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Three Approaches to Iterated Belief Contraction

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Abstract In this paper we investigate three approaches to iterated contraction, namely: the Moderate (or Priority) contraction, the Natural (or Conservative) contraction, and the Lexicographic contraction. We characterise these three contraction functions using certain, arguably plausible, properties of an iterated contraction function. While we provide the characterisation of the first two contraction operations using rationality postulates of the standard variety for iterated contraction, we found doing the same for the Lexicographic contraction more challenging. We provide its characterisation using a variation of Epistemic ranking function instead.

Keywords Belief contraction · State contraction · Iterated belief contraction · Degrees of belief

1 Introduction

It is now taken for granted that any rational theory of belief change, such as the classic AGM theory [1] and those that extended it in various directions, must provide accounts of how new beliefs are added (*belief expansion*), old beliefs are removed (*belief contraction*), as well as how current beliefs are modified in light of new information that is deemed accepted (*belief revision*).

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One of the problems that the classical AGM account has been extended to deal with is the problem of iterated belief change – the problem of how an agent is supposed to continually modify its beliefs in light of a sequence of observations made (or pieces of information received). However, for some reason or other, the emphasis in this context has historically been on iterated revision; the problem of iterated contraction having received a rather step-motherly treatment. In more recent times, however, the research community appears to be taking measures to redress this inequity [11, 16, 18, 20]. Three approaches to such an account examined by Nayak and colleagues [16, 18] are called the *Natural* Contraction, the *Priority* Contraction and the *Lexicographic* Contraction.[1] The accounts provided of these operations in the literature are by and large semantic; and the attempt made to characterize the Lexicographic Contraction via rationality postulates in [16] is rather incomplete. In this paper we aim to characterize these three operations. Unlike the Natural and Priority contractions, we found Lexicographic contraction difficult to characterize in the expected fashion using rationality postulates of the sort well known in the literature starting from [1]. We instead use a more "information-rich" measure that we call *degree of belief* for this purpose.

Throughout this paper we will assume a propositional object language $\mathcal{L}$ generated from a finite alphabet, whose sentences, denoted by lower case Greek letters such as α and β, with or without decorations, will be used to represent individual beliefs. Sets of such sentences will be represented by uppercase Latin letters, such as $\mathcal{A}, \mathcal{B}, \ldots$; in particular, $\mathcal{K}$ with or without decorations, will be used to denote sets of beliefs. For the sake of simplicity we take the background logic governing this language to be the classical propositional logic, identified with the classical deducibility relation $\vdash$.[2] Given a set $\mathcal{A}$ of sentences, we will represent by $[\mathcal{A}]$ the set of models of $\mathcal{A}$; and for readability, given a sentence α, the set of models $[\{\alpha\}]$ will be represented simply by $[\alpha]$. Other such measures will be adopted for readability where no confusion is imminent; for instance, for some function f and sentence α, the expression $f([\alpha])$ will be simplified to $f[\alpha]$, whose official representation is in fact $f([\{\alpha\}])$. We will be using the words "interpretation" and "world" interchangeably, with the understanding that models of some sentence α are the worlds that satisfy α; individual worlds would be denoted by ω with possible decorations, the set of all worlds will be denoted by Ω, and subsets of Ω by upper case Greek letters such as Γ and Δ.

As is standard, we will assume that a *belief set*, that is the body of beliefs of an epistemic agent, is represented by a set of sentences, typically $\mathcal{K}$ that is closed under $\vdash$, i.e., $\mathcal{K} = \{\alpha \in \mathcal{L} \mid \mathcal{K} \vdash \alpha\}$. A *belief set* such as $\mathcal{K}$ is distinguished from a *belief state* such as $\mathbf{K}$: the latter is a richer representation of the relevant

[1] Please note that the *Natural* Contraction and the *Priority* Contraction are respectively termed the *Conservative* Contraction and the *Moderate* Contraction in [20].

[2] Alternatively, as standard in the literature, we can take the background logic to be a propositional, supra-classical logic satisfying the deduction theorem and compactness.

information. In particular, we assume that the belief set $\mathcal{K}_{\mathbf{K}}$ associated with a belief state $\mathbf{K}$ can be extracted from the latter using some appropriate operation, say, *bel*; thus, $bel(\mathbf{K}) = \mathcal{K}_{\mathbf{K}}$. Furthermore, we assume that the belief state $\mathbf{K}$ incorporates a relevant belief (set) contraction operation, say $-_{\mathbf{K}}$ that determines the outcome $\mathcal{K}' = -_{\mathbf{K}}(\mathcal{K}_{\mathbf{K}}, \alpha)$ of removing some information α from the associated belief set $\mathcal{K}_{\mathbf{K}}$. In general, in response to the removal of α, the belief state $\mathbf{K}$ will undergo modification to, say, $\mathbf{K}' = -(\mathbf{K}, \alpha)$ such that the new belief state $\mathbf{K}'$ and the new belief set $\mathcal{K}'$ are appropriately aligned, that is, $bel(\mathbf{K}') = \mathcal{K}'$ such that further belief removal can be carried out as and when necessary.

In the literature, belief states have been represented in many different ways, including a relational epistemic entrenchment measure [8], a numerical possibility measure [5] and an ordinal ranking function [23]. In this paper we will represent a belief state as a total preorder (i.e., a reflexive, connected and transitive relation) $\sqsubseteq$ over Ω, with the understanding that $\omega \sqsubseteq \omega'$ means ω is at least as plausible as ω'. It is well known in the literature that a total preorder such as $\sqsubseteq$ can be directly translated to Grove's *system of spheres* as propounded in [9] and vice versa. Furthermore, the plausibility measure $\sqsubseteq$ over Ω and the entrenchment measure $\preceq$ over sentences of $\mathcal{L}$ are inter-translatable.[3] We will denote by $\sqsubset$ the strict part of $\sqsubseteq$, and by $\approx$ its symmetric part. It is worth noting that on occasion we will need to refer to modified plausibility preorders such as $\sqsubseteq^-_\alpha$; and in such cases we will refer to their strict part and symmetric part by $\sqsubset^-_\alpha$ and $\approx^-_\alpha$ respectively.

Given any set of worlds $\Delta \subseteq \Omega$ we will denote by $\min_{\sqsubseteq}(\Delta) = \{\omega \in \Delta \mid \omega \sqsubseteq \omega', \text{ for all } \omega' \in \Delta\}$ the set of $\sqsubseteq$-minimal worlds of Δ that the epistemic agent considers most plausible among those in Δ. In particular, $\min_{\sqsubseteq}(\Omega)$ will represent the set of most plausible worlds among all possible worlds as viewed by the agent and $\min_{\sqsubseteq^-_\alpha}(\Omega)$ will represent the most plausible worlds after it has removed information α from its belief state (set). The beliefs of the agent are those that are true in all these most plausible worlds. This is captured in the following equation that we name after Grove. Note that from here onwards, the set of beliefs $bel(\sqsubseteq)$ extracted from the belief state $\sqsubseteq$ will be denoted by $\mathcal{K}_{\sqsubseteq}$.

$$bel(\sqsubseteq) = \mathcal{K}_{\sqsubseteq} = \{\alpha \in \mathcal{L} \mid \min_{\sqsubseteq}(\Omega) \subseteq [\alpha]\} \qquad \text{(G1)}$$

[3] Epistemic entrenchment is a relational measure, roughly indicating how hard it is to remove a given belief. It is typically defined as a binary relation $\preceq$ over sentences of a language satisfying certain standard conditions such as those provided in [8]. How an entrenchment relation can be obtained from a given plausibility preorder is well-known in the literature: for all sentences $\alpha, \beta \in \mathcal{L}$, $\alpha \preceq \beta$ iff $\omega \sqsubseteq \omega'$ for all $\omega \in min_{\sqsubseteq}[\neg\alpha]$ and $\omega' \in min_{\sqsubseteq}[\neg\beta]$. The other side, how the plausibility preorder can be obtained from a given entrenchment relation can be found on page 252 of [19] in a slightly different framework. In our notation, it would be: for all worlds $\omega, \omega' \in \Omega$, $\omega \sqsubseteq \omega'$ iff for every sentence $\alpha \in \mathcal{L}$ such that $\omega \models \neg\alpha$, there exists a sentence $\alpha' \in \mathcal{L}$ such that both $\omega' \models \neg\alpha'$ and $\alpha \preceq \alpha'$. The proof is easily verified. For ($\Rightarrow$), assume $\omega \sqsubseteq \omega'$, $\omega \in [\neg\alpha]$ and set $[\neg\alpha'] = \{\omega'\}$. For ($\Leftarrow$), let $[\neg\alpha] = \{\omega\}$, $\alpha \preceq \alpha'$ and $\omega' \in [\neg\alpha']$. The proof will use the well known definition of $\preceq$ via $\sqsubseteq$ just mentioned.

At this point we seek to clarify certain notational device that is potentially confusing. The contraction operation $-$ that we are concerned with is a *state contraction operation*: given a belief state $\sqsubseteq$ and a sentence α to be removed from it, this operation returns a belief state $\sqsubseteq_\alpha^-$ in which, in general, α is not believed. The corresponding propositional content of these two belief states, that is the associated belief sets, are $\mathcal{K}_\sqsubseteq$ and $\mathcal{K}_{\sqsubseteq_\alpha^-}$. When the prior state $\sqsubseteq$ can be contextually determined, and the intended reading is clear, for the sake of simplicity we will refer to these belief sets as $\mathcal{K}$ and $\mathcal{K}_\alpha^-$ instead. In a similar fashion, $(\mathcal{K}_\alpha^-)_\beta^-$ will refer to the belief set associated with the belief state $(\sqsubseteq_\alpha^-)_\beta^-$. In other words, the same symbol $-$ is used for both state contraction operation as in $\mathcal{K}_{\sqsubseteq_\alpha^-}$, as well as the corresponding set contraction operation $-_\sqsubseteq$ (as in $\mathcal{K}_\alpha^-$, which should officially be written as: $\mathcal{K}_\alpha^{-_\sqsubseteq}$), but since the belief set and belief state in question are assumed to be appropriately co-related, it is not problematic.

An inconsistent belief state is represented by an empty relation $\sqsubseteq_\perp$: for any two distinct worlds $\omega, \omega' \in \Omega, \omega \not\sqsubseteq_\perp \omega'$.[4] On the other hand $\sqsubseteq_\top$ denotes the *full* pre-order relation: for every $\omega,\ \omega' \in \Omega, \omega \sqsubseteq_\top \omega'$. The relation $\sqsubseteq_\top$ denotes the belief state where the agent believes only in logical tautologies and the relation $\sqsubseteq_\perp$ denotes the state where the agent believes in every sentence.

A notion that we would be using throughout this paper is that of a *chain of worlds*. Given a belief state $\sqsubseteq$, and a set of worlds $\Delta \subseteq \Omega$, a *chain of worlds in* Δ is a sequence of worlds in Δ ordered by the strict part $\sqsubset$ of $\sqsubseteq$, e.g., $\omega_0 \sqsubset \omega_1 \sqsubset \ldots \sqsubset \omega_n$. Based on this notion, we define a *complete chain of worlds in* Δ as follows. A chain of worlds $\omega_0 \sqsubset \omega_1 \sqsubset \ldots \sqsubset \omega_n$ in Δ is said to be *complete* iff ω_0 is a $\sqsubseteq$-minimal world of Δ and for any $\omega_{i-1} \sqsubset \omega_i$ $(1 \leq i \leq n)$ there does not exist any world ω' in Δ such that $\omega_{i-1} \sqsubset \omega' \sqsubset \omega_i$. The length of a complete chain of worlds $\mathcal{C} = \omega_0 \sqsubset \omega_1 \sqsubset \ldots \sqsubset \omega_n$ is defined to be n, and is denoted by $\|\mathcal{C}\|$. Two special cases of this notion are of interest: when Δ is the universal set Ω, and when Δ is exactly the set of models $[\alpha]$ of some sentence α.

2 Three Contraction Functions

The modified preorder $\sqsubseteq_\alpha^-$ represents the result of contracting the prior belief state $\sqsubseteq$ by a sentence α. From Eq. G1 it is evident that, if α is to be successfully removed from the corresponding belief set in this process, then there must exist at least one model of $\neg\alpha$ among the minimal worlds in $\sqsubseteq_\alpha^-$. This would mean that $min_{\sqsubseteq_\alpha^-}(\Omega)$ is not contained in $[\alpha]$ and hence α is not retained. We will say that a belief state contraction operation $-$ is *AGM rational* just in case the corresponding belief set contraction operation satisfies the standard AGM postulates [1]. Combining the semantic account of belief (set) contraction provided in [4, 8, 9] with the account of belief (state) contraction we are

[4]Here by empty preorder we mean a reflexive and transitive relation which is "completely disconnected", i.e., no two distinct worlds are related to each other.

espousing, we get that a belief state contraction operation $-$ is AGM rational when

$$\min_{\sqsubseteq^-_\alpha}(\Omega) = \min_{\sqsubseteq}(\Omega) \cup \min_{\sqsubseteq}[\neg\alpha] \tag{G2}$$

Henceforth in this paper, any reference to an AGM-rational contraction function will refer either to a state contraction function which satisfies Eq. G2, or a belief set contraction operation appropriately obtained from such a state contraction function. The context should sufficiently disambiguate the usage.

