J Philos Logic (2014) 43:333–363
DOI 10.1007/s10992-012-9265-z

The Fundamental Theorem of World Theory

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Received: 25 April 2012 / Accepted: 17 December 2012 / Published online: 30 January 2013

Abstract The fundamental principle of the theory of possible worlds is that a proposition p is possible if and only if there is a possible world at which p is true. In this paper we present a valid derivation of this principle from a more general theory in which possible worlds are defined rather than taken as primitive. The general theory uses a primitive modality and axiomatizes abstract objects, properties, and propositions. We then show that this general theory has very small models and hence that its ontological commitments—and, therefore, those of the fundamental principle of world theory—are minimal.

Keywords Possible worlds · Modality · Modal logic · Object theory

1 Introduction

The fundamental principle of the theory of possible worlds can be expressed as follows, where p stands for a sentence or proposition:

The Equivalence Principle
It is possible that p if and only if there is a possible world at which p is true.

The left-to-right direction of the Equivalence Principle effectively requires that every metaphysical possibility is realized at some world. It therefore constitutes a sort of

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plenitude principle that ensures there are "no gaps in logical space...where a world might have been, but isn't" (Lewis [8], 86). In the presence of modal claims such as that there might have been talking donkeys or that there might have been million carat diamonds, the left-to-right direction guarantees the existence of worlds where *there are* talking donkeys or million carat diamonds. This direction, therefore, allows one to derive the existence of non-actual possible worlds from claims of the form: p is false but possibly true.[1] The right-to-left direction of the Equivalence Principle seals the connection between worlds and possibilities by ensuring that anything true at some world is in fact a genuine metaphysical possibility.

We can express the Equivalence Principle in a formally precise way if we use the modal language of a hybrid logic containing primitive symbols $p, q, \ldots$, a necessity operator ($\Box$), variables w, v, ... ranging over worlds, and sentences of the form '$w \models p$' that assert *p is true at w*.[2] For the moment, it doesn't really matter whether the symbols $p, q, \ldots$ are sentence letters or variables ranging over propositions. What matters is that statements of the form $w \models p$ are governed by an axiom of *Coherence* which asserts that the negation of p is true at w if and only if it is not the case that p is true at w:

Co $(w \models \neg p) \leftrightarrow (\neg w \models p)$

If we now add to this basis the usual definition of the possibility operator '$\Diamond$', we can then express the Equivalence Principle formally as follows:

EP $\Diamond p \leftrightarrow \exists w(w \models p)$

Note that, given Coherence and some basic modal and propositional logic, the Equivalence Principle is equivalent to:

The Leibniz Principle
It is necessary that p if and only if p is true at every possible world.

More formally, in terms of the language at hand:

LP $\Box p \leftrightarrow \forall w(w \models p)$

[1]Assuming, of course, (a) that whenever q is true at w but not at w', then $w \neq w'$, and (b) that whenever q is false, then q is false at the actual world w^*. Given these assumptions, we can prove that there are nonactual possible worlds if we consider some false but possibly true sentence or propositions p. Since p is possibly true, there is a world, say w_1, at which it is true, by the left-to-right direction of the Equivalence Principle. But since p is false, then by (b), it is false at w^*. So by (a), w_1 isn't the actual world w^*.

[2]In this paper, the symbol '$\models$' is used both as a metalinguistic symbol with its usual model theoretic meaning as well as an object language symbol indicating truth at a world. It is always abundantly clear from the surrounding context which is intended.

Given this equivalence between the two principles,[3] one can take either principle as a basic axiom and derive the other. In what follows, however, our focus will be on **EP** rather than **LP**, as the former involves an explicit existence claim about possible worlds that is independent of Coherence.

One of the most interesting and important philosophical questions is: Independent of any particular modal beliefs about what is or is not possible, what are the ontological commitments of **EP**? Since **EP** does not wear its commitments on its sleeve, a natural way of approaching the question is to reframe it thus: What are the smallest models in which **EP** is true?

If the variables $p, q, \ldots$ are interpreted as sentential letters for which one can substitute complex sentences φ, then it is already known that **EP** is true in any model of any standard language for a hybrid modal logic that contains a modal operator and world quantifiers (see, e.g., [3]). Taking the semantic values of sentences as usual to be sets of possible worlds, all that is needed is a single primitive possible world $\mathbf{w}$ so that $\{\mathbf{w}\}$ can serve as the value of every true sentence and the empty set $\varnothing$ as the value of every false sentence. Sentences $w \models \varphi$ are then interpreted to be true just in case the semantic value of 'w' is a member of the semantic value of φ. So endorsing **EP** commits one only to an ontology with a single possible world, although of course the domain of worlds might grow significantly if we add our modal beliefs as assumptions to the logic.

When the symbols $p, q, \ldots$ are interpreted as variables ranging over propositions, then the smallest models in which **EP** is true include a domain of propositions. Of course, if the domain of propositions is allowed to be empty, then since **EP** is, under this interpretation, an (implicit) universally quantified claim, it would be vacuously true. The smallest model in which **EP** is non-vacuously true requires a domain with just two propositions $\mathbf{p}$ and $\neg\mathbf{p}$ (assuming the domain of propositions is closed under negation). We can then easily construct a 3-element model of **EP** containing two propositions and one possible world $\mathbf{w}$: take $\mathbf{p}$ to be true at $\mathbf{w}$ (and hence $\neg\mathbf{p}$ to be false at $\mathbf{w}$) and the extension of '$\models$' to be $\{\langle \mathbf{w}, \mathbf{p} \rangle\}$.

Consequently, no matter how we interpret the symbols $p, q, \ldots$, the ontological commitments of the Equivalence Principle *per se* are meager. This, indeed, is part of the philosophical attraction of the principle. It expresses a fundamental relationship between possibilities and worlds that when spelled out formally, doesn't entail any significant ontological claims in the absence of the data (i.e., in the absence of our modal beliefs about what is possibly true).

[3] Here is a derivation of **EP** from **LP**:

1. $\Box\neg p \leftrightarrow \forall w(w \models \neg p)$	Instance of **LP**, with $\neg p$ substituted for p.
2. $\Box\neg p \leftrightarrow \forall w \neg(w \models p)$	From 1 and **Co**.
3. $\neg\Box\neg p \leftrightarrow \neg\forall w \neg(w \models p)$	From 2 by basic propositional logic.
4. $\Diamond p \leftrightarrow \exists w(w \models p)$	From 3 and the interdefinability of $\Box/\Diamond$, $\forall/\exists$.

To show the converse and, hence, that **EP** is equivalent to **LP**, substitute $\neg p$ for p in **EP** and follow reasoning similar to the above.

However, there are two ways in which one can endorse the Equivalence Principle. The first way is to take the Equivalence Principle in one of its forms as fundamental or axiomatic. Thus far, we've been examining the ontological implications of such a position. The second way is to *derive* the Equivalence Principle *as a theorem* from a more general theory in which possible worlds are defined rather than primitive. Our interest in what follows is in examining the resources needed to do this. Note that we are not talking about deriving the Equivalence Principle from one of its equivalent forms; nor are we talking about deriving it from axioms that already quantify over primitive possible worlds. Rather, what interests us here is finding *more fundamental* principles that imply the Equivalence Principle in one of its forms as a consequence. If that can be done, then the focus of the question of ontological commitment moves from **EP** proper to the more general theory.

Most possible world theorists take one of the above forms of the Equivalence Principle as basic and give no thought whatsoever to the idea of deriving it as a theorem of a more general theory. Thus, Kripke [6, 7] takes the Leibniz Principle as the fundamental insight underlying his interpretation of modal languages with sentential letters and alethic modal operators. But he doesn't introduce a hybrid language containing both modal operators and quantifiers over worlds in the attempt to derive the Leibniz Principle. Lewis asserts the left-to-right direction of the Equivalence Principle using 'ways a world could be' instead of propositions, for he says "absolutely every way that a world could possibly be is a way that some world is" ([8], 2, 71, 86). But there is no derivation of this principle from his other principles; rather, as Lewis acknowledges (87), under his identification of ways worlds could be with worlds themselves, the principle is rendered trivial.[4]

Most other philosophers who work with possible worlds take some form of the Equivalence Principle to be such a truism that they rarely bother to explicitly endorse it, much less attempt to derive it. This is true, for example, of almost all of the abstractionists about possible worlds, such as Adams [1], Plantinga [15, 44–46], Stalnaker [20], Chisholm [5], Pollock [17], Prior [18], and Sider [19, 299]. A notable exception is the attempted derivation in Plantinga [16] though, unfortunately, his attempt failed in various ways.[5] The basic problem for the abstractionists about worlds is that, in order to prove the existence of the actual world, one has to ensure the existence of

[4] More specifically, it becomes the principle that every world is identical with some world. Likewise it is rendered trivial if ways are identified with propositions, which in turn are identified with sets of worlds—it becomes the principle that every nonempty set of worlds is identical with some nonempty set of worlds. By contrast, the Equivalence Principle seems to postulate a substantive connection between genuine metaphysical possibilities and the existence of possible worlds, and what makes the connection between the two substantive is their conceptual independence.

[5] Plantinga's attempted derivation rests on: (a) an unspecified theory of propositions that includes at least one strong existence principle (namely, that for any set S of propositions, there is a proposition, $\wedge S$, that is the *conjunction* of the propositions in S); (b) no formal identity conditions for propositions, which in particular means there is no guarantee that there is a unique actual world [9]; (c) a fragment of set theory that includes the axioms of Pairing, Union, and Choice (which entail an infinite ontology of sets); (d) the (highly problematic) thesis that for any proposition p, there is a set A_p of propositions that are possible and entail p; and (e) an unjustified modal principle (namely, that the conjunction $\wedge B$ of any "maximal" chain B of propositions in A_p is possible). For further details regarding (c)–(e) see Menzel [11].

some sort of construct—a large conjunction or set of propositions, for example—that implies all and only the true propositions. And to ensure that there is a distinct possible world corresponding to each distinct possibility, one has to have a mechanism in place for generating similar constructs, each of which implies all and only those propositions that would have been true had things been otherwise in some way. As soon as one asserts principles strong enough to guarantee the existence of such constructs, there are issues to confront: in the case of conjunctive propositions, issues about the existence and identity of such propositions, and in the case where sets are employed, issues concerning the strength of the set-theoretic principles needed, such as whether they commit one to an infinite domain or raise the specter of Russellian paradoxes concerning sets of propositions.[6]

To the best of our knowledge, the literature contains only one successful attempt to prove **EP**. Using the resources of his theory of abstract objects, Zalta [21, 84] derives **LP** and, in [22] (109), offers a one-line proof sketch of **EP** as a corollary to **LP**, citing only "contraposition and modal negation". It should be noted that in those theorems, the symbols $p, q, \ldots$ were construed as propositional variables, not sentence letters.

However, in the works just cited, several interesting research issues are not addressed:

- No direct proof of **EP** is ever developed, and the proof of **LP** is, at best, a sketch that takes some shortcuts.
- The derivation of **LP** takes place in a context in which the full resources of the theory of abstract objects are available—no attempt is made to isolate only those resources needed for the proof of **EP**, thus leaving open the question of which minimal group of axioms are needed for a direct proof of **EP**.
- No attempt is made to identify the smallest model of those axioms needed for the proof of **EP**, thus leaving open the question of the minimum ontological commitments of the theory.

The goals of the present paper, therefore, are to improve and advance this research in several ways:

- We produce a direct proof of **EP**, in which the symbols $p, q, \ldots$ are interpreted as variables ranging over propositions.
- We extract from the proof of **EP** a list of only the axioms required for the proof.
- We develop a minimal theory based upon those axioms and investigate the smallest models of the theory, thereby identifying its minimal ontological commitments.

These results prepare the ground for future research. For one of the fundamental questions of the theory of possible worlds is, what is the epistemological justification for the Equivalence Principle? Though our attempt to answer this question will be reserved for another occasion, the present investigations will enhance one's ability

[6]See Menzel [14]. See also Chihara [4], in particular regarding the significant threat of paradox implicit in Plantinga's world theory.

to develop an answer and evaluate the various alternatives. The developments in this paper will bring out into the open the minimal resources needed for a proof of **EP**. When such resources are clarified, philosophers will be able to compare the present approach to the theory of possible worlds with that of others.

2 Object Theory and Possible Worlds

Our derivation of **EP** will be presented in detail in Section 3. But since we already know what axioms are used in the derivation, we present in this section only the the core theory containing those axioms (and the language and definitions needed to express them). This theory constitutes a monadic subtheory of the axiomatic theory of abstract objects of Zalta [21, 23]. For purposes here, the theory divides naturally into two parts, a logical core, which we will refer to as *monadic object theory*, or *MOT*, for short, and the addition of a comprehension schema (**OC**) to this core. The theory MOT + **OC** is called MOTC. In Section 4, we construct models that reveal the minimal ontological commitments of MOTC by laying out its model theory and showing that the theory has very small models.

