revert, for the balance of this paper, to our standing assumption that *all COMs are finite-dimensional*. In this case, the weak-$*$ density assumption above simply says that $A^{\#} = A^*$ as vector spaces. Even in this situation, however, the chosen cone $A^{\#}_+$ will generally be smaller than the dual cone A^*_+, so the positive cone $(A^{\#})^*_+$ will in general be larger than A_+. It is useful to regard normalized elements of the former cone as *mathematically consistent* probability assignments on the effects in $A^{\#}$, from which the model singles out those in A_+ as *physically* possible. In the special case in which $A^{\#}_+ = A^*_+$—as, e.g., in the case of quantum systems—we shall say that the COM A is *saturated*.

Example 5 Let E be a finite set, thought of as the outcome set for a discrete classical experiment. Take $A = \mathbb{R}^E$, with A_+ the cone of non-negative functions on E, and let $u_A(f) = \sum_{x \in E} f(x)$. Then $\Omega = u^{-1}(1)$ is simply the set of probability weights on E. Geometrically, this last is a *simplex*. In finite dimensions, every simplex has this form. Accordingly, we say a COM is *classical* iff its normalized state space is a simplex.

Example 6 Let $\mathbf{H}$ be a finite-dimensional complex Hilbert space, and let $A = \mathcal{L}_h(\mathbf{H})$, the space of Hermitian operators $a : \mathbf{H} \to \mathbf{H}$, with the usual positive cone, i.e, A_+ consists of all Hermitian operators of the form $a^\dagger a$. Let $u_A(a) = \mathrm{Tr}(a)$. Then Ω_A is the convex set of density operators on $\mathbf{H}$, i.e., the usual space of mixed quantum states.

Example 7 Let (X, Σ, p) be any Mackey triple. Construct the state-space Ω and the associated ordered Banach space $V(\Omega)$ as described above. Letting $A^{\#}_+$ be the cone in $V^*(\Omega)$ generated by the evaluation functionals a_x, $x \in X$, we have a convex operational model. This will be finite-dimensional iff the span of (the image of) Σ in $\mathbb{R}^X$ is finite dimensional.

3.1 Processes as Positive Mappings

Definition 8 A *morphism of COMs* from $(A, A^{\#}, u_A)$ to $(B, B^{\#}, u_B)$ is a positive linear map $\phi : A \to B$ such that the usual linear adjoint map $\phi^* : B^* \to A^*$ is positive with respect to the designated cones $A^{\#}_+$ and $B^{\#}_+$.

The set of morphisms of COMs from A to B is clearly a sub-cone of $\mathcal{L}_+(A, B)$. It is clear that the composition of two mappings of COMs is again a mapping of COMs, so that COMs form a concrete category.

Definition 9 Let A and B be COMs. A *process* from A to B is a morphism $\phi : A \to B$ such that, for every state $\alpha \in \Omega_A$, $u_B(\phi(\alpha)) \leq 1$, or, equivalently, if $\phi^*(u_B) \leq u_A$.

If $\phi : A \to B$ is a process, we can regard $u_B(\phi(\alpha))$ as the *probability* that the process represented by ϕ occurs. If we regard $\mathbb{R}$ as an COM with $u_{\mathbb{R}}$ the identity mapping on $\mathbb{R}$, this is consistent with our understanding of $a(\alpha)$ as the probability of the effect $a : A \to \mathbb{R}$ occurring. Notice that a positive linear map $\phi : \mathbb{R} \to A$ is a process if and only if $\phi(1)$ is a sub-normalized state, while a positive functional $f : A \to \mathbb{R}$ is a process if and only if $f \in A^{\#}_{+}$ and $f \leq u_A$—in other words, if and only if f is an effect. Finally, since Ω_A is compact, $u_A(\phi(\alpha))$ attains a maximum value, say M on Ω_A. $M^{-1}\phi$ is a process, so every morphism of COMs is a positive multiple of a process.

3.2 Bipartite States and Composite Systems

Given two separate systems, represented by COMs A and B, we should expect that any state of the composite system AB will induce a joint probability assignment $p(a, b)$ on pairs of effects $a \in A^{\#}$, $b \in B^{\#}$. If the two systems can be prepared independently, we should also suppose that, for any two states $\alpha \in A$ and $\beta \in B$, the product state $\alpha \otimes \beta$, given by $(\alpha \otimes \beta)(a, b) = \alpha(a)\beta(b)$, will be a legitimate joint state. Finally, if the two systems do not interact, the choice of measurement made on A ought not to influence the statistics of measurement outcomes on B, and vice versa. This latter "no-signaling" condition is equivalent [44] to the condition that the joint probability assignment p extends to a bilinear form on $A^{\#} \times B^{\#}$, normalized so that $p(u_A, u_B) = 1$. Abstractly, then, one makes the following definition.

Definition 10 A (normalized, non-signaling) *bipartite state* between convex operational models A and B is a bilinear form $\omega : A^{\#} \times B^{\#} \to \mathbb{R}$ that is *positive*, in the sense that $\omega(a, b) \geq 0$ for all effects $a \in A^{\#}$ and $b \in B^{\#}$, and *normalized* (satisfies $\omega(u_A, u_B) = 1$).

Implicit in this definition is the assumption, lately called *local tomography* [18], that a joint state is determined by the joint probabilities it assigns to measurement outcomes associated with the local systems A and B. As has been pointed out by many authors, e.g. [6, 15, 33], this condition is violated in both real and quaternionic quantum theory, and can therefore be made to serve as an axiom separating standard complex QM from these. A more general notion of non-signaling bipartite state would merely associate, rather than identify, each such state with a positive bilinear form on $A^{\#} \times B^{\#}$. See the remarks following Definition 11 for more on this.

It is clear that any product $\omega = \alpha \otimes \beta$ of normalized states $\alpha \in A$ and $\beta \in B$ defines a non-signaling state; hence, so do convex combinations of product states. Non-signaling states arising in this way, as mixtures of product states, are said to be *separable* or *unentangled*. An *entangled* non-signaling state is one that is *not* a convex combination of product states. Many of the basic properties of entangled *quantum* states actually hold for entangled states in this much more general setting [8, 33].