It is to be noted that the condition G2 on $\sqsubseteq^-_\alpha$ does not guarantee a unique way of changing $\sqsubseteq$ to $\sqsubseteq^-_\alpha$. Different ways of changing $\sqsubseteq$ have been proposed, each giving a different contraction function. Three such contraction functions, namely the moderate contraction, the natural contraction and the lexicographic contraction have been proposed in [17, 18]. In this section we briefly outline these functions.

2.1 Moderate Contraction Function

This contraction function is referred to as *moderate contraction* function in [20] and as *priority contraction* function in [17, 18]. The basic idea behind moderate contraction is simple. Suppose we want to remove a non-trivial belief α (i.e., $\not\models \alpha$). When we remove α from a belief set, under usual circumstances, α becomes a non-belief. Every sentence of the form $\beta \to \alpha$ can be viewed as contributing to the agent's epistemic attitude towards α. This demands that out of every pair $\beta \to \alpha$ and $\neg\beta \to \alpha$, at least one be removed. The question is, what happens to those sentences $\beta \to \alpha$ which are not removed from the belief set in the process. Moderate contraction is based on the intuition that even if some sentence $\beta \to \alpha$ is not removed via removal of α, its entrenchment should be reduced, and its status be demoted.[5] The degree of entrenchment of $\beta \to \alpha$ for some particular β is reflected by the most plausible worlds in $[\beta] \cap [\neg\alpha]$ in the belief state $\sqsubseteq$. By promoting all the worlds in $[\neg\alpha]$ the entrenchment of every $\beta \to \alpha$ is reduced. This is captured by the condition **MC3** in the following formalization.

Given a total pre-order relation $\sqsubseteq$ on Ω, and a sentence α to be removed from the associated belief set, a moderate contraction function $-$ changes the relation $\sqsubseteq$ to a new relation $\sqsubseteq^-_\alpha$. If α is not believed in the state represented by the prior $\sqsubseteq$, or it is a tautology, then no change is effected in $\sqsubseteq$, i.e., $\sqsubseteq^-_\alpha = \sqsubseteq$. In the principal, non-trivial case, where $\alpha \in \mathcal{K}$ and $\not\vdash \alpha$, the new total pre-order $\sqsubseteq^-_\alpha$ satisfies the following conditions: for any $\omega_1, \omega_2 \in \Omega$,

MC1 When $\omega_1 \models \alpha$ and $\omega_2 \models \alpha$ then $\omega_1 \sqsubseteq^-_\alpha \omega_2$ if and only if $\omega_1 \sqsubseteq \omega_2$.
MC2 When $\omega_1 \models \neg\alpha$ and $\omega_2 \models \neg\alpha$ then $\omega_1 \sqsubseteq^-_\alpha \omega_2$ if and only if $\omega_1 \sqsubseteq \omega_2$.

[5]How this idea can be formally captured is debatable. Since the entrenchment relation is changing in the process, it is not immediately obvious how to capture it in a relational setup. In Section 3 we define *degree of belief* on the sentences of the language. We can show that the degree of belief of the sentences of the form $\beta \to \alpha$ decreases when α is removed using moderate contraction.

MC3 When $\omega_1 \models \alpha, \omega_1 \notin \min_{\sqsubseteq}(\Omega)$ and $\omega_2 \models \neg\alpha$ then $\omega_2 \sqsubset^-_\alpha \omega_1$.
MC4 When $\omega_1 \in \min_{\sqsubseteq}(\Omega)$ or $\omega_1 \in \min_{\sqsubseteq}[\neg\alpha]$, then $\omega_1 \sqsubseteq^-_\alpha \omega_2$, for any $\omega_2 \in \Omega$.

It is clear from conditions **MC1** and **MC2** that the relative plausibility of worlds in $[\alpha]$ (and respectively in $[\neg\alpha]$) are not affected. We come across such invariance a number of times in this paper, and need a name for it. Accordingly we introduce the notion of *Order Preservation* in Section 2.4 which is closely related to the postulates of iterated revision (CR1 and CR2) as presented in [4]. Condition **MC3** captures the uniform promotion of worlds in $[\neg\alpha]$ and condition **MC4** reflects Eq. G2. The change to the preorder relation as prescribed here is presented pictorially in Fig. 1a. In Fig. 1 preorder relations are depicted as systems of spheres since they are equivalent, and representation of preorders as systems of spheres is well understood.

2.2 Natural Contraction Function

Boutilier [2] introduced *natural revision* in an attempt to account for iterated belief revision while remaining true to the dictum of "minimal change" to the belief state. Upon receiving information γ, natural revision changes the belief state $\sqsubseteq$ such that only the $\min_{\sqsubseteq}[\gamma]$ worlds become the minimal worlds in the revised belief state; no other change is made to the relation $\sqsubseteq$. Formally the natural revision function is defined as follows: considering the non-trivial cases, when $[\gamma] \neq \emptyset$ and $\sqsubseteq \neq \sqsubseteq_\bot$,

NR1 If $\omega_1 \in \min_{\sqsubseteq}[\gamma]$, and $\omega_2 \notin \min_{\sqsubseteq}[\gamma]$, then $\omega_1 \sqsubset^*_\gamma \omega_2$.
NR2 If $\omega_1 \notin \min_{\sqsubseteq}[\gamma]$ and $\omega_2 \notin \min_{\sqsubseteq}[\gamma]$, then $\omega_1 \sqsubseteq^*_\gamma \omega_2$ iff $\omega_1 \sqsubseteq \omega_2$.
NR3 If $\omega_1, \omega_2 \in \min_{\sqsubseteq}[\gamma]$ then $\omega_1 \approx^*_\gamma \omega_2$.

Natural contraction follows natural revision in changing the relation on Ω only with respect to the most plausible worlds in $[\neg\alpha]$ when contracting by α. Given a total pre-order relation $\sqsubseteq$ on Ω, and a sentence α, the changed total pre-order relation $\sqsubseteq^-_\alpha$ satisfies the following conditions: for any $\omega_1, \omega_2 \in \Omega$,

NC1 If $\omega_1 \in \min_{\sqsubseteq}(\Omega)$ or $\omega_1 \in \min_{\sqsubseteq}[\neg\alpha]$, then $\omega_1 \sqsubseteq^-_\alpha \omega_2$, for any $\omega_2 \in \Omega$.
NC2 If ω_1,$\omega_2 \notin \min_{\sqsubseteq}(\Omega)$ and $\omega_1, \omega_2 \notin \min_{\sqsubseteq}[\neg\alpha]$ then $\omega_1 \sqsubseteq^-_\alpha \omega_2$ iff $\omega_1 \sqsubseteq \omega_2$.

Natural contraction is pictorially depicted in Fig. 1b. It is clear that since a natural contraction function satisfies Eq. G2, the belief set corresponding to the contracted belief state will satisfy the standard AGM contraction postulates given in [1, 8]. The relative orderings of worlds in $[\alpha]$ (and respectively in $[\neg\alpha]$) are not changed; thus it satisfies *Order Preservation* (which we discuss in Section 2.4) just like the moderate contraction. It should be noted that Rott [20] has provided an equivalent definition of natural contraction when the belief state of the agent is given by an epistemic entrenchment relation.

2.3 Lexicographic Contraction Function

The Harper identity gives a relation between contraction and revision [10]. It states that $\mathcal{K}^-_\alpha = \mathcal{K} \cap \mathcal{K}^*_{\neg\alpha}$ [8]. Semantically the Harper identity may be taken to say that the $\sqsubseteq$-minimal worlds of Ω and the $\sqsubseteq$-minimal worlds of $[\neg\alpha]$ are to be given equivalent status in the state resulting from the contraction of $\sqsubseteq$ by α. We present below a mildly modified version of the generalized Harper identity that was presented in [18]:[6]

Generalized Harper Identity. In the trivial case, where $\alpha \notin \mathcal{K}_\sqsubseteq$, the preorder $\sqsubseteq$ does not change under contraction by α. As to the principal case, let B_i, $0 \leq i \leq n-1$ be the n bands ($\sqsubseteq$-equivalence classes) of worlds generated by the pre-contraction state $\sqsubseteq$, where B_0 consists of the $\sqsubseteq$-minimal worlds in Ω and $\omega \sqsubset \omega'$ for all $\omega \in B_i$, $\omega' \in B_j$ and $i < j$. Let C_i, $0 \leq i < k \leq n$ be the k $\sqsubseteq$-equivalent classes of worlds in $[\neg\alpha]$, i.e., $\bigcup_{i=0}^{k-1} C_i = [\neg\alpha]$, and $\omega \sqsubset \omega'$ for all $\omega \in C_i$, $\omega' \in C_j$ and $i < j$. Similarly, let C'_i, $0 \leq i < k' \leq n$ be the k' $\sqsubseteq$-equivalent classes of worlds in $[\alpha]$, i.e., $\bigcup_{i=0}^{k'-1} C'_i = [\alpha]$, and $\omega \sqsubset \omega'$ for all $\omega \in C'_i$, $\omega' \in C'_j$ and $i < j$. Define $C_{i+1} = \emptyset$ for $k-1 \leq i < k'-1$, if $k < k'$; otherwise, if $k > k'$, then define $C'_{i+1} = \emptyset$ for $k'-1 \leq i < k-1$. The bands in $\sqsubseteq^-_x$ are given by $D_i = C_i \cup C'_i$ for $0 \leq i < max(k, k') - 1$.

It can be easily noted that neither moderate nor natural contraction function satisfies the generalized Harper identity. On the other hand, lexicographic contraction is defined to satisfy the generalized Harper identity. Given a total pre-order relation $\sqsubseteq$ on Ω and a belief $\alpha \in \mathcal{K}$, when contracting α from $\mathcal{K}$, the lexicographic contraction function $-$ changes the total pre-order to $\sqsubseteq^-_\alpha$. The changed relation $\sqsubseteq^-_\alpha$ is given as follows:

LC1 If $\omega \models \alpha$ and $\omega' \models \alpha$, then $\omega \sqsubseteq^-_\alpha \omega'$ iff $\omega \sqsubseteq \omega'$

LC2 If $\omega \models \neg\alpha$ and $\omega' \models \neg\alpha$, then $\omega \sqsubseteq^-_\alpha \omega'$ iff $\omega \sqsubseteq \omega'$

LC3 Let χ be one member of $\{\alpha, \neg\alpha\}$ and $\overline{\chi}$ the other. If $\omega \models \chi$ and $\omega' \models \overline{\chi}$, then $\omega \sqsubseteq^-_\alpha \omega'$ iff the length of a complete chain of worlds in $[\chi]$ which ends in ω is less than or equal to the length of a complete chain of worlds in $[\overline{\chi}]$ which ends in ω'.[7,8]

For the special case where $\alpha \notin \mathcal{K}$ or $\vdash \alpha$, the lexicographic contraction results in an unchanged pre-order relation, $\sqsubseteq^-_\alpha = \sqsubseteq$. Conditions **LC1** and **LC2** state that the prior ordering between any two worlds ω, ω' that both satisfy α (or $\neg\alpha$) is not changed upon contraction by α. **LC3** states that the worlds of $[\neg\alpha]$ are simultaneously shifted so that the worlds in $[\neg\alpha]$ and worlds in $[\alpha]$ which are ranked equally within the respective sets (after removing the empty

[6]The modification in question deals with certain emptysets that were not properly dealt with in the version presented in [18].

[7]Please note that empty "layers" are ignored in the process.

[8]This is equivalent to the original condition presented in [16].

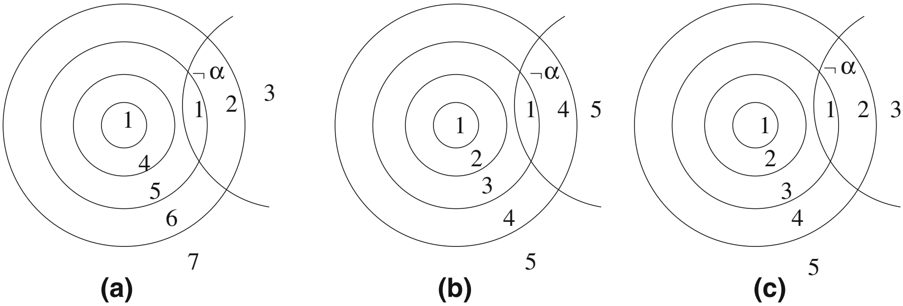

Fig. 1 **a** Priority/Moderate contraction, **b** Natural/Conservative contraction, and **c** Lexicographic contraction. The numbers indicate how the "system of spheres" should be constructed after contraction by α. The cells numbered *1* would jointly constitute the centre; the ones numbered *2* will be the next layer, and so on

sets) are equally placed in the preorder resulting from contraction. A pictorial representation of lexicographic contraction is given in Fig. 1c.