2.1 The Languages of Monadic Object Theory

Languages for MOT A language $\mathcal{L}$ for MOT contains the usual logical apparatus of monadic second-order quantified modal logic including the logical operators $\neg$, $\rightarrow$, $\forall$, and $\Box$ and denumerably many individual variables, denumerably many 0-place predicate variables, and denumerably many 1-place predicate variables; the operators $\wedge$, $\vee$, $\leftrightarrow$, $\exists$, and $\Diamond$ are defined as usual. Informally, the three classes of variable range over objects, propositions, and properties, respectively. The actual shapes of these variables are irrelevant; the metavariables x, y, and z (possibly with numerical subscripts) will range over individual variables; likewise p, q, r, and F, G, H will range over 0- and 1-place predicates, respectively. Lower case Greek letters may be used as metavariables as well, typically when a variable is needed to range over more than one syntactic class. Additionally, $\mathcal{L}$ contains a distinguished 1-place predicate constant $A!$ which, intuitively, expresses the property of being an abstract object. $\mathcal{L}$ may also contain any (countable) number of individual constants and individual predicates which, for purposes here, we will indicate by means of the lower case metavariables a and P. (We will also use boldface variables for semantic entities, but these latter don't make an appearance until Section 4.) Henceforth we shall assume that $\mathcal{L}$ refers to a specific language for MOT.

A Grammar for the Languages In addition to this more or less standard lexicon, the grammar for $\mathcal{L}$ introduces a rich array of complex predicates that, intuitively, denote logically complex propositions and properties. However, only those formulas deemed *predicable* can be used to form such predicates—indeed, such formulas will themselves serve as the complex 0-place predicates; more standard λ-notation will be used to form complex 1-place predicates, where some notion of variable binding is needed. Object theory also introduces a new primitive mode of predication, called *encoding*. Like exemplification, encoding is not expressed by means of an explicit predicate but structurally by means of a new type of atomic formula; specifically, in addition to familiar formulas like Fx, $\mathcal{L}$ also includes formulas like xF. Those of the former sort can be read as "x exemplifies F" and those of the latter sort as "x encodes F".[7] These features force us to define the grammar for $\mathcal{L}$ rather more delicately than for most standard higher-order languages; notably, our grammar must define six notions—*term*, *predicate*, *formula*, *predicable*, *subformula*, and *free in*—by means of a simultaneous recursion. As the last two are ancillary only, we separate the clauses in their definition from those of the first four for the sake of readability.

1. Every individual constant or individual variable is a *term*.
   - If x is an individual variable, then the occurrence of x itself in x is *free in* x.
2. Every 0-place (1-place) predicate variable or predicate constant π is a 0-place (1-place) (*primitive*) *predicate*.
   - If π is a predicate variable, then the occurrence of π itself in π is *free in* π.
3. If π is a 0-place primitive predicate, then π is an (*atomic*) *formula* and π is *predicable*. If τ is a term and π is a 1-place predicate, then $\pi\tau$ and $\tau\pi$ are (atomic) formulas and $\pi\tau$ is predicable.
   - If τ is an individual variable, then (i) the rightmost occurrence of τ in $\pi\tau$ is free in $\pi\tau$ and the leftmost occurrence of τ is free in $\tau\pi$, and (ii) every free occurrence of a variable in π is a free occurrence in $\pi\tau$ and $\tau\pi$.
   - Every formula is a *subformula* of itself.
4. If φ is predicable, then φ is a 0-place predicate.
5. If φ and ψ are (predicable) formulas, then $\neg\varphi$, $\Box\varphi$, and $(\varphi \rightarrow \psi)$ are (predicable) formulas.[8]
   - Every occurrence of a variable that is free in φ is free in $\neg\varphi$ and $\Box\varphi$; likewise, every occurrence of a variable that is free in φ or ψ is free in $(\varphi \rightarrow \psi)$.
   - φ and its subformulas are subformulas of $\neg\varphi$ and $\Box\varphi$; φ and ψ and their subformulas are subformulas of $(\varphi \rightarrow \psi)$.
6. If φ is a formula and α any variable, then $\forall\alpha\varphi$ is a (*quantified*) formula. If in addition (i) φ is predicable, (ii) α is an individual variable and (iii) there is no free occurrence of α in any 1-place predicate occurring in φ or in any quantified subformula of φ, then $\forall\alpha\varphi$ is predicable.
   - Every free occurrence of a variable other than α in φ is a free occurrence of that variable in $\forall\alpha\varphi$.
   - φ and its subformulas are subformulas of $\forall\alpha\varphi$.

[7] In full object theory with n-place predicates and n-place exemplification formulas, encoding formulas are always monadic and the predicate in a well-formed encoding formula is always unary.

[8] We will follow standard practice and drop outer parentheses when the conditional is the main connective.

7. If (i) φ is predicable, (ii) x is any individual variable, and (iii) there is no free occurrence of x either in any 1-place predicate occurring in φ or in any quantified subformula of φ, then $[\lambda x\ \varphi]$ is a 1-place (λ-)predicate.
   - Every free occurrence of a variable other than x in φ is a free occurrence in $[\lambda x\ \varphi]$.
8. Nothing is predicable, or free in something, or a term, a predicate, a formula, or a subformula of something unless it can be so demonstrated by the clauses above.[9]

Given the presence of complex predicates, the notion of substitutability that is needed for stating a number of axiom schemas has to be expressed a bit more generally than usual. Towards this end, let us say that two expressions are *of the same type* if both are terms, both are 0-place predicates, or both are 1-place predicates, and let φ^{α}_{τ} be the result of replacing every free occurrence of α in φ with an occurrence of τ. Given this, say that a term or predicate τ is *substitutable for* the variable α in φ if (a) τ and α are of the same type, and (b) every free occurrence of a variable β in τ is still free in φ^{α}_{τ}.

2.2 MOT: Logical Axioms, Definitions, and Proofs

Basic Logical Axioms The basic logical axioms of MOT consist of the axioms of classical S5 modal propositional logic and classical monadic second-order

[9]The exclusion of encoding formulas in clause 3 is required to avoid the paradoxes of object theory (see [21], pp. 158–160) and the restriction to individual variables in clauses 6(ii) and 7(ii) and the restrictions concerning free occurrences of individual variables in λ-predicates in clauses 6(iii) and 7(iii) all arise out of certain properties of the logical structure of relations. We believe both of these restrictions can be justified on philosophical grounds (for the latter, see [13]). By contrast, the restriction on free occurrences of variables in quantified subformulas in clauses 6(iii) and 7(iii) is forced upon on us by our monadic framework and might appear to impose severe expressive restrictions on our framework, as they rule out such formulas as $\forall x \forall y(Fy \wedge Gx)$ from serving as (proposition denoting) predicates and such 1-place predicates as $[\lambda x\ \forall y(Fy \rightarrow Gx)]$. More generally, say that a formula φ satisfies the *scope condition* if neither φ itself nor any of its subformulas is a quantified formula containing a free occurrence of a variable. Then we can put the matter thus: many useful and seemingly innocuous formulas fail to satisfy the scope condition and, hence, are neither predicable nor can be used to form predicates. The clauses might therefore appear at first sight to impose a serious (and somewhat embarrassing) limitation on the expressiveness of our framework. But the situation is not so dire. In the case of formulas involving no modal operators, at least, in virtue of our λ-conversion principle and well-known normal form theorems of propositional and predicate logic, it is always possible to to a convert a formula θ that violates the scope condition into a logically equivalent formula that does not. (For example, the non-predicable formula above is equivalent to $\forall y Fy \wedge \forall x Gx$ and the formula in the illegitimate λ-predicate above is equivalent to $\exists y Fy \rightarrow Gx$.) Moreover, in virtue of the validity of the Barcan schema and its converse the basic logic of MOT and basic principles of modal propositional logic, the same sort of conversion is possible for many modal formulas. (The proof of this is tedious but straightforward.) Only those formulas in which a universal (existential) quantifier is in the scope of a possibility (necessity) operator is the conversion not always possible in virtue of the general invalidity of $\Diamond \forall x \varphi \leftrightarrow \forall x \Diamond \varphi$ ($\Box \exists x \varphi \leftrightarrow \exists x \Box \varphi$). (Our thanks to a referee for pointing out that we had omitted consideration of the modal case in an earlier draft.) Note, however, that these restrictions on occurrences of free variables disappear entirely in full object theory.

quantification theory (without identity). For clarity, we express the universal instantiation axiom explicitly—every instance of the following is an axiom:

UI $\forall\alpha\varphi \rightarrow \varphi^{\alpha}_{\tau}$, where α is any variable, and τ is substitutable for α in φ.

Recall that all predicable formulas are 0-place predicates and so can be substituted for universally quantified propositional variables.

The Logic of Abstraction In general, abstraction principles say that the n-place relation ($n \geq 0$) defined by a certain condition φ is true of n objects $y_1, ..., y_n$ just in case the condition holds of (alternatively, is satisfied by) those objects:

Λ $[\lambda x_1 ... x_n\ \varphi]y_1 ... y_n \leftrightarrow \varphi^{x_1 ... x_n}_{y_1 ... y_n}$, where each y_i substitutable for x_i in φ.

This principle—often also known as λ-*conversion*—is in fact included in full object theory for all $n \geq 0$. In the case where $n = 0$, the principle reduces to $[\lambda\ \varphi] \leftrightarrow \varphi$, which asserts that the proposition $[\lambda\ \varphi]$ is true just in case φ.[10] In the language $\mathcal{L}$ of MOT developed here, where our concern is primarily with the derivation of **EP** and the minimal commitments of that derivation, we do not need to quantify over n-place relations generally, but only propositions and properties. Hence, we need only 0- and 1-place complex predicates. However, it is also the case that, for our purposes here, complex 0-place predicates have simply been *identified* with the conditions that define them—viz., predicable formulas—which renders the 0-place abstraction principle trivial. Consequently, we only need the 1-place case:

$\mathbf{\Lambda_1}$ $[\lambda x\ \varphi]y \leftrightarrow \varphi^{x}_{y}$, where y is substitutable for x in φ.

Informally, that is, $\mathbf{\Lambda_1}$ says that an object y exemplifies the property *being such that* φ just in case φ holds of y.

Definition of Identity for Objects As noted, $\mathcal{L}$ does not include identity as a primitive; rather, identity is a defined notion in object theory. In fact, there are is a separate definition for each of the three basic logical types in MOT: objects, properties and propositions. We first define identity for objects.

Abstract objects can be thought of as *pure objects of thought*—the properties they encode are the ones by which we conceive of them. Thus, different objects of thought have to differ in some respect. Hence, abstract objects, *qua* pure objects of thought, are taken to be identical just in case they encode the same properties:

$\mathbf{Id_{A!}}$ $x =_{A!} y\ =_{df}\ A!x \wedge A!y \wedge \Box\forall F(xF \leftrightarrow yF)$

The distinction between ordinary and abstract objects does not play a role in the derivation of **EP**. However, as the identity conditions for ordinary objects in object

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[10] We introduce 'is true' into the reading because truth is the 0-place case of exemplification. In full object theory, these 0-place λ-predicates prove useful in applications other than the theory of truth, e.g., in the theory of belief.

theory are quite different than those for abstract objects, for the sake of completeness once again it is good to state them explicitly. To this end we introduce a defined predicate ‘$O!$′ which, intuitively, expresses the property of being an “ordinary” object:[11]

O! $O!x =_{df} \neg A!x$

Ordinary objects are then defined to be identical just in case they necessarily exemplify all of the same properties:

$\mathbf{Id}_{\boldsymbol{O!}}$ $x =_{O!} y =_{df} O!x \wedge O!y \wedge \Box\forall F(Fx \leftrightarrow Fy)$

Identity for objects generally can now be defined as the disjunction of these $\mathbf{Id}_{\boldsymbol{A!}}$ and $\mathbf{Id}_{\boldsymbol{O!}}$:

Id $x = y =_{df} x =_{A!} y \vee x =_{O!} y$

Definition of Identity for Properties and Propositions One of monadic object theory’s virtues is its ability to provide identity conditions for properties and propositions that do not require them to be identical if necessarily coextensive. To state the definitions, note that there is no condition on λ-predicates $[\lambda x\, \varphi]$ requiring x to occur free in φ. Thus, in particular, for every proposition p, there is the *propositional property* $[\lambda x\, p]$ of p expressing, intuitively, the property *being such that p is true*. Given this, we have the following definitions:

$\mathbf{Id_1}$ $F = G =_{df} \Box\forall x(xF \leftrightarrow xG)$
$\mathbf{Id_0}$ $p = q =_{df} [\lambda y\, p] = [\lambda y\, q]$

$\mathbf{Id_1}$ tells us that properties are identical if encoded by the same abstract objects. The intuition here is that, if properties F and G are distinct, then there is a pure object of thought that encodes the one but not the other. And if there isn’t a pure object that encodes F without encoding G, then there is nothing in their nature to distinguish them and, hence, F and G must be identical. $\mathbf{Id_0}$, in turn, tells us that propositions are identical if their property correlates are.

Principles of Identity It is straightforward to prove that, on the above definitions, the reflexivity of identity falls out as a theorem:[12]

Ref $\forall\alpha(\alpha = \alpha)$, for any variable α.

[11] This departs from previous developments of object theory, which almost always start with a primitive predicate ”$E!$” (expressing the property *being concrete*) and which define an ordinary object as one which is possibly concrete and an abstract object as one that couldn’t possibly be concrete. However, for the present development, it simplifies matters to simply take $A!$ as primitive and define ordinary objects as those that are not abstract, thereby eliminating the need for a concreteness predicate $E!$.

