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Intuitionistic Epistemic Logic, Kripke Models and Fitch's Paradox

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Abstract The present work is motivated by two questions. (1) What should an intuitionistic epistemic logic look like? (2) How should one interpret the *knowledge* operator in a Kripke-model for it? In what follows we outline an answer to (2) and give a model-theoretic definition of the operator K. This will shed some light also on (1), since it turns out that K, defined as we do, fulfills the properties of a necessity operator for a *normal* modal logic. The interest of our construction also lies in a better insight into the intuitionistic solution to Fitch's paradox, which is discussed in the third section. In particular we examine, in the light of our definition, DeVidi and Solomon's proposal of formulating the verification thesis as $\phi \to \neg\neg K\phi$. We show, as our main result, that this definition excapes the paradox, though it is validated only under restrictive conditions on the models.

Keywords Intuitionistic logic · Epistemic logic · Fitch's paradox · Kripke models

1 Introduction

Epistemic concepts are deeply entrenched in intuitionistic logic. They are at the heart of the usual explanation of truth as *provability by an ideal reasoner*. Intuitionistic Kripke models (IKM) provide a model-theoretic characterization of these logics, usually presented in epistemic words. More specifically, these models are based on partially ordered structures, intuitively representing the

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evolutionary process of the (monotonically) growing informational state of an ideal agent.

Considering that IKMs are based on simple relational structures similar to the models of classical modal logic, it is puzzling why the seemingly natural issue of investigating an intuitionistic epistemic logic based on these models has not been deeply explored thus far.[1]

This question is made more critical by the debate over Fitch's paradox of knowability, which constitutes the main philosophical thrust of this paper (Section 4). Fitch's paradox is usually presented as a *reductio* of the verificationist claim that *every truth is knowable* to the much more problematic claim that *every truth is known*. It has been argued from many sides that this *reductio* can be blocked, in a principled way, if one reasons according to the rules of intuitionistic logic.[2] An IKM-based approach will help to clarify the rationale behind this claim and better explore its consequences. Such a claim has already been made and investigated by DeVidi and Solomon [3], inspiring the present work. However, DeVidi and Solomon's analysis focuses mainly on the behaviour of the logical constants in IKMs; they do not formulate any explicit semantics for the knowledge operator K. The question of how knowledge of an agent should be interpreted remains open, as well as the question of how K interacts with logical operators.

The present work fills this gap by defining, in Section 3, a semantics for the K operator that reads $K\phi$ as "ϕ is the case in all the informational expansions of the agent's present state that are consistent with its evolution". The agent is supposed to be able to discard, as possible epistemic alternatives, (i) all the states containing less information than the actual one and (ii) all the states in which information has been acquired in a different order. Eliminating the (i) states amounts to supposing that the agent is fully aware of the information she holds (but not necessarily of whatever happens in the world), while discarding the (ii) states amounts to perfect recall. K will represent the knowledge of an ideal agent (who is logically omniscient). The most important thing to stress is that, even in this limit case of perfection, the collapse of knowability into actual knowledge can be blocked in a principled way, and relevant claims on the intuitionistic notion of knowledge can be model-theoretically justified. Although I am not persuaded of the decisiveness of the intuitionistic answer to Fitch's paradox, nevertheless the technical achievement of the present paper should offer clarifications, and a more stable basis for discussion and for further refinements.

The second section will be devoted to investigating several, more general questions. The first is whether, and to what extent, using the explicit knowledge operator K is useful in intuitionistic logic and reasoning. The second is whether

[1] This is probably due to the attitude of the intuitionistic tradition, favoring the more "constructive" proof-theoretic semantics. This attitude is reflected as well in the philosophical applications of intuitionistic epistemic logics.

[2] The paradox is derived by means of classical epistemic logic.

IKMs are a correct semantic tool to express the meaning of a knowledge operator, as well as whether they can be a useful explanatory tool. The third is whether the structure of an intuitionistic epistemic logic should be a *normal* one. I will show that the way I define the accessibility relation underlying K satisfies the conditions for normality as isolated in Bozic and Dosen [1] (with further refinements by Wolter and Zakharyaschev [13]). In this way, all the soundness and completeness results for a normal modal intuitionistic calculus will also hold for my definition.

In the Appendix, I present a short summary of all the basic model-theoretic definitions and results that are needed to make this work self-contained.

2 The Ideal Reasoner and the Empirical World

The intuitionistic notion of truth is often presented, in a Brouwerian account, as *provability in principle by an ideal mathematician or reasoner*. Here, I wish to consider the case of the ideal reasoner confronted by a world where contingent facts occur. One can then ask the basic question whether *being true* and *being known* should be understood as two different concepts for her, where truth and meaning are characterized by the intuitionistic logical operators. Tennant [7] has classified a negative answer to this question as a form of *hard anti-realism*. Nevertheless, it seems intuitive that, if intuitionistic logic should serve for modeling not only mathematical discourse but also empirical reasoning,[3] then such a distinction is unavoidable.

Intuitionistic truth is intended, I have said, as provability in principle or, in empirical discourse, as verifiability in principle. Such a notion of truth has the epistemic flavour of implicit knowledge: something is true whenever the ideal reasoner can verify it, given unbounded resources of calculating power, time, memory, etc. Implicit knowledge is to be distinguished from explicit. Explicit knowledge, even for an ideal reasoner, concerns data the reasoner already has; it cannot be about contingent future facts or unobserved ones.[4] Even for an ideal reasoner, one cannot equate explicit knowledge and truth.

This contrast, between the explicit and implicit information available to an ideal reasoner, seems to be deeply entrenched in IKMs. These models consist of partially ordered sets of points that represent possible informational states.[5] Some formulae in these states are satisfied (or better forced; see the

[3] A large part of the anti-realist tradition coming out of M. Dummett's works in the '70s foreshadows these applications.

[4] In a recent paper [9] J. van Benthem also hints at these two different notions of implicit/explicit knowledge to be found in IKMs. He suggests that the second one could be regarded as a notion of "seeing that".

[5] Strictly speaking, IKMs are *pre-orders*: i.e., they are *reflexive* and *transitive*. For the purposes of the present paper, I will focus on *partial* orders, where the relation is also anti-symmetric. These states contain all the (intuitionistic) logical theorems, and they are closed under intuitionistic logical consequence.

definition in the Appendix) only by reference to the present informational state. For other formulae, satisfaction is tested with respect to other (upper) states. Atomic propositions are formulae of the first kind. Formulae involving negations or implications are formulae of the second kind. Propositional atoms can easily be regarded as the basic factual information that an agent can know by an explicit act of seeing.

The partial ordering of the states in these models intuitively represents the evolutionary process of the monotonically increasing informational states of an ideal reasoner. Moreover, the ordering of informational states in an empirical context can be seen as a kind of temporal process: upper states are possible future states and there might also be, in a partial ordering, unrelated states representing alternative, non-actual situations that the agent could have faced.

What about the knowledge operator and its semantics? If one defines it in the usual way (ϕ is known by an ideal reasoner whenever it is either satisfied or forced in all of the states that are epistemically accessible to her), then the generalization rule is sound, and it will give rise to a normal modal logic that validates the Kripke axiom **K** ($\Box(\phi \to \psi) \to \Box\phi \to \Box\psi$). Such features make the agent logically omniscient, but this need not be a big problem in this case: after all, we are dealing with an ideal reasoner.[6]

Another important question is whether explicit knowledge should validate other, usual axioms of classical epistemic logic such as **T** ($K\phi \to \phi$), **4** ($K\phi \to KK\phi$), and **5** ($\neg K\phi \to K\neg K\phi$). In the case of **T** and of **4** (also known as the *axiom of positive introspection*), the answer is affirmative: the explicit knowledge of an agent in some given state is likely to imply truth at the same time and be positively introspective. Since axioms **T** and **4** also express these conditions in the intuitionistic semantics (see the Appendix), they must be valid. The case of **5**, the axiom of *negative introspection*, is more complicated, but one thing is clear: due to the presence of negation, the informal reading of this axiom does not correspond to the semantic condition intended by *negative introspection* in a classical framework (if $K\phi$ is not forced [in a state], then it is not forced in any accessible state) but to a stronger condition: under a temporal reading, it says intuitively that if ϕ will never be known, then it is *known* that it will never be known.[7] The definition of the epistemic operator offered here will invalidate both **5** and the weaker standard semantic condition for *negative*

[6]The problem of logical omniscience is a major issue in classical epistemic logic and is the source of a vast literature. In the typical case of an agent having only limited computational resources, the agent is unlikely to know all logical theorems and all of the logical consequences of her knowledge base. However, one can suppose, as suggested by J. Hintikka and reaffirmed by others, that logical omniscience is not a real problem when dealing with an ideal agent possessing unlimited computational resources. In our case, the ideal agent can be regarded as a sort of perfect, instantaneous, intuitionistic theorem-prover, whose only indecision lies with unestablished factual content.