The space $\mathfrak{B}(A^{\#}, B^{\#})$ of all bilinear forms on $A^{\#} \times B^{\#}$, ordered by the cone of all positive bilinear forms, is the *maximal tensor product*, $A \otimes_{max} B$, of A and B. This notation is reasonable, since (in finite dimensions), $\mathfrak{B}(A^{\#}, B^{\#})$ is one model of the tensor product $(A^{\#})^* \otimes (B^{\#})^*$—thus, of the vector-space tensor product $A \otimes B$.[7] Ordering $A \otimes B$ instead by the generally much smaller cone of unentangled states, that is, the cone generated by the product states, gives the *minimal tensor product*, $A \otimes_{min} B$. It is important to note that these coincide only when A or B is classical [8]. If A and B are quantum state spaces, then the cone of bipartite density matrices for the composite system lies properly between the maximal and minimal cones. This indicates the need for something more general:

Definition 11 A (locally tomographic) *composite* of COMs A and B is a convex operational model $(AB, (AB)^{\#}, u_{AB})$, such that $AB \subseteq \mathfrak{B}(A^{\#}, B^{\#})$, with $u_{AB} = u_A \otimes u_B$, $\alpha \otimes \beta \in (AB)_+$ for all $\alpha \in A_+$, $\beta \in B_+$, and $a \otimes b \in (AB)^{\#}_+$ for all $a \in A^{\#}_+$, $b \in B^{\#}_+$.

It is worth stressing that there are perfectly reasonable theories that are *not* locally tomographic. Indeed, one of these is quantum mechanics over the real, rather than complex, scalars. We might more generally define a *composite in the wide sense* of COMs A and B to be a COM $(AB, (AB)^{\#}, u_{AB})$, together with (i) a positive linear embedding (injection) $i : A \otimes_{min} B \to AB$, and (ii) a positive map $r : AB \to A \otimes_{max} B$, surjective as a *linear* map, such that for all $a \in A^{\#}$, $b \in B^{\#}$,

$$r(i(\alpha \otimes \beta))(a, b) = a(\alpha)b(\beta),$$

i.e., $r \circ i$ is the canonical embedding of $A \otimes_{min} B$ in $A \otimes_{max} B$. However, we shall make no use of this extra generality here. Accordingly, we assume henceforth that *all composites are locally tomographic*, as per Definition 11 above.

3.3 Conditioning and Remote Evaluation

A bipartite state ω on A and B gives rise to a positive linear mapping $\widehat{\omega} : A^{\#} \to B$ with

$$b(\widehat{\omega}(a)) = \omega(a, b)$$

for all $a \in A^{\#}$ and $b \in B^{\#}$; dually, a bipartite effect $f \in (AB)^{\#}$ gives rise to a linear map $\widehat{f} : A \to B^{\#}$, given by $\widehat{f}(\alpha)(\beta) = f(\alpha \otimes \beta)$, and subject to $f(\alpha) \leq u_B$ for all $\alpha \in \Omega_A$.

The *marginals* of ω are given by $\omega_B = \widehat{\omega}(u_A)$ and $\omega_A = \widehat{\omega}^*(u_B)$. Note that ω is normalized iff $u_B(\widehat{\omega}(u_A)) = 1$. We can define the *conditional states* of A and B given (respectively) effects $b \in [0, u_B]$ and $a \in [0, u_A]$ by

$$\omega_{A|b} := \frac{\widehat{\omega}^*(b)}{\omega_B(b)} \text{ and } \omega_{B|a} := \frac{\widehat{\omega}(a)}{\omega_A(a)}$$

provided the marginal probabilities $\omega_B(b)$ and $\omega_A(a)$ are non-zero. Accordingly, we refer to $\widehat{\omega}(a)$ as the *un-normalized conditional state*. Notice that the linear adjoint, $\widehat{\omega}^* : B^{\#} \to A$, of $\widehat{\omega}$ represents the same state, but evaluated in the opposite order: $\widehat{\omega}^*(b)(a) = \widehat{\omega}(a)(b) = \omega(a, b)$.

[7] This is a straightforward extension of the definition in [13] to the context of possibly non-saturated models.

Lemma 12 (Remote Evaluation 1) *Let A,B and C be convex operational models. For any bipartite effect on $f \in (AB)^*$ and any bipartite state $\omega \in BC$, and for any state $\alpha \in A$,*

$$(f \otimes -)(\alpha \otimes \omega) = \widehat{\omega}(\widehat{f}(\alpha)).$$

Proof It is straightforward that this holds where f and ω are a product effect and a product state, respectively. Since these generate $(AB)^*$ and AB, the result follows. □

Operationally, this says that one can *implement* the transformation $\widehat{\omega} \circ \widehat{f}$ by *preparing* the tripartite system ABC in state $\alpha \otimes \omega$, where $\alpha \in A$ is the "input" state to be processed, and then making a measurement on AB, of which f is a possible outcome: the un-normalized conditional state of C, given the effect f on AB, is exactly $\widehat{\omega}(\widehat{f}(\alpha))$. Thus, the process $\phi := \widehat{\omega} \circ \widehat{f} : A \to C$ becomes a special case of *conditioning*. In [9], we have called this protocol *remote evaluation*. Note that conclusive, or post-selected, teleportation arises as the special case of remote evaluation in which, up to some specified isomorphism, $C \simeq A$ and $\widehat{\omega} \circ \widehat{f} \simeq 1_A$. We shall return to this point below.

4 Categories of Convex Operational Models

We now wish to chart some connections between the two approaches outlined above. In the first place, we will bring some category-theoretic order to the concepts developed in the preceding section.

Since morphisms of COMs compose, we can define a category **Com** of all convex operational models and COM morphisms. As described in Section 3.1, the hom-sets $\mathbf{Com}(A, B)$ are themselves cones. Let I denote the COM $\mathbb{R}$ with its standard cone and order unit, i.e. $I = (\mathbb{R}, \mathbb{R}, 1)$.

Definition 13 A *category of COMs* is a subcategory $\mathcal{C}$ of **Com** such that: (i) $\mathcal{C}(A, B)$ is a (regular) sub-cone of the cone $\mathbf{Com}_+(A, B)$; (ii) $\mathcal{C}$ contains the distinguished COM I; (iii) $\mathcal{C}(I, A) \simeq A$; and (iv) $\mathcal{C}(A, I) \simeq A^\#$. We call a such a category *finite-dimensional* if all state spaces $A \in \mathcal{C}$ are finite dimensional.