[12] The proof is by cases. In the first case, when α is the variable x, then use a disjunctive syllogism starting with the fact that $A!x \vee O!x$, i.e., by definition **O!**, $A!x \vee \neg A!x$. The second and third cases, when α is the variable F or α is the variable p, the proof is trivial.

The indiscernibility of identicals, restricted to ordinary objects, are also theorems of MOT,[13] However, it is convenient to state the principle generally for all entities:

Ind $\alpha = \beta \rightarrow (\varphi \rightarrow \varphi')$, where β is substitutable for α in φ, and φ' is the result of replacing zero or more free occurrences of α in φ with occurrences of β.

We note that an instance of **Ind** for propositions only is used in the derivation of **EP**.

We also include a "reducibility" schema for λ-predicates that avoids intuitively unnecessary multiplication of properties.

Red $[\lambda x\ Fx] = F$

Logical Axioms for Encoding Recall that the intuition behind abstract objects is that they are objects of pure thought; the properties such an object encodes are thus *constitutive* of the object. One aspect of this idea has been captured in the definition $\mathbf{Id}_{A!}$ of identity for abstract objects. A second aspect, however, is modal: it cannot be a mere matter of happenstance that an abstract object encodes the properties it does. Otherwise put, encoding is *rigid*; any property an abstract object happens to encode is one that it encodes necessarily:

RE $xF \rightarrow \Box xF$

Moreover, *being* an abstract object cannot itself be a mere matter of happenstance; thus:

$\Box$**A!** $A!x \rightarrow \Box A!x$

Finally, whereas both abstract objects and non-abstract, or ordinary, objects such as those typically given in experience exemplify properties, only abstract encode them. This property of abstract objects is in fact not needed in the derivation of **EP** but we include it here for the sake of completeness:[14]

AE $xF \rightarrow A!x$

Proofs and Theorems A *proof* in MOT is understood as usual as a sequence of formulas consisting of either logical axioms (as given in this Section 2.2) or formulas that follow from preceding formulas in the sequence by a rule of inference: Modus Ponens, Generalization, and Necessitation in the following form:

RN $\Box\varphi$ follows from φ.

A formula φ is *a theorem of MOT* ($\vdash_{\text{MOT}} \varphi$) if there is a proof in MOT whose last member is φ. Note that, where α is any variable, all instances of the first- and second-order Barcan schema $\Diamond\exists\alpha\varphi \rightarrow \exists\alpha\Diamond\varphi$ and the Buridan schema $\Diamond\forall\alpha\varphi \rightarrow \forall\alpha\Diamond\varphi$ are theorems of MOT; indeed, they are derivable in the basic logic alone.

[13] A proof sketch of the principle for ordinary objects is given in fn 31.

[14] In previous versions of object theory, where abstract objects are defined as $\neg\Diamond E!x$ and ordinary objects, $O!x$, are defined as $\Diamond E!x$, the following was taken as an axiom: $O!x \rightarrow \Box\neg\exists F xF$. This axiom is equivalent to **AE**.

For any set Γ of formulas of $\mathcal{L}$, we will say that φ is *provable in MOT from* Γ (written $\Gamma\vdash_{\text{MOT}}\varphi$) if there are formulas $\psi_1, ..., \psi_n \in \Gamma$ such that

$$\vdash_{\text{MOT}} (\psi_1 \wedge ... \wedge \psi_n) \rightarrow \varphi.$$

For purposes below we note the following two theorems of MOT:[15]

(1) $\Diamond xF \rightarrow \Box xF$
(2) $\Diamond A!x \rightarrow A!x$

2.3 MOTC—MOT with Object Comprehension

The fundamental principle of object theory is *Object Comprehension*. This is a sort of plenitude principle for abstract objects: it captures the idea that *any* possible conceptualization corresponds exactly to a (unique) abstract object. More exactly: necessarily, for any condition φ on properties, there is an abstract object that encodes exactly the properties satisfying φ:

OC $\Box\exists x(A!x \wedge \forall F(xF \leftrightarrow \varphi))$, where x not free in φ.

We have not here counted **OC** among the logical principles of MOT for two reasons: The question of logical status of comprehension principles (notably, Frege's Axiom V) is a controversial one, to say the least. In fact, we believe one can reasonably argue for **OC**'s logicality but we will not contest the matter here. More to the point for present purposes, however, **OC** is not logically valid in the rather simplified model theory for $\mathcal{L}$ that we develop in Section 4.1. Thus, for present purposes, we present Object Comprehension as a non-logical, or *proper*, axiom schema.

Theorems of MOTC Let *MOTC* be MOT+**OC**, i.e., MOT supplemented with the Object Comprehension schema. In the special case of the provability of a formula φ from Γ where Γ consists of zero or more instances of **OC**, we say simply that φ is *provable in MOTC*, or that φ is a *theorem of MOTC*, and we may alternatively write $\vdash_{\text{MOTC}}\varphi$.

2.4 World Theory

A simple but powerful theory of possible worlds falls out of the axioms of object theory by means of a few definitions. As noted above, $\mathcal{L}$ contains predicates of the form $[\lambda x\ p]$—intuitively, expressing the propositional property *being such that p is true*. By Object Comprehension (**OC**) there will be abstract objects that encode only such properties; these are the *situations*:

Sit $Situation(x) =_{df} A!x \wedge \forall F(xF \rightarrow \exists p(F=[\lambda y\ p]))$

[15]For (1), note that, by **RE** and **RN**, we have $\Box(xF \rightarrow \Box xF)$ and by basic modal logic $\Diamond xF \rightarrow \Diamond\Box xF$. By the characteristic S5 schema we have $\Diamond\Box xF \rightarrow \Box xF$. So (1) follows by a hypothetical syllogism. (2) is derived similarly, albeit with an application of the T schema as well.

Next we say that a proposition p is *true at* a situation (or other abstract object) x just in case x encodes $[\lambda y\ p]$:

Tr $x \models p =_{df} x[\lambda y\ p]$

Finally, we say that a situation x is a *possible world* if it could be the case that all and only the truths are true at x:

PW $World(x) =_{df} Situation(x) \wedge \Diamond\forall p(x \models p \leftrightarrow p)$.[16]

Since worlds are situations, they are abstract objects (by **Sit**), and so the identity of worlds reduces to the identity of abstract objects—they are identical whenever they encode the same properties. Since they are situations, and hence encode only propositional properties, they are identical whenever the same propositions are true at them (by **Sit** and **Tr**).

3 Deriving the Equivalence Principle

We begin this section by noting that, in MOT, the use of restricted world variables is defined notation; specifically:

$\forall w\varphi =_{df} \forall x(World(x) \rightarrow \varphi)$
$\exists w\varphi =_{df} \exists x(World(x) \wedge \varphi)$

3.1 The Derivation

MOTC is the minimal general theory that is required to systematize the expressions and inferences used in the derivation of **EP**. To derive **EP** in MOTC, we first derive the left-to-right direction and then the right-to-left direction.

(⟹) We prove the left-to-right direction $\Diamond p \rightarrow \exists w(w \models p)$ in MOTC by hypothetical syllogism in two stages:

Stage A: Show that $\vdash_{\text{MOTC}} \Diamond p \rightarrow \Diamond\exists w(w \models p)$.
Stage B: Show that $\vdash_{\text{MOTC}} \Diamond\exists w(w \models p) \rightarrow \exists w(w \models p)$.

Stage A Our strategy is first to show that $\Box\Phi \rightarrow \Box(p \rightarrow \exists w(w \models p))$ is a theorem of MOT, where $\Box\Phi$ is a particular instance of **OC**. By basic modal logic, it will follow that $\Box\Phi \rightarrow (\Diamond p \rightarrow \Diamond\exists w(w \models p))$ is a theorem of MOT and, hence, by definition, that $\Diamond p \rightarrow \Diamond\exists w(w \models p)$ is a theorem of MOTC.

[16] To remove an ambiguity, we take $\models$ to bind more tightly than the connectives. Thus, $x \models r \leftrightarrow r$ is to be parsed as $(x \models r) \leftrightarrow r$. To represent the claim that x makes the proposition $r \leftrightarrow r$ true, we would write $x \models (r \leftrightarrow r)$.

We begin with the following assumption:

Φ: $\exists x(A!x \wedge \forall F(xF \leftrightarrow \exists q(q \wedge F = [\lambda y\, q])))$

Φ asserts that there exists an abstract object that encodes all and only the "true" propositional properties, i.e., only those properties F such that, for some true proposition q, F is the property *being such that q is true*. Our first task is to show that, from this assumption, $p \rightarrow \exists w(w \models p)$ follows.

So assume p. Let a be an arbitrary object instantiating Φ; that is, assume:

(3) $A!a \wedge \forall F(aF \leftrightarrow \exists q(q \wedge F = [\lambda y\, q]))$

We will show that a is a possible world where p is true. To do so, the definitions **PW** and **Tr** tell us we must establish:

(4) $Situation(a)$
(5) $\Diamond\forall q(a \models q \leftrightarrow q)$
(6) $a \models p$

To establish (4), the definition **Sit** requires that we establish $A!a \wedge \forall F(aF \rightarrow \exists q(F = [\lambda y\, q]))$. We've already established the left conjunct, $A!a$, since it is the first conjunct of (3). Now assume aG, for conditional proof. By (3), $\exists q(q \wedge G = [\lambda y\, q])$. *A fortiori*, $\exists q(G = [\lambda y\, q])$. So by conditional proof, $aG \rightarrow \exists q(G = [\lambda y\, q])$. By Generalization, we infer the right conjunct.

To establish (5), we first establish $\forall q(a \models q \leftrightarrow q)$ and then apply the $\Diamond$ version of the T schema (i.e., $\chi \rightarrow \Diamond\chi$). Assume $a \models r$ (i.e., $a[\lambda y r]$), where r is an arbitrary proposition. Then by the right conjunct of (3), $\exists q(q \wedge [\lambda y\, r] = [\lambda y\, q])$. Let s be an arbitrary such proposition. Then we know that s and $[\lambda y\, r] = [\lambda y\, s]$, and so by definition $\mathbf{Id_0}$, $r = s$. But since s is true, we know by **Ind** that r is.[17] Hence, we have established $a \models r \rightarrow r$. Now assume r. By **Ref**, $[\lambda y\, r] = [\lambda y\, r]$, so we have $r \wedge [\lambda y\, r] = [\lambda y\, r]$. So $\exists q(q \wedge [\lambda y\, r] = [\lambda y\, q])$. Hence, by the right conjunct of (3), it follows that $a[\lambda y\, r]$, i.e., $a \models r$. Hence, we have established $r \rightarrow a \models r$. So we may conclude $a \models r \leftrightarrow r$ and so, as r was arbitrary, $\forall q(a \models q \leftrightarrow q)$. Thus, by the T schema, $\Diamond\forall q(a \models q \leftrightarrow q)$.

To establish (6), we simply note that it follows from the combination of our assumption that p and the claim that $\forall q(a \models q \leftrightarrow q)$, which we established as an intermediate step in the argument for (5).

So from our assumption (3) we have established (4), (5), and (6) and, hence, from them, that $World(a) \wedge a \models p$ and, therefore, that $\exists x(World(x) \wedge x \models p)$, i.e., $\exists w(w \models p)$. So by conditional proof, $p \rightarrow \exists w(w \models p)$. Since we've proved this conditional from an instance of Φ, and the conditional doesn't contain an occurrence of 'a', it follows from Φ. Therefore, by conditional proof again, we have shown:

(7) $\Phi \rightarrow (p \rightarrow \exists w(w \models p))$

[17] A bit more exactly, we are using the (derivable) instance $s = r \rightarrow (s \rightarrow r)$ of **Ind** for propositions here.

By **RN** we infer:

(8) $\Box(\Phi \rightarrow (p \rightarrow \exists w(w \models p)))$

and thence, by some basic modal logic,[18] we have:

(9) $\Box\Phi \rightarrow (\Diamond p \rightarrow \Diamond\exists w(w \models p))$

But, as noted above, $\Box\Phi$ is an instance of **OC** and, hence, we have shown that $\Diamond p \rightarrow \Diamond\exists w(w \models p)$ is a theorem of MOTC. This concludes Stage A.

Stage B We begin this stage by assuming $\Diamond\exists w(w \models p)$; our goal is to show $\exists w(w \models p)$. Eliminating the restricted variable w in our assumption, we have $\Diamond\exists x(World(x) \wedge x \models p)$. By the Barcan Formula, it follows that $\exists x\Diamond(World(x) \wedge x \models p)$. Let a be such an object; that is assume

(10) $\Diamond(World(a) \wedge a \models p)$.

Since the conjuncts of a possibly true conjunction are possible, it follows that $\Diamond World(a) \wedge \Diamond a \models p$. We now establish that each possibility is a non-modal fact.