[7]The antecedent of **5** does not say what may happen if $K\phi$ is not forced in a particular state but is at a later one. In other words, in principle **5** could be valid even if the classical condition for negative introspection is not satisfied.

introspection. However, it is better to postpone a more detailed discussion of this point until after the definition of K.

3 IKMs and Knowledge

Given the previous interpretation of what is represented in a Kripke model, epistemic accessibility may be defined in many ways. In Section 3.2, I will introduce my proposal. Before that, it will be useful to recapitulate the basic model-theoretic semantics for intuitionistic modal logic. The reader may also refer to Bozic and Dosen [1] or Wolter and Zakharyaschev [13].

3.1 Frames and Models

The class **HK** of frames for intuitionistic modal logic contains, as its elements, bi-relational expansions of the mono-relational Kripke frames for intuitionistic propositional calculus.

Definition 3.1 (Frames and Models) A frame for **HK** is a triple $\mathcal{F} = (W, R_{\leq}, R_M)$, where W is a non-empty domain and $R_{\leq}$ and R_M are subsets of $W \times W$ fulfilling the following conditions:

(1) $R_{\leq}$ is a pre-order on W (i.e., it is reflexive and transitive).
(2) $R_{\leq} R_M \subseteq R_M R_{\leq}$.[8]

(2) is a necessary and sufficient condition to ensure *monotonicity* (see the Appendix), guaranteeing that our models extend (monotonic) IKMs for propositional calculus. A model is a couple $\mathcal{M} = (\mathcal{F}, V)$, where $\mathcal{F}$ is a frame for **HK** and V is a function from the set of propositional variables into $\mathcal{P}(W)$ such that:

(3) For all $w, v \in W$ such that $w R_{\leq} v$ and for all propositional variables p, if $w \in V(p)$ then $v \in V(p)$.

A non-classical notion of satisfaction $\models^i$, called the *forcing relation*, is defined as follows:[9]

1. $w \models^i p$ iff $w \in V(p)$.
2. $w \models^i \phi \wedge \psi$ iff $w \models^i \phi$ and $w \models^i \psi$.
3. $w \models^i \phi \vee \psi$ iff $w \models^i \phi$ or $w \models^i \psi$.
4. $w \models^i \phi \rightarrow \psi$ iff for all v such that $w R_{\leq} v$, if $v \models^i \phi$ then $v \models^i \psi$.
5. $w \models^i \neg\phi$ iff for all v such that $w R_{\leq} v$, $v \not\models^i \phi$.
6. $w \models^i \Box\phi$ iff for all v such that $w R_M v$, $v \models^i \phi$.

[8] Expressions of the form $R_1 R_2$ indicate composition, also written as $R_1 \circ R_2$: i.e., $w R_1 R_2 v$ iff there is some z such that $w R_1 z$ and $z R_2 v$.

[9] I will adopt the superscript *i* ("intuitionistic") in order to distinguish this notion from the classical one. I will omit superscripts wherever no confusion is possible.

The clause for $\Box$ gives the truth conditions for any universal modal operator the language may contain, such as *metaphysical necessity*, *knowledge*, *belief* or other.

3.2 The Knowledge Operator

Many alternatives are available for defining the knowledge operator K based on this general semantics. A naive one identifies the epistemic accessibility relation with the pre-ordering relation $R_{\leq}$, thus yielding the following semantic definition:

$(K1)\quad w \models^i K\phi$ iff for all w', such that $w R_{\leq} w'$, $w' \models^i \phi$.

In this case, truth and knowledge are conflated due to *monotonicity* (i.e. if $w R_{\leq} v$ and $w \models^i \phi$ then $v \models^i \phi$), as the *hard* anti-realist would desire.

Truth and knowledge are not conflated only if some epistemically accessible states are not accessible by the $R_{\leq}$ relation. When one reads $R_{\leq}$ as a temporal relation, there should be some epistemically accessible states that are *not* possible future states. A stronger claim would be that the class of epistemically accessible states is wider than the class of temporally (or causally) accessible states, even for an ideal reasoner. This seems reasonable enough. Given that informational states cannot take into account all the external features of the world, some might be conceivable as consistent with the actual world, while still not attainable, because the course of events has, unnoticed to the agent, already dismissed them .

My suggestion is that the agent should consider as epistemic alternatives *all* the informational upgrades consistent with her state (not only actual future states) except for those having a different history: after all, the agent is supposed to be able to keep track of the way and the order in which she acquired information.

In order to implement such an intuition, one must first consider the notion of *informational equivalence among states*, defined as:

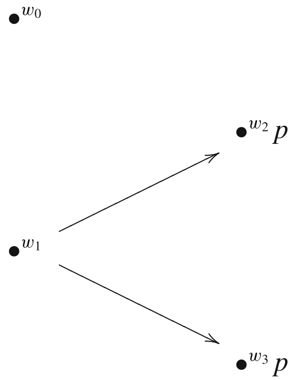
Definition 3.2 (Informational Equivalence $\equiv$) Two states in a model, w and v, are informationally equivalent ($w \equiv v$) iff for all atomic p, $w \in V(p)$ iff $v \in V(p)$.

In other words, two states are informationally equivalent when they force the same basic propositions. Let $Pr(w)$ denote the set of $R_{\leq}$-predecessors of state w, including w. Then the informal intuition can be rephrased more precisely as:

Definition 3.3 (R_K) $w R_K v$, iff there is a bijective function f from $Pr(w)$ into a *downward closed* (w.r.t. $R_{\leq}$)[10] subset of $Pr(v)$, such that, for all $z \in Pr(w)$, $f(z) \equiv z$.

[10] One can say that a set X is *downward closed* iff whenever $x \in X$ and $y R_{\leq} x$, $y \in X$.

Fig. 1 The failure of **5**



One can then give, based on R_K, the usual semantic clause for the K operator.

(K) $\quad w \models^i K\phi$ iff, for all w' such that $w R_K w'$, $w' \models^i \phi$.

Now, the only epistemic alternatives for a state w are those which may be reached by prolongating a path isomorphic to the one which actually leads to w. As noted earlier, the order in which information is acquired counts,[11] and this imposes *perfect recall*. Weaker mnemonic capacities can be attributed to the agent by relaxing the condition of similarity between histories. (Isomorphism is, indeed, a strong one.)

It is easy to verify that the relation R_K, thus defined, is such that $R_\leq \subseteq R_K$ and satisfies the basic property of an R_M-relation for intuitionistic normal modal logic, as defined in Section 3.1. The only thing still to be verified is $R_\leq R_K \subseteq R_K R_\leq$ (see Lemmas 1 and 2, Appendix). Suppose that $w R_\leq R_K v$. Then, for some z, $w R_\leq z R_K v$. This implies that $Pr(z)$ is isomorphic to a downward closed subset of $Pr(v)$. Since $w R_\leq z$, the same obviously holds for $Pr(w)$, and thus $w R_K v$. By the reflexivity of $R_\leq$, the result is established.[12]

With this definition in hand, one can return to the additional axioms for epistemic logic (**D**, **T**, **4**, and **5**). It is easy to verify that the relation R_K is serial, reflexive, and transitive, and thus validates axioms **D**, **T**, and **4** (see Appendix). It follows that this notion of knowledge is factive (Axiom **T**) and satisfies positive introspection (Axiom **4**). On the other hand, Axiom **5** fails. This can be illustrated by the failure of the canonical property defined by this axiom (see Appendix A.1);[13] or, more simply, by the example shown in Fig. 1.