A more general definition would require only that $\mathcal{C}(A, B)$ be *some* set of *processes*, in the sense of Definition 9, between A and B. However, we should like to be able to construct random mixtures of processes, so $\mathcal{C}(A, B)$ should at least be convex. Allowing for the taking of limits as a reasonable idealization, it is plausible to take $\mathcal{C}(A, B)$ also to be closed. Finally, one should require that, if $\phi : A \to B$ is a physically valid process, then so is $t\phi$ for any $t \in [0, 1]$–this reflecting the possibility of *attenuating* a process (as, for instance, by some filter that admits only a fraction t of incident systems, but otherwise leaves systems unchanged, or by in some other way conditioning its occurrence on an event assigned a probability less than 1). This much given, the physically meaningful processes between two systems should generate a closed, convex, pointed cone of positive mappings, as per Definition 13.

In [8, 9, 12], a *probabilistic theory* is defined, rather loosely, to be any class of COMs (or “probabilistic models”) that is equipped with some device, or devices, for forming composite systems. Tightening this up considerably, we make the following definition.

Definition 14 A *monoidal category of COMs* is a category of COMs equipped with a monoidal structure, such that (i) the monoidal unit is the COM I; (ii) for every $A, B \in \mathcal{C}$, $A \otimes B$ is a non-signaling composite in the sense of Definition 11.

While Definition 14 does not require it, in the rest of the paper we will assume that all monoidal categories of COMs are *symmetric* monoidal, and (in accordance with our standing assumtion), finite dimensional.

As an example, the category **FDCom** of all finite-dimensional convex operational models and positive mappings can be made into a monoidal category in two ways, using either the maximal or the minimal tensor product. Another example is the “box-world” considered, e.g., in [22, 41]: here, state spaces are constructed by forming maximal tensor products of basic systems, the normalized state spaces of which are two-dimensional squares. Another example is afforded by the category of quantum-mechanical systems, represented as the self-adjoint parts of complex matrix algebras. Here, the appropriate monoidal product of two systems A and B is what is sometimes referred to as the “spatial” tensor product, obtained by forming tensor products of the Hilbert spaces on which the A and B act, and taking the self-adjoint operators on this space.

4.1 Remote Evaluation Again

We now reformulate the conditioning maps and remote evaluation protocol discussed above in purely categorical terms. In fact, both make sense in any symmetric monoidal category $\mathcal{C}$. Suppose, then, that $\omega : I \to B \otimes A$ is a “bipartite state”, i.e, a state of the composite system $B \otimes A$. Then there is a canonical mapping $\widehat{\omega} : \mathcal{C}(B, I) \to \mathcal{C}(I, A)$ given by

$$\begin{array}{ccc} I & \xrightarrow{\ \omega\ } & B \otimes A \\ & \searrow^{\widehat{\omega}(b)} & \downarrow{\scriptstyle b \otimes 1_A} \\ & & A \end{array} \tag{4}$$

Dually, if $f \in \mathcal{C}(A \otimes B, I)$, there is a natural mapping $\widehat{f} : \mathcal{C}(I, A) \to \mathcal{C}(B, I)$ given by

$$\begin{array}{ccc} B & \xrightarrow{\ \alpha \otimes 1_B\ } & A \otimes B\ . \\ & \searrow^{\widehat{f}(\alpha)} & \downarrow{\scriptstyle f} \\ & & I \end{array} \tag{5}$$

Note that if $\mathcal{C}$ is *already* a category of COMs, then $\widehat{\omega}$ and $\widehat{f}$ are exactly the maps discussed in the last section. This has a simple but important corollary, namely, that these maps are indeed morphisms of COMs. Another consequence is that any monoidal category of COMs is *closed under conditioning* – that is, if ω is a normalized bipartite state of such a theory, belonging, say, to a composite system *AB*, then for every effect a on A and b on B, the composite states $\omega_{B|a}$ and $\omega_{A|b}$ are indeed states of A and B, respectively (as opposed to merely being elements of $(A^{\#})^*$ and $(B^{\#})^*$).[8]

The remote evaluation protocol of Lemma 12 also has a purely category-theoretic formulation:

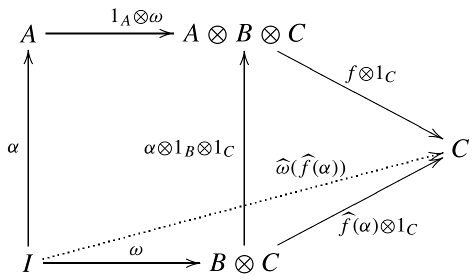
Lemma 15 (Remote Evaluation 2) *Let* $\omega : I \to B \otimes C$ *and* $f : A \otimes B \to I$ *in* $\mathcal{C}$. *Then*

$$\widehat{\omega}(\widehat{f}(\alpha)) = (f \otimes 1_C) \circ (1_A \otimes \omega) \circ \alpha = (f \otimes 1_C) \circ (\alpha \otimes \omega) \tag{6}$$

for all $\alpha \in \mathcal{C}(I, A)$. *Dually, for every* $\beta \in \mathcal{C}(I, B)$, *we have*

$$\widehat{\omega}^*(\widehat{f}^*(\beta)) = (1_A \otimes f) \circ (\omega \otimes \beta). \tag{7}$$

Proof We prove (6), the proof of (7) being similar. Tensoring the diagram (5) with C (on the right) gives the right-hand triangle in the diagram below. Applying (4) to compute $\widehat{\omega}(\widehat{f}(\alpha))$ gives the lower triangle. The square commutes by the bifunctoriality of the tensor.



Chasing around the diagram gives the desired result. □

Suppose that, in the preceding lemma, $\omega(1) \in A \otimes B$ is a normalized state, and $f : B \otimes C \to I$ is an effect, i.e, $0 \leq f \leq u_{BC}$. Then, in operational terms, the Lemma says that the mapping $\widehat{\omega} \circ \widehat{f}$ is represented, within the category $\mathcal{C}$, by the composite morphism $(f \otimes 1_C) \circ (1_A \otimes \omega)$.[9] In other words: preparing *BC* in joint state ω, and then measuring *AB* and obtaining *f*, guarantees that the "un-normalized conditional state" of C is $\widehat{\omega}(\widehat{f}(\alpha))$, where α is the state of A.