To see that $\Diamond World(a)$ implies $World(a)$, assume the former. Then, by **PW** and **Sit**, $\Diamond(A!a \wedge \forall F(aF \rightarrow \exists p(F=[\lambda y\, p])) \wedge \Diamond\forall p(a \models p \leftrightarrow p))$. Since the conjuncts of a possibly true conjunction are possible, it follows that:

(11) $\Diamond A!a \ \wedge\ \Diamond\forall F(aF \rightarrow \exists p(F=[\lambda y\, p])) \ \wedge\ \Diamond\Diamond\forall p(a \models p \leftrightarrow p)$

To derive $World(a)$ from (11), we need to show, by the definitions **PW** and **Sit**, that:

(12) $A!a$
(13) $\forall F(aF \rightarrow \exists p(F=[\lambda y\, p]))$
(14) $\Diamond\forall p(a \models p \leftrightarrow p)$

(12) follows from the first conjunct of (11), by our theorem (2). To derive (13), consider the second conjunct of (11). By the Buridan schema, the second conjunct of (11) immediately implies $\forall F\Diamond(aF \rightarrow \exists p(F=[\lambda y\, p]))$; call this statement Ω. Now let G be an arbitrary property and assume aG, for conditional proof. $\Box aG$ follows by **RE**. By instantiating Ω to G, it follows that $\Diamond(aG \rightarrow \exists p(G = [\lambda y\, p]))$. Hence, applying some basic modal logic to the two preceding results we have $\Diamond\exists p(G = [\lambda y\, p])$. It is separately provable in MOT that, for any property H, $\Diamond\exists p(H = [\lambda y\, p]) \rightarrow \exists p(H = [\lambda y\, p])$.[19] Hence, from the preceding result, $\exists p(G = [\lambda y\, p])$ follows. Thus, by conditional proof, we infer that $aG \rightarrow \exists p(G = [\lambda y\, p])$. As G was arbitrary, we may conclude: $\forall F(aF \rightarrow \exists p(F=[\lambda y\, p]))$. Finally, note that (14) follows from the third conjunct of (11) by the characteristic schema of S4, which is derivable in S5. So we have established $\Diamond\forall p(a \models p \leftrightarrow p)$.

[18] Specifically, the theorems $\Box(q \rightarrow (r \rightarrow s)) \rightarrow (\Box q \rightarrow \Box(r \rightarrow s))$ and $\Box(r \rightarrow s) \rightarrow (\Diamond r \rightarrow \Diamond s)$.

[19] The consequent follows quickly from the antecedent by applying, in order, the Barcan formula, the definition $\mathbf{Id_1}$ of property identity, and the characteristic S5 schema.

So from $\Diamond World(a)$ we have established (12), (13), and (14) and, hence, $World(a)$.

Next we show that $\Diamond a \models p \rightarrow a \models p$. Note that $\Diamond a \models p$, by definition **Tr**, means $\Diamond a[\lambda y\ p]$. By (1) it follows that $\Box a[\lambda y\ p]$. And by the T schema, it follows that $a[\lambda y\ p]$, i.e., $a \models p$.

So, from (10), we've established $World(a) \wedge a \models p$ and, hence, we may infer $\exists x(World(x) \wedge x \models p)$. And, once again, as this result does not involve our arbitrary instance a, we may infer that it follows from (10)'s generalization $\exists x \Diamond(World(x)x \models p)$ which, recall, we had inferred from $\Diamond\exists x(World(x) \wedge x \models p)$, i.e., reintroducing our restricted variable, $\Diamond\exists w(w \models p)$. By conditional proof we conclude that $\Diamond\exists w(w \models p) \rightarrow \exists w(w \models p)$. Combining Stages A and B, we have shown that $\Diamond p \rightarrow \exists w(w \models p)$ is a theorem of MOTC.

($\Longleftarrow$) We now show that the right-to-left direction of **EP** is a theorem of MOT (hence of MOTC). So assume $\exists w(w \models p)$, i.e., $\exists x(World(x) \wedge x \models p)$. Let a be such an object:

(15) $World(a) \wedge a \models p$

From the left conjunct we have by definition **PW** that $\Diamond\forall q(a \models q \leftrightarrow q)$. By the Buridan schema, we have $\forall q \Diamond(a \models q \leftrightarrow q)$ and hence, in particular, $\Diamond(a \models p \leftrightarrow p)$ and so, *a fortiori*, $\Diamond(a \models p \rightarrow p)$. But by (15) we have $a \models p$ and hence, by **RE**, $\Box a \models p$. Since it is a theorem of basic modal logic that $(\Diamond(\varphi \rightarrow \psi) \wedge \Box\varphi) \rightarrow \Diamond\psi$, we have $\Diamond p$. So we have deduced $\Diamond p$ from (15). As this conclusion does not involve the arbitrary world a, we may conclude that $\Diamond p$ follows from $\exists x(World(x) \wedge x \models p)$. By conditional proof it follows that $\exists w(w \models p) \rightarrow \Diamond p$. We note that our reasoning was entirely in MOT (since we invoked no instances of **OC**) and, hence, trivially, in MOTC. Putting together our proofs of the left-to-right and right-to-left directions have shown that **EP** is a theorem of MOTC.

Inspection of the above derivation shows that MOTC offers two special axioms that play a key role in the proof of **EP**: the logical axiom **RE** and an instance of the principle of Object Comprehension **OC**. The other axioms presented in Section 2.2 that are used in the proof can be found in any second-order quantified modal logic with identity, propositions (defined as 0-place relations), and λ-expressions.[20] Interestingly, although the properties denoted by λ-expressions play critical roles in the proof, the λ-abstraction principle $\mathbf{\Lambda_1}$ that governs those concepts is not itself used in the proof. But we have included this principle because we want to systematize the concepts that play these critical roles.

3.2 Other Consequences

Given that **EP** is a theorem of MOTC, we can prove that there are non-actual possible worlds with the following two steps. First we define:

$$Actual(x) =_{df} \forall p(x \models p \rightarrow p)$$

[20] The only qualification that needs to be made here is that our formulation of **Ind**, though identical in form to the usual principle of identity substitution, is stated in terms of defined notions of identity.

Second, we assert that there are propositions that are false but possibly true:

$$\exists p(\neg p \; \& \; \Diamond p)$$

This last claim is not provable in MOTC, for reasons that we discuss in more detail in the next section. (Specifically, it will be shown that MOTC is true in a model with just one primitive possible world and two propositions. In such models, all true propositions are necessarily true and all false propositions are necessarily false.)

Once we have the definition $Actual(x)$ and the claim that there are contingently false propositions, it follows from **EP** that:

$$\exists x(World(x) \wedge \neg Actual(x)).$$

For if q is some false but possibly true proposition, then by **EP** there is a world, say w_1, where it is true, i.e., such that $w_1 \models q$. But by hypothesis, q is false, and so w_1 is not actual.

Our derivation of **EP** in the previous subsection offers some evidence that the other theorems of world theory derived in Zalta [21, 22] are still derivable in the more limited context of MOTC. For example, it is of significant philosophical interest to verify that one can still derive the claim that there is a unique actual world.[21] It is also provable that every world w is *maximal*, i.e., that $\forall p(w \models p \vee w \models \neg p)$, and that every world w is *consistent*, i.e., that $\neg\exists p(w \models p \wedge w \models \neg p)$. From these two theorems, it is easy to establish that every world w is *coherent*, i.e., that $\forall p(w \models \neg p \leftrightarrow \neg w \models p)$.[22] Since truth at a world ($\models$) is coherent and the 0-place predicate '$\neg q$' also denotes the negation of the proposition q, we can derive the equivalence of **EP** and **LP** as we did in footnote 3, by universally instantiating $\neg q$ for p in the first line of both directions of the proof. So, our proof of **EP** yields **LP** as a corollary.

4 MOTC and Ontological Commitment

Our proof of **EP** in the previous section appears to use some heavy-duty logical and metaphysical machinery. But appearances can be deceiving. We now turn to the question: What are the smallest models of MOTC? After a preliminary definition, we lay out the model theory of our language $\mathcal{L}$. We then construct the smallest model of MOTC. Finally, we contruct the smallest non-trivial model of the theory. These

[21] The derivation proceeds from the following instance of **OC**:

$$\exists x(A!x \wedge \forall F(xF \leftrightarrow \exists p(p \wedge F = [\lambda y \; p])))$$

To complete the proof, call such an object b and then show that b is a world, that b is actual, and that anything else y that is an actual world is identical to b.

[22] Here's how. ($\rightarrow$) Let w and q be any world and proposition, respectively, and assume $w \models \neg q$. It follows by w's consistency that $\neg(w \models q)$. ($\leftarrow$) Assume $\neg w \models q$. Then by w's maximality, it follows that $w \models \neg q$.

models reveal the minimal ontological commitments of MOTC and, hence, the minimal ontological commitments needed to derive **EP** as a theorem.

4.1 Model Theory for $\mathcal{L}$

An interpretation $\mathcal{I}$ for $\mathcal{L}$ can be thought of as a 7-tuple $\langle \mathbf{D}, \mathbf{W}, \mathbf{P}, \mathbf{Op}, \mathbf{ex}, \mathbf{en}, \mathbf{V}\rangle$ such that:

1. **D** and **W** are non-empty sets ("objects" and "worlds", respectively) where the latter contains a distinguished element $\mathbf{w}^*$ (the "actual" world). **D** is the union of two mutually disjoint sets **A** (the domain of abstract objects) and **O** (the domain of ordinary objects); **A** must be nonempty.
2. **P** is the union of two mutually disjoint, nonempty sets $\mathbf{P}_0$ (the domain of propositions) and $\mathbf{P}_1$ (the domain of properties), the latter of which contains a distinguished element $\mathbf{p}^*$.
3. **Op** is a set of logical operations **neg**, **cond**, **univ**, **nec**, **vac**, **plug** described more fully below.
4. The exemplification extension function, **ex**, is a total function on $\mathbf{W} \times \mathbf{P}$ that maps $\mathbf{W} \times \mathbf{P}_0$ into $\{0, 1\}$ and $\mathbf{W} \times \mathbf{P}_1$ into $\wp(\mathbf{D})$. In particular, we set the exemplification extension of the distinguished property $\mathbf{p}^*$ to be the set **A** at every world: $\mathbf{ex}(\mathbf{w}, \mathbf{p}^*) = \mathbf{A}$, for all $\mathbf{w} \in \mathbf{W}$. (**ex** is subject to further constraints described below.)
5. The encoding extension function, **en**, maps $\mathbf{P}_1$ into $\wp(\mathbf{A})$ in such a way that, (i) for distinct $\mathbf{a}_1, \mathbf{a}_2 \in \mathbf{A}$, there is a $\mathbf{p}_1 \in \mathbf{P}_1$ such that $\mathbf{a}_1 \in \mathbf{en}(\mathbf{p}_1)$ iff $\mathbf{a}_2 \notin \mathbf{en}(\mathbf{p}_1)$; and (ii) for distinct $\mathbf{p}_1, \mathbf{p}_2 \in \mathbf{P}_1$, $\mathbf{en}(\mathbf{p}_1) \neq \mathbf{en}(\mathbf{p}_2)$. (Condition (i) ensures that distinct abstract objects cannot encode the same properties and condition (ii) ensures that distinct properties cannot be encoded by the same abstract objects.)
6. The valuation function **V** maps each term of $\mathcal{L}$ to a member of **D**, each 0-place primitive predicate of $\mathcal{L}$ to a member of $\mathbf{P}_0$, and each 1-place primitive predicate of $\mathcal{L}$ to a member of $\mathbf{P}_1$;[23] in particular, we stipulate that $\mathbf{V}(A!) = \mathbf{p}^*$.

Intuitively, $\mathbf{P} \cup \mathbf{D}$ and **Op** together can be thought of as an algebra, where the elements of **P** are generated from an initial set of primitive properties, propositions, and objects by the operations in **Op** [2, 10, 21]. All of these operations (with the exception of **vac**) correspond semantically to the syntactic operations whereby complex formulas are constructed from the primitive lexicon of $\mathcal{L}$. Specifically, the operation $\mathbf{plug} : \mathbf{P}_1 \times \mathbf{D} \longrightarrow \mathbf{P}_0$ corresponds to the formation of an atomic formula from a 1-place predicate; thus, intuitively, $\mathbf{plug}(\mathbf{r}, \mathbf{a})$ is the atomic "singular" proposition **that a exemplifies r**. For $0 \leq i \leq 1$, the operations $\mathbf{neg} : \mathbf{P}_i \longrightarrow \mathbf{P}_i$, $\mathbf{cond} : \mathbf{P}_i \times \mathbf{P}_i \longrightarrow \mathbf{P}_i$, $\mathbf{univ} : \mathbf{P}_1 \longrightarrow \mathbf{P}_0$, and $\mathbf{nec} : \mathbf{P}_i \longrightarrow \mathbf{P}_i$ are semantic counterparts of the usual logical operators of quantified modal logic. And for each

[23] To avoid variable assignments, we are treating variables as "quantifiable constants" and assigning them fixed values via **V**. This does not substantially affect the metatheory. See, e.g., [12].

proposition $\mathbf{r}$, the operation $\mathbf{vac} : \mathbf{P}_0 \longrightarrow \mathbf{P}_1$—which is stipulated to be one-to-one—generates the "propositional property" **being such that r**. These latter properties, as we've seen, are critical to the definition of possible worlds in object theory.