[11] Indeed, suppose that a state w forces both p and q, but has just one predecessor forcing q but not forcing p. Then, those informational extensions of w whose predecessors force p before forcing q are automatically excluded.

[12] For completeness sake, one should also note that this property is trivially satisfied in the case where $R_K = R_\leq$, for it consists in the identity $R_\leq R_\leq = R_\leq R_\leq$.

[13] Our relation is also not *Euclidean*: that is to say, it does not satisfy the canonical property defined by **5** in classical modal logic.

In the model presented in Fig. 1, (i) $w_0 \models^i \neg Kp$ (since $w_0 \not\models^i Kp$ and w_0 has no strict successors). However, (ii) $w_2 \not\models^i \neg Kp$ (since $w_2 \models^i Kp$), and (iii) $w_0 R_K w_2$. Together, (ii) and (iii) imply (iv) $w_0 \not\models^i K\neg Kp$. Thus, by (i) and (iv), one obtains $w_0 \not\models^i \neg Kp \to K\neg Kp$.[14]

As hinted earlier (Section 2), this axiom is not strictly related to the standard semantic condition for negative introspection.[15] This condition, i.e. "if Kϕ is not forced at some state, then it is not forced at any accessible state", also fails in my model, for the simple reason that the agent considers some informational upgrades as open possibilities. Failure of negative introspection is a sensible concern, given that we are considering an ideal reasoner; but that failure is justified in the case of explicit knowledge. There is, indeed, an asymmetry with respect to *positive introspection*, which prescribes that the agent should be aware of what she explicitly sees. This seems reasonable. Being aware that one does not *explicitly* see something is a more complicated matter, for one may believe it anyway or find it very likely.[16]

4 Knowability and Fitch's Paradox

One of the most debated topics linking intuitionistic logic to the notions of knowledge and knowability has been, for the last two decades, Fitch's paradox of knowability. It consists of a modal argument leading from the apparently innocent assumptions that *every truth is possibly known* **(VT)** $\forall\phi(\phi \to \diamond K\phi)$, and that *there is some unknown truth* **(NO)** $\exists\phi(\phi \wedge \neg K\phi)$, to the stronger and counterintuitive conclusion that *every truth is known* **(AK)** $\forall\phi(\phi \to K\phi)$.[17] The use of propositional quantifiers is dispensable in formulating the paradox, nonetheless it is helpful in order to understand the scope of the intuitionistic solution (see Williamson [11]). I will take the license to use quantifiers also later on (e.g. on p. 16), where needed for the sake of explanation, even if they are not part our modal propositional language.

[14]One may easily observe from this counterexample the failure of the **B** axiom $\phi \to K\neg K\neg\phi$. Nevertheless, the failure of a *Brouwerian* axiom in an intuitionistic system is less problematic than it may seem at first sight (see footnote 20 for an explanation). One may further notice that the defined relation is clearly not symmetric: i.e., it does not satisfy the canonical property defined by **B** in classical modal logic.

[15]In order to produce a counterexample to **5**, one needs to find a case where p is not known and will never be known in any possible future *informational upgrade*, even while this fact remain hidden to the ideal knower, who considers as an open possibility a state in which she knows p. The reader will note that this model still does not satisfy the condition for **5**: i.e., condition (d) in the Appendix. According to this condition, if one wants to secure **5** at state w, then every R_M-accessible state should have R_M access back to an $R_\leq$ successor to w—which does not happen in my model. This condition seems unrelated with being Euclidian: i.e., the property defined by **5** in classical modal logic.

[16]For the same reason, Hintikka [6] discards the axiom of negative introspection.

[17]**(VT)** is shorthand for *verification thesis*. **(NO)** stands for *non-omniscience* and **(AK)** for *actual knowledge*.

4.1 Intuitionistic Perspectives on the Fitch Paradox

The paradox is usually presented in the following derivation.

1.	$\exists\phi(\phi \wedge \neg K\phi)$	**(NO)**
2.	$\forall\phi\,(\phi \rightarrow \diamond K\phi)$	**(VT)**
3.	$(p \wedge \neg Kp)$	instantiation of **(NO)**
4.	$((p \wedge \neg Kp) \rightarrow \diamond K(p \wedge \neg Kp))$	(by 2 and 3)
5.	$\diamond K(p \wedge \neg Kp)$	MP
6.	$\diamond(Kp \wedge K\neg Kp)$	distribution of K over $\wedge$
7.	$\diamond(Kp \wedge \neg Kp)$	by **T** on the second conjunct
8.	$\bot$	by 7 and $\diamond\bot \leftrightarrow \bot$
9.	$\neg\exists p\,(p \wedge \neg Kp)$	$\neg$-intro, discharge 1
10.	$\forall p\,\neg(p \wedge \neg Kp)$	definition of $\exists$
11.	$\forall p\,(p \rightarrow Kp)$	definition of $\rightarrow$

This derivation is generally perceived as presenting a paradox for verificationism, when **(VT)** is taken to be a correct representation of the verificationist principle according to which every truth is knowable.[18]

T. Williamson (see Williamson [10, 11, 12]) inaugurated the revision of Fitch's derivation via intuitionistic logic in the 1980s. The usual strategy for this approach involves blocking one or more steps of the derivation for a simple logico-philosophical reason: a verificationist (or anti-realist) logic *stricto sensu* should be weaker than classical logic, which lacks certain fundamental constructive aspects. The formulation of the verification thesis should then be considered meaningful only within a constructive framework, of which intuitionistic logic is one. However, Fitch's derivation adopts classical logic, along with its theorems and inference rules. The crucial point raised by Williamson is that the step from 10 to 11 is not intuitionistically justified. Therefore, the conclusion (11) ought not to be considered a blow for verificationism.

The usual objections to this type of solution claim that other, paradoxical consequences can be derived if one is using only intuitionistically sound inference rules. Intuitionistic counter-objections claim that all these other consequences are paradoxical or counterintuitive only under a classical reading but are perfectly acceptable under an intuitionistic one. It is not my purpose here to go into the whole discussion of possible counter-objections, going to the *n*th degree of refinement. Far more important is that amind all the controversy, there is no common ground nor shared semantics of the modal operators involved, especially the knowledge operator. I think that the model-theoretic interpretation of K illustrated in the previous section can be a useful tool for clarification. As mentioned in the introduction, DeVidi and Solomon [3] sketched a move in this direction, but without defining K.

[18]The equation of the formal **(VT)** with the verificationist principle is hotly debated. I myself find it very problematic. However, a discussion at this point would lead too far away from the principal focus here.

In analyzing Fitch's derivation, the intuitionist agrees *in toto* with what concerns the derivation of a contradiction and the consequent discharging of **(NO)** $\exists\psi\ (\psi \wedge \neg K\psi)$, which generates $\neg\exists\psi\ (\psi \wedge \neg K\psi)$ and, subsequently, $\forall\psi\neg(\psi \wedge \neg K\psi)$. Thus, the intuitionistically minded philosopher is disposed to agree for all proposition ϕ that, given the intuitionistic sense of the negation (and of the conditional), it is not the case that ϕ is true and *not* known. What she counters is the derivation, from this, of the thesis of *actual knowledge* **(AK)** $\forall\psi\ (\psi \rightarrow K\psi)$. She claims that, instead, we can only derive

(AK') $\forall\psi\ (\psi \rightarrow \neg\neg K\psi)$

which is classically, but not intuitionistically, equivalent to **(AK)**.

The classically-minded reply claims that **(AK')** is just as problematic as **(AK)**, for it would mean that, for all ϕ, if ϕ is true, then it is not true that ϕ is not known. For them, this is counterintuitive to the same degree. This is not the only problem: other intuitionistically derivable formulae seem likewise highly problematic:

1. $\neg K\phi \rightarrow \neg\phi$.
2. $\neg K\phi \leftrightarrow \neg\phi$.
3. $\neg(\neg K\phi \wedge \neg K\neg\phi)$.