Remark 16 An important point here is that any process that factors as $\widehat{\omega} \circ \widehat{f}$ can be simulated by a remote evaluation protocol, using what amounts to *classical conditioning*–in particular, without need to invoke any mysterious "collapse" of the state, or for that matter, any other *physical* dynamics at all.

[8] This is closely related to the notion of *regular composite* introduced in [9].

[9] Technically we are relying on the isomorphisms between $A \cong \mathcal{C}(I, A)$ and $A^{\#} \cong \mathcal{C}(A, I)$ to guarantee that the internal representation of $\widehat{\omega} \circ \widehat{f}$ defines the right linear map.

4.2 Teleportation, Conditional Dynamics and Compact Closure

Suppose that, in the remote evaluation protocol of Lemma 15, we have $C = A$. Suppose further that the mapping $\widehat{\omega} : B^{\#} \to A$ has a right inverse — that is, suppose there exists a positive linear map $\widehat{r} : A \to B^{\#}$ such that $\widehat{\omega} \circ \widehat{r} = 1_A$. Then we can re-scale $\widehat{r}$ to obtain an effect f on $A \otimes B$ by

$$f(\alpha, \beta) = c\widehat{r}(\alpha)(\beta),$$

for a small enough positive constant c. Upon obtaining the result f in a measurement on $A \otimes B$ when the composite system is in state $\alpha \otimes \omega$, the un-normalized conditional state of C is:

$$\widehat{\omega}(\widehat{f}(\alpha)) = c\alpha.$$

The *normalized* conditional state will be exactly α. This is what is meant, in quantum-information theory, by a conclusive, correction-free *teleportation protocol*. Adopting this language, we will say that it is possible to *teleport system A through system B* if and only if there exists such a pair $\widehat{\omega}, \widehat{r}$.

If $\widehat{\omega}$ is in fact an isomorphism $A^{\#} \cong B$, then $\widehat{r} = \widehat{\omega}^{-1}$, and system B can also be teleported through system A. When this is the case, Lemma 15 tells us that $\omega : I \to B \otimes A$ and $f : A \otimes B \to I$ with $\widehat{f} = \widehat{\omega}^{-1}$, provide respectively a unit and co-unit making (B, ω, f) a dual for A. Thus, a compact closed category of COMs is exactly one in which every system A is paired with a second system $B = A'$, in such a way that each system can be teleported through the other.

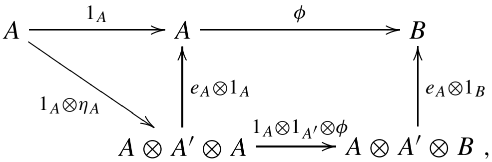
Proposition 17 *Let $\mathcal{C}$ be a monoidal category of COMs. The following are equivalent.*

(a) *$\mathcal{C}$ is compact closed.*
(b) *Every $A \in \mathcal{C}$ can be teleported through some $B \in \mathcal{C}$, which in turn can be teleported through A.*
(c) *Every morphism in $\mathcal{C}$ has the form $\widehat{\omega} \circ \widehat{f}$ for some bipartite state ω and bipartite effect f.*

Proof The equivalence of (a) and (b) is clear from the preceding discussion. To see that these are in turn equivalent to (c), suppose first that (a) holds, and let (A', η_A, e_A) be the dual for A. Suppose that $\phi : A \to B$ is a morphism in $\mathcal{C}$, and define $\omega_\phi = (1_{A'} \otimes \phi) \circ \eta_A$. By Remote Evaluation (Lemma 15), we have

$$\widehat{\omega_\phi}(\widehat{e_A}(\alpha)) \;=\; (e_A \otimes 1_B) \circ (1_A \otimes \omega_\phi) \circ \alpha$$

for every $\alpha \in \mathcal{C}(I, A)$. Since $\mathcal{C}$ is compact closed, the following diagram commutes:



and hence $\widehat{\omega_\phi}(\widehat{e_A}(\alpha)) = \phi(\alpha)$. Since $\mathcal{C}(I, A) \cong A$ we have $\phi = \widehat{\omega_\phi} \circ \widehat{e_A}$ as required. Conversely, if (c) holds, then for each A, the identity mapping 1_A factors as $\widehat{\omega}_A \circ \widehat{f_A}$ for some $\omega_A \in B \otimes A$ and some $f \in A \otimes B$. It follows that $\widehat{\omega}_A = \widehat{f_A}^{-1}$, so this gives us a compact closed structure. □

5 Weakly Self-Dual Theories

In a compact closed category $\mathcal{C}$, the internal adjoint $' : \mathcal{C} \to \mathcal{C}$ described in Section 3.1 establishes an isomorphism $\mathcal{C} \simeq \mathcal{C}^{op}$. In particular, for every object A in the category, understood as a "physical system", there is a distinguished isomorphism between the system's state space $\mathcal{C}(I, A)$ and the space $\mathcal{C}(A, I)$ of effects.

In contrast, a convex operational model A is not generally isomorphic to its dual. Indeed, there is a type issue: A has, by definition, a distinguished unit functional $u_A \in A^\#$; in order for $A^\#$ to be treated as a COM, one would need to privilege a *state* $\alpha_o \in A$ to serve as an order unit on $A^\#$. Only in special cases is there a natural way of doing so.[10] Beyond this, there is the more fundamental problem that, geometrically, the cones A_+ and $A^\#_+$ are generally not isomorphic. This said, those convex operational models that *are* order-isomorphic to their duals are of considerable interest – not only because both classical and quantum systems exhibit this sort of self-duality, but because it appears to be a strong constraint, in some measure *characteristic* of these theories.

5.1 Weak vs Strong Self-Duality

A finite-dimensional ordered vector space A (or its cone, A_+) is said to be *self-dual* iff there exists an inner product–that is, a positive-definite, hence also symmetric and non-degenerate, bilinear form $\langle\ ,\ \rangle$ – on A such that

$$A_+ = A^+ := \{a \in A | \langle a, x\rangle \geq 0\ \forall x \in A_+\}.$$

In this case, we have $A \simeq A^*$, as ordered spaces, via the canonical isomorphism $a \mapsto \langle a, .\rangle$. A celebrated theorem of Koecher and Vinberg [21, 28, 42] states that if A_+ is both self-dual and *homogeneous*, meaning that the group of order-automorphisms of A acts transitively on the interior of A_+, then A is isomorphic, as an ordered space, to a formally real Jordan algebra ordered by its cone of squares. The Jordan-von Neumann-Wigner [26] classification of such algebras then puts us within hailing distance of quantum mechanics.