2.4 Comparison of the Three Contraction Functions

Moderate, natural and lexicographic contraction functions offer three different ways of changing the total preorder on Ω. They share some similarities. The three contraction functions behave in the same way when the sentence α (which is being contracted) is either a logical tautology or is not an existing belief of the agent. In the non-trivial case, where α is a contingent belief, i.e., both $\alpha \in \mathcal{K}$ and $\nvdash \alpha$, all three of them preserve the ordering of the worlds in $[\alpha]$ after contraction. Furthermore, in all the three cases of contraction, ordering of worlds in $[\neg\alpha]$ is also preserved. We call these properties as *Order Preservation* in $[\alpha]$ and *Order Preservation* in $[\neg\alpha]$.

Definition 1 (Order Preservation) Let $\sqsubseteq$ be a total preorder on Ω representing the belief state, and α be a sentence. A contraction function $-$ is said to obey *Order Preservation in Δ, for any $\Delta \subseteq \Omega$, if and only if, for every $\omega, \omega' \in \Delta$, $\omega \sqsubseteq^-_\alpha \omega'$ iff $\omega \sqsubseteq \omega'$.*

When $-$ obeys *Order Preservation* in Δ upon contraction by α, we write this in short-hand by $\mathcal{OP}_\alpha(\Delta)$. It is worth noting that *Order Preservation* has been studied in the literature in different contexts. Order Preservation property was proposed in the context of iterated belief revision in [4]. Chopra and colleagues [3] provide order preservation properties based on iterated belief change in the context of presenting variations of the *recovery axiom* [1].[9] We study order preservation purely in the context of iterated contraction.

[9]The recovery axiom: $\mathcal{K} \subseteq (\mathcal{K}^-_\alpha)^+_\alpha$.

When contracting a belief α, the contraction function $-$ might preserve the ordering of worlds in two different sets $[\alpha]$ and $[\neg\alpha]$. The following two lemmas show that if the contraction function is an AGM-rational state contraction function that satisfies both $\mathcal{OP}_\alpha[\alpha]$ and $\mathcal{OP}_\alpha[\neg\alpha]$, then the result of consecutive contractions using this function follows a predictable pattern under special circumstances. We provide the proofs of these and other claims in Appendix A.

Lemma 1 *Let $\sqsubseteq$ be a consistent belief state and $\mathcal{K}$ its associated belief set. An AGM-rational state contraction function $-$ satisfies $\mathcal{OP}_\alpha[\alpha]$ for any sentence α iff for every sentence β such that $\vdash \alpha \vee \beta$, $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\to\beta}$.*

Lemma 2 *Let $\sqsubseteq$ be a consistent belief state and $\mathcal{K}$ its associated belief set. An AGM-rational state contraction function $-$ satisfies $\mathcal{OP}_\alpha[\neg\alpha]$ for any sentence α iff for every sentence β such that $\vdash \alpha \to \beta$, $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta}$.*

Lemmas 1 and 2 highlight the similarity between the three contraction functions. To illustrate the differences between the three contraction functions, we draw attention to an example provided in [17] which is a variation of the well known example given in [4].

Example 1 Let the agent believe that *Mr. Craig is rich and Mr. Craig is smart. The result of removing the belief* smart *followed by the removal of the belief* rich *in terms of the three state contraction functions is given in Fig. 2.*

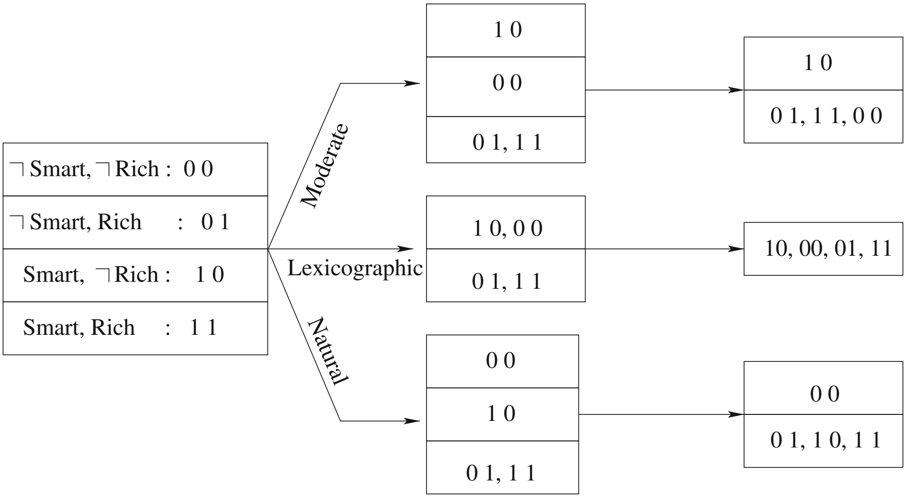

Fig. 2 This figure shows the result of contracting a belief state first by *smart* and then by *rich*. The preorder relation is represented as *boxes*. The *lowest box* denotes the set of most plausible worlds, the *box above* it denotes the set of the next most plausible worlds, and so on

In this example it is shown that given an initial belief state represented as a total preorder on Ω, shown as the four-layered box on the leftmost column, all the three contraction functions retain the same set of beliefs after first contraction, represented in Fig. 2 as the lowest layers in the three boxes in the central column (consisting of worlds 01 and 11). But in the process of contracting the belief, the three contraction functions change the belief state in different ways, thus resulting in different belief states. Upon second contraction, according to moderate contraction function, the agent will end up believing *if smart then rich*. When the agent uses conservative contraction function, the agent will end up believing *either smart or rich*. On the other hand, if the contraction function used is lexicographic contraction, the agent will end up believing only in tautologies. The difference in how the non-minimal $[\alpha]$ and $[\neg\alpha]$ worlds are shifted relative to each other by each contraction function gives rise to different results upon iterated contraction. In Section 4, we study the effect of different changes to the preorder relation on iterated contraction.

2.5 Principled Factored Insertion

In their attempt to characterize lexicographic contraction functions, the authors in [16] studied *Principled Factored Insertion* which is derived from *Qualified Intersection* [19] and *Factoring* [1].

Principled Factored Insertion (PFI) Given $\beta \in \mathcal{K}^-_\alpha$

1. If $\alpha \to \beta \in (\mathcal{K}^-_\alpha)^-_\beta$, then $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta}$
2. If $\beta \vee \alpha \in (\mathcal{K}^-_\alpha)^-_\beta$, then $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\to\beta}$
3. If neither $\alpha \to \beta \in (\mathcal{K}^-_\alpha)^-_\beta$ nor $\alpha \vee \beta \in (\mathcal{K}^-_\alpha)^-_\beta$, then $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta} \cap \mathcal{K}^-_{\alpha\to\beta}$.

Any AGM contraction function that satisfies PFI is said to be a *principled iterated contraction operation*. In [16] it was shown that every lexicographic contraction function is a principled iterated contraction operation. However, it was also observed that even moderate contraction function is a principled iterated contraction operation. Here we present sufficiency conditions for a contraction function to satisfy PFI.

Theorem 1 *Every contraction function – satisfying Eq. G2, $\mathcal{OP}_\alpha[\alpha]$ and $\mathcal{OP}_\alpha[\neg\alpha]$ satisfies the principled factored insertion(PFI).*

We have already seen that moderate, natural and lexicographic contraction functions satisfy $\mathcal{OP}_\alpha[\alpha]$ and $\mathcal{OP}_\alpha[\neg\alpha]$. They also satisfy Eq. G2. Hence all the three contraction functions satisfy PFI, and therefore, they are all principled iterated contraction operations. These contraction functions differ from each other in how the ordering between the worlds of $[\alpha]$ and $[\neg\alpha]$ are changed relative to each other. To characterize these contraction functions we need to identify properties that capture these changes. We investigate such properties in Section 4. But before that, we present a measure called *degree of belief* based on the belief state of the agent. This measure would be helpful in investigating the characteristics of different ways of changing the preorder on Ω.

3 Degrees of Belief

All the beliefs of an agent need not be considered to be equally important. Prioritization of beliefs has been studied in detail in many works, [6, 7, 12, 21]. The quality or strength of beliefs is captured in the literature in different ways, as 'incorrigibilty' of a belief [13, 14], degrees of potential surprise or disbelief [22], entrenchment relations [8], possibility measures [6] and ordinal ranking functions [23]. These offer a measure of gradation among the beliefs of an agent, be it qualitative or quantitative. Here we present a quantitative measure that we call *degree of belief*, and denote it by the function $d(\cdot)$.

Before we define the function d, we recall that a chain of worlds $\omega_0 \sqsubset \omega_1 \sqsubset \ldots \sqsubset \omega_n$ is said to be *complete* iff ω_0 is a $\sqsubseteq$-minimal world of Ω and for any $\omega_{i-1} \sqsubset \omega_i$ $(1 \leq i \leq n)$ there does not exist any world ω' such that $\omega_{i-1} \sqsubset \omega' \sqsubset \omega_i$. Degree of belief, d is a function that takes the sentences in the language to a non-negative integer. To every sentence α, the function d assigns a value based on the belief state $\sqsubseteq$.

Definition 2 Given a consistent belief state $\sqsubseteq$ (i.e., $\sqsubseteq \neq \sqsubseteq_\perp$), the degree of belief d for any sentence $\alpha \in \mathcal{L}$ is defined as

1. $d(\alpha) = \infty$ when $\vdash \alpha$,
2. $d(\alpha) = 0$ when $\alpha \vdash \perp$,
3. $d(\alpha) = \min\{\|\mathcal{C}\| : \mathcal{C}$ is a complete chain of worlds in Ω ending in some model of $\neg\alpha\}$, for any α such that $\nvdash \alpha$ and $\alpha \nvdash \perp$.

One way to get a better grasp of this notion of *degree of belief* is the following. Imagine that the belief state $\sqsubseteq$ that typically is viewed as a system of spheres in fact represents a graduated cup [15], shown in Fig. 3.[10] The central sphere (i.e., $\min_{\sqsubseteq}(\Omega)$) constitutes the base of this cup and other subsequent $\sqsubseteq$-equivalence classes correspond to different levels as marked in the cup. The set of worlds $[\neg\alpha]$ then will correspond to some "hole" in this cup and $[\alpha]$ to a graduated cup with a hole in it. Given this picture, the measure $d(\alpha)$ in fact specifies the lowest point of the hole(s) represented by $[\neg\alpha]$, and hence, equivalently, tells how much liquid the damaged graduated cup $[\alpha]$ can hold. Alternatively, we may say $d(\alpha)$ specifies where the belief α leaks, and to that effect specifies the degree of belief in α. Definition 2 above refers only to the case where the belief state is consistent. An inconsistent belief state is represented by $\sqsubseteq_\perp$ which is totally disconnected. Then the set of minimal

[10] In [15], the notion of graduated cup was used to motivate the semantics of epistemic entrenchment (footnote 12 in Section 1.4.3). Here we use it to motivate the degree of belief which is similar.

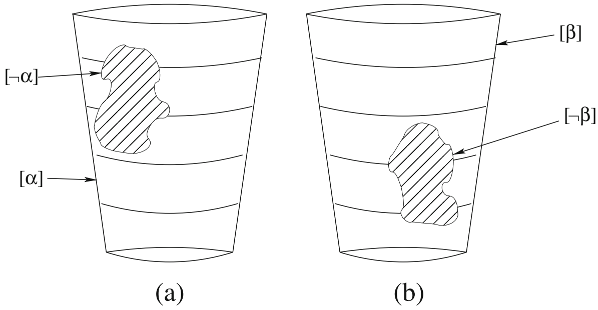

Fig. 3 The cup and the hole (denoted by the *shaded region*) together form the set of all possible worlds. The cup in (**a**) holds more liquid than the cup in (**b**); hence the degree of belief in the proposition α is higher than that of β

worlds $min_{\sqsubseteq_\perp}(\Omega)$ is an empty set. This is a special case, and needs to be dealt with as such. Hence we postulate that the degree of belief of every sentence in an inconsistent belief state is ∞. Alternatively, when the belief state is denoted by a *full preorder* relation $\sqsubseteq_\top$, we get $min_{\sqsubseteq_\top}(\Omega) = \Omega$. In this case, for every contingent sentence there is a model of its negation which is present in $min_{\sqsubseteq}(\Omega)$. By our definition, every contingent sentence is, therefore, assigned *zero* as degree of belief. This is as expected, since when a belief state is represented by a full preorder every contingent sentence is a non-belief and the only beliefs are tautologies.