On the other hand, DeVidi and Solomon [3] write that these problems are due to a faulty reading of negation, so that **(AK')** should rather be understood, by a constructive reading, as saying that it is not possible to find some truth which is unknowable; or else, by a proof-theoretic turn, that, for all ϕ, given a proof of ϕ, an absurdity follows from $K\phi$ implying an absurdity. Under the same reading, (1), (2), and (3) all become plausible.[19] DeVidi and Solomon suggested that, if we consider Kripke models and their semantics, it is immediately obvious that $K\phi$ and $\neg\neg K\phi$ have two, very different meanings. Indeed, given the model-theoretical meaning of double negation in this context, we can even consider **(AK')**, by itself, as a plausible formulation of the verification thesis.[20]

One can analyze these claims using the definition of the knowledge operator that I have offered. First, note that $\phi \rightarrow K\phi$ can easily be falsified. Indeed, in

[19] The constructive reading (1) corresponds to "if knowing ϕ is absurd (i.e., impossible) then ϕ is also absurd (i.e., impossible)", which is more acceptable than the classical reading, according to which ignorance entails falsity. The converse of (1) is even more plausible, making (2) also acceptable. Meanwhile, (3) does not say that one should either know ϕ or its negation, but rather that it is absurd (impossible) that both knowing ϕ and knowing $\neg\phi$ entails an absurdity.

[20] Indeed, if we interpret $R_{\leq}$ as a modal relation, then double negation should be read classically as $\Box\Diamond$, and thus $\neg\neg\phi$ stands for the impossibility of the impossibility of ϕ. Under this reading of double negation, Axiom **B** of modal logic $\phi \rightarrow \Box\Diamond\phi$ corresponds to the intuitionistically valid law $\phi \rightarrow \neg\neg\phi$. This is precisely the reason why O. Becker named this modal axiom after L.E.J. Brouwer (see Goldblatt [5], p. 315). Double negation may thus be seen as a *strong possibility operator*.

the preceding section $\neg Kp \rightarrow K\neg Kp$, which is an instance of this schema, was falsified. Second, note that $\phi \rightarrow \neg\neg K\phi$ can also be falsified.[21]

In general, if R_K were arbitrarily defined, the conditions for the validity of **(AK')** would be given by the following

Theorem 1 *Let* HAK′ *be the class of Kripke frames for intuitionistic modal logic fulfilling the condition*

$$\forall w \forall v \left(w R_{\leq} v \rightarrow \exists z \left(v R_{\leq} z \wedge \forall y \left(z R_K y \rightarrow w R_{\leq} y\right)\right)\right)$$

Then for any frame $\mathcal{F}$

$$\mathcal{F} \models^i \phi \rightarrow \neg\neg K\phi \textit{ iff } \mathcal{F} \in \texttt{HAK}'$$

Proof See Appendix A.3. □

But in the case of the R_K we are working with the situation is more complicated, since this relation is defined on a pre-given IKM and thus the question now is whether there is a class *of models* defining **(AK')**. Finding a necessary and sufficient condition for validity is more complex in this case.[22] Nevertheless, it is possible to isolate a class MAK′ of models in which $\phi \rightarrow \neg\neg K\phi$ is valid but $\phi \rightarrow K\phi$ is not. Theorem 2 gives a sufficiency condition for **(AK')** to be satisfied and for the anti-realist claim to be justified. This condition is motivated by the first-order translation of **(AK')**; by the counterexample offered in footnote 21; and, in particular, by the fact that this counterexample seems at least as strong as J.P. Burgess' *discovery principle*. The *discovery principle* was so named in Burgess [2] and represents the temporal version of **(VT)**: i.e., $\phi \rightarrow FK\phi$, which says that every truth will, sooner or later, be known. The result stated in the theorem is based on the following lemma.

Lemma 1 *Let w be a point in a model such that, for every point v, if wR_Kv then $v \equiv w$. Then, for every formula ϕ:*

(1) $w \models^i \phi$ *iff* $\forall v$ *such that* wR_Kv, $v \models^i \phi$
(2) $w \not\models^i \phi$ *iff* $\forall v$ *such that* wR_Kv, $v \not\models^i \phi$

Proof See Appendix A.3. □

From this lemma, the following theorem is derived.

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[21]One just needs to consider a two-dimensional model based on the domain $\mathbb{N} \times \mathbb{N}$, where $(n, k) R_{\leq} (n', k')$ iff $n = n'$ and $k \leq_N k'$ ($\leq_N$ is the usual ordering on natural numbers). Given a propositional letter p, define $V(p)$ as the set of (n, m) such that $n \geq_N 1$ and $m \geq_N n$. Then, clearly, $(0, 0) \models \neg p$. At the same time, one can see that $(0, 0) \not\models \neg\neg K\neg p$, since, at every $(0, i)$, the alternative $(i + 1, i + 1)$ is accessible, although clearly $(i + 1, i + 1) \not\models \neg p$ (for $(i + 1, i + 2) \models p$).

[22]The hard task consists in showing that a given condition is *necessary* for (AK') to be valid. In the case of frames, e.g. in Theorem 1, this is done by proving that when the condition fails in a frame one can define a falsifying model based on the same frame. But when working on a class of given *models* one cannot so easily find a falsifying valuation.

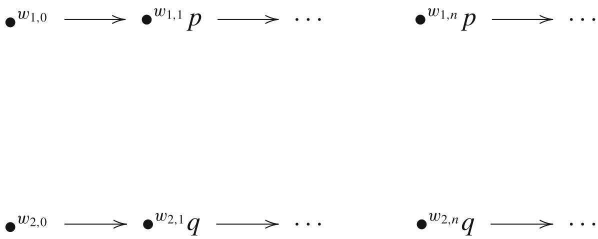

Fig. 2 Falsification of $\phi \rightarrow K\phi$

Theorem 2 *Let* $\mathtt{MAK'}$ *be the subclass of models fulfilling the following condition.*

$$\forall w \forall v \left(w R_{\leq} v \rightarrow \exists z \left(v R_{\leq} z \wedge \forall z' \left(z R_K z' \rightarrow z' \equiv z \right) \right) \right)$$

Then

$$\models^{i}_{MAK'} \phi \rightarrow \neg\neg K\phi$$

Proof See Appendix A.3. □

It is possible to show that the schema $\phi \rightarrow K\phi$ can be falsified within the class $\mathtt{MAK'}$. This is the case of the model presented in Fig. 2. This model fulfills the conditions of Theorem 2 for the simple reason that all successors of $w_{1,1}$ and $w_{2,1}$ are informationally equivalent to these states, and there are no epistemic alternatives. One may observe that $w_{1,0} \models^i \neg q$ and, since $w_{1,0} R_K w_{2,0}$ and $w_{2,0} \not\models^i \neg q$, then $w_{1,0} \not\models^i K\neg q$. But, as I showed via the preceding result (although it is also directly verifiable), one concludes that $w_{1,0} \models^i \neg\neg K\neg q$.

As I mentioned, the condition on $\mathtt{MAK'}$ is very strong. Despite the fact that it is not necessary for **(AK')** to hold (see Appendix),[23] available counterexamples are cases in which R_K stops branching at some point. These too are cases in which every sort of indecision about empirical facts ends up being settled, and that seems, as an assumption, at least as optimistic as the *discovery principle*. As stressed by Burgess, this last is rather a theological principle and is far from the anti-realist spirit. Finding a necessary condition for validating **(AK')**, and not **(AK)**, would thus be relevant for testing its possible acceptance by an anti-realist.

Inside the class $\mathtt{MAK'}$, one can also reconsider the other "bad" consequences: i.e.,

1. $\neg K\phi \rightarrow \neg\phi$.
2. $\neg K\phi \leftrightarrow \neg\phi$.
3. $\neg(\neg K\phi \wedge \neg K\neg\phi)$.

[23]The necessity of this condition for (AK') has been an open question until the last stages of the review process. I owe to Sebastian Enqvist a negative answer and the counterexample given in the Appendix.

In the semantics presented here, the condition imposed on MAK′ is also a sufficient condition for (1): i.e., states in a MAK′-model where, for all informational upgrades, $K\phi$ is never forced are states in which ϕ is also never forced.