Definition 18 A COM $(A, A^\#, u_A)$ is *weakly self-dual* (WSD) iff there exists an order-isomorphism $\phi : A \simeq A^\#$. We shall say that A is *symmetrically* self-dual iff ϕ can be so chosen that $\phi(\alpha)(\beta) = \phi(\beta)(\alpha)$ for all $\alpha, \beta \in A$.

[10] When the state space is sufficiently symmetric, there may be a natural choice of state invariant under the symmetry group. For example, if the base-preserving automorphisms act transitively on the pure states, the state obtained by group-averaging is the natural choice.

Note that, for a given linear map $\phi : A \to A^{\#}$, the bilinear form $\langle \alpha, \beta \rangle := \phi(\alpha)(\beta)$ is non-degenerate iff ϕ is a linear isomorphism, and symmetric iff $\phi = \phi^*$. If we don't require saturation, *any* finite-dimensional ordered linear space A can serve as the state space for a weakly self-dual COM, simply by setting $(A^{\#})_+ = \phi(A_+)$ for some nonsingular positive linear mapping $A \to A^*$, and taking any point in the interior of $(A^{\#})_+$ for u_A. However in the saturated case, weak self-duality is a real constraint on the geometry of the state cone, although strong self-duality is an even stronger one.

Note that, for a given linear map $\phi : A \to A^{\#}$, the bilinear form $\langle \alpha, \beta \rangle := \phi(\alpha)(\beta)$ is non-degenerate iff ϕ is a linear isomorphism, and symmetric iff $\phi = \phi^*$. Thus, A will be self-dual, in the classical sense described above, iff $\langle \; , \; \rangle$ is positive-definite, *and* $A^{\#} = A^*$, i.e, A is saturated. To emphasize the distinction, we shall henceforth refer to this situation as *strong* self-duality.

If $\phi : A \simeq A^{\#}$ is an order-isomorphism implementing A's self-duality, then $\phi^{-1} : A^{\#} \simeq A$ defines an un-normalized bipartite non-signaling state γ in $A \otimes_{max} A$ with $\phi^{-1} = \widehat{\gamma}$ —that is, $\gamma(a, b) = \phi^{-1}(a)(b)$. Following [12], we shall call such a state an *isomorphism state*. It is shown in [12] that such a state is necessarily pure in $A \otimes_{max} B$.[11] In this language, *A is WSD iff $A \otimes_{max} B$ contains an isomorphism state*.

Example 19 Let A be the convex operational model of a basic quantum-mechanical system, i.e., the space of self-adjoint operators associated with the system's Hilbert space $\mathbf{H}$. The standard *maximally entangled state* on $A \otimes A$ is the pure state associated with the unit vector

$$\Psi = \frac{1}{\sqrt{d}} \sum_i x_i \otimes x_i$$

where $\{x_1, ..., x_n\}$ is an orthonormal basis for $\mathbf{H}$. Using this, one has a mapping

$$R : T \mapsto R_T := (T \otimes 1) P_\Psi$$

taking operators $T : \mathbf{H} \to \mathbf{H}$ to operators $\mathcal{B}(\mathbf{H} \otimes \mathbf{H})$. It is a basic result, due to Choi and, independently, Jamiolkowski, that this is a linear isomorphism, taking the cone of completely positive maps on $\mathcal{B}(\mathbf{H})$ onto the cone of positive operators on $\mathbf{H} \otimes \mathbf{H}$. Note that R^{-1} maps [31] ρ to T_ρ where the latter is given by

$$\langle x | T_\rho(\sigma) y \rangle = d \mathrm{Tr}(\rho((|y\rangle\langle x|) \otimes \sigma^T))$$

where $|y\rangle\langle x|$ is the operator $z \mapsto \langle z, x \rangle y$, and the transpose is defined relative to the chosen orthonormal basis. This gives us a state $\gamma \in A \otimes A$ with $\widehat{\gamma} : A^* \simeq A$, namely,

$$\widehat{\gamma}(a)(b) = \mathrm{Tr}(P_\Psi(a \otimes b)) = \mathrm{Tr}(P_\Psi(a \otimes 1)(1 \otimes b)).$$

5.2 Categories of Self-dual COMs

Let $\mathcal{C}$ be a monoidal category of COMs, as described in Section 4. There is a distinction between requiring that a state space $A \in \mathcal{C}$ be weakly self-dual, which implies

[11] Strictly speaking, [12] deals with the case in which A and B are saturated, but the proof is easily extended to the general case.

only that there exist an order-isomorphism $A^{\#} \simeq A$ —an isomorphism in **Ordlin**— and requiring that this correspond to an (un-normalized) state $\gamma \in (AA)_+$, hence, to an element of $\mathcal{C}(I, A \otimes A)$. We now focus on categories in which this latter condition holds for every system.

Definition 20 A symmetric monoidal category $\mathcal{C}$ of COMs is *weakly self-dual* (WSD) iff for every $A \in \mathcal{C}$, there exists a pair (γ_A, f_A) consisting of a bipartite state $\gamma_A \in A \otimes A$ and a positive functional $f_A \in (A \otimes A)^{\#} = \mathcal{C}(A \otimes A, I)$ (a multiple of an effect) such that (i) γ_A is an isomorphism state, and (ii) $\widehat{f_A} = \widehat{\gamma_A}^{-1}$. If γ_A can be chosen to be symmetric for every $A \in \mathcal{C}$, we shall say that $\mathcal{C}$ is *symmetrically self-dual* (SSD).