The following properties of d are easily established. Given two sentences α and β in the language:

1. $d(\alpha \wedge \beta) = \min\{d(\alpha), d(\beta)\}$.
2. $d(\beta) \geq d(\alpha)$, when α and β are such that $\alpha \vdash \beta$.
3. $d(\beta) = d(\alpha)$, when α and β are equivalent under $\vdash$.

Further, we have defined the degree of belief of $\perp$ to be zero. By observing these properties, we realize that our definition of degree of belief bears a striking resemblance to the *Entrenchment Ranking Function* [21]. In fact, our definition is a translation of the notion of entrenchment ranking function in to a system of spheres framework. We now extend our definition of degree of belief to some *restricted belief states*.

3.1 Conditional Degrees of Belief

The degree of belief of a sentence is the dual of a measure of doubt or uncertainty on that sentence. The higher the degree of belief of a sentence the lower is its measure of doubt. When a sentence has maximum degree of belief, there is no trace of doubt attached to it. The agent assigns a degree of belief zero to any sentence it maximally doubts. In general, tautologies are the only sentences to have maximum degree of belief. However, it is quite conceivable that an agent unreservedly believes a sentence which is not a tautology. For instance a highly opinionated agent might refuse to see the fallibility of some of its beliefs. Alternatively, a reasonable agent might take a certain physical law such as the Second Law of Thermodynamics[11] as a given, and beyond the reach of any doubt. In that case the agent considers the sentence in question to have maximum degree of belief. We propose to capture such phenomena by a restriction of the belief state such that a given contingent sentence has maximum degree of belief.

Let $\sqsubseteq$ be the initial belief state of the agent (where only tautologies have maximum degree of belief). Suppose that the agent decides that a set of contingent sentences are true beyond any doubt. We denote this set of sentences by $\mathcal{R}$. The agent then assigns a maximum degree of belief to the sentences in $\mathcal{R}$. To this end, we restrict the initial belief state $\sqsubseteq$ of the agent to $[\mathcal{R}]$, denoting the resultant state by $\sqsubseteq_{\mathcal{R}}$. In other words, for any two possible worlds ω_1 and ω_2, $\omega_1 \sqsubseteq_{\mathcal{R}} \omega_2$ if and only if $\omega_1,\ \omega_2 \in [\mathcal{R}]$ and $\omega_1 \sqsubseteq \omega_2$. We call this restriction of the belief state by $\mathcal{R}$ as *conditionalization* of the belief state.[12] We denote the degree of belief in the conditionalized belief state as $d_{\mathcal{R}}$ or $d(.|\mathcal{R})$.

To gain a better understanding of the issue at hand, let us consider a very simple case. Suppose that the set $\mathcal{R}$ is a singleton set, $\mathcal{R} = \{\alpha\}$. There are three cases to consider while conditionalizing by α.

Case 1. When α is an existing belief of the agent, every $\sqsubseteq$-minimal world of Ω is a model of α. Upon conditionalization, all these worlds remain to be minimal worlds of $[\mathcal{R}]$ (now $[\mathcal{R}] = [\alpha]$) based on $\sqsubseteq_{\mathcal{R}}$. Therefore the set of beliefs of the agent remains the same after conditionalization. However, the degree of belief assigned to sentences have changed. For instance, degree of belief of α is changed from some non-zero value to $d_{\mathcal{R}}(\alpha) = \infty$.

Case 2. Suppose neither α nor $\neg\alpha$ is believed in the initial belief state. The set of $\sqsubseteq_{\mathcal{R}}$-minimal worlds is given by the set of all $\sqsubseteq$-minimal worlds of Ω which are also models of α. Therefore, from Eq. G1 we see that no beliefs are lost but the belief set is expanded to include α. The degree of belief of sentence α is changed from zero to ∞ upon conditionalization while the degree of belief of $\neg\alpha$ remains zero.

Case 3. Suppose the agent initially believes in $\neg\alpha$. Conditionalisation by α in this case changes the set of beliefs. Some beliefs are lost and some beliefs are gained. The minimal worlds of $[\alpha]$ based on the relation $\sqsubseteq$ become the minimal worlds based on the relation $\sqsubseteq_{\mathcal{R}}$. The degree of belief of α changes from zero to ∞ and the degree of belief of $\neg\alpha$ changes from non-zero positive value to zero.

When the set $\mathcal{R}$ is empty, this reduces to the case where only logical tautologies are the sentences with maximum degree of belief. It can be seen that $d(\alpha) = d(\alpha|\top)$, where d is the degree of belief based on $\sqsubseteq$. On the other

[11] This law says the the overall entropy of a closed system will never decrease.

[12] This is very similar to conditional probability. In probabilistic conditionalization, the posterior state ignores the simple events that are inconsistent with the evidence; similarly here worlds outside $[\mathcal{R}]$ are ignored.

hand suppose $\mathcal{R}$ contains an inconsistent sentence. Conditionalization by $\mathcal{R}$ in this case results in an inconsistent belief state. We have already defined the degree of belief of any sentence in an inconsistent state to be ∞. Hence the degree of belief of every sentence upon conditionalization by an inconsistent sentence becomes ∞. Some of the properties of conditional degrees of belief are as follows:

1. $d(\beta|\alpha) = \infty$ for every β such that $\alpha \vdash \beta$. In particular:
   a) $d(\alpha|\alpha) = \infty$.
   b) $d(\top|\alpha) = \infty$.
   c) $d(\beta|\bot) = \infty$ for every β.
2. $d(\beta|\alpha) = 0$ for every β such that $\alpha \vdash \neg\beta$. As special cases, we have:
   a) $d(\bot|\alpha) = 0$.
   b) $d(\neg\alpha|\alpha) = 0$.

Although the degree of belief is similar to epistemic entrenchment, it is richer than a pure relational measure. In a certain sense, it allows us to "compare" the effects of different epistemic changes on one or more beliefs. For instance, intuitively, I would more firmly believe that kangaroos are man eaters if I observe a killer kangaroo than if I observe a grass munching kangaroo. While this intuition cannot be captured via relational measures such as epistemic entrenchment, it can be captured using the degree of belief. We will make use of such conditional degrees of belief in characterising lexicographic contraction in Section 5. But before that we need to discuss various cases that arise in sequential contraction of α followed by β; and that is the topic we take up in the next section.

4 Plausible Properties of Iterable Contraction Functions

As we saw in Example 1, the difference between the three contraction functions is evident in the resultant sets of beliefs only after repeated contractions. Therefore we aim to completely characterize these state contraction functions with the help of the properties based on iterated contractions. Towards this goal, in this section we study various cases that arise when contracting two beliefs α and β one after the other. From Eq. G2 it is evident that when the belief being withdrawn is not present in the belief set, the result of contraction by any of these three contraction functions does not change the resulting set of beliefs. Hence in the following discussion, we will assume that $\alpha,\ \beta \in \mathcal{K}$ and also $\beta \in \mathcal{K}^-_\alpha$. Sentences $\alpha \vee \beta$ and $\alpha \rightarrow \beta$ are two important factors to be considered when removing β following removal of α. Let us suppose $\alpha \vee \beta$ survives the individual removal of α and β, i.e., $\alpha \vee \beta \in \mathcal{K}^-_\alpha$ and $\alpha \vee \beta \in \mathcal{K}^-_\beta$.

In such a case one might want $\alpha \vee \beta \in (\mathcal{K}_\alpha^-)_\beta^-$. Given the assumption that $\beta \in \mathcal{K}_\alpha^-$, we get:[13]

$$\alpha \vee \beta \in \mathcal{K}_\beta^- \Rightarrow (\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\to\beta}^-. \tag{1}$$

As a special case we have α and β such that $\vdash \alpha \vee \beta$. Thus,

$$\vdash \alpha \vee \beta \Rightarrow (\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\to\beta}^-. \tag{2}$$

When the belief $\alpha \to \beta$ is preferred to $\alpha \vee \beta$, the agent might retain $\alpha \to \beta$ in $\mathcal{K}_\beta^-$, at the cost of $\alpha \vee \beta$. Since both $\alpha \to \beta \in \mathcal{K}_\alpha^-$ and $\alpha \vee \beta \in \mathcal{K}_\alpha^-$ (courtesy the assumption that β in $\mathcal{K}_\alpha^-$), the agent might make the same choice when contracting β from $\mathcal{K}_\alpha^-$; i.e., $\alpha \to \beta \in (\mathcal{K}_\alpha^-)_\beta^-$. Therefore $\alpha \to \beta \in \mathcal{K}_\beta^-$ suggests that $\alpha \to \beta \in (\mathcal{K}_\alpha^-)_\beta^-$ and hence from PFI

$$\alpha \to \beta \in \mathcal{K}_\beta^- \Rightarrow (\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\vee\beta}^-. \tag{3}$$

As a special case of Eq. 3, we get:

$$\vdash \alpha \to \beta \Rightarrow (\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\vee\beta}^-. \tag{4}$$

We recall that Lemmas 1 and 2 concern Eqs. 2 and 4 respectively. Furthermore, as a dual of Eq. 2 we get:

$$\nvdash \alpha \vee \beta \Rightarrow (\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\vee\beta}^-. \tag{5}$$

Lemma 3 *An AGM-rational contraction function $-$ satisfies Eq. 5 iff for every sentence α, both:* (a) *$-$ satisfies $\mathcal{OP}_\alpha[\neg\alpha]$, and* (b) *$\omega \sqsubset_\alpha^- \omega'$ for every $\omega, \omega' \in \Omega$ such that $\omega \in [\neg\alpha]$ and $\omega' \in [\alpha] \backslash min_{\sqsubseteq}[\alpha]$.*

Finally, Eqs. 1 and 3 jointly suggest:

$$\alpha \vee \beta \notin \mathcal{K}_\beta^-, \alpha \to \beta \notin \mathcal{K}_\beta^- \Rightarrow (\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\vee\beta}^- \cap \mathcal{K}_{\alpha\to\beta}^-. \tag{6}$$

Lemma 4 *For any arbitrary beliefs α and β such that $\beta \in \mathcal{K}_\alpha^-$, an AGM-rational contraction function $-$ satisfies Eqs. 1, 3 and 6 if and only if $\forall\omega, \omega' \notin (min_{\sqsubseteq}(\Omega) \cup min_{\sqsubseteq}[\neg\alpha])$, $\omega \sqsubseteq_\alpha^- \omega'$ iff $\omega \sqsubseteq \omega'$.*

The Eqs. 1, 3 and 6 have been presented as *Naive factored insertion* in [16]. It is worth noting that there is a certain amount of conflict among the Eqs. 1–6. For instance, one can imagine cases where the preconditions of Eqs. 1 and 5 are jointly satisfied, but their consequences are not. In fact, it may be argued that these equations naturally fall into two exclusive sets: {(2) and (5)} is one and {(1), (3) and (6)} is the other, since their respective preconditions partition the hypothesis space in different manner.

[13] Note that the symbol $\Rightarrow$ in the properties listed henceforth is not a logical connective. For readability, we use this symbol rather than natural language "if …, then …".

4.1 Properties Based on Degrees of belief

We now present a translation of Eqs. 1–6 in terms of degrees of belief. As we have already seen, $\alpha \vee \beta$ and $\alpha \rightarrow \beta$ are two important factors to be considered when removing β following the removal of α. The choice between contracting by $\alpha \vee \beta$ and $\alpha \rightarrow \beta$ can be resolved with the help of degrees of belief. When the degree of belief in $\alpha \vee \beta$ is greater than the degree of belief in $\alpha \rightarrow \beta$, then the agent contracts $\alpha \rightarrow \beta$, and similarly it contracts $\alpha \vee \beta$ when the degree of belief in $\alpha \vee \beta$ is less than that of $\alpha \rightarrow \beta$. When both have equal degrees of belief then both are contracted. This can be formalized in terms of the following:

$$d(\alpha \vee \beta) > d(\alpha \rightarrow \beta) \Rightarrow (\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha \rightarrow \beta}. \tag{7}$$

$$d(\alpha \vee \beta) < d(\alpha \rightarrow \beta) \Rightarrow (\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha \vee \beta}. \tag{8}$$

$$d(\alpha \vee \beta) = d(\alpha \rightarrow \beta) \Rightarrow (\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha \rightarrow \beta} \cap \mathcal{K}^-_{\alpha \vee \beta}. \tag{9}$$

With the help of following observation we note that Eqs. 7, 8 and 9 are equivalent to Eqs. 1, 3 and 6 respectively when the contraction function involved is AGM-rational.

OBSERVATION 1 Let $-$ be an AGM-rational contraction function, and the degree of belief function d is appropriately related with the presumed belief state $\sqsubseteq$. Then:

(a) Equation 1 is satisfied iff Eq. 7 is,
(b) Equation 3 is satisfied iff Eq. 8 is, *and*
(c) Equation 6 is satisfied iff Eq. 9 is.