Point (2) is an immediate consequence of (1) and of the converse of factivity **T**; and it holds also in this class of models. More importantly, (3) is a consequence of (2), since $(\neg K\phi \wedge \neg K\neg\phi)$ would be, by substitution of equivalents, equivalent to the contradiction $(\neg\phi \wedge \neg\neg\phi)$. One can prove that (3) is also valid in the class MAK′. Consider an arbitrary point w in a MAK′-model. There are two possible options. In the first, for some v such that $wR_{\leq}v$, $v \models^i \phi$. In this case, by the defining condition of MAK′-models, there is also a v' such that $vR_{\leq}v'$ and $v' \models^i K\phi$. Thus it follows that $w \not\models^i \neg K\phi$ (since $wR_{\leq}v$) and, more generally, (i) $w \not\models^i \neg K\neg\phi \wedge \neg K\phi$. In the second, for all v such that $wR_{\leq}v$, $v \not\models^i \phi$. This implies, by the definition of negation, that $w \models^i \neg\phi$ and, by the condition on MAK′ models, that there is a z such that $wR_{\leq}z$ and $z \models^i K\neg\phi$. Thus, $w \not\models^i \neg K\neg\phi$ (since $wR_{\leq}z$), and then (i) $w \not\models^i \neg K\neg\phi \wedge \neg K\phi$.

In both cases w satisfies (i) and, since w was arbitrarily chosen, one concludes that, for all v in every MAK′ model, $v \not\models^i \neg K\neg\phi \wedge \neg K\phi$. Finally, this implies that $\neg(\neg K\neg\phi \wedge \neg K\phi)$ is valid in MAK′.

4.2 Building a Bridge

When considering two essentially different paradigms, such as classical and intuitionistic logic, it is easy to take King Solomon's position, following the Quinean motto according to which changes in logic are changes in meaning. At the same time, mutual comprehension is often possible, especially when translations are available; and this is the case with Gödel's translation of intuitionistic propositional logic into classical modal logic. Moreover, Kripke models represent a common semantic basis of interpretation for both languages. The Appendix includes a well-known translation (*) between the intuitionistic modal language with only one modal operator (taken to be the K operator) and the language of classical modal logic with two operators.[24]

Since I interpreted the relation $R_{\leq}$ of intuitionistic Kripke models as a temporal relation, it is natural to read $\Box_{\leq}$ (see the Appendix) as the temporal operator G (the dual of F), signifying "it will always be the case that". In the Appendix, I introduce the system **S4S4**, for which the translation (*) preserves the theorems of the intuitionistic epistemic logic: i.e.,

$$\vdash_{HS4} \phi \text{ iff } \vdash_{S4S4} \phi^*$$

[24]For the purposes of this paper, the translation of interest is the one given in Bozic and Dosen [1]. More advanced and general mathematical results about classical *bi-modal companions of intuitionistic modal logics* can be found in a work specifically dedicated to this topic: Wolter and Zakharyaschev [13].

Given canonicity, it also preserves deducibility for an arbitrary set Γ of formulae

$$\Gamma \vdash_{HS4} \phi \text{ iff } \Gamma^* \vdash_{S4S4} \phi^*$$

This technical result has an important consequence for the debate between the intuitionist and the classical logician: every hypothesis ϕ formulated by the intuitionist logician can be rephrased by the classical logician as ϕ^* in a temporal-epistemic language given by the following fragment $\mathcal{L}^*$ of the bi-modal language.

$$\mathcal{L}^* \ ::= Gp \mid G\neg\psi \mid \phi \wedge \psi \mid \phi \vee \psi \mid G(\phi \rightarrow \psi) \mid K\psi$$

Moreover, if one translates back from classical into intuitionistic language, the only propositions that the intuitionistic logician would accept as valid are the theorems of **S4S4** which belong to $\mathcal{L}^*$. In order to obtain an intuitionistically sound argument, one should translate the premises of the classical argument in the same way.

Thus, if one formulates the *verification thesis* as $\phi \rightarrow \neg\neg K\phi$ then, for every ϕ, one should (by classical logic) accept his translation, i.e. the schema

$$G\left(\phi^* \rightarrow GFK\phi^*\right)$$

For example, given an atomic p, applying the knowability principle to it yields $G(Gp \rightarrow GFKGp)$. The translation process can also shed light on the *non-omniscience hypothesis* in Fitch's derivation. In fact, that fundamental premise cannot be formulated as **(NO)** $\phi \wedge \neg K\phi$. If non-omniscience were represented by this formula, this would constitute a possible objection to the intuitionistic solution, since even the new version of the verification thesis presented here $(\phi \rightarrow \neg\neg K\phi)$ would lead to a contradiction, if taken in conjunction with $\psi \wedge \neg K\psi$. More precisely, it can easily be seen that substitution yields $(\psi \wedge \neg K\psi) \rightarrow \neg\neg K(\psi \wedge \neg K\psi)$, and that the consequent is likewise contradictory. The intuitionistic logician would thus be forced to deny non-omniscience.

This problem is discussed in Williamson [11, 12]. Williamson's suggestion is that the intuitionistic logician should claim that $\exists\psi(\psi \wedge \neg K\psi)$ is not a good formulation for non-omniscience in Fitch's derivation. He suggests replacing it with:

(NO') $\neg\forall\phi(\phi \rightarrow K\phi)$.

This formulation is, on a classical but not on an intuitionistic approach, equivalent to the first. The two formulations are also clearly distinguishable from the point of view of Kripke semantics. If one interprets **(NO)** as saying that there is a formula ϕ such that $\phi \wedge \neg K\phi$, and **(NO')** as saying that it is not always the case that $\phi \rightarrow K\phi$, then it is clear that **(NO)** has **(NO')** as a consequence. Conversely, **(NO')** is verified in the class of MAK′ models (i.e., there can be a point w and a formula ψ such that $w \models \phi$ and $w \not\models K\phi$), whereas there is no w and no ϕ in these models such that $w \models \phi$ and $w \models \neg K\phi$. Thus, **(NO')** does *not* have **(NO)** as a consequence.

Given the translation (*), we should translate $\phi \wedge \neg K\phi$ by:

$$\phi^* \wedge G\neg K\phi^*$$

and, in the special case of atomic propositions, by:

$$Gp \wedge G\neg KGp$$

Such a formula represents, for the classical logician, a strong assumption about knowability. It means that there is a formula which will always be true, such that the agent will never know that it will always be true. While this assumption implies non-omniscience, it is much stronger than it and should rather be read as claiming a less obvious *epistemic pessimism*.

Because of that, a problem remains for the intuitionistic solution: how, intuitionistically, to formulate the non-omniscience assumption, according to which there is an unknown truth: say, ϕ. It seems that one can only formulate this meta-linguistically, by saying that ϕ is forced at some point where $K\phi$ is not forced. This fact will be problematic for many. It is probably one of the reasons why J. van Benthem wrote, in van Benthem [8], that trying to solve this paradox by weakening the logic is almost like turning down the volume of the radio so as not to hear the bad news.

5 Conclusions

The goal of this paper was to provide a plausible semantics for the knowledge operator, based on IKMs. I argued that the K operator introduced is well-behaved, in some relevant sense, with respect to the features one may reasonably attribute to an ideal intuitionistic reasoner. The agent's knowledge turns out to be robust in relation to basic informational data but does not collapse into truth. It fulfills the general conditions of a normal intuitionistic modal logic. As explained in the beginning, the agent is assumed to be equipped with perfect recall and unlimited computational abilities and resources. This makes sense from the perspective of an ideal reasoner and that reasoner's informational state: having a perfect memory (in the sense of Definition 2.3) makes it plausible, in the case of IKMs, that all the future temporal alternatives are also epistemic ones. Discarding such a requirement would possibly allow one to discard certain temporal alternatives from the set of epistemic ones and so deviate to a non-normal modal logic. This will be desirable for those who find that ideal features imposed by normality, like *logical omniscience*, are highly problematic. Possible weakenings of these ideal conditions could be an interesting topic for further research.

Nevertheless, the study of idealized limit cases is useful, at least in some circumstances, one of which is the discussion on Fitch's paradox. I was able to show that, even under idealized capacities of the agent (given suitable assumptions relating temporal ordering and epistemic accessibility), knowability, if expressed by **(AK')**, need not entail actual knowledge **(AK)**. The question remains open whether all of that makes a case for the intuitionistic approach to Fitch's paradox; but it can be, nevertheless, a useful means of clarification. Indeed, my definition of K, together with a previously existing translation of intuitionistic epistemic logic into classical temporal-epistemic logic (Section 4 and Appendix), allows a mutual understanding, within a common semantics, for both the classical/realist logician and the intuitionistic/anti-realist one.