Note that this is stronger than merely requiring every COM $A \in \mathcal{C}$ to be weakly self-dual. Equivalently, we may say that category $\mathcal{C}$ of COMs is WSD iff every system can be equipped with a designated conclusive, correction-free teleportation protocol $\gamma_A \in A \otimes A$, $f_A \in (A \otimes A)^{\#}$, whereby A can be teleported "through itself". Thus, by Proposition 17, we have:

Theorem 21 *A monoidal category $\mathcal{C}$ of convex operational models is weakly self-dual iff it is compact closed, and can be equipped with a compact structure such that $A' = A$ for all objects $A \in \mathcal{C}$.*

Recall that any morphism ϕ in a compact closed category has a categorial adjoint, ϕ'. In the context of a WSD category of COMs, this has a useful interpretation in terms of the linear adjoint, ϕ^*:

Lemma 22 *Let $\mathcal{C}$ be any WSD category of ASPs, regarded as compact closed as above. Let $\phi : A \to B$. Then the canonical adjoint mapping $\phi' : B \to A$ is given by*

$$\phi' = (\widehat{f_B} \circ \phi \circ \widehat{\gamma_A})^* = \widehat{\gamma}_A^* \circ \phi^* \circ \widehat{f}_B^*$$

Proof We must show that, for any $\beta \in B$ – that is, any $\beta \in \mathcal{C}(I, B)$ – we have $\phi'(\beta) := \phi^* \circ \beta = \widehat{\gamma}_A^*(\phi^*(\widehat{f}_B^*(\beta)))$, where $\phi^* : B^{\#} \to A^{\#}$ is the *linear* adjoint. Let $\omega := (1_A \otimes \phi) \circ \gamma_A : I \to A \otimes B$. Then we have

$$\begin{aligned}
\phi'(\beta) &= \phi' \circ \beta \\
&= (1_A \otimes f_B) \circ (1_A \otimes \phi \otimes 1_B) \circ (\gamma_A \otimes 1_B) \circ \beta \\
&= (1_A \otimes f_B) \circ (1_A \otimes \phi \otimes \beta) \circ \gamma_A \\
&= (1_A \otimes (f_B \circ (1_B \otimes \beta))) \circ ((1_A \otimes \phi) \circ \gamma_A) \\
&= (1_A \otimes \widehat{f}^*(\beta)) \circ \omega) \\
&= \widehat{\omega}^*(\widehat{f}_B^*(\beta)).
\end{aligned}$$

Now, for any $b : B \to I$, we have

$$\begin{aligned}\widehat{\omega}^*(b) &= (\widehat{\sigma_{B,B} \circ \omega})(b)\\ &= (1_A \otimes b) \circ \omega\\ &= (1_A \otimes b) \circ (1_A \otimes \phi) \circ \gamma_A\\ &= (1_A \otimes (b \circ \phi)) \circ \gamma_A\\ &= (1_A \otimes \phi^*(b)) \circ \gamma_A\\ &= \widehat{\gamma}_A^*(\phi^*(b)).\end{aligned}$$

With $b = \widehat{f}_B^*(\beta)$, this gives the desired result. □

Corollary 23 *For all $A \in \mathcal{C}$, $f_A' = \sigma_{A,A} \circ \gamma_A$.*

Proof Note first that $f_A^*(1) = f_A \in (A \otimes A)^\#$. Thus, the preceding Lemma gives us

$$f_A'(1) = (\widehat{\gamma}_{A\otimes A}^* \circ f_A^* \circ \widehat{f}_I^*)(1)$$

Since $f_I = f_I^* = 1_I$, we have

$$f_A'(1) = (\widehat{\gamma}_A^* \otimes \widehat{\gamma}_A^*)(f_A).$$

Thus, for every $a, b \in A^\#$, we have

$$\begin{aligned}f_A'(1)(a, b) &= (\widehat{\gamma}_A^* \otimes \widehat{\gamma}_A^*)(f_A)(a, b)\\ &= f_A(\widehat{\gamma}_A(a), \widehat{\gamma}_A(b))\\ &= \widehat{f}_A^*(\widehat{\gamma}_A(b))(\widehat{\gamma}_A^*(a))\\ &= b(\widehat{\gamma}_A^*(a))\\ &= \gamma_A(b, a)\\ &= (\sigma \circ \gamma_A)(a, b).\end{aligned}$$

□

Theorem 24 *Let $\mathcal{C}$ be a symmetrically self-dual monoidal category of COMs. Then $\mathcal{C}$ is dagger compact with † given by the canonical adjoint $'$: $\mathcal{C}^{op} \to \mathcal{C}$.*

Proof By Theorem 21, $\mathcal{C}$ is degenerate compact closed, with a compact structure on $A \in \mathcal{C}$ given by (A, γ_A, f_A), where γ_A is symmetric. Define $(\cdot)^\dagger : \mathcal{C}^{\rm op} \to \mathcal{C}$ by $(\cdot)^\dagger = (\cdot)'$; then $(\cdot)^\dagger$ is a monoidal functor which is the identity on objects. Since $\sigma_{A,B}' = \sigma_{A,B}^{-1}$ in any compact closed category, $\mathcal{C}$ is dagger-monoidal. That $f_A^\dagger = \sigma_{A,A} \circ \gamma_A$ is immediate from Corollary 23. □

Lemma 25 *For all A, let $\tau_A : A \to A$ be the order-isomorphism given by*

$$\tau_A := \widehat{\gamma}_A \circ \widehat{f}_A^*.$$

Then, for all $\phi \in \mathcal{C}(A, B)$, we have

$$\phi'' = \tau_B^{-1} \circ \phi \circ \tau_A.$$

Proof Notice first that $\tau_B^{-1} = (\widehat{\gamma}_B \circ \widehat{f}_B^*)^{-1} = \widehat{\gamma}_B^* \circ \widehat{f}_B$. By Lemma 22, we have

$$\begin{aligned}\phi'' = (\widehat{f}_A \circ \phi' \circ \widehat{\gamma}_B)^* &= (\widehat{f}_A \circ (\widehat{f}_B \circ \phi \circ \widehat{\gamma}_A)^* \circ \widehat{\gamma}_B)^* \\ &= \widehat{\gamma}_B^* \circ (\widehat{f}_B \circ \phi \circ \widehat{\gamma}_A)^{**} \circ \widehat{f}_A^* \\ &= (\widehat{\gamma}_B^* \circ \widehat{f}_B) \circ \phi \circ (\widehat{\gamma}_A \circ \widehat{f}_A^*) \\ &= \tau_B^{-1} \circ \phi \circ \tau_A\end{aligned}$$

□

Corollary 26 *Let $\phi : A \to B$ with $\phi'' = \phi$. Then $\phi \circ \tau_A = \tau_B \circ \phi$.*