Given the logical structure of properties and propositions determined by these operations, **ex**, in turn, must assign exemplification extensions systematically in a way that reflects this structure. Specifically, for $\mathbf{r}_0, \mathbf{s}_0 \in \mathbf{P}_0$ and $\mathbf{r}_1, \mathbf{s}_1 \in \mathbf{P}_1$:

E1. $\mathbf{ex}(\mathbf{w}, \mathbf{plug}(\mathbf{r}_1, \mathbf{a})) = 1$ iff $\mathbf{a} \in \mathbf{ex}(\mathbf{w}, \mathbf{r}_1)$

E2. $\mathbf{ex}(\mathbf{w}, \mathbf{neg}(\mathbf{r}_0)) = 1 - \mathbf{ex}(\mathbf{w}, \mathbf{r}_0)$
$\mathbf{ex}(\mathbf{w}, \mathbf{neg}(\mathbf{r}_1)) = \mathbf{D} \setminus \mathbf{ex}(\mathbf{w}, \mathbf{r}_1)$

E3. $\mathbf{ex}(\mathbf{w}, \mathbf{cond}(\mathbf{r}_0, \mathbf{s}_0)) = max\{1 - \mathbf{ex}(\mathbf{w}, \mathbf{r}_0), \mathbf{ex}(\mathbf{w}, \mathbf{s}_0)\}$
$\mathbf{ex}(\mathbf{w}, \mathbf{cond}(\mathbf{r}_1, \mathbf{s}_1)) = (\mathbf{D} \setminus \mathbf{ex}(\mathbf{w}, \mathbf{r}_1)) \cup \mathbf{ex}(\mathbf{w}, \mathbf{s}_1)$

E4. $\mathbf{ex}(\mathbf{w}, \mathbf{nec}(\mathbf{r}_0)) = min\{\mathbf{ex}(\mathbf{w}', \mathbf{r}_0) \mid \mathbf{w}' \in \mathbf{W}\}$
$\mathbf{ex}(\mathbf{w}, \mathbf{nec}(\mathbf{r}_1)) = \bigcap\{\mathbf{ex}(\mathbf{w}', \mathbf{r}_1) \mid \mathbf{w}' \in \mathbf{W}\}$

E5. $\mathbf{ex}(\mathbf{w}, \mathbf{univ}(\mathbf{r}_1)) = 1$ iff $\mathbf{ex}(\mathbf{w}, \mathbf{r}_1) = \mathbf{D}$

E6. $\mathbf{ex}(\mathbf{w}, \mathbf{vac}(\mathbf{r}_0)) = \begin{cases} \mathbf{D} & \text{if } \mathbf{ex}(\mathbf{w}, \mathbf{r}_0) = 1 \\ \varnothing & \text{otherwise} \end{cases}$

In contrast to these conditions on the exemplification extension function **ex**, the encoding extension function **en** has two features: (a) it is not relativized to worlds, and (b) there are no systematic connections between the encoding extensions of properties and their logical structure—e.g., an object can encode the conditional property $\mathbf{cond}(\mathbf{r}_1, \mathbf{s}_1)$ without encoding either $\mathbf{neg}(\mathbf{r}_1)$ or $\mathbf{s}_1$.

Valuation and Truth The valuation function $\mathbf{V}$ for terms and primitive predicates of $\mathcal{L}$ determines a unique function $\overline{\mathbf{V}}$ that extends $\mathbf{V}$ so as to assign semantic values to the non-primitive predicates of $\mathcal{L}$ recursively in accordance with their form. Specifically, for terms and primitive predicates α, $\overline{\mathbf{V}}(\alpha) = \mathbf{V}(\alpha)$; and for the rest:

V1. $\overline{\mathbf{V}}([\lambda x\ \rho x]) = \overline{\mathbf{V}}(\rho)$, for 1-place predicates ρ of $\mathcal{L}$
$\overline{\mathbf{V}}(\pi\tau) = \mathbf{plug}(\overline{\mathbf{V}}(\pi), \overline{\mathbf{V}}(\tau))$

V2. $\overline{\mathbf{V}}(\neg\varphi) = \mathbf{neg}(\overline{\mathbf{V}}(\varphi))$
$\overline{\mathbf{V}}([\lambda x\ \neg\varphi]) = \mathbf{neg}(\overline{\mathbf{V}}([\lambda x\ \varphi]))$, if x occurs free in φ

V3. $\overline{\mathbf{V}}(\varphi \to \psi) = \mathbf{cond}(\overline{\mathbf{V}}(\varphi), \overline{\mathbf{V}}(\psi))$
$\overline{\mathbf{V}}([\lambda x\ \varphi \to \psi]) = \mathbf{cond}(\overline{\mathbf{V}}([\lambda x\ \varphi]), \overline{\mathbf{V}}([\lambda x\ \psi]))$, if x is free in $\varphi \to \psi$

V4. $\overline{\mathbf{V}}(\Box\varphi) = \mathbf{nec}(\overline{\mathbf{V}}(\varphi))$
$\overline{\mathbf{V}}([\lambda x\ \Box\varphi]) = \mathbf{nec}(\overline{\mathbf{V}}([\lambda x\ \varphi]))$, if x is free in φ

V5. $\overline{\mathbf{V}}(\forall x\varphi) = \mathbf{univ}(\overline{\mathbf{V}}([\lambda x\ \varphi]))$[24]

V6. $\overline{\mathbf{V}}([\lambda x\ \varphi]) = \mathbf{vac}(\overline{\mathbf{V}}(\varphi))$, if x is not free in φ.

[24]Note that the coordination betweeen condition 6(iii) on the construction of predicable quantified formulas and condition 7(iii) on the construction of λ-predicates in the grammar for $\mathcal{L}$ in Section 2.1 guarantees that $\forall x\varphi$ is predicable if and only if $[\lambda x\ \varphi]$ is a 1-place predicate of $\mathcal{L}$. This clause would be illegitmate otherwise.

The truth of a formula φ at a world $\mathbf{w}$ under an interpretation $\mathcal{I} = \langle \mathbf{D}, \mathbf{W}, \mathbf{P}, \mathbf{Op}, \mathbf{ex}, \mathbf{en}, \mathbf{V} \rangle$, written $\mathbf{w} \models_{\mathcal{I}} \varphi$, is defined more or less as usual in a possible world semantics with a fixed domain of individuals, except that the truth conditions for 0- and 1-place atomic formulas are given in terms of the relevant extensions of the *denotations* of their component predicates:

T1. $\mathbf{w} \models_{\mathcal{I}} \pi$ iff $\mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}(\pi)) = 1$
$\mathbf{w} \models_{\mathcal{I}} \rho\tau$ iff $\overline{\mathbf{V}}(\tau) \in \mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}(\rho))$
$\mathbf{w} \models_{\mathcal{I}} \tau\rho$ iff $\overline{\mathbf{V}}(\tau) \in \mathbf{en}(\overline{\mathbf{V}}(\rho))$.

The clauses for the Boolean and modal operators are as usual:

T2. $\models_{\mathcal{I}} \neg\varphi$ iff, $\mathbf{w} \not\models_{\mathcal{I}} \varphi$
T3. $\mathbf{w} \models_{\mathcal{I}} \varphi \rightarrow \psi$ iff $\mathbf{w} \not\models_{\mathcal{I}} \varphi$ or $\mathbf{w} \models_{\mathcal{I}} \psi$
T4. $\models_{\mathcal{I}} \Box\varphi$ iff, for all $\mathbf{w} \in \mathbf{W}$, $\mathbf{w} \models_{\mathcal{I}} \varphi$.

As we are doing without separate variable assignments the quantificational clauses take on a slightly different form than in most definitions of truth. If α is a variable and $\mathbf{e} \in \mathbf{D} \cup \mathbf{P}$, let $\mathbf{V}^{\alpha}_{\mathbf{e}}$ be the valuation function that differs from $\mathbf{V}$ (at most) in that it assigns entity $\mathbf{e}$ to the variable α. That is, $\mathbf{V}^{\alpha}_{\mathbf{e}}(\beta) = \mathbf{V}(\beta)$ for terms and primitive predicates $\beta \neq \alpha$ and $\mathbf{V}^{\alpha}_{\mathbf{e}}(\alpha) = \mathbf{e}$. Now let $\mathcal{I}^{\alpha}_{\mathbf{e}} = \langle \mathbf{D}, \mathbf{W}, \mathbf{P}, \mathbf{Op}, \mathbf{ex}, \mathbf{en}, \mathbf{V}^{\alpha}_{\mathbf{e}} \rangle$. Then we have:

T5. $\mathbf{w} \models_{\mathcal{I}} \forall x\varphi$ iff, for all $\mathbf{a} \in \mathbf{D}$, $\mathbf{w} \models_{\mathcal{I}^{x}_{\mathbf{a}}} \varphi$
$\mathbf{w} \models_{\mathcal{I}} \forall F\varphi$ iff, for all $\mathbf{p}_i \in \mathbf{P}_i$, $\mathbf{w} \models_{\mathcal{I}^{F}_{\mathbf{p}_i}} \varphi$, for i-place predicate variables F ($i \in \{0, 1\}$).

With these definitions, we may define the truth of a formula φ under an interpretation $\mathcal{I}$, written $\models_{\mathcal{I}} \varphi$, as $\mathbf{w}^* \models_{\mathcal{I}} \varphi$. φ is then *logically* true, written $\models \varphi$, if and only if $\models_{\mathcal{I}} \varphi$, for all interpretations $\mathcal{I}$.

4.2 The Smallest Models of MOTC

The abstraction principle $\mathbf{\Lambda_1}$ together with the definitions $\mathbf{Id_0}$ and $\mathbf{Id_1}$ of identity for propositions and properties, respectively, are consistent both with the thesis that necessarily equivalent properties and propositions are identical and with the thesis that they are distinct. Our own philosophical intuitions lean toward the latter. However, because **EP** makes no assumptions either way on this issue, in the *smallest* models of the fragment of object theory needed to derive **EP**, necessarily coextensional properties and propositions are identified. In the Appendix we show that all instances of the schema $\mathbf{\Lambda_1}$ are logically true.

Note also that for any given interpretation $\mathcal{I}$ of $\mathcal{L}$, there is no condition on its set $\mathbf{A}$ of "abstract objects" beyond non-emptiness. There is therefore no guarantee that, for any condition φ on properties, there will be an abstract object in $\mathbf{A}$ that encodes (i.e., that is in the encoding extension of) exactly the properties satisfying φ. Consequently, in contrast to the axioms of MOT, not all instances of **OC** are logically true relative to our model theory.

With this in mind, we can construct a smallest model of MOTC, and thus a smallest interpretation of $\mathcal{L}$, in which all instances of **OC** *are* true. Such a model contains only one world, two properties (complements of each other), two propositions (negations of each other), and four abstract objects (one for each of the four sets of properties). This is because the smallest interpretation of $\mathcal{L}$ requires that there be at least two properties (the universal property and the empty property) and at least two propositions (the True and the False). **OC** in turn requires that there be four abstract objects—intuitively, for each set of properties, the object that encodes exactly the properties in that set.

We don't plan to define these smallest models formally, as they trivialize modality—since there is only one possible world, the modal operators are rendered otiose. That is, $\Box\varphi$, $\Diamond\varphi$, and φ are all equivalent in the model, for all φ. In addition, these models collapse materially equivalent properties and materially equivalent propositions. These facts explain why the claim $\exists p(\neg p \;\&\; \Diamond p)$ is not true in the smallest model of MOTC, since $\neg p$ and $\Box\neg p$ have the same truth value.

But once we add this latter claim, MOTC can only be true in *non-trivial* models, that is, models in which necessary truth and necessary falsity do not collapse into mere truth and falsity. By adding the assertion that there are contingently false propositions, non-trivial models are forced to contain both contingently true and contingently false propositions, as well as necessarily true and necessarily false propositions. Thus, such models will contain nonactual possible worlds. Moreover, non-trivial models will also include properties that are contingently true (false) of everything and properties necessarily true (false) of everything. Thus, such models will include as many abstract objects as there are expressible sets of properties. (This will make **OC** true.)

Although the general model theory of $\mathcal{L}$ doesn't force us to identify properties and propositions whenever they are necessarily equivalent, this is something one can do to define the *smallest* non-trivial models of MOTC. Specifically, any such model contains:

- four propositions: one of which is contingently true, one contingently false, one necessarily true, and one necessarily false;
- four corresponding properties: one contingently true of everything, one contingently false of everything, one necessarily true of everything, and one necessarily false of everything;
- two possible worlds, one of which is nonactual; and
- sixteen abstract objects.

This, we claim, is all that is (non-trivially) presupposed by MOTC. In particular, we do not include any contingent objects in the model, as the existence of contingent beings is not required by logic. Note also that our work earlier in the paper establishes that the two possible worlds can be identified with certain abstract objects. However, for the model-theoretic purposes of this section, we don't make this identification explicit. (See Section 5, especially footnote 29, for the explicit identification.)