One of the main philosophical objections faced by an intuitionistic approach to Fitch's paradox (and similar problems) concerns its expressiveness. Intuitionistic semantics seem too poor to model certain aspects of common discourse. First of all, true propositions that are verified at any point remain true forever. This is reasonable when one speaks about the informational data of an ideal agent; or if one supposes that propositions, once true, are true forever. It may, however, be a problem if one wishes to deal with contingent or ephemeral[25] truths, such as "it is raining". Such truths are in common, even ubiquitous, linguistic use. A related problem is the intuitionistic meaning of negation: the reading of $\neg\phi$ as "ϕ will never be true", or "it is impossible for ϕ to be true", or "ϕ implies a contradiction" makes sense in mathematical discourse; but it seems inappropriate for most everyday uses of negation in propositions like "there is no milk". From the point of view of possible applications, one may hope for more refined semantic approaches to be articulated, able to model many finer aspects of a constructivist empirical reasoning.

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Appendix

A.1 Intuitionistic Modal Logic

A system **HK** of intuitionistic normal modal logic, with an operator $\Box$, was presented in Bozic and Dosen [1], based on the following language $\mathcal{L}$, where Φ is a set of propositional variables:

$$\mathcal{L} = \Phi \mid \neg\phi \mid \phi \vee \psi \mid \phi \wedge \psi \mid \phi \rightarrow \psi \mid \Box\phi$$

The operator $\Diamond$ is not introduced, nor is it defined as $\neg\Box\neg$.[26]

[25] The use of this adjective here reflects that of Burgess [2].

[26] Indeed, the two modal operators $\Box$ and $\Diamond$ are not interdefinable in a way that preserves their usual universal/existential meaning, due to the peculiar role of negation in intuitionistic logic. For this reason Bozic and Dosen [1] deals separately with axiomatic systems $\mathbf{HK}_\Box$ with $\Box$ (our **HK**), $\mathbf{HK}_\Diamond$ with $\Diamond$, and $\mathbf{HK}_{\Box\Diamond}$ with both $\Box$ and $\Diamond$ as primitive operators.

The system **HK** consists of a first group of axiom schemata H1–H10 of Heyting's intuitionistic propositional calculus (H) along with the additional modal axiom K.

(H1)	$\phi \to (\psi \to \phi)$
(H2)	$(\phi \to (\psi \to \zeta)) \to ((\phi \to \psi) \to (\phi \to \zeta))$
(H3)	$(\zeta \to \phi) \to ((\zeta \to \psi) \to (\zeta \to \phi \wedge \psi))$
(H4)	$\phi \wedge \psi \to \phi$
(H5)	$\phi \wedge \psi \to \psi$
(H6)	$\phi \to \phi \vee \psi$
(H7)	$\psi \to \phi \vee \psi$
(H8)	$(\phi \to \zeta) \to ((\psi \to \zeta) \to (\phi \vee \psi \to \zeta))$
(H9)	$(\phi \to \neg\psi) \to (\psi \to \neg\phi)$
(H10)	$\neg\phi \to (\phi \to \psi)$
(K)	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

with the following inference rules:

(Subst)	If $\vdash \phi$ then $\vdash \phi^\sigma$
(MP)	If $\vdash \phi \to \psi$ and $\vdash \phi$ then $\vdash \psi$
($\Box$-gen)	If $\vdash \phi$ then $\vdash \Box\phi$

where ϕ^σ is any formula obtained from ϕ by uniform substitution. The abbreviation $\vdash_{HK} \phi$ (or, for an arbitrary system Λ, $\vdash_\Lambda \phi$) means that ϕ is a theorem of **HK** (of Λ). Moreover, given a logic Λ and a set of formulae $\Gamma \cup \{\phi\}$, we say that ϕ is Λ-*deducible* from Γ (written $\Gamma \vdash_\Lambda \phi$) iff ϕ is a theorem of Λ, ($\vdash_\Lambda \phi$), or there are formulae $\psi_1, \ldots, \psi_n \in \Gamma$ such that $\vdash_\Lambda (\psi_1 \wedge \cdots \wedge \psi_n) \to \phi$.

A set of formulae Γ is Λ-consistent if not every formula is Λ-deducible from it. Otherwise it is Λ-inconsistent. It is not difficult to show that Γ is Λ-inconsistent iff $\Gamma \vdash_\Lambda \bot$, where $\bot$ stands for the negation of a theorem: e.g., $\neg(\phi \to \phi)$ (we will also use $\top$, standing for $\neg\bot$).

Given the semantics introduced in Section 3.2, it can be shown, as in the propositional case, that the crucial condition of monotonicity (also called *intuitionistic heredity*) holds.[27]

Lemma 2 (Monotonicity) *In every model* $\mathcal{M} = (W, R_\leq, R_M, V)$*, for every* $w, v \in W$*, if* $w \models^i \phi$ *and* $w R_\leq v$*, then* $v \models^i \phi$

Proof See Bozic and Dosen [1]. □

[27]More specifically, the condition (2) $R_\leq R_M \subseteq R_M R_\leq$ is necessary and sufficient to preserve monotonicity.

The notions of *validity* and *logical consequence*, with respect to the forcing relation $\models^i$, are defined as in classical logic. In particular, if S is a class of frames contained in **HK**, then one should write $\models^i_{\mathtt{S}} \phi$ for "ϕ is valid in $\mathtt{S}$" and $\Gamma \models^i_{\mathtt{S}} \phi$ for "ϕ is a consequence, in the class $\mathtt{S}$, of the set Γ of formulae". The following completeness result was proven in Bozic and Dosen [1].

Theorem 3 (Completeness) *Consider the set of formulae* Γ *and a formula* ϕ. *Then, the following holds:*

$$\Gamma \vdash_{HK} \phi \text{ if and only if } \Gamma \models^i_{HK} \phi$$

The right-to-left part of the demonstration is a usual *completeness-via-canonicity* one. The points in the canonical model are saturated sets of formulae here. (They need not be *maximally consistent* sets as in the classical case.) The following definitions introduce some key concepts.

Definition 5.1 (Deductive Closure) Let $Cl(\Sigma) := \{\phi \mid \Sigma \vdash_\Lambda \phi\}$. A set Σ of formulae is *deductively closed* w.r.t. a logic Λ iff $Cl(\Sigma) = \Sigma$.

Definition 5.2 (Disjunction Property) A set of formulae Σ has the *disjunction property* iff for every couple of formulae ϕ, ψ, if $\phi \vee \psi \in \Sigma$, then either $\phi \in \Sigma$ or $\psi \in \Sigma$.

Definition 5.3 (Saturation) A set of formulae Σ is *saturated* iff:

(i) Σ is consistent.
(ii) Σ is deductively closed.
(iii) Σ has the disjunction property.

An important result, analogous to *Lindenbaum's Lemma*, is the following:

Lemma 3 (Saturability) *Let* Λ *be a logic containing H1-H10,* Σ *be a set of formulae, and* ϕ *be a formula such that* $\Sigma \nvdash_\Lambda \phi$. *Then, there is a saturated set* Σ^+ *such that* $\Sigma \subseteq \Sigma^+$ *and* $\Sigma^+ \nvdash_\Lambda \phi$.

We can then introduce the fundamental notion of a canonical model.

Definition 5.4 The *canonical model* for **HK** is the couple $\mathcal{M}^{HK} = (\mathcal{F}^{HK}, V^{HK})$, where $\mathcal{F}^{HK} = (W^{HK}, R^{HK}_{\leq}, R^{HK}_{M})$ is defined as follows:

(1) $W^{HK} := \{\Sigma \mid \Sigma$ is saturated w.r.t. **HK**$\}$.
(2) $\Gamma R^{HK}_{\leq} \Delta$ iff $\Gamma \subseteq \Delta$.
(3) $\Gamma R^{HK}_{M} \Delta$ iff $\Gamma_\Box \subseteq \Delta$ where $\Gamma_\Box := \{\phi \mid \Box\phi \in \Gamma\}$

and $V^{HK}(p) = \{\Gamma \mid p \in \Gamma\}$.