Property 5, which states that when $\alpha \vee \beta$ is not a tautology the contraction by β after contraction by α is given by the meet of contraction by α and $\alpha \vee \beta$, can be captured in terms of degrees of belief by

$$d(\alpha \vee \beta) < \infty \Rightarrow (\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha \vee \beta}. \tag{10}$$

The property 2, on the other hand, can be translated as follows:

$$d(\alpha \vee \beta) = \infty \Rightarrow (\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha \rightarrow \beta}. \tag{11}$$

By using conditional degrees of belief, we can list some more properties for iterated contraction. The degree of belief of $\alpha \vee \beta$ in the belief state conditionalised with respect to $\neg\alpha$ can be taken as indicating the degree to which the belief $\alpha \vee \beta$ is independent of α. Similarly the conditional degree of belief $d(\alpha \rightarrow \beta|\alpha)$ can be considered as indicating the degree of independence of belief $\alpha \rightarrow \beta$ from $\neg\alpha$. Suppose we have $d(\alpha \rightarrow \beta|\alpha) > d(\alpha \vee \beta|\neg\alpha)$, we can interpret this as $\alpha \rightarrow \beta$ is more "self-sufficient" than $\alpha \vee \beta$ and hence there is more reason to retain the belief in $\alpha \rightarrow \beta$ after iterated contraction of α

followed by β. Going back to PFI in case the agent decides to retain $\alpha \rightarrow \beta$ in $(\mathcal{K}_\alpha^-)_\beta^-$, we say that the result of iterated contraction is given by the meet of contraction by α and $\alpha \vee \beta$. Similarly when $d(\alpha \vee \beta|\neg\alpha) > d(\alpha \rightarrow \beta|\alpha)$, the result of iterated contraction can be derived from PFI to be the meet of contraction by α and $\alpha \rightarrow \beta$. If $d(\alpha \vee \beta|\neg\alpha) = d(\alpha \rightarrow \beta|\alpha)$, then it is indicated that both $\alpha \vee \beta$ and $\alpha \rightarrow \beta$ are equally independent of α and $\neg\alpha$, respectively. With equal preference, the agent could decide not to retain either of them and hence the result of the iterated contraction could be the combined meet of contraction by α, $\alpha \vee \beta$ and $\alpha \rightarrow \beta$. We list these in the form of following properties.

$$d(\neg\alpha \vee \beta|\alpha) > d(\alpha \vee \beta|\neg\alpha) \Rightarrow (\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\vee\beta}^- \tag{12}$$

$$d(\alpha \vee \beta|\neg\alpha) > d(\neg\alpha \vee \beta|\alpha) \Rightarrow (\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\rightarrow\beta}^- \tag{13}$$

$$d(\alpha \vee \beta|\neg\alpha) = d(\neg\alpha \vee \beta|\alpha) \Rightarrow (\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\rightarrow\beta}^- \cap \mathcal{K}_{\alpha\vee\beta}^-. \tag{14}$$

We give two lemmas that connect the properties based on degrees of belief and properties 1–6 described earlier in Section 4.

Lemma 5 *Any AGM-rational contraction function − that satisfies Eq. 13 also satisfies Eq. 2.*

Lemma 6 *Any AGM-rational contraction function − that satisfies Eq. 12 also satisfies Eq. 4.*

5 Representation Results

In this section we will provide the representation results which axiomatically characterize moderate, natural and lexicographic contraction functions based on these properties.

Theorem 2 *An AGM-rational contraction function is a moderate contraction function iff it satisfies properties 2 and 5.*

The above result gives a very simple characterisation of the moderate contraction function in that, it identifies the AGM-rational contraction functions that also satisfy the equations:

$$\vdash \alpha \vee \beta \Rightarrow (\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\rightarrow\beta}^- \tag{2}$$

$$\nvdash \alpha \vee \beta \Rightarrow (\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\vee\beta}^-. \tag{5}$$

to be exactly the moderate contraction functions. In terms of degrees of belief, any AGM-rational contraction function is a moderate contraction function iff it satisfies Eqs. 10 and 11.

Theorem 3 *An AGM-rational contraction function is a natural contraction function iff it satisfies properties 1, 3 and 6.*

Theorem 3 identifies the necessary and sufficiency conditions for a contraction function to be qualified as a natural contraction function. Keeping in mind how AGM-rational contraction functions are constructed, we claim that a belief (set) contraction function is generated from a natural belief state contraction function iff it satisfies the AGM contraction postulates along with

$$\alpha \vee \beta \in \mathcal{K}^-_\beta \Rightarrow (\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\to\beta} \tag{1}$$

$$\alpha \to \beta \in \mathcal{K}^-_\beta \Rightarrow (\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta} \tag{3}$$

$$\alpha \vee \beta, \alpha \to \beta \notin \mathcal{K}^-_\beta \Rightarrow (\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta} \cap \mathcal{K}^-_{\alpha\to\beta} \tag{6}$$

Since Eqs. 1, 3 and 6 form the Naive factored insertion [16], we have that an AGM-rational contraction function is a natural contraction function iff it satisfies naive factored insertion. The natural contraction function can also be characterized in terms of degrees of belief based on a given belief state. Every AGM-rational contraction function is a natural (or conservative) contraction function if and only if it satisfies the properties 7, 8 and 9.

Theorem 4 *An AGM-rational contraction function is a lexicographic contraction function iff it satisfies properties 12, 13 and 14.*

The above theorem gives the necessary and sufficient conditions for a contraction function to be a lexicographic contraction function. Thus we have that an AGM-rational contraction function is lexicographic iff it satisfies the properties

$$d(\neg\alpha \vee \beta|\alpha) > d(\alpha \vee \beta|\neg\alpha) \Rightarrow (\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta} \tag{12}$$

$$d(\alpha \vee \beta|\neg\alpha) > d(\neg\alpha \vee \beta|\alpha) \Rightarrow (\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\to\beta} \tag{13}$$

$$d(\alpha \vee \beta|\neg\alpha) = d(\neg\alpha \vee \beta|\alpha) \Rightarrow (\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\to\beta} \cap \mathcal{K}^-_{\alpha\vee\beta} \tag{14}$$

6 Concluding Remarks

In this paper we examined three different iterable contraction functions, namely, the moderate (or priority) contraction, the natural (or conservative) contraction and the lexicographic contraction. The proposals for these contraction functions were originally couched in semantic terms, and their properties have not been studied before. We showed that all these three functions satisfy the *Principled factored insertion* which was studied in [16] in relation to lexicographic contraction. We presented and examined a list of plausible properties (similar to rationality postulates) that one may expect an iterable contraction function to satisfy. Using these properties, we provided representation results for the natural and moderate contraction functions.

In order that we can characterise lexicographic contraction, we introduced a quantitative measure that we call the *degrees of belief*, and its derivative notion called the *conditional degree of belief*. This notion allows simple representation of plausible properties of iterated contraction. Using different sets of such properties we characterised lexicographic, moderate as well as natural contraction functions. Nonetheless, we find it somewhat unsatisfying that a standard characterisation of lexicographic contraction still eludes us. It will be nice if such a characterisation can be given, or it can be established that such standard characterisation of lexicographic contraction is not possible. We leave this task to a future occasion.

A Proofs

Lemma 1 *Let $\sqsubseteq$ be a consistent belief state and $\mathcal{K}$ its associated belief set. An AGM-rational state contraction function $-$ satisfies $\mathcal{OP}_\alpha[\alpha]$ for any sentence α iff for every sentence β such that $\vdash \alpha \vee \beta$, $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\to\beta}$.*

Proof Let $-$ be a contraction function satisfying Eq. G2, $\sqsubseteq$ a belief state and $\mathcal{K}$ given by Eq. G1. Let α be any arbitrary sentence. We need to show that the contraction function $-$ satisfies $\mathcal{OP}_\alpha[\alpha]$ iff $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\to\beta}$ for every sentence β such that $\vdash \alpha \vee \beta$.

(Left to Right). Assume that $-$ satisfies $\mathcal{OP}_\alpha[\alpha]$. Let β be an arbitrary sentence such that $\vdash \alpha\vee\beta$. Since $\vdash \alpha\vee\beta$ we have $[\neg\beta] \subseteq [\alpha]$. Therefore $\min_{\sqsubseteq^-_\alpha}[\neg\beta] = \min_{\sqsubseteq^-_\alpha}[\alpha\wedge\neg\beta]$. Since $-$ satisfies $\mathcal{OP}_\alpha[\alpha]$, we further have $\min_{\sqsubseteq^-_\alpha}[\alpha\wedge\neg\beta] = \min_{\sqsubseteq}[\alpha\wedge\neg\beta]$. Therefore $\min_{\sqsubseteq}(\Omega) \cup \min_{\sqsubseteq}[\neg\alpha] \cup \min_{\sqsubseteq^-_\alpha}[\neg\beta] = \min_{\sqsubseteq}(\Omega) \cup \min_{\sqsubseteq}[\neg\alpha] \cup \min_{\sqsubseteq}[\alpha\wedge\neg\beta]$, i.e., $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\to\beta}$.

(Right to Left). Assume that $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\to\beta}$ for every sentence β such that $\vdash \alpha \vee \beta$. Let ω, ω' be two arbitrarily chosen worlds in $[\alpha]$. Let us assume that $\omega \sqsubseteq \omega'$. There are two cases : (1) Suppose $\omega \in \min_{\sqsubseteq}(\Omega)$. Then from Eq. G2 we can deduce that $\omega \in \min_{\sqsubseteq^-_\alpha}(\Omega)$, i.e., $\omega \sqsubseteq^-_\alpha \omega'$. (2) Suppose $\omega \notin \min_{\sqsubseteq}(\Omega)$. Let β be a sentence such that $[\neg\beta] = \{\omega, \omega'\}$. Since $[\neg\beta] \subseteq [\alpha]$ we have $\vdash \alpha \vee \beta$. Therefore $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\to\beta}$, whereby, from Eq. G2 $\min_{\sqsubseteq}(\Omega) \cup \min_{\sqsubseteq}[\neg\alpha] \cup \min_{\sqsubseteq^-_\alpha}[\neg\beta] = \min_{\sqsubseteq}(\Omega) \cup \min_{\sqsubseteq}[\neg\alpha] \cup \min_{\sqsubseteq}[\alpha \wedge \neg\beta]$. However $\omega \notin \min_{\sqsubseteq}(\Omega)$, $\omega \notin \min_{\sqsubseteq}[\neg\alpha]$ and $\omega \in \min_{\sqsubseteq}[\alpha \wedge \neg\beta]$. Therefore $\omega \in \min_{\sqsubseteq^-_\alpha}[\neg\beta]$, i.e., $\omega \sqsubseteq^-_\alpha \omega'$. Similarly assuming $\omega \sqsubset \omega'$ yields $\omega \sqsubset^-_\alpha \omega'$. Hence $-$ satisfies $\mathcal{OP}_\alpha[\alpha]$. □

Lemma 2 *Let $\sqsubseteq$ be a consistent belief state and $\mathcal{K}$ its associated belief set. An AGM-rational state contraction function $-$ satisfies $\mathcal{OP}_\alpha[\neg\alpha]$ for any sentence α iff for every sentence β such that $\vdash \alpha \to \beta$, $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta}$.*

Proof This proof is similar to that of Lemma 1. Let $-$ be a contraction function satisfying Eq. G2, $\sqsubseteq$ a belief state and $\mathcal{K}$ the corresponding belief set. Let α be an arbitrary sentence. We need to show that $-$ satisfies $\mathcal{OP}_\alpha[\neg\alpha]$ iff $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta}$ for every sentence β such that $\vdash \alpha \to \beta$.

(Left to Right). Let $-$ satisfy $\mathcal{OP}_\alpha[\neg\alpha]$. Let β be a sentence such that $\vdash \alpha \to \beta$. Since $\vdash \alpha \to \beta$ we have $[\alpha] \subseteq [\beta]$. Therefore $\min_{\sqsubseteq^-_\alpha}[\neg\beta] \subseteq \min_{\sqsubseteq^-_\alpha}[\neg\beta \wedge \neg\alpha]$. Since $-$ satisfies $\mathcal{OP}_\alpha[\neg\alpha]$ we further have $\min_{\sqsubseteq^-_\alpha}[\neg\beta \wedge \neg\alpha] = \min_{\sqsubseteq}[\neg\beta \wedge \neg\alpha]$. Therefore $\min_{\sqsubseteq}(\Omega) \cup \min_{\sqsubseteq}[\neg\alpha] \cup \min_{\sqsubseteq^-_\alpha}[\neg\beta] = \min_{\sqsubseteq}(\Omega) \cup \min_{\sqsubseteq}[\neg\alpha] \cup \min_{\sqsubseteq}[\neg\alpha \wedge \neg\beta]$, i.e., $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta}$.