4.3 The Smallest Non-Trivial Models of MOTC

A smallest non-trivial model of MOTC, in a language $\mathcal{L}$, is an interpretation $\mathcal{I}^* = \langle \mathbf{D}, \mathbf{W}, \mathbf{P}, \mathbf{Op}, \mathbf{ex}, \mathbf{en}, \mathbf{V} \rangle$ for $\mathcal{L}$ such that:

- $\mathbf{D} = \mathbf{A} \cup \mathbf{O}$, where $\mathbf{O} = \varnothing$, $\mathbf{A} = \wp(\mathbf{P}_1)$ and $\mathbf{P}_1$ is defined as below;
- $\mathbf{W} = \{\mathbf{w_0}, \mathbf{w_1}\}$ (i.e., two primitive "possible worlds") and $\mathbf{w}^* = \mathbf{w_0}$;
- $\mathbf{P} = \mathbf{P}_0 \cup \mathbf{P}_1$, where $\mathbf{P}_0 = \{\mathbf{p}_0, \overline{\mathbf{p}_0}, \mathbf{q}_0, \overline{\mathbf{q}_0}\}$, $\mathbf{P}_1 = \{\mathbf{p}_1, \overline{\mathbf{p}_1}, \mathbf{q}_1, \overline{\mathbf{q}_1}\}$,[25] and $\mathbf{p}^* = \mathbf{p}_1$;
- $\mathbf{Op}$ is as specified below;
- $\mathbf{ex}(\mathbf{w}, \mathbf{p}_0) = 1$ and $\mathbf{ex}(\mathbf{w}, \overline{\mathbf{p}_0}) = 0$, for $\mathbf{w} \in \mathbf{W}$
  $\mathbf{ex}(\mathbf{w}_0, \mathbf{q}_0) = \mathbf{ex}(\mathbf{w}_1, \overline{\mathbf{q}_0}) = 1$; $\mathbf{ex}(\mathbf{w}_1, \mathbf{q}_0) = \mathbf{ex}(\mathbf{w}_0, \overline{\mathbf{q}_0}) = 0$
  $\mathbf{ex}(\mathbf{w}, \mathbf{p}_1) = \mathbf{D}$ and $\mathbf{ex}(\mathbf{w}, \overline{\mathbf{p}_1}) = \varnothing$, for $\mathbf{w} \in \mathbf{W}$
  $\mathbf{ex}(\mathbf{w}_0, \mathbf{q}_1) = \mathbf{ex}(\mathbf{w}_1, \overline{\mathbf{q}_1}) = \mathbf{D}$; $\mathbf{ex}(\mathbf{w}_1, \mathbf{q}_1) = \mathbf{ex}(\mathbf{w}_0, \overline{\mathbf{q}_1}) = \varnothing$;
- $\mathbf{en}(\mathbf{r}) = \{\mathbf{a} \in \mathbf{A} \mid \mathbf{r} \in \mathbf{a}\}$;
- $\mathbf{V}$ is any mapping on the terms and primitive predicates of $\mathcal{L}$ that comports with clause 6 in the definition of an interpretation.

Thus, our model contains two worlds in which different sets of propositions are true (notably, $\mathbf{q}_0$ is true at $\mathbf{w}_0$—by stipulation, the "actual" world w^* of the model—and false in $\mathbf{w}_1$) and hence it is non-trivial. $\mathbf{P}$, as noted, contains four properties and four propositions. Intuitively (and as reflected by the definition of $\mathbf{ex}$), $\mathbf{p}_0$ is a necessarily true proposition (indeed, the only one), $\mathbf{q}_0$ is a contingent proposition, and $\overline{\mathbf{p}_0}$ and $\overline{\mathbf{q}_0}$ are their complements. Thus, $\overline{\mathbf{p}_0}$ is impossible and $\overline{\mathbf{q}_0}$ is also contingent but, at any world, is true if and only if $\mathbf{q}_0$ is false. Likewise, $\mathbf{p}_1$ is a property that necessarily holds of everything, $\mathbf{q}_1$ a property that contingently holds (or fails to hold) of everything, and $\overline{\mathbf{p}_1}$ and $\overline{\mathbf{q}_1}$ are their complements.

As noted, there are no ordinary objects in the model; the domain $\mathbf{D}$ consists solely of abstract objects, which are themselves represented simply as sets of properties—each abstract object is simply identified with the set of properties it encodes (as reflected in the definition of $\mathbf{en}$). We believe this comports well with $\mathcal{I}^*$'s being a simplest non-trivial model, as we do not believe that the existence of contingent individuals is a matter of *logic* and hence such individuals can be omitted from a simplest model. $\mathbf{D}$, then, consists of the sixteen abstract objects there can be, given our initial stock of four properties. Note that the non-world-relative definition of $\mathbf{en}$ ensures that encoding is rigid and hence the truth of the principle **RE**. The fact that abstract objects are simply sets in the model ensures that both condition (i)—that distinct abstract objects do not encode exactly the same properties—and condition (ii)—that distinct properties are not encoded by exactly the same abstract objects—of the definition of $\mathbf{en}$ are met. Moreover, because there are no ordinary objects, we can identify the property $\mathbf{p}_1$ with the property of being abstract. For $\mathbf{p}_1$ holds of everything—hence, of exactly the abstract objects—at every world. This is reflected in the definition of $\mathbf{V}$.

[25] As will be seen below, $\bar{\mathbf{r}}$ indicates the negation of the property or proposition $\mathbf{r}$.

Finally, we need to specify the operators in **Op**. The central challenge here is to specify the operators so that they satisfy the constraints imposed by the definition of an interpretation for $\mathcal{L}$. Specifically, we need to show that (i) every non-primitive predicate denotes a property or proposition in **P** whose logical form comports with the grammatical form of the predicate, and (ii) that the extension of every property or proposition is determined appropriately by its logical form.

To begin, then, note that three of our operators—**neg**, **cond**, and **nec**—are defined on all of **P**. Accordingly, for $i \in \{0, 1\}$, we have:

- $\mathbf{neg}(\mathbf{r}_i) = \overline{\mathbf{r}_i}$, for $\mathbf{r}_i \in \{\mathbf{p}_i, \mathbf{q}_i\}$
  $\mathbf{neg}(\overline{\mathbf{r}_i}) = \mathbf{r}_i$, for $\mathbf{r}_i \in \{\mathbf{p}_i, \mathbf{q}_i\}$.
- $\mathbf{nec}(\mathbf{p}_i) = \mathbf{p}_i$
  $\mathbf{nec}(\mathbf{q}_i) = \mathbf{nec}(\overline{\mathbf{q}_i}) = \mathbf{nec}(\overline{\mathbf{p}_i}) = \overline{\mathbf{p}_i}$.[26]
- $\mathbf{cond}(\mathbf{p}_i, \mathbf{r}_i) = \mathbf{r}_i$, for $\mathbf{r}_i \in \mathbf{P}_i$
  $\mathbf{cond}(\overline{\mathbf{p}}_i, \mathbf{r}_i) = \mathbf{p}_i$, for $\mathbf{r}_i \in \mathbf{P}_i$
  $\mathbf{cond}(\mathbf{q}_i, \mathbf{p}_i) = \mathbf{cond}(\mathbf{q}_i, \mathbf{q}_i) = \mathbf{p}_i$
  $\mathbf{cond}(\mathbf{q}_i, \overline{\mathbf{p}_i}) = \mathbf{cond}(\mathbf{q}_i, \overline{\mathbf{q}_i}) = \overline{\mathbf{q}_i}$
  $\mathbf{cond}(\overline{\mathbf{q}_i}, \mathbf{p}_i) = \mathbf{cond}(\overline{\mathbf{q}_i}, \overline{\mathbf{q}_i}) = \mathbf{p}_i$
  $\mathbf{cond}(\overline{\mathbf{q}_i}, \overline{\mathbf{p}_i}) = \mathbf{cond}(\overline{\mathbf{q}_i}, \mathbf{q}_i) = \mathbf{q}_i$

Unlike the preceding operations, the remaining operations—**vac**, **univ**, and **plug**—yield values in domains other than the domains of their arguments. To facilitate their definition, for our properties $\mathbf{p}_1$, $\overline{\mathbf{p}_1}$, $\mathbf{q}_1$, $\overline{\mathbf{q}_1}$, respectively, let us say that the *corresponding propositions* are $\mathbf{p}_0$, $\overline{\mathbf{p}_0}$, $\mathbf{q}_0$, $\overline{\mathbf{q}_0}$, respectively. Then, where $\mathbf{r}_1$ is any of our properties and $\mathbf{r}_0$ its corresponding proposition, we have:

- $\mathbf{vac}(\mathbf{r}_0) = \mathbf{r}_1$;
- $\mathbf{univ}(\mathbf{r}_1) = \mathbf{r}_0$;
- $\mathbf{plug}(\mathbf{r}_1, \mathbf{a}) = \mathbf{r}_0$, for all $\mathbf{a} \in \mathbf{D}$.

That is, the property $\mathbf{r}_1$ can be identified with the property $\mathbf{vac}(\mathbf{r}_0)$ of **being such that** $\mathbf{r}_0$. (Note that this means that **vac** is one-to-one, as required.) The proposition $\mathbf{r}_0$ can be identified with the proposition $\mathbf{univ}(\mathbf{r}_1)$ that everything exemplifies the property $\mathbf{r}_1$. And, given how we have assigned exemplification extensions to our four properties, for all $\mathbf{a} \in \mathbf{D}$, the proposition $\mathbf{plug}(\mathbf{r}_1, \mathbf{a})$ that **a** exemplifies $\mathbf{r}_1$ can be identified, for every **a**, with the corresponding proposition $\mathbf{r}_0$.[27]

To illustrate the construction, consider the following complex predicate:

(16) $[\lambda x\, \forall y P y \rightarrow \neg Q x]$,

[26]That is, the proposition that the necessarily true proposition is necessary is the necessarily true proposition; the proposition that **r** is necessary, where **r** is either of our contingent propositions or the impossible proposition, is simply the impossible proposition; analogously for properties.

[27]This element of the construction in fact reflects an important theorem of object theory, namely, that there are distinct abstract objects that exemplify all the same properties. In our simplest model, this in fact happens to be true of *all* pairs of distinct abstract objects.

Then, where $\mathbf{V}(P) = \mathbf{p}_1$ and $\mathbf{V}(Q) = \mathbf{q}_1$, we may apply our definition of $\overline{\mathbf{V}}$ for λ-predicates to identify the denotation of this predicate as follows:

$$\begin{aligned}
\overline{\mathbf{V}}([\lambda x\, \forall y P y \rightarrow \neg Q x]) &= \mathbf{cond}(\overline{\mathbf{V}}([\lambda x\, \forall y P y]), \overline{\mathbf{V}}([\lambda x\, \neg Q x])) \\
&= \mathbf{cond}(\mathbf{vac}(\overline{\mathbf{V}}([\lambda\, \forall y P y])), \mathbf{neg}(\overline{\mathbf{V}}([\lambda x\; Q x]))) \\
&= \mathbf{cond}(\mathbf{vac}(\mathbf{univ}(\overline{\mathbf{V}}([\lambda y\; P y]))), \mathbf{neg}(\overline{\mathbf{V}}(Q))) \\
&= \mathbf{cond}(\mathbf{vac}(\mathbf{univ}(\overline{\mathbf{V}}(P))), \mathbf{neg}(\mathbf{q}_1)) \\
&= \mathbf{cond}(\mathbf{vac}(\mathbf{univ}(\mathbf{p}_1)), \overline{\mathbf{q}_1}) \\
&= \mathbf{cond}(\mathbf{vac}(\mathbf{p}_0), \overline{\mathbf{q}_1}) \\
&= \mathbf{cond}(\mathbf{p}_1, \overline{\mathbf{q}_1}) \\
&= \overline{\mathbf{q}_1}
\end{aligned}$$

We have therefore shown that our construction $\mathcal{I}^*$ is an interpretation of $\mathcal{L}$. All seven elements of an interpretation have been specifically identified and, as our example above should sufficiently illustrate, every complex 1-place predicate of our language denotes one of the four properties in the interpretation and every complex 0-place predication of our language denotes one of the four propositions.

Since we have shown in the Appendix that all the axioms of MOT are valid, it follows that they are all true in $\mathcal{I}^*$. It therefore only remains to be shown that all instances of **OC** are also true in $\mathcal{I}^*$. But this is immediate. For **OC** says that there is a unique abstract object for any definable collection of properties. But, in our construction, *every* collection of properties determines a unique abstract object, since the set of abstract objects is simply identified with the set of all sets of properties.[28]

5 Concluding Observations

In the foregoing, we have derived **EP**, the fundamental principle of world theory, from the general principles of (a minimal version of) object theory. Within object theory, worlds have a clearly defined nature that is given by the definition **PW**, which reveals them to be abstract objects that encode properties. As abstract objects, they also have clear identity conditions as given by $\mathbf{Id}_{A!}$ and clear existence conditions as given by **EP**. The proof of **EP** utilizes the comprehension principle **OC** and we included the abstraction principle $\mathbf{\Lambda_1}$ in our minimal object theory because it systematizes the properties that play a crucial role in the proof. All of these principles might seem to have serious ontological commitments when considered jointly. But our work shows that this is not the case. The general principles of object theory have minimal ontological commitments. Indeed, given our object-theoretic definition

[28]Note that paradox is avoided here because, in our model, properties are primitive entities and are not identified with sets of objects in the domain of the model. Hence, there can be fewer properties than there are sets of objects.

of possible worlds, we may suppose that in the smallest model of MOTC, the single possible world is one of the four abstract objects, and in the smallest non-trivial models of MOTC, the two possible worlds are among the sixteen abstract objects.[29] This further reduces the ontological commitments of MOTC and, hence, of **EP**. So we have a proof of **EP** that preserves it as an unrestricted plenitude principle committed only to small, finite domain, no matter whether one takes it as an axiom as most world theorists do or derives it from more general principles as we have done.