It is not difficult to verify that $\mathcal{F}^{HK}$ is a HK-frame and that the definition of V^{HK} is a good one: i.e., it respects monotonicity.

(a) If $\Gamma R_{\leq}^{HK} \Delta$, then $\Gamma \in V^{HK}(p)$ implies $\Delta \in V^{HK}(p)$.

The completeness proof goes through the following *truth lemma*.

Lemma 4 (Truth Lemma) *In the canonical model for HK, for every Γ and every ϕ one has:*

$$\mathcal{M}^{HK}, \Gamma \models^i \phi \; iff \; \phi \in \Gamma$$

Completeness results for some extensions of **HK** are given in Dosen [4]. We consider some axioms holding in most systems of classical modal logic, when $\Box$ is taken as an epistemic operator.

(D) $\Box\phi \to \neg\Box\neg\phi$
(T) $\Box\phi \to \phi$
(4) $\Box\phi \to \Box\Box\phi$
(5) $\neg\Box\phi \to \Box\neg\Box\phi$

along with the following systems:

1 **HD = HK + D**
2 **HT = HK + T**
3 **HS4 = HK + T + 4**
4 **HS5 = HK + T + 4 + 5**

Completeness results for these systems are given by the following theorem.

Theorem 4 (Completeness) ***HD, HT, HS4*** *and* ***HS5*** *are sound and complete respectively in the following classes of frames* HD, HT, HS4 *and* HS5*:*

(a) HD *is the class of frames where R_M is serial.*
(b) HT *is the class of frames where R_M is reflexive.*
(c) HS4 *is the class of frames where R_M is reflexive and transitive.*
(d) HS5 *is the class of frames where R_M is reflexive, transitive, and satisfies the first-order condition $\forall x, y(xR_M R_{\leq} y \to \exists t(xR_{\leq} t \wedge yR_M R_{\leq} t))$.*

A.2 Embedding HK into Classical Bimodal Logic

It is a well-known result that intuitionistic propositional logic **H** can be embedded into the system **S4** of classical modal logic, via a translation (*) from the language of propositional logic into the language of modal logic such that:

$$\vdash_H \phi \text{ iff } \vdash_{S4} \phi^*$$

This translation can easily be extended into one that translates from intuitionistic modal language to classical bi-modal language, as was shown by Bozic and Dosen [1] and developed further in Wolter and Zakharyaschev [13].

The intuition behind it is that a Kripke frame $\mathcal{F} = (W, R_{\leq}, R_M)$ for intuitionistic modal logic can also serve as the basis for a model $\mathcal{M} = (W, R_{\leq}, R_M, V^c)$ of classical modal logic with two modalities—call them $\Box_{\leq}$ and $\Box_M$—where V^c is a classical valuation on possible worlds that can be extended to a classical satisfaction relation $\models^c$ (the superscript c is adopted in order to distinguish the classical from the intuitionistic satisfaction relation), where the satisfaction clauses for modalities are, as usual:

(a) $w \models^c \Box_{\leq}\phi$ iff for all w', such that $wR_{\leq}w'$, $w' \models^c \phi$
(b) $w \models^c \Box_M\phi$ iff for all w', such that wR_Mw', $w' \models^c \phi$

The following result defines (in classical terms) the class of the **HK**-frames among other Kripke frames.

Theorem 5 (HK-Frames) *Let $\mathcal{F} = (W, R_{\leq}, R_M)$ be a bi-relational frame. Then:*

$$\mathcal{F} \models^c \Box_M\Box_{\leq}\phi \to \Box_{\leq}\Box_M\phi \textit{ iff } R_{\leq}R_M \subseteq R_MR_{\leq}$$

Proof Easy (see Bozic and Dosen [1]). □

We are mainly interested in the following system **S4S4**:

P	All tautologies of classical propositional calculus.
$\Box_{\leq}1$	$\Box_{\leq}(\phi \to \psi) \to \Box_{\leq}\phi \to \Box_{\leq}\psi$
$\Box_{\leq}2$	$\Box_{\leq}\phi \to \phi$
$\Box_{\leq}3$	$\Box_{\leq}\phi \to \Box_{\leq}\Box_{\leq}\phi$
$\Box_M1$	$\Box_M(\phi \to \psi) \to \Box_M\phi \to \Box_M\psi$
S4K	$\Box_M\Box_{\leq}\phi \to \Box_{\leq}\Box_M\phi$
$\Box_M2$	$\Box_M\phi \to \phi$
$\Box_M3$	$\Box_M\phi \to \Box_M\Box_M\phi$

with the usual rules of modus ponens, substitution, and generalization for both modalities.

This system is easily shown to be classically sound and complete with respect to the class of HS4-frames. This result can be derived from Theorem 5 and the canonicity of the axioms of **S4**.[28] Now, consider the following translation *:

$$p^* = \Box_{\leq}p$$

$$(\phi \wedge \psi)^* = \phi^* \wedge \psi^*$$

$$(\phi \vee \psi)^* = \phi^* \vee \psi^*$$

$$(\phi \to \psi)^* = \Box_{\leq}(\phi^* \to \psi^*)$$

$$(\neg\phi)^* = \Box_{\leq}\neg\phi^*$$

$$(\Box\phi)^* = \Box_M\phi^*$$

[28] This result is a simple extension of the characterization of **HK**-frames given in Bozic and Dosen [1].

In the following lemmata, $\mathcal{M}$ will be a **HK**-model, and $\mathcal{M}^*$ will be a classical bi-relational model based on the same frame.

Lemma 5 *If, for every w and every p, $\mathcal{M}, w \models^i p$ iff $\mathcal{M}^*, w \models^c p^*$, then for every ϕ $\mathcal{M}, w \models^i \phi$ iff $\mathcal{M}^*, w \models^c \phi^*$*

Proof The proof is done by induction on the complexity of formulae (see Bozic and Dosen [1]). □

Given this, the following theorem can be proven (see Bozic and Dosen [1]):

Theorem 6 *Consider the class **HK** of bi-relational frames for intuitionistic modal logic. Then:*

$$\models^i_{HK} \phi \text{ } iff \text{ } \models^c_{HK} \phi^*$$

This result can be extended in a straightforward way in order to obtain:

Corollary 1 *Given the class* `HS4`*:*

$$\models^i_{HS4} \phi \text{ } iff \text{ } \models^c_{HS4} \phi^*$$

Given these equivalences along with the completeness results for intuitionistic and classical modal logic, the following result is immediate:

Theorem 7

$$\vdash_{HS4} \phi \text{ } iff \text{ } \vdash_{S4S4} \phi^*$$

If we take $\Box_\leq$ to be the temporal operator G and $\Box_M$ to be the epistemic K this theorem provides the translation discussed in Section 4.

A.3 Theorems of Section 4.1

Proof of Theorem 1 The right-to-left direction can be proved by contraposition. Suppose that $\phi \to \neg\neg K\phi$ is not valid. Then, for some w, (a) $w \models \phi$ and $w \not\models \neg\neg K\phi$. This means that for some v such that $wR_\leq v$, $v \models \neg K\phi$. Thus, for all z such that $vR_\leq z$, $z \not\models K\phi$, i.e. that for some y such that $zR_K y$, (b) $y \not\models \phi$. But, by the condition on `HAK′` we have that, for at least some of these y, (c) $wR_\leq y$ and a contradiction would follow from (a), (b), (c) and the condition of monotonicity.

For the left-to-right direction, suppose that the condition of `HAK′` does not hold. This means that there are w and v such that $wR_\leq v$ and for all z such that $vR_\leq z$ (this set is non empty because $R_\leq$ is reflexive) there is a y such that $zR_K y$ and y is not a $R_\leq$-successor of w. We can define $V(p) := \{x \mid x \text{ is a } R_\leq\text{-successor of } w\}$ ($V(p)$ respects the condition 3 of Section 3.1, so the model obtained is an IKM). It is straightforward to check that $w \models p$ but $w \not\models \neg\neg Kp$, because $v \models \neg Kp$. So the formula $\phi \rightarrow \neg\neg K\phi$ is not valid. □

Proof of Lemma 1 The proof works by induction from the complexity of ϕ. The direction ($\Leftarrow$) of (1) and (2) is already given by the reflexivity of R_K. For the direction ($\Rightarrow$) one may proceed as follows.