(Right to Left). Assume that $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta}$ for every β such that $\vdash \alpha \to \beta$. Let ω, ω' be two worlds in $[\neg\alpha]$. Assume further that $\omega \sqsubseteq \omega'$. Two cases arise here: (1) Suppose $\omega \in \min_{\sqsubseteq}(\Omega)$. Then $\omega \in \min_{\sqsubseteq^-_\alpha}(\Omega)$, i.e., $\omega \sqsubseteq^-_\alpha \omega'$. (2) On the other hand, suppose $\omega \notin \min_{\sqsubseteq}(\Omega)$. Let β be a sentence such that $[\neg\beta] = \{\omega, \omega'\}$. Since $[\neg\beta] \subseteq [\neg\alpha]$ we have $\vdash \alpha \to \beta$. Therefore $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta}$ which, via Eq. G2, gives $\min_{\sqsubseteq}(\Omega) \cup \min_{\sqsubseteq}[\neg\alpha] \cup \min_{\sqsubseteq^-_\alpha}[\neg\beta] = \min_{\sqsubseteq}(\Omega) \cup \min_{\sqsubseteq}[\neg\alpha] \cup \min_{\sqsubseteq}[\neg\alpha \wedge \neg\beta]$. We have $\omega \in \min_{\sqsubseteq}[\neg\alpha \wedge \neg\beta]$ but $\omega \notin \min_{\sqsubseteq}(\Omega)$. Now, if $\omega \in \min_{\sqsubseteq}[\neg\alpha]$ then from Eq. G2 we have $\omega \in \min_{\sqsubseteq^-_\alpha}(\Omega)$, i.e., $\omega \sqsubseteq^-_\alpha \omega'$. On the other hand, if $\omega \notin \min_{\sqsubseteq}[\neg\alpha]$, then $\omega \in \min_{\sqsubseteq^-_\alpha}[\neg\beta]$, i.e., $\omega \sqsubseteq^-_\alpha \omega'$. Similarly assuming $\omega \sqsubset \omega'$ yields $\omega \sqsubset^-_\alpha \omega'$. Hence $-$ satisfies $\mathcal{OP}_\alpha[\neg\alpha]$. □

Theorem 1 *Every contraction function $-$ satisfying Eq. G2, $\mathcal{OP}_\alpha[\alpha]$ and $\mathcal{OP}_\alpha[\neg\alpha]$ satisfies the principled factored insertion(PFI).*

Proof Let $-$ be a contraction function that satisfies Eq. G2, $\mathcal{OP}_\alpha[\alpha]$ and $\mathcal{OP}_\alpha[\neg\alpha]$. Let $\mathcal{K}$ be the belief set obtained from the belief state $\sqsubseteq$ by Eq. G1. We need to show that for any arbitrary beliefs α and β in $\mathcal{K}$ such that $\beta \in \mathcal{K}^-_\alpha$, $-$ satisfies PFI.

Suppose that $\alpha \vee \beta \in (\mathcal{K}^-_\alpha)^-_\beta$. Equation G2 gives that $\min_{\sqsubseteq^-_\alpha}(\Omega) = \min_{\sqsubseteq}(\Omega) \cup \min_{\sqsubseteq}[\neg\alpha]$. Now applying Eq. G2 again (for contraction by β) we get $\min_{(\sqsubseteq^-_\alpha)^-_\beta}(\Omega) = \min_{\sqsubseteq^-_\alpha}(\Omega) \cup \min_{\sqsubseteq^-_\alpha}[\neg\beta]$. From $\alpha \vee \beta \in (\mathcal{K}^-_\alpha)^-_\beta$ we get that $\min_{(\sqsubseteq^-_\alpha)^-_\beta}(\Omega) \subseteq [\alpha \vee \beta]$, i.e., $\min_{\sqsubseteq^-_\alpha}[\neg\beta] \subseteq [\alpha]$ whereby $\min_{\sqsubseteq^-_\alpha}[\neg\beta] = \min_{\sqsubseteq^-_\alpha}[\neg\beta \wedge \alpha]$. Since $-$ satisfies $\mathcal{OP}_\alpha[\alpha]$ we have $\min_{\sqsubseteq^-_\alpha}[\neg\beta] = \min_{\sqsubseteq^-_\alpha}[\neg\beta \wedge \alpha] = \min_{\sqsubseteq}[\neg\beta \wedge \alpha]$. Therefore Eq. G2 gives $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\to\beta}$ as desired.

Similarly, if $\alpha \to \beta \in (\mathcal{K}^-_\alpha)^-_\beta$ then we get $\min_{(\sqsubseteq^-_\alpha)^-_\beta}(\Omega) \subseteq [\alpha \to \beta]$, i.e., $\min_{\sqsubseteq^-_\alpha}[\neg\beta] \subseteq [\neg\alpha]$ whereby $\min_{\sqsubseteq^-_\alpha}[\neg\beta] = \min_{\sqsubseteq^-_\alpha}[\neg\beta \wedge \neg\alpha]$. Now since $-$ satisfies $\mathcal{OP}_\alpha[\neg\alpha]$ we have $\min_{\sqsubseteq^-_\alpha}[\neg\beta \wedge \neg\alpha] = \min_{\sqsubseteq}[\neg\beta \wedge \neg\alpha]$ from which we can deduce $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta}$.

Similarly when neither $\alpha \vee \beta$ nor $\alpha \to \beta$ belong to $(\mathcal{K}^-_\alpha)^-_\beta$ we have $\min_{\sqsubseteq^-_\alpha}[\neg\beta] = \min_{\sqsubseteq^-_\alpha}[\neg\beta \wedge \neg\alpha] \cup \min_{\sqsubseteq^-_\alpha}[\neg\beta \wedge \alpha]$. Since $-$ satisfies both $\mathcal{OP}_\alpha[\alpha]$

and $\mathcal{OP}_\alpha[\neg\alpha]$, we have $\min_{\sqsubseteq^-_\alpha}[\neg\beta] = \min_\sqsubseteq[\neg\beta \wedge \neg\alpha] \cup \min_\sqsubseteq[\neg\beta \wedge \alpha]$. Hence we get, $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\to\beta} \cap \mathcal{K}^-_{\alpha\to\beta}$. Thus $-$ satisfies PFI. □

Lemma 3 *An AGM-rational contraction function $-$ satisfies Eq. 5 iff for every sentence α, both: (a) $-$ satisfies $\mathcal{OP}_\alpha[\neg\alpha]$, and (b) $\omega \sqsubset^-_\alpha \omega'$ for every $\omega, \omega' \in \Omega$ such that $\omega \in [\neg\alpha]$ and $\omega' \in [\alpha] \backslash min_\sqsubseteq[\alpha]$.*

Proof Let $-$ be a contraction function which satisfies Eq. G2. Let $\sqsubseteq$ denote a consistent belief state and $\mathcal{K}$ the corresponding belief set.

(Left to Right). Assume that $-$ satisfies Eq. 5. We need to show that when contracting by an arbitrary sentence α, $-$ changes the belief state $\sqsubseteq$ to $\sqsubseteq^-_\alpha$ such that $-$ satisfies $\mathcal{OP}_\alpha[\neg\alpha]$ and for all $\omega, \omega' \in \Omega$ such that $\omega \in [\neg\alpha]$ and $\omega' \in [\alpha] \backslash \min_\sqsubseteq[\alpha]$, we have $\omega \sqsubset^-_\alpha \omega'$.

Note that if $\vdash \alpha$, the set of worlds $[\neg\alpha]$ is empty and then $-$ trivially satisfies the requirements (a) and (b). We consider the non-trivial case of $\nvdash \alpha$ in detail.

PART (a): First we show that $-$ satisfies $\mathcal{OP}_\alpha[\neg\alpha]$. Consider $\omega, \omega' \in [\neg\alpha]$. Suppose that $\omega \sqsubseteq \omega'$. Since the language $\mathcal{L}$ is finitely generated, there exists a sentence $\beta' \in \mathcal{L}$ such that $[\neg\beta'] = \{\omega, \omega'\}$. Therefore we have $\vdash \alpha \to \beta'$ and $\nvdash \alpha \vee \beta'$. Suppose $\omega \in \min_\sqsubseteq(\Omega)$ or $\omega \in \min_\sqsubseteq[\neg\alpha]$ then from Eq. G2 we have $\omega \in \min_{\sqsubseteq^-_\alpha}(\Omega)$ and hence $\omega \sqsubseteq^-_\alpha \omega'$. Alternatively, suppose that $\omega \notin \min_{\sqsubseteq^-_\alpha}(\Omega)$. We know that $\omega \in \min_\sqsubseteq[\neg\beta' \wedge \neg\alpha]$. Since $-$ satisfies Eq. 5, $(\mathcal{K}^-_\alpha)^-_{\beta'} = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta'}$. Hence $\min_{(\sqsubseteq^-_\alpha)^-_{\beta'}}(\Omega) = \min_\sqsubseteq(\Omega) \cup \min_\sqsubseteq[\neg\alpha] \cup \min_\sqsubseteq[\neg\beta' \wedge \neg\alpha]$. Therefore from Eq. G2, $\omega \in \min_{\sqsubseteq^-_\alpha}[\neg\beta']$. This gives $\omega \sqsubseteq^-_\alpha \omega'$. We can similarly show that if $\omega \sqsubset \omega'$ then $\omega \sqsubset^-_\alpha \omega'$. Hence $-$ satisfies $\mathcal{OP}_\alpha[\neg\alpha]$.

PART (b): Now we show that for all $\omega, \omega' \in \Omega$ such that $\omega \in [\neg\alpha]$ and $\omega' \in [\alpha] \backslash \min_\sqsubseteq[\alpha]$, we have $\omega \sqsubset^-_\alpha \omega'$. Consider $\omega \in [\neg\alpha]$ and $\omega' \in [\alpha] \backslash \min_\sqsubseteq[\alpha]$. As the language $\mathcal{L}$ is finitely generated, there exists a sentence $\beta' \in \mathcal{L}$ such that ω and ω' are the only models of its negation. Two possibilities arise here:

Case (1) $\omega \in \min_\sqsubseteq[\neg\alpha]$. Then we will have $\omega \in \min_{\sqsubseteq^-_\alpha}(\Omega)$ from which we conclude that $\omega \sqsubset^-_\alpha \omega'$.

Case (2) $\omega \notin \min_\sqsubseteq[\neg\alpha]$. It is clear that β' is such that $\nvdash \alpha \vee \beta'$. Since $-$ satisfies Eq. 5, $(\mathcal{K}^-_\alpha)^-_{\beta'} = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta'}$. Therefore with the aid of Eq. G2 we get $\min_{(\sqsubseteq^-_\alpha)^-_{\beta'}}(\Omega) = \min_\sqsubseteq(\Omega) \cup \min_\sqsubseteq[\neg\alpha] \cup \min_\sqsubseteq[\neg\beta' \wedge \neg\alpha]$. We know that since $[\neg\beta' \wedge \neg\alpha]$ is a singleton set, $\omega \in \min_\sqsubseteq[\neg\beta' \wedge \neg\alpha]$. But we have assumed that $\omega \notin \min_\sqsubseteq(\Omega)$ and $\omega \notin \min_\sqsubseteq[\neg\alpha]$. This shows that $\omega \in \min_{\sqsubseteq^-_\alpha}[\neg\beta']$. Also since $\omega' \notin \min_\sqsubseteq[\neg\beta' \wedge \neg\alpha]$ and $\omega' \notin \min_{\sqsubseteq^-_\alpha}(\Omega)$, we have $\omega' \notin \min_{\sqsubseteq^-_\alpha}[\neg\beta']$. This gives $\omega \sqsubset^-_\alpha \omega'$, as desired.

(Right to Left). Let $-$ be a contraction function that satisfies Eq. G2. For every belief α, assume that $-$ satisfies $\mathcal{OP}_\alpha[\neg\alpha]$ and $\omega \sqsubset^-_\alpha \omega'$ for every $\omega \in [\neg\alpha]$ and $\omega' \in [\alpha] \backslash \min_\sqsubseteq[\alpha]$, . We need to show that $-$ satisfies Eq. 5 that is when α is a belief of the agent and β is an arbitrary belief such that $\nvdash \alpha \vee \beta$ and $\beta \in \mathcal{K}^-_\alpha$ then $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta}$. In fact we will prove a stronger result by showing that $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta}$ even when $\beta \notin \mathcal{K}^-_\alpha$.