Of course, when we *apply* the above theory to our modal beliefs, the ontology of properties, propositions, and abstract objects, and thus, possible worlds, will grow. It is only by committing ourselves to a large body of data—specifically, a large body of false but possibly true propositions—that we become committed to the existence of a large body of nonactual possible worlds. But, of course, this is no fault of our theory. Indeed, it is precisely when we add those beliefs that our results become epistemologically significant. For in light of our work, we don't need, for each possible world in the ontology, special evidence for the existence of that world. Instead, we can cite **EP** as the principle that justifies our belief in the nonactual worlds that correspond to false, but possibly true, propositions. In turn, the justification of **EP** is grounded in the axioms of MOTC, and in particular, **OC** and **RE**. Thus, the epistemological justification for belief in possible worlds rests on two special principles of MOTC.

We conclude with one final observation, namely, that metaphysical questions concerning such matters as the ontological commitments of **EP**, the nature of possible worlds and what it means for a proposition to be true at a world simply have no definite meaning until one has a theory precise enough to answer them. In this paper we have provided such a theory. As other theories of possible worlds are founded upon similarly rigorous bases, philosophers will be in a better position to develop meaningful comparisons between them.

Acknowledgments We would like to thank Steve Awodey, Ole Hjortland, Hannes Leitgeb, Godehard Link, Paul E. Oppenheimer, Florian Steinberger as well as the members of the Departamento de Lóxica e Filosofía Moral at the Universidade de Santiago de Compostela, for comments on presentations and drafts of this paper. Our thanks also go to an anonymous reviewer for this *Journal* for helpful comments on an earlier version. Finally, we thank Elia Zardini for extraordinarily detailed and insightful written comments on an earlier draft, which led to substantial revisions and improvements. We would also like to acknowledge the Munich Center for Mathematical Philosophy (MCMP), where both authors spent time as visiting fellows during the 2011–12 academic year, and to the Alexander von Humboldt Foundation for its support of the MCMP.

[29] Specifically, in the non-trivial model, the actual world $\mathbf{w}^*$ is the abstract object represented by $\{\mathbf{p}_1, \mathbf{q}_1\}$ and the nonactual possible world is the abstract object represented by $\{\mathbf{p}_1, \overline{\mathbf{q}_1}\}$, where $\mathbf{p}_1 = \mathbf{vac}(\mathbf{p}_0)$, $\mathbf{q}_1 = \mathbf{vac}(\mathbf{q}_0)$, and $\overline{\mathbf{q}_1} = \mathbf{neg}(\mathbf{q}_1) = \mathbf{neg}(\mathbf{vac}(\mathbf{q}_0))$. The actual world is $\{\mathbf{p}_1, \mathbf{q}_1\}$ because it encodes the two propositional properties constructed out of the two propositions true at $\mathbf{w}^*$ (= $\mathbf{w}_0$), and the non-actual possible world is $\{\mathbf{p}_1, \overline{\mathbf{q}_1}\}$ because it encodes the two propositional properties constructed out of the two propositions true at $\mathbf{w}_1$.

Appendix: A Soundness Theorem for MOT

In this Appendix it will be shown that MOT is sound, i.e., that all instances of the schema $\mathbf{\Lambda_1}$ and all of the remaining logical axioms and rules of MOT found in Section 2.2 are true in every interpretation of $\mathcal{L}$. (As noted above, we are not arguing here that **OC** is a logical truth and hence we have not added conditions to the model theory for $\mathcal{L}$ that guarantee its validity.)

The Validity of the Basic Logic As our model theory is classical, our basic apparatus of classical propositional logic and second-order monadic quantification theory is unproblematically valid. That all the axioms of S5 are valid follows from the fact that no accessibility restrictions are placed on worlds in an interpretation. Moreoever, it is easy to verify by a straightforward induction that, if φ is valid, i.e., true at the actual world of every interpretation, then φ is true at every world of every interpretation of $\mathcal{L}$. Hence, if φ is valid, so is $\Box\varphi$. Consequently, the rule of Necessitation **RN** is sound.

The Validity of λ-Conversion Next we show that all instances of our 1-place abstraction principle $\mathbf{\Lambda_1}$ are valid. Actually, however, we will show something stronger, namely, that every instance of $\mathbf{\Lambda_1}$ is true at every world of every interpretation. More exactly, where $\mathbf{W}_\mathcal{I}$ is the set of "worlds" of an interpretation $\mathcal{I}$, we will show:

Lemma 1 *For every 1-place predicate* $[\lambda x\ \varphi]$ *and term* τ *of* $\mathcal{L}$, *and for every interpretation* $\mathcal{I}$ *of* $\mathcal{L}$, $\mathbf{w} \models_\mathcal{I} [\lambda x\ \varphi]\tau$ *iff* $\mathbf{w} \models_\mathcal{I} \varphi^x_\tau$, *for every* $\mathbf{w} \in \mathbf{W}_\mathcal{I}$.

We need no corresponding lemma for 0-place predicates, of course, because 0-place predicates are also formulas of $\mathcal{L}$. This eliminates the need for λ-predicates of the form $[\lambda\ \varphi]$ and, hence, the need to prove the validity of a 0-place abstraction principle, $[\lambda\ \varphi] \leftrightarrow \varphi$. However, because of the semantic interplay of 0- and 1-place predicates, particularly in condition V6, there is still a corresponding model theoretic fact about 0-place predicates φ that we need to establish in concert with Lemma 1, viz., that such predicates are semantically "harmonious", i.e., that the truth value of the proposition that a 0-place predicate φ denotes *qua* predicate walks in lockstep with the truth value of φ *qua* formula from world to world. More exactly, say that a 0-place predicate φ is *harmonious in* an interpretation $\mathcal{I}$ of $\mathcal{L}$ just in case $\mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}(\varphi)) = 1$ iff $\mathbf{w} \models_\mathcal{I} \varphi$, for all $\mathbf{w} \in \mathbf{W}_\mathcal{I}$. Then we need also to show:

Lemma 0 *All 0-place predicates are harmonious in all interpretations of* $\mathcal{L}$.

So let $\mathcal{I} = \langle \mathbf{W}, \mathbf{P}, \mathbf{D}, \mathbf{Op}, \mathbf{ex}, \mathbf{en}, \mathbf{V} \rangle$ be an arbitrary interpretation. The first task is to show that every predicate of $\mathcal{L}$ has a well-defined denotation of the appropriate arity. That denotations have the appropriate arity follows from the fact that, if an i-place predicate π ($i \in \{0, 1\}$) has a denotation $\overline{\mathbf{V}}(\pi)$ at all, it is a member of $\mathbf{P_i}$. That all such predicates do in fact have unique denotations follows from the fact that (i) every λ-predicate fits exactly one of the semantic clauses V1–V6 in the specification of $\overline{\mathbf{V}}$ in Section 4.1, (ii) the denotations of the primitive predicates of $\mathcal{L}$ are well-defined, and (iii) all of the logical functions in terms of which the denotations of predicates are defined are total on their given domains. These facts are easily, if somewhat tediously, verified.

Given that the predicates of $\mathcal{L}$ all denote appropriately, we now need to show that our two lemmas hold. Given the semantic interplay of 0- and 1-place predicates just noted, we must prove this by induction on predicable formulas φ simultaneously for both lemmas. We begin with a simple fact about 1-place predicates with vacuous λ-operators[30] that we appeal to at several points below:

Fact 1 *Let* $[\lambda x\ \varphi]$ *be a 1-place predicate such that* x *does not occur free in* φ *and suppose that* φ *is harmonious in* $\mathcal{I}$. *Then, for any term* τ, $\mathbf{w} \models_{\mathcal{I}} [\lambda x\, \varphi]\tau$ *iff* $\mathbf{w} \models_{\mathcal{I}} \varphi^x_\tau$, *for all* $\mathbf{w} \in \mathbf{W}$. □

Proof Fact 1 follows directly by the semantics of the **vac** operator: $\mathbf{w} \models_{\mathcal{I}} [\lambda x\, \varphi]\tau$ iff $\overline{\mathbf{V}}(\tau) \in \mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}([\lambda x\, \varphi]))$ (by T1) iff $\overline{\mathbf{V}}(\tau) \in \mathbf{ex}(\mathbf{w}, \mathbf{vac}(\overline{\mathbf{V}}(\varphi)))$ (by V6). But the latter is the case only if $\mathbf{ex}(\mathbf{w}, \mathbf{vac}(\overline{\mathbf{V}}(\varphi))) \neq \varnothing$; and by E6, that means $\mathbf{ex}(\mathbf{w}, \mathbf{vac}(\overline{\mathbf{V}}(\varphi))) = \mathbf{D}$, in which case $\overline{\mathbf{V}}(\tau) \in \mathbf{ex}(\mathbf{w}, \mathbf{vac}(\overline{\mathbf{V}}(\varphi)))$. So the latter is the case iff $\mathbf{ex}(\mathbf{w}, \mathbf{vac}(\overline{\mathbf{V}}(\varphi))) = \mathbf{D}$ which, by E6 again, is so iff $\mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}(\varphi)) = 1$ and hence, given that φ is harmonious, iff $\mathbf{w} \models_{\mathcal{I}} \varphi$, i.e., as $\varphi = \varphi^x_\tau$ (since x does not occur in φ), iff $\mathbf{w} \models_{\mathcal{I}} \varphi^x_\tau$. □

Now for the proof of our lemmas. We first prove that atomic formulas φ are harmonious. In the case where φ is simply a 0-place atomic formula (i.e., a primitive 0-place predicate) π, the result is immediate by T1. For 1-place atomic formulas $\rho\tau$, we have: $\mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}(\rho\tau)) = 1$ iff $\mathbf{ex}(\mathbf{w}, \mathbf{plug}(\overline{\mathbf{V}}(\rho), \overline{\mathbf{V}}(\tau))) = 1$ (by V1) iff $\overline{\mathbf{V}}(\tau) \in \mathbf{ext}(\mathbf{w}, \overline{\mathbf{V}}(\rho))$ (by E1) iff $\mathbf{w} \models_{\mathcal{I}} \rho\tau$ (by T1).

Now for the atomic case of Lemma 1. We first consider the case where φ is either a 0-place atomic formula π or a 1-place atomic formula $\rho\sigma$, where σ is not the variable x. Then in either case—the latter by clause 7 of the grammar for $\mathcal{L}$—if $[\lambda x\, \varphi]$ is a 1-place predicate, x does not occur free in ρ and hence in φ. But we have just established that, in either case, φ is harmonious. Hence, by Fact 1, $\mathbf{w} \models_{\mathcal{I}} [\lambda x\, \varphi]\tau$ iff $\mathbf{w} \models_{\mathcal{I}} \varphi^x_\tau$. So suppose instead that φ is a 1-place atomic formula ρx. Then we have: $\mathbf{w} \models_{\mathcal{I}} [\lambda x\, \rho x]\tau$ iff $\overline{\mathbf{V}}(\tau) \in \mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}([\lambda x\, \rho x]))$ (by E1) iff $\overline{\mathbf{V}}(\tau) \in \mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}(\rho))$ (by V1) iff $\mathbf{w} \models_{\mathcal{I}} \rho\tau$ (by T1), i.e., $\mathbf{w} \models_{\mathcal{I}} \rho x^x_\tau$.

Assuming now φ is of the form $\neg\psi$ and that the lemmas hold for ψ: $\mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}(\neg\psi)) = 1$ iff $\mathbf{ex}(\mathbf{w}, \mathbf{neg}(\overline{\mathbf{V}}(\psi))) = 1$ (by V2) iff $\mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}(\psi)) = 0$ (by E2) iff $\mathbf{w} \not\models_{\mathcal{I}} \psi$ (by our induction hypothesis) iff $\mathbf{w} \models_{\mathcal{I}} \neg\psi$, i.e., $\neg\psi$ is harmonious. And for the case of Lemma 1: If x does not occur free in ψ, then, as we have just shown that $\neg\psi$ is harmonious, our result is immediate by Fact 1. So suppose x does occur free in ψ. Then $\mathbf{w} \models_{\mathcal{I}} [\lambda x\ \neg\psi]\tau$ iff $\overline{\mathbf{V}}(\tau) \in \mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}([\lambda x\ \neg\psi]))$ iff $\overline{\mathbf{V}}(\tau) \in \mathbf{ex}(\mathbf{w}, \mathbf{neg}(\overline{\mathbf{V}}([\lambda x\ \psi])))$ (by V2) iff $\overline{\mathbf{V}}(\tau) \in \mathbf{D} \setminus \mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}([\lambda x\ \psi]))$ (by E2)

[30] Such predicates, recall, denote "propositional" properties, which are critical to the object theoretic analysis of worlds.

iff $\overline{\mathbf{V}}(\tau) \notin \mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}([\lambda x\ \psi]))$ iff $\mathbf{w} \not\models_{\mathcal{I}} [\lambda x\ \psi]\tau$ (by T1) iff (by our hypothesis) $\mathbf{w} \not\models_{\mathcal{I}} \psi^x_\tau$ iff $\mathbf{w} \models_{\mathcal{I}} \neg\psi^x_\tau$ (by T2).