Theorem 27 *For any object A in a weakly self-dual category $\mathcal{C}$ of convex operational models, the following are equivalent: (i) $\phi'' = \phi$ for all $\phi \in \mathcal{C}(A, A)$, (ii) $\tau_A = 1_A$, and (iii) f_A and γ_A are symmetric as a bilinear forms.*

Proof (i) implies (ii): From (i), and the fact that the morphisms in $\mathcal{C}(A, B)$ are a basis for $\mathcal{L}(A, B)$, $(\cdot)''$ is the identity map. Let E_i, F_j be bases of the spaces $\mathcal{L}(A, A), \mathcal{L}(B, B)$ of linear maps on vector spaces *A,B* respectively. Then the maps $X \mapsto F_j X E_i$, where $X \in \mathcal{L}(A, B)$, are a basis for the space of linear maps from $\mathcal{L}(A, B)$ to itself. Using this fact, we can expand the map $(\cdot)'' : \phi \mapsto \tau_B^{-1} \circ \phi \circ \tau_A$, which is a map from $\mathcal{L}(A, B)$ to itself, in a basis $M_{ij} : \phi \mapsto F_j X E_i$ where $E_0 = 1_B, F_0 = 1_A$. By the uniqueness of expansions in bases and the fact that $(\cdot)''$ is the identity map, we get $\tau_A = 1_A, \tau_B = 1_B$.
(ii) implies (iii): By (ii) we have $\tau_A := \widehat{\gamma}_A \circ \widehat{f}_A^* = 1_A$, so $\widehat{f}_A^* = \widehat{\gamma}_A^{-1} = \widehat{f}_A$. Since $f(a, b) \equiv \widehat{f}(a)(b) \equiv \widehat{f}^*(b)(a)$, $\widehat{f}_A^* = \widehat{f}_A$ is equivalent to symmetry of f_A. Symmetry of γ_A then follows from the fact that $\gamma_A = f_A^{-1}$.
(iii) implies (i): If f_A (hence, also γ_A) are symmetric, then we have $f_A^* = f_A$, whence, $\tau_A = \gamma_A \circ \widehat{f}_A = 1_A$; thus, by Lemma 25, $\phi'' = \phi$ for all $\phi \in \mathcal{C}(A, A)$. □

Applying Theorem 24, we now have the

Corollary 28 *A WSD monoidal category of COMs is dagger compact with respect to the canonical adjoint $'$: $\mathcal{C}^{op} \to \mathcal{C}$, if and only if it is* symmetrically *self-dual.*

Theorem 27 tells us that if $\mathcal{C}$ is a *saturated* weakly self-dual theory in which $'$ is an involution, then the interior of A_+ is a domain of positivity in the sense of Koecher [30]. If we further suppose that every irreducible state space in $\mathcal{C}$ is *homogeneous*, meaning that $G(A_+)$ acts transitively on the interior of A_+ for every $A \in \mathcal{C}$ (a condition one can motivate physically in several ways, e.g., [12, 45]), then we are close to requiring that every state space in $\mathcal{C}$ be a formally real (also called Euclidean) Jordan algebra [28, 42]. This line of thought will be pursued in a sequel to this paper.

6 Conclusion

We began with the observation that compact closure, and still more, dagger compactness, represent strong constraints on a physical theory, thought of as a symmetric monoidal category of "systems" and "processes". Our results cast some light on the operational (or, if one prefers, the physical) content of these assumptions in the concrete—but still very general—context of probabilistic (or convex operational) theories. In particular, we have seen that, in probabilistic theories *qua* categories of COMs, compact closure amounts to the condition that all processes–that is, all dynamics–can be induced by the kind of conditioning that occurs in a teleportation-like protocol. Indeed, in such a theory, a process between systems A and B *amounts to* a choice of bipartite state on $A \otimes B$. Finally, we have established that for weakly self-dual theories *symmetric* weak self-duality implies the existence of a dagger compatible with the compact closed structure. As in the special case of quantum mechanics, a dagger amounts to reversing the order of conditioning.

Several directions for further study suggest themselves. It would be interesting to identify necessary and sufficient conditions for the COM representations discussed in Section 4.1 to yield *finite dimensional* models–and, equally, one would like to know how far the other results of Sections 4 and 5 extend to infinite-dimensional systems. Our definition of category of weakly self-dual state spaces assumes the existence of a state that induces, by conditioning, an isomorphism from the state cone to the effect cone, and an effect inducing its inverse; but as we noted, it would be interesting to investigate conditions under which this follows just from weak self-duality of the objects. As mentioned at the end of Section 5, the consequences of homogeneity of the state-spaces should also be explored.

Perhaps the most urgent task, though, is to identify operational and category-theoretic conditions equivalent to the strong self-duality of a probabilistic theory.[12]

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References

1. Abramsky, S., & Coecke, B. (2004). A categorical semantics of quantum protocols. In *Proceedings of the 19th annual IEEE symposium on logic in computer science: LICS 2004* (pp. 415–425). IEEE Computer Science Press. Also arXiv:0402130v5 [quant-ph/].
2. Abramsky, S., & Coecke, B. (2008). Categorical quantum mechanics. In K. Engesser, D. Gabbay, D. Lehman (Eds.), *Handbook of quantum logic and quantum structures II*. Amsterdam: Elsevier.

[12] See [37, 46] for some recent progress on this question.