Case 1: Consider $\beta \notin \mathcal{K}_\alpha^-$. Then we have $\min_{\sqsubseteq_\alpha^-}[\neg\beta] \subseteq \min_{\sqsubseteq_\alpha^-}(\Omega)$. Therefore from Eq. G2 we get $\min_{(\sqsubseteq_\alpha^-)_\beta^-}(\Omega) = \min_{\sqsubseteq_\alpha^-}(\Omega)$. Since we have $\beta \in \mathcal{K}$ we can say that $\min_\sqsubseteq[\neg\alpha] = \min_\sqsubseteq[\neg\alpha] \cup \min_\sqsubseteq[\neg\alpha \wedge \neg\beta]$. Therefore $(\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\vee\beta}^-$.

Case 2: Consider $\beta \in \mathcal{K}_\alpha^-$. This gives $\min_{\sqsubseteq_\alpha^-}(\Omega) \subseteq [\beta]$. Note that $\alpha \in K$, and hence $\min_\sqsubseteq[\alpha] = \min_\sqsubseteq(\Omega)$. Therefore there exists no model in $\min_\sqsubseteq[\alpha]$ or in $\min_\sqsubseteq[\neg\alpha]$ which is also a model of $\neg\beta$. Since $\nvdash \alpha \vee \beta$ there is a world $\omega \in [\neg\alpha]$ which models $\neg\beta$. Therefore $\omega \sqsubset_\alpha^- \omega'$ for every ω' which is a model of $\alpha \wedge \neg\beta$. This gives $\min_{\sqsubseteq_\alpha^-}[\neg\beta] \subseteq [\neg\beta \wedge \neg\alpha]$. By our hypothesis the contraction function $-$ satisfies $\mathcal{OP}_\alpha[\neg\alpha]$. Therefore $\min_{\sqsubseteq_\alpha^-}[\neg\beta] = \min_{\sqsubseteq_\alpha^-}[\neg\beta \wedge \neg\alpha] = \min_\sqsubseteq[\neg\beta \wedge \neg\alpha]$. Hence we have $\min_{(\sqsubseteq_\alpha^-)_\beta^-}(\Omega) = \min_\sqsubseteq(\Omega) \cup \min_\sqsubseteq[\neg\alpha] \cup \min_\sqsubseteq[\neg\alpha \wedge \neg\beta]$ which gives $(\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\vee\beta}^-$. Therefore $-$ satisfies Eq. 5. □

Lemma 4 *For any arbitrary beliefs α and β such that $\beta \in \mathcal{K}_\alpha^-$, an AGM-rational contraction function $-$ satisfies Eqs. 1, 3 and 6 if and only if $\forall\omega, \omega' \notin (min_\sqsubseteq(\Omega) \cup min_\sqsubseteq[\neg\alpha])$, $\omega \sqsubseteq_\alpha^- \omega'$ iff $\omega \sqsubseteq \omega'$.*

Proof Let $-$ be a contraction function that satisfies Eq. G2. Let $\mathcal{K}$ be the belief set associated with a consistent belief state $\sqsubseteq$. Let $\alpha,\ \beta$ be two arbitrary beliefs such that $\beta \in \mathcal{K}_\alpha^-$. We need to show that $-$ satisfies Eqs. 1, 3 and 6 if and only if $\forall\ \omega,\ \omega' \in \Omega$ such that $\omega, \omega' \notin (\min_\sqsubseteq(\Omega) \cup \min_\sqsubseteq[\neg\alpha])$ we have $\omega \sqsubseteq_\alpha^- \omega'$ iff $\omega \sqsubseteq \omega'$.

(Left to Right). Let $-$ satisfy Eqs. 1, 3 and 6. Let $\omega, \omega' \in \Omega$ be such that $\omega, \omega' \notin \min_\sqsubseteq(\Omega)$ and $\omega, \omega' \notin \min_\sqsubseteq[\neg\alpha]$. Since $\mathcal{L}$ is finitely generated, there is a sentence $\beta' \in \mathcal{L}$ such that ω and ω' are the only models of its negation. By our assumption of ω, ω' we get that $\beta' \in \mathcal{K}_\alpha^-$.

Case 1: Suppose $\omega \models \alpha$ and $\omega' \models \alpha$. Then $\vdash \alpha \vee \beta'$. Therefore $\alpha \vee \beta' \in \mathcal{K}_{\beta'}^-$ and hence from Eq. 1 we get $(\mathcal{K}_\alpha^-)_{\beta'}^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\to\beta'}^-$. From Eq. G2, we can write this as $\min_{\sqsubseteq_\alpha^-}(\Omega) \cup \min_{\sqsubseteq_\alpha^-}[\neg\beta'] = \min_\sqsubseteq(\Omega) \cup \min_\sqsubseteq[\neg\alpha] \cup \min_\sqsubseteq[\neg\beta' \wedge \alpha]$. However, since $\beta' \in \mathcal{K}_\alpha^-$ and $\vdash \alpha \vee \beta'$ we have $\min_{\sqsubseteq_\alpha^-}[\neg\beta'] = \min_\sqsubseteq[\neg\beta' \wedge \alpha]$. Therefore $\omega \sqsubseteq_\alpha^- \omega'$ iff $\omega \sqsubseteq \omega'$.

Case 2: Suppose $\omega \models \neg\alpha$ and $\omega' \models \neg\alpha$. Along the same lines as presented above, since $-$ satisfies Eq. 3 we have $\omega \sqsubseteq_\alpha^- \omega'$ iff $\omega \sqsubseteq \omega'$.

Case 3: Suppose $\omega \models \alpha$, $\omega' \models \neg\alpha$ and $\omega \sqsubset \omega'$. Therefore $\omega \in \min_\sqsubseteq[\neg\beta']$. From Eq. G2 $\min_{\sqsubseteq_{\beta'}^-}(\Omega) = \min_\sqsubseteq(\Omega) \cup \min_\sqsubseteq[\neg\beta']$. Since we know that α is a belief in $\mathcal{K}$, we have that $\min_{\sqsubseteq_{\beta'}^-}(\Omega) \subseteq [\alpha]$. From this we can derive that $\alpha \vee \beta' \in \mathcal{K}_{\beta'}^-$. The contraction function $-$ satisfies Eq. 1. Therefore $(\mathcal{K}_\alpha^-)_{\beta'}^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\to\beta'}^-$. From Eq. G2 we can write this as $\min_{\sqsubseteq_\alpha^-}(\Omega) \cup \min_{\sqsubseteq_\alpha^-}[\neg\beta'] = \min_\sqsubseteq(\Omega) \cup \min_\sqsubseteq[\neg\alpha] \cup \min_\sqsubseteq[\neg\beta' \wedge \alpha]$. Since $\beta' \in \mathcal{K}_\alpha^-$, we get $\min_{\sqsubseteq_\alpha^-}[\neg\beta'] = \min_\sqsubseteq[\neg\beta' \wedge \alpha]$. This gives that $\omega \sqsubset_\alpha^- \omega'$.

Case 4: Suppose $\omega \models \alpha, \omega' \models \neg\alpha$ and $\omega' \sqsubset \omega$. Along the same lines as in case 3 we can show that, since $-$ satisfies Eq. 3 we have $\omega' \sqsubset_\alpha^- \omega$.

Case 5: Suppose $\omega \models \alpha$ and $\omega' \models \neg\alpha$. Also let $\omega \approx \omega'$. This case too follows in similar lines as in case 3 and 4. In this case neither $\alpha \vee \beta'$ nor $\alpha \to \beta'$ belong to $\mathcal{K}_{\beta'}^-$. Since $-$ satisfies Eq. 6 it follows that $\omega \approx_\alpha^- \omega'$.

(Right to Left). Let $-$ be a contraction function that satisfies Eq. G2. Given a belief state $\sqsubseteq$, upon contraction by an arbitrary belief α, let $-$ change the belief state to $\sqsubseteq_\alpha^-$ where for every $\omega, \omega' \in \Omega$ such that $\omega, \omega' \notin (\min_\sqsubseteq(\Omega) \cup \min_\sqsubseteq[\neg\alpha])$ we have $\omega \sqsubseteq_\alpha^- \omega'$ iff $\omega \sqsubseteq \omega'$. Let β be an arbitrary belief such that $\beta \in \mathcal{K}_\alpha^-$.

Case 1: $\vdash \beta$. Hence both $\alpha \vee \beta$ and $\alpha \to \beta$, being theorems, belong to $\mathcal{K}_\beta^-$. Therefore this case satisfies the pre-conditions of both Eqs. 1 and 3, but not of Eq. 6. Thus Eq. 6 is trivially satisfied. As to Eqs. 1 and 3, note that $[\beta] = [\alpha \vee \beta] = [\alpha \to \beta] = \Omega$, and hence, $\min_\sqsubseteq[\neg\beta] = \min_\sqsubseteq[\neg\alpha \wedge \neg\beta] = \min_\sqsubseteq[\alpha \wedge \neg\beta] = \emptyset$. The desired result easily follows from it.

Case 2: $\vdash \alpha$ and $\nvdash \beta$. It follows that $\alpha \vee \beta$ is a theorem, and hence belongs to $\mathcal{K}_\beta^-$. Now $\alpha \to \beta$ is logically equivalent to β; hence $\mathcal{K}_\beta^- = \mathcal{K}_{\alpha\to\beta}^-$. Furthermore, since $\vdash \alpha$, we get $\sqsubseteq_\alpha^- = \sqsubseteq$, whereby, $\mathcal{K}_\alpha^- = \mathcal{K}$. Hence $[(\mathcal{K}_\alpha^-)_\beta^-] = \min_\sqsubseteq[\neg\beta] = \min_\sqsubseteq[\alpha \wedge \neg\beta]$ wherefrom the desired result $(\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_{\alpha\to\beta}^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\to\beta}^-$ follows.

Case 3: Let $\nvdash \alpha$ and $\nvdash \beta$. Three subcases arise here:

Case (3a): Suppose α and β are such that $\alpha \vee \beta \in \mathcal{K}_\beta^-$. From Eq. G2 we can write $\min_{\sqsubseteq_\beta^-}(\Omega) \subseteq [\alpha \vee \beta]$. Therefore $\min_\sqsubseteq[\neg\beta] \subseteq [\alpha]$, i.e. $\min_\sqsubseteq[\neg\beta] = \min_\sqsubseteq[\neg\beta \wedge \alpha]$. Since $\beta \in \mathcal{K}_\alpha^-$, if ω is a model of $\neg\beta$ then $\omega \notin (\min_\sqsubseteq(\Omega) \cup \min_\sqsubseteq[\neg\alpha])$. By our hypothesis $-$ preserves ordering within $[\neg\beta]$, i.e., $-$ satisfies $\mathcal{OP}_\alpha[\neg\beta]$. Hence $\min_{\sqsubseteq_\alpha^-}[\neg\beta] = \min_\sqsubseteq[\neg\beta]$. This gives $\min_{\sqsubseteq_\alpha^-}[\neg\beta] = \min_\sqsubseteq[\neg\beta \wedge \alpha]$. Therefore when $\alpha \vee \beta \in \mathcal{K}_\beta^-$ we have $(\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\to\beta}^-$, i.e., $-$ satisfies Eq. 1.

Case (3b): Suppose α and β are such that $\alpha \to \beta \in \mathcal{K}_\beta^-$. Following along the same lines as presented for Case (3a) we can show that $-$ satisfies Eq. 3.

Case (3c): Suppose α and β are such that $\alpha \vee \beta \notin \mathcal{K}_\beta^-$ and $\alpha \to \beta \notin \mathcal{K}_\beta^-$. With similar arguments as in Case (3a) we can show that $-$ satisfies Eq. 6. □

OBSERVATION 1 Let $-$ be an AGM-rational contraction function, and the degree of belief function d is appropriately related with the presumed belief state $\sqsubseteq$. Then:

(a) Equation 1 is satisfied iff Eq. 7 is,
(b) Equation 3 is satisfied iff Eq. 8 is, and
(c) Equation 6 is satisfied iff Eq. 9 is.

Proof Let us make the general assumption that $-$ is an AGM-rational contraction function and $\sqsubseteq$, $\mathcal{K}$ denote a consistent belief state and the corresponding belief set. Also assume that d is appropriately related to $\sqsubseteq$. We provide the proofs of the three different parts of this observation separately.

Part (a)

Consider two arbitrary beliefs α and β. It will be sufficient to show that $\alpha \vee \beta \in \mathcal{K}_\beta^-$ is equivalent to $d(\alpha \vee \beta) > d(\alpha \to \beta)$.

(Left to Right). Let $\alpha \vee \beta \in \mathcal{K}^-_\beta$. This implies that $\min_{\sqsubseteq^-_\beta}(\Omega) \subseteq [\alpha \vee \beta]$, whereby, $\min_{\sqsubseteq}[\neg\beta] \subseteq [\alpha]$. Therefore for any world ω' in $[\neg\beta \wedge \neg\alpha]$ and $\omega \in \min_{\sqsubseteq}[\neg\beta \wedge \alpha]$ we have $\omega \sqsubset \omega'$. In terms of degrees of belief we have $d(\alpha \vee \beta) > d(\alpha \rightarrow \beta)$.