Assuming φ is of the form $\psi \rightarrow \theta$ and our lemmas hold for ψ and θ: $\mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}(\psi \rightarrow \theta)) = 1$ iff $\mathbf{ex}(\mathbf{w}, \mathbf{cond}(\overline{\mathbf{V}}(\psi), \overline{\mathbf{V}}(\theta))) = 1$ (by V3) iff $max\{1 - \mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}(\psi)), \mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}(\theta))\} = 1$ (by E3) iff $\mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}(\psi)) = 0$ or $\mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}(\theta)) = 1$ (by T3) iff $\mathbf{w} \not\models_{\mathcal{I}} \psi$ or $\mathbf{w} \models_{\mathcal{I}} \theta$ (by our induction hypothesis) iff $\mathbf{w} \models_{\mathcal{I}} \psi \rightarrow \theta$, i.e., $\psi \rightarrow \theta$ is harmonious. For the case of Lemma 1: Given the preceding and Fact 1, our result is immediate if x does not occur free in $\psi \rightarrow \theta$, so suppose it does: $\mathbf{w} \models_{\mathcal{I}} [\lambda x\ \psi \rightarrow \theta]\tau$ iff $\overline{\mathbf{V}}(\tau) \in \mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}([\lambda x\ \psi \rightarrow \theta]))$ iff $\overline{\mathbf{V}}(\tau) \in \mathbf{ex}(\mathbf{w}, \mathbf{cond}(\overline{\mathbf{V}}([\lambda x\ \psi]), \overline{\mathbf{V}}([\lambda x\ \theta])))$ (by V3) iff $\overline{\mathbf{V}}(\tau) \in (\mathbf{D} \setminus \mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}([\lambda x\ \psi]))) \cup \mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}([\lambda x\ \theta]))$ (by E3) iff $\overline{\mathbf{V}}(\tau) \notin \mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}([\lambda x\ \psi]))$ or $\overline{\mathbf{V}}(\tau) \in \mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}([\lambda x\ \theta]))$ iff $\mathbf{w} \not\models_{\mathcal{I}} [\lambda x\ \psi]\tau$ or $\mathbf{w} \models_{\mathcal{I}} [\lambda x\ \theta]\tau$ iff (by our hypothesis) $\mathbf{w} \not\models_{\mathcal{I}} \psi^x_\tau$ or $\mathbf{w} \models_{\mathcal{I}} \theta^x_\tau$ iff $\mathbf{w} \models_{\mathcal{I}} \psi^x_\tau \rightarrow \theta^x_\tau$ (by T3) iff $\mathbf{w} \models_{\mathcal{I}} (\psi \rightarrow \theta)^x_\tau$.

Assuming φ is of the form $\Box\psi$ and that the lemmas hold for ψ: $\mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}(\Box\psi)) = 1$ iff $\mathbf{ex}(\mathbf{w}, \mathbf{nec}(\overline{\mathbf{V}}(\psi))) = 1$ (by V4) iff $min\{\mathbf{ex}(\mathbf{u}, \overline{\mathbf{V}}(\psi)) \mid \mathbf{u} \in \mathbf{W}\}) = 1$ (by E4) iff, for all $\mathbf{u} \in \mathbf{W}$, $\mathbf{ex}(\mathbf{u}, \overline{\mathbf{V}}(\psi)) = 1$ iff, for all $\mathbf{u} \in \mathbf{W}$, $\mathbf{u} \models_{\mathcal{I}} \psi$ (by our induction hypothesis) iff $\mathbf{w} \models_{\mathcal{I}} \Box\psi$ (by T4). For the case of Lemma 1, assuming again, given Fact 1 and the harmoniousness of $\Box\psi$ just established, that x occurs free in ψ: $\mathbf{w} \models_{\mathcal{I}} [\lambda x\ \Box\psi]\tau$ iff $\overline{\mathbf{V}}(\tau) \in \mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}([\lambda x\ \Box\psi]))$ iff $\overline{\mathbf{V}}(\tau) \in \mathbf{ex}(\mathbf{w}, \mathbf{nec}(\overline{\mathbf{V}}([\lambda x\ \psi])))$ (by V4) iff $\overline{\mathbf{V}}(\tau) \in \bigcap\{\mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}([\lambda x\ \psi])) : \mathbf{w} \in \mathbf{W}\}$ (by E4) iff, for all $\mathbf{u} \in \mathbf{W}$, $\overline{\mathbf{V}}(\tau) \in \mathbf{ex}(\mathbf{u}, \overline{\mathbf{V}}([\lambda x\ \psi]))$ iff, for all $\mathbf{u} \in \mathbf{W}$, $\mathbf{u} \models_{\mathcal{I}} [\lambda x\ \psi]\tau$ iff (by our induction hypothesis), for all $\mathbf{u} \in \mathbf{W}$, $\models_{\mathcal{I}} \psi^x_\tau$ iff $\mathbf{w} \models_{\mathcal{I}} \Box\psi^x_\tau$ (by T4).

Finally, we have the quantifier case. Assuming that φ is of the form $\forall y\psi$ and the lemmas hold for ψ and formulas of equal or lesser complexity: $\mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}(\forall y\psi)) = 1$ iff $\mathbf{ex}(\mathbf{w}, \mathbf{univ}(\overline{\mathbf{V}}([\lambda y\ \psi]))) = 1$ (by V5) iff $\mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}([\lambda y\ \psi])) = \mathbf{D}$ (by E5) iff, for all $\mathbf{a} \in \mathbf{D}$, $\mathbf{a} \in \mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}}([\lambda y\ \psi]))$ iff, for all $\mathbf{a} \in \mathbf{D}$, $\overline{\mathbf{V}^y_{\mathbf{a}}}(y) \in \mathbf{ex}(\mathbf{w}, \overline{\mathbf{V}^y_{\mathbf{a}}}([\lambda y\ \psi]))$ iff, for all $\mathbf{a} \in \mathbf{D}$, $\mathbf{w} \models_{\mathcal{I}^y_{\mathbf{a}}} [\lambda y\ \psi]y$ iff for all $\mathbf{a} \in \mathbf{D}$, $\mathbf{w} \models_{\mathcal{I}^y_{\mathbf{a}}} \psi$ (by our induction hypothesis) iff $\mathbf{w} \models_{\mathcal{I}} \forall y\psi$ (by T5), i.e., $\forall y\psi$ is harmonious. But now the needed instance of Lemma 1 follows as well by Fact 1. For by clause 7 of the grammar for $\mathcal{L}$, x cannot occur free in $\forall y\psi$ in a 1-place predicate of the form $[\lambda x\ \forall y\psi]$. So there is nothing more to prove in this case.

So we have shown that Lemmas 0 and 1 hold. From the latter, it follows that, for all $\mathbf{w} \in \mathbf{W}$, $\mathbf{w} \models_{\mathcal{I}} [\lambda x\ \varphi]\tau \leftrightarrow \varphi^x_\tau$ and so in particular, $\mathbf{w}^* \models_{\mathcal{I}} [\lambda x\ \varphi]\tau \leftrightarrow \varphi^x_\tau$. Since our interpretation $\mathcal{I}$ was chosen arbitrarily, we conclude that all instances of $\mathbf{\Lambda_1}$ are valid.

The Validity of the Identity Principles The schema **Ind** has instances of several forms depending on whether α and β are object variables or predicate variables. In the former case, when we unpack the definition **Id**, instances of **Ind** are of the form:

(17) $(x =_{A!} y \vee x =_{O!} y) \rightarrow (\varphi \rightarrow \varphi')$.

We shall therefore prove the validity of (all instances of) (17) by proving the validity of the schemas:

(18) $x =_{O!} y \rightarrow (\varphi \rightarrow \varphi')$

(19) $x =_{A!} y \rightarrow (\varphi \rightarrow \varphi')$.

(φ' in each case here is of course to be understood appropriately for the given schema.) We shall then demonstrate the validity of **Ind** generally by proving the validity of those instances where α and β are predicate variables, i.e.,

(20) $F = G \rightarrow (\varphi \rightarrow \varphi')$
(21) $p = q \rightarrow (\varphi \rightarrow \varphi')$.

All instances of (18) are in fact theorems of MOT, a proof of which is left for a footnote.[31] Beyond axioms of classical quantification theory and our underlying propositional modal logic, the proof appeals only to the axiom **AE**, whose validity is established below. It follows that (18) is valid.

The validity of (19) is trivial. Suppose '$x =_{A!} y$' is true in some interpretation and the values of those variables are abstract objects **a** and **b**, respectively. By $\mathbf{Id}_{A!}$, this means that **a** and **b** encode the same properties. But clause 5(i) of the definition of an interpretation in Section 4.1 for $\mathcal{L}$ guarantees that abstract objects that encode the same properties are genuinely identical. Hence, as the denotations of variables are fixed in all contexts, by clause 6 of the definition of an interpretation, variables denoting the same abstract object can be substituted one for the other *salva veritate*. (19), therefore, is valid. Hence, so is (17).

The validity of (20) is also trivial. For suppose '$F = G$' holds for arbitrary properties $\mathbf{p_1}$ and $\mathbf{q_1}$, respectively. By $\mathbf{Id_1}$ this means that $\mathbf{p_1}$ and $\mathbf{q_1}$ are encoded by the same abstract objects. But clause 5(ii) guarantees that properties that are encoded by the same abstract objects are genuinely identical. So again variables denoting the same property can be substituted one for the other *salva veritate*. (20), therefore, is valid.

Given the validity of (20), the validity of (21) follows directly. Suppose '$p = q$' holds in a given interpretation $\mathcal{I}$, for arbitrary propositions $\mathbf{p_0}$ and $\mathbf{q_0}$, respectively. By definition $\mathbf{Id_0}$, this means that '$[\lambda x\, p] = [\lambda x\, q]$' holds. By clause V6 in the definition of the denotation function for $\mathcal{I}$ and the fact shown in the preceding paragraph that

[31] The proof is by induction on the complexity of φ. The atomic exemplification case follows immediately from the definitions **O!** and $\mathbf{Id}_{O!}$ and the atomic encoding case follows from **O!** and **AE** which together yield $\neg zF$ for all ordinary objects z. The boolean cases are straightforward. So assume that **Ind** holds for formulas of complexity less than that of $\varphi = \forall\alpha\psi$. Then in particular $x =_{O!} y \rightarrow (\psi \rightarrow \psi')$. If α is x or y then, in either case, the only instances of **Ind** for $\forall\alpha\psi$ are those in which $\forall\alpha\psi = (\forall\alpha\psi)'$, rendering **Ind** trivial in those cases. So assume α is neither. Then by the rule of Generalization we have $\forall\alpha(x =_{O!} y \rightarrow (\psi \rightarrow \psi'))$. Since α does not occur free in $x =_{O!} y$ (see its definition $\mathbf{Id}_{O!}$), we have $x =_{O!} y \rightarrow \forall\alpha(\psi \rightarrow \psi'))$ by a simple theorem of classical quantification theory and so by $\forall$-distribution we have $x =_{O!} y \rightarrow (\forall\alpha\psi \rightarrow \forall\alpha\psi')$. Since $\forall\alpha\psi' = (\forall\alpha\psi)'$ given that α is neither x nor y, our result follows. For the modal case, assuming once again that **Ind** holds for formulas of complexity less than that of $\varphi = \Box\psi$, we have in particular $x =_{O!} y \rightarrow (\psi \rightarrow \psi')$. By **RN** and two applications of $\Box$-distribution, we have $\Box(x =_{O!} y) \rightarrow (\Box\psi \rightarrow \Box\psi')$. By the definition $\mathbf{Id}_{O!}$ of $=_{O!}$, the antecedent here, unpacked, is $\Box(O!x \wedge O!y \wedge \Box\forall F(Fx \leftrightarrow Fy))$, which by basic modal logic is equivalent to $\Box O!x \wedge \Box O!y \wedge \Box\Box\forall F(Fx \leftrightarrow Fy)$. By **O!** and (2) we have as a theorem $O!x \rightarrow \Box O!x$. From this and a bit of modal logic, in particular, instances of the T and S4 schemas, the preceding conjunction is equivalent to $O!x \wedge O!y \wedge \Box\forall F(Fx \leftrightarrow Fy)$, i.e., $x =_{O!} y$. Thus, substituting for $\Box(x =_{O!} y)$ above, we have $x =_{O!} y \rightarrow (\Box\psi \rightarrow \Box\psi')$ and hence, as $\Box\psi' = (\Box\psi)'$, we have our result.

'=' indicates genuine identity for properties, this means that $\mathbf{vac}(\mathbf{p_0})$ and $\mathbf{vac}(\mathbf{q_0})$ are identical. By the condition in the definition of an interpretation that the **vac** operation is one-to-one, it follows that $\mathbf{p_0}$ and $\mathbf{q_0}$ are themselves genuinely identical. Hence, 'p' and 'q' will be substitutable *salve veritate* in $\mathcal{I}$, i.e., (21) is valid. Since therefore, we have established the validity of (17), (20), and (21), we have thereby established the validity of **Ind**.

Finally, the validity of the reducibility principle **Red** is immediate from clause V1 in the definition of the denotation function.

The Validity of the Logical Principles for Abstract Objects The validity of the principle **RE** follows in virtue of clause T1 in the definition of truth in an interpretation in Section 4.1 and the fact that the encoding extension function **en** is not defined relative to worlds. The validity of $\Box A!$ is guaranteed by clause 4 in the definition of an interpretation, which stipulates that the extension of the distinguished property $\mathbf{p}^*$ at every world is the set **A** of abstract objects, and clause 6, which stipulates that $\mathbf{p}^*$ is the denotation of $A!$. And, finally, the validity of **AE** is guaranteed by the condition in clause 5 in the definition of an interpretation, which stipulates that the encoding function maps each property to a subset of the set **A** of abstract objects.

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