3. Abramsky, S., & Duncan, R. (2006). A categorical quantum logic. *Mathematical Structures in Computer Science*, *16*, 486–489.
4. Alfsen, E.M. (1971). *Compact convex sets and boundary integrals*. Springer.
5. Alfsen, E.M., & Shultz, F.W. (2002). *State spaces of operator algebras: basic theory, orientations, and C^*-products*. Boston: Birkhauser.
6. Araki, H. (1980). On a characterization of the state space of quantum mechanics. *Communications in Mathematical Physics*, *75*(1980), 1–24.
7. Baez, J. (2004). Quantum quandaries: a category-theoretic perspective. quant-ph/0404040.
8. Barnum, H., Barrett, J., Leifer, M., Wilce, A. (2007). Generalized no-broadcasting theorem. *Physical Review Letters*, *99*, 24051.
9. Barnum, H., Barrett, J., Leifer, M., Wilce, A. (2008). Teleportation in general probabilistic theories. In S. Abramsky & M. Mislove (Eds.), *Mathematical foundations of information flow.* American Mathematical Society. 2012 also arXiv:0805.3553.
10. Barnum, H., Dahlsten, O., Leifer, M., Toner, B. (2008). Nonclassicality without entanglement enables bit commitment. In *Proc. IEEE information theory workshop, Porto, May 2008* (pp. 386–390). Also arXiv: 0803.1264.
11. Barnum, H., Fuchs, C., Renes, J., Wilce, A. (2005). Influence-free states on compound quantum systems. quant-ph/0507108.
12. Barnum, H., Gaebler, P., Wilce, A. (2009). Ensemble steering, weak self-duality and the structure of probabilistic theories. arXiv:0912.5532.
13. Barnum, H., & Wilce, A. (2011). Information processing in convex operational theories. *Electronic Notes in Theoretical Computer Science*, *270*, 3–15. Also arXiv:0908.2352.
14. Barnum, H., & Wilce, A. (2009). Ordered linear spaces and categories as frameworks for information-processing characterizations of quantum and classical theory. arXiv:0908.2354.
15. Barrett, J. (2007). Information processing in general probabilistic theories. *Physical Reviews A*, *75*, 032304. Also arXiv:0.508211 [quant-ph/].
16. Beltrametti, E., & Cassinelli, G. (1980). *The logic of quantum mechanics*. Academic Press.
17. Coecke, B., Paquette, E.O., Pavlovic, D. (2009). Classical and quantum structuralism. In I. Mackie & S. Gay (Eds.), *Semantic techniques for quantum computation* (pp. 29–69). Cambridge University Press.
18. d'Ariano, G.M. (2006). How to derive the Hilbert-space formulation of quantum mechanics from purely operational axioms. arXiv.org:0603011 [quant-ph/].
19. Davies, E.B., & Lewis, J.T. (1970). An operational approach to quantum probability. *Communications in Mathematical Physics*, *17*, 239–260.
20. Edwards, C.M. (1970). The operational approach to algebraic quantum theory I. *Communications in Mathematical Physics*, *16*, 207–230.
21. Faraut, J., & Korányi, A. (1994). *Analysis on symmetric cones*. Oxford: Oxford University Pres.
22. Gross, D., Müller, M., Colbeck, R., Dahlsten, O.C.O. (2010). All reversible dynamics in maximally nonlocal theories are trivial. *Physical Reviews and Letters*, *104*, 080402.
23. Hanche-Olsen, H. (1983). On the structure and tensor products of JC algebras. *Canadian Journal of Mathematics*, *35*, 1059–1074.
24. Hanche-Olsen, H. (1985). JB-algebras with tensor products are C^*-algebras. *Lecture Notes in Mathematics (Springer)*, *1132*, 223–229.
25. Heunen, C. (2009). *Categorical quantum models and logics*. Doctoral thesis, Radboud University, Nijmegen. Amsterdam: Pallas Publications.
26. Jordan, P., von Neumann, J., Wigner, E.P. (1934). On an algebraic generalization of the quantum-mechanical formalism. *Annals of Mathematics*, *35*, 29–64.
27. Kläy, M. (1988). Einstein-Podolsky-Rosen experiments: the structure of the sample space. *Foundations of Physics Letters*, *1*, 205–244.
28. Koecher, M. (1958). Die geodätischen von Positivitätsbereichen. *Mathematische Annalen*, *135*, 192–202.
29. Koecher, M. (1962). On real Jordan algebras. *Bulletin of the American Mathematical Society*, *68*, 374–377.
30. Koecher, M. (1999). The minnesota notes on Jordan algebras and their applications. In A. Krieg & S. Walcher (Eds.), *Physical Review Letters* (Vol. 1710). Springer Verlag.
31. Keyl, M., & Werner, R. (2007). Channels and maps. In D. Bruss & G. Leuchs (Eds.), *Lectures on quantum information* (pp. 73–86). Wiley.

32. Kelly, G.M., & Laplaza, M.L. (1980). Coherence for compact closed categories. *Journal of Pure and Applied Algebra*, *19*, 193–213.
33. Klay, M., Randall, C.H., Foulis, D.J. (1987). Tensor products and probability weights. *International Journal of Theoretical Physics*, *26*, 199–219.
34. Ludwig, G. (1983). *Foundations of quantum mechanics I*. Springer.
35. Mackey, G. (1963). *Mathematical foundations of quantum mechanics*. Addison-Wesley.
36. MacLane, S. (1997). *Categories for the working mathematician*, 2nd ed. Springer.
37. Mueller, M., & Ududec, C. (2012). *Physical Reviews and Letters*, *108*, 130401. Also arXiv:1110.3516.
38. Selinger, P. (2007). Dagger compact closed categories and completely positive maps (extended abstract). *Electronic Notes in Theoretical Computer Science*, *170*, 139–163.
39. Selinger, P. (2004). Towards a semantics for higher-order quantum computation. In *Proceedings of the 2nd international workshop on quantum programming languages, Turku, Finland* (pp. 127–143). TUCS General Publication No. 33.
40. Selinger, P. (2010). Autonomous categories in which $A \cong A^*$. In *Proceedings of the 7th international workshop on quantum physics and logic (QPL 2010)* (pp. 151–160). Oxford. Available electronically at: www.cs.ox.ac.uk/people/bob.coecke/QPLproceedings.html.
41. Short, A.J., & Barrett, J. (2601). Strong non-locality: a tradeoff between states and measurements. arXiv:0909.2601.
42. Vinberg, E.B. (1960). Homogeneous cones. *Doklady Akademii Nauk SSSR*, *141*, 270–273. English translation: *Soviet Mathematics Doklady*, *2*, 1416–1619 (1961).
43. von Neumann, J. (1932). *Mathematische Grundlagen der Quantenmechanik*. Berlin: Springer. English translation: *Mathematical foundations of quantum mechanics*. Princeton: Princeton University Press (1955).
44. Wilce, A. (1992). Tensor products in generalized measure theory. *International Journal of Theoretical Physics*, *31*, 1915.
45. Wilce, A. (2012). Four and a half axioms for finite dimensional quantum mechanics In Y. Ben-Menachem & M. Hemmo (Eds.), *Probability in physics*. Springer.
46. Wilce, A. (2012). Conjugates, correlation and quantum mechanics. arXiv:1206.2897.