(a) If $\phi = p$, then the result follows immediately from the definition of R_K and from the condition on the accessible states.

(b) If $\phi = \neg\psi$ then, for what concerns (1), if $w \models^i \phi$ then $w \not\models^i \psi$; and thus, by the induction hypothesis, for every v such that wR_Kv, one finds that $v \not\models^i \psi$. It follows from the definition of R_K that, for every v', if $vR_{\leq}v'$ then wR_Kv'; and, by consequence, $v' \not\models^i \psi$. This implies by the clause on negation that $v \models^i \neg\psi$; and so the result is proved.

For what concerns (2), if $w \not\models^i \neg\psi$, there are two possible cases. In the first, $w \models^i \psi$ and, by the induction hypothesis, for every v such that wR_Kv, $v \models^i \psi$. Thus, it holds *a fortiori* that $v \not\models^i \neg\psi$. In the second, $w \not\models^i \psi$, which implies, by the induction hypothesis, that, for every v such that wR_Kv, $v \not\models^i \psi$. Since $w \not\models^i \neg\psi$, one must also conclude that there is some z such that $wR_{\leq}z$ and $z \models^i \psi$. Since $R_{\leq} \subseteq R_K$, one also concludes that wR_Kz, and so one derives a contradiction.

(c) $\phi = \psi \wedge \zeta$. Immediate.

(d) $\phi = \psi \vee \zeta$. Immediate.

(e) $\phi = \psi \rightarrow \zeta$. If $w \models^i \psi \rightarrow \zeta$, then we could have $w \models^i \psi$ and $w \models^i \zeta$. This will be the case, by the induction hypothesis, for every v which is R_K-accessible, and which should then satisfy $v \models^i \psi \rightarrow \zeta$. Otherwise $w \not\models^i \psi$; but this implies, by the induction hypothesis, that, for every v which is R_K-accessible, and thus for every one of its successors, $v \not\models^i \psi$. This further implies, by the forcing clause, that $v \models^i \psi \rightarrow \zeta$.

For what concerns (2), if $w \not\models^i \psi \rightarrow \zeta$, there are two possible cases. The first is that $w \models^i \psi$ and $w \not\models^i \zeta$; this implies, by the induction hypothesis, that, for all R_K-accessible v, $v \models^i \psi$ and $v \not\models^i \zeta$. Thus, $v \not\models^i \psi \rightarrow \zeta$. On the other hand, if $w \not\models^i \psi$ and $w \not\models^i \zeta$, then we have, by the induction hypothesis, that for all v which are R_K-accessible from w, $v \not\models^i \psi$; this implies *a fortiori* that, for all v which are $R_{\leq}$-accessible from w, $v \not\models^i \psi$ and $w \models^i \psi \rightarrow \zeta$, which contradicts the hypothesis.

(f) $\phi = K\psi$. For what concerns (1), if $w \models^i K\psi$ then, by Axiom **4**, $w \models^i KK\psi$ and, for all v such that wR_Kv, $w \models^i K\psi$.

In the case of (2), if $w \not\models^i K\psi$ then, again, there are two possible cases. The first is that $w \models^i \psi$; but then, by the induction hypothesis, every accessible v is such that $v \models^i \psi$; and this implies, by the satisfaction clause of K, that $w \models^i K\psi$, thereby yielding a contradiction. Otherwise $w \not\models^i \psi$; then; for every accessible v, $v \not\models^i \psi$ and, since the accessibility relation is reflexive, $v \not\models^i K\psi$. This is what we set out to prove. □

From this the following result follows

Proof of Theorem 2 Given a point w and a formula ϕ, there are two possible cases. In the first, $w \models^i \phi$; then, from the supposed condition, for all v such that $w R_{\leq} v$, there exists some z such that $v R_{\leq} z$ and such that every point R_K-accessible from z is equivalent to z. Moreover, $z \models^i \phi$ by the monotonicity condition, and thus, from point (1) of the preceding lemma, one finds that $z \models^i K\phi$. Thus one proves that every successor v of w has a successor z satisfying $K\phi$; and this implies, by the satisfaction clause of negation, that $w \models^i \neg\neg K\phi$.

Otherwise, if $w \not\models^i \phi$, then either it is the case that every $R_{\leq}$-successor v is such that $v \not\models^i \phi$, from which it follows that $w \models^i \phi \rightarrow \neg\neg K\phi$; or else there is some v such that $v \models^i \phi$. One can then reproduce the same reasoning from the preceding point in order to find some $R_{\leq}$-successor z such that $z \models^i K\phi$, thus implying that $w \models^i \neg\neg K\phi$. Thus the proof is complete. □

The last proof, due to Sebastian Enqvist, is a counterexample showing a model not in `MAK′` which validates $\phi \rightarrow \neg\neg K\phi$ (but not $\phi \rightarrow K\phi$).

The condition on `MAK′` *is not necessary for* ***(AK')*** Let $p_1, p_2 \ldots$ be an enumerated set of propositional variables. Consider a model $\mathcal{M}$ consisting of the following components:

- W consists of all pairs $(0, 0), (0, 1), (0, 2) \ldots$ and all pairs $(1, 0), (1, 1), (1, 2) \ldots$
- $R_{\leq}$ is defined by setting $(i, j) R_{\leq} (k, l)$ iff $i = k$ and $j \leq l$
- The valuation V makes true, at each world $(0, n)$ with $n \geq 1$, precisely the propositional variables in the set $\{p_1, \ldots, p_n\}$, and makes true no propositional variables at all at any point either of the form $(1, n)$ or $(0, 0)$.

This model is not in the class `MAK′`: for we have $(0, 0) R_{\leq} (0, 0)$ but for any $n \geq 0$ there is some n', say $n + 1$, such that $(0, n) R_K (0, n')$ but clearly not $(0, n) \equiv (0, n')$. However, the formula $\phi \rightarrow \neg\neg K\phi$ is valid in this model. Indeed, supposing that $(i, n) \models \phi$, we can show that $(i, n) \models \neg\neg K\phi$ and the result follows *a fortiori*. There are two cases to consider.

(a) $i = 0$. Then $(0, m) R_{\leq} (0, m + 1)$ and it is easy to see that if $(0, m + 1)$ $R_K(z, z')$ then we must have $z = 0$ and $z' \geq m + 1$ (since p_{m+1} is true at $(0, m)$ but false at any world with first component 1). So $(0, m + 1)$ $R_{\leq}(z, z')$ and then $(0, n) R_{\leq} (z, z')$. Thus, by monotonicity $(z, z') \models \phi$. This shows that $(0, m + 1) \models K\phi$ and so, since $(0, m) R_{\leq} (0, m + 1)$ it is not the case that $(0, m) \models \neg K\phi$, which is what we wanted to prove.

(b) $i = 1$. Then $(1, m) R_{\leq} (1, m + 1)$, and it is easy to show that if $(1, m + 1)$ $R_K(z, z')$ we must have $z = 1$ and $z' \leq m + 1$. We shall use the fact that in any model, if $x R_K y$, then y cannot have fewer predecessors than x. Since $m + 1 > 0$, $Pr((1, m + 1))$ has at least two members, hence we cannot have $z = 0$ since any predecessors of (z, z') would then have to have first component 0, and there is only one such world in W that satisfies the same atomic formulas as $(1, m + 1)$, namely $(0, 0)$. Also, if we had $z' < m + 1$, then $(z, z') = (1, z')$ would have to have fewer predecessors than $(1, m + 1)$. So we know that $z = 1$ and $z' \geq m + 1$. Hence $(1, m + 1) R_{\leq} (z, z')$ and so $(1, m) R_{\leq} (z, z')$. So, by monotonicity, we have $(z, z') \models \phi$. This shows that $(1, m + 1) \models K\phi$, and so it is not the case that $(1, m) \models \neg K\phi$, which is what we wanted to prove.

So the model validates $\phi \to \neg\neg K\phi$ even though it does not satisfy the condition of the class `MAK'`. However, it does not validate $\phi \to K\phi$: $\neg p_1$ is true at $(1, 0)$, but $(1, 0) R_K (0, 0)$ and $\neg p_1$ is false at $(0, 0)$, so $K\neg p_1$ is false at $(1, 0)$. □

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