(Right to Left). For the reverse, we just need to trace our step backwards in the proof for the ($\Rightarrow$) part.

Part (b)

For the proof for the ($\Rightarrow$) part, we begin by assuming that $\alpha \rightarrow \beta \in \mathcal{K}^-_\beta$. This implies $\min_{\sqsubseteq}[\neg\beta] \subseteq [\neg\alpha]$ which leads us to conclude $d(\alpha \rightarrow \beta) > d(\alpha \vee \beta)$. The proof for the ($\Leftarrow$) part can be obtained by simply tracing our steps in the ($\Rightarrow$) part backwards.

Part (c)

For the proof for the ($\Rightarrow$) part, we begin by assuming that neither $\alpha \rightarrow \beta$ nor $\alpha \vee \beta$ are retained in $\mathcal{K}^-_\beta$. This implies that $\min_{\sqsubseteq}[\neg\beta] = \min_{\sqsubseteq}[\neg\beta \wedge \alpha] \cup \min_{\sqsubseteq}[\neg\beta \wedge \neg\alpha]$ and from the definition of degrees of belief we deduce that $d(\alpha \rightarrow \beta) = d(\alpha \vee \beta)$. Again the proof for the ($\Leftarrow$) part can be obtained by simply tracing the above steps backwards. □

Lemma 5 *Any AGM-rational contraction function $-$ that satisfies Eq. 13 also satisfies Eq. 2.*

Proof Let $-$ be an AGM-rational contraction function that satisfies Eq. 13. Let $\sqsubseteq$ be the presumed belief state and $\mathcal{K}$ be its associated belief set that is consistent. We need to show that for any two beliefs α and β such that $\vdash \alpha \vee \beta$ we have $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\rightarrow\beta}$.

Case 1: Suppose α is a sentence such that $\vdash \alpha$. Since $-$ is an AGM-rational contraction function we have $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\rightarrow\beta}$. Hence $-$ satisfies Eq. 2 trivially.

Case 2: Suppose $\vdash \beta$. Based on similar arguments as presented in case 1, we can say that $-$ trivially satisfies Eq. 2.

Case 3: Suppose $\nvdash \alpha, \nvdash \beta$ but $\vdash \alpha \vee \beta$. By Property 1 of conditional degrees of belief we have $d(\alpha \vee \beta|\neg\alpha) = \infty$. However $d(\alpha \rightarrow \beta|\alpha) < \infty$ since, the alternative, $d(\alpha \rightarrow \beta|\alpha) = \infty$ would yield $\vdash \alpha \rightarrow \beta$ which is not possible given our assumptions $\vdash \alpha \vee \beta$ and $\nvdash \beta$. Therefore we have $d(\alpha \vee \beta|\neg\alpha) > d(\alpha \rightarrow \beta|\alpha)$. From Eq. 13 we get $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\rightarrow\beta}$. Hence $-$ satisfies Eq. 2. □

Lemma 6 *Any AGM-rational contraction function $-$ that satisfies Eq. 12 also satisfies Eq. 4.*

Proof Let $-$ be an AGM-rational contraction function that satisfies Eq. 12. Let $\sqsubseteq$ be the underlying belief state; and $\mathcal{K}$ its associated, consistent belief set, and d the relevant degree of belief function. Assume two beliefs α and β such that $\vdash \alpha \rightarrow \beta$. We need to show that $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta}$.

Case 1: Assume $\vdash \beta$. With arguments similar to those in case 1, we can say that $-$ trivially satisfies Eq. 4. Since $-$ is AGM-rational, we have $(\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^-$. Furthermore, since $\vdash \alpha \vee \beta$, we have $\mathcal{K}_{\alpha\vee\beta}^- = \mathcal{K}$. Hence we get the desired result, $(\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\vee\beta}^-$.

Case 2: Assume $\vdash \alpha$. Since by assumption $\vdash \alpha \rightarrow \beta$, it follows that$\vdash \beta$. Thus it reduces to Case 1.

Case 3: Suppose $\nvdash \alpha, \nvdash \beta$ but $\vdash \alpha \rightarrow \beta$. Now, by Property 1 of conditional degrees of belief, we have $d(\alpha \rightarrow \beta|\alpha) = \infty$. However, $d(\alpha \vee \beta|\neg\alpha) < \infty$ because $d(\alpha \vee \beta|\neg\alpha) = \infty$ would give $d(\alpha \vee \beta) = \infty$. That would lead to $\vdash \beta$. Therefore we have $d(\alpha \rightarrow \beta|\alpha) > d(\alpha \vee \beta|\neg\alpha)$. From 12 we get $(\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\vee\beta}^-$, as desired. □

Theorem 2 *An AGM-rational contraction function is a moderate contraction function iff it satisfies properties 2 and 5.*

Proof Let $-$ be an AGM-rational contraction function. Consider the belief set $\mathcal{K}$ which is obtained from a given consistent belief state $\sqsubseteq$ by Eq. G1. We need to show that the contraction function $-$ is a moderate contraction function iff it satisfies Eqs. 2 and 5 for any two arbitrary beliefs α, β in $\mathcal{K}$.

(Left to Right). Let us assume that $-$ is a moderate contraction function. By definition of a moderate contraction function we know that $-$ satisfies $\mathcal{OP}_\alpha[\alpha]$ for any arbitrary belief α. Hence from Lemma 1 it is clear that $-$ satisfies Eq. 2.

Furthermore, consider any two ω and ω' such that $\omega \in [\neg\alpha]$ and $\omega' \in [\alpha]\setminus \min_{\sqsubseteq}(\Omega)$. Since α is a belief in $\mathcal{K}$, we have $\min_{\sqsubseteq}(\Omega) = \min_{\sqsubseteq}[\alpha]$. Furthermore, since $\omega \in [\neg\alpha]$ and $\omega' \in [\alpha]\setminus \min_{\sqsubseteq}[\alpha]$, we get $\omega \sqsubset_\alpha^- \omega'$. Since we also have that $-$ satisfies $\mathcal{OP}_\alpha[\neg\alpha]$, Lemma 3 leads us to conclude that $-$ satisfies Eq. 5.

(Right to Left). Let $-$ be an AGM-rational contraction function that satisfies Eqs. 2 and 5 when successively contracting two arbitrary beliefs α and β. We need to show that $-$ is a moderate contraction function.

Since $-$ satisfies Eq. 2, we can conclude from Lemma 1 that $-$ satisfies $\mathcal{OP}_\alpha[\alpha]$. By our assumption that $-$ satisfies Eq. 5 and from Lemma 3, we also have that $-$ satisfies $\mathcal{OP}_\alpha[\neg\alpha]$. We can also conclude that for any two worlds ω and ω' such that $\omega \in [\neg\alpha]$ and $\omega' \in [\alpha]\setminus \min_{\sqsubseteq}[\alpha]$, we have $\omega \sqsubset_\alpha^- \omega'$. Since α is a belief of the agent, we also have $\min_{\sqsubseteq}[\alpha] = \min_{\sqsubseteq}(\Omega)$. Therefore we have that $\omega \sqsubset_\alpha^- \omega'$ when $\omega \in [\neg\alpha]$ and $\omega' \in [\alpha]\setminus \min_{\sqsubseteq}(\Omega)$. Hence $-$ is a moderate contraction function, satisfying conditions MC1–MC4 as presented in the Section 2.1. □

Theorem 3 *An AGM-rational contraction function is a natural contraction function iff it satisfies properties 1, 3 and 6.*

Proof The proof of this theorem follows directly from Lemma 4. □

Theorem 4 *An AGM-rational contraction function is a lexicographic contraction function iff it satisfies properties 12, 13 and 14.*

Proof Let $-$ be an AGM-rational contraction function. Let $\sqsubseteq$ represent a consistent belief state and $\mathcal{K}$ denote the corresponding belief set. Let α and β be any arbitrary beliefs. We need to show that the contraction function $-$ is a lexicographic contraction function iff it satisfies 12, 13 and 14.

(Left to Right). Let $-$ be a lexicographic contraction function. We need to show that for any arbitrary beliefs α, β, $-$ satisfies Eqs. 12, 13 and 14. We do that by analysing four possible cases.

Case 1: Let both $\vdash \alpha$ and $\vdash \beta$. Then $\mathcal{K}^-_\alpha = \mathcal{K} = (\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_{\alpha\vee\beta} = \mathcal{K}^-_{\alpha\to\beta}$. Hence the conditions 12, 13 and 14 are trivially satisfied.

Case 2: Let $\vdash \alpha$ and $\nvdash \beta$. By the properties of conditional degrees of belief, we have $d(\alpha \vee \beta|\neg\alpha) = \infty$ (since $\vdash \alpha \vee \beta$) and $d(\alpha \to \beta|\alpha) = d(\beta)$ which is less than ∞. Therefore $d(\alpha \vee \beta|\neg\alpha) > d(\alpha \to \beta|\alpha)$ which satisfies the precondition for Eq. 13. Clearly then, the preconditions of Eqs. 12 and 14 are not satisfied, whereby Eqs. 12 and 14 are trivially satisfied. As to Eq. 13, since $-$ is a lexicographic contraction function it satisfies $\mathcal{OP}_\alpha[\alpha]$ whereby $\sqsubseteq^-_\alpha = \sqsubseteq$. Therefore we have $\min_{\sqsubseteq^-_\alpha}[\neg\beta] = \min_\sqsubseteq[\neg\beta] = \min_\sqsubseteq[\neg\beta \wedge \alpha]$. Therefore from Eq. G2 we get $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\to\beta}$, as desired.

Case 3: Let $\nvdash \alpha$ and $\vdash \beta$. Since $-$ satisfies Eq. G2 we note that the second contraction in $(\mathcal{K}^-_\alpha)^-_\beta$ is vacuous. From this we can see that $-$ trivially satisfies Eqs. 12, 13 and 14.

Case 4: Let both $\nvdash \alpha, \nvdash \beta$. We discuss two subcases here.

Sub-case (4a): Suppose $\beta \notin \mathcal{K}^-_\alpha$. From Eq. G2 and since β is a belief in $\mathcal{K}$, we see that there is a model of $\neg\beta$ in $\min_\sqsubseteq[\neg\alpha]$. Any complete chain of worlds in $[\neg\alpha]$ begin with some world in $\min_\sqsubseteq[\neg\alpha]$. Since there exists a model of $\neg\beta$ in $\min_\sqsubseteq[\neg\alpha]$ we have $d(\beta|\neg\alpha) = 0$, i.e., $d(\alpha \vee \beta|\neg\alpha) = 0$. From $\beta \in \mathcal{K}$ we get that $d(\beta|\alpha) > 0$, i.e., $d(\alpha \to \beta|\alpha) > 0$. Hence $d(\alpha \to \beta|\alpha) > d(\alpha \vee \beta|\neg\alpha)$. Note hence that this satisfies the precondition of Eq. 12, and hence Eqs. 13 and 14 are trivially satisfied. We need only to show that Eq. 12 is satisfied. Now, $\min_{(\sqsubseteq^-_\alpha)^-_\beta}(\Omega) = \min_{\sqsubseteq^-_\alpha}(\Omega) \cup \min_{\sqsubseteq^-_\alpha}[\neg\beta]$. Since $\beta \notin \mathcal{K}^-_\alpha$ we have $\min_{\sqsubseteq^-_\alpha}[\neg\beta] \subseteq \min_{\sqsubseteq^-_\alpha}(\Omega)$. Therefore $\min_{(\sqsubseteq^-_\alpha)^-_\beta}(\Omega) = \min_{\sqsubseteq^-_\alpha}(\Omega)$, i.e., $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha$. Again from $\beta \in \mathcal{K}$ and $\beta \notin \mathcal{K}^-_\alpha$ we get $\min_\sqsubseteq[\neg\alpha] \cap [\neg\beta] \neq \emptyset$. We can therefore write $\min_\sqsubseteq[\neg\alpha] = \min_\sqsubseteq[\neg\alpha] \cup \min_\sqsubseteq[\neg\alpha \wedge \neg\beta]$. From Eq. G2 we get $\min_{\sqsubseteq^-_\alpha}(\Omega) = \min_\sqsubseteq(\Omega) \cup \min_\sqsubseteq[\neg\alpha] = \min_\sqsubseteq(\Omega) \cup \min_\sqsubseteq[\neg\alpha] \cup \min_\sqsubseteq[\neg\alpha \wedge \neg\beta]$. Therefore $(\mathcal{K}^-_\alpha)^-_\beta = \mathcal{K}^-_\alpha \cap \mathcal{K}^-_{\alpha\vee\beta}$, as desired.

Sub-case (4b): Suppose $\beta \in \mathcal{K}^-_\alpha$. We will show that Eq. 13 is satisfied, and leave out Eqs. 12 and 14 since the proofs are analogous. Let us assume that $d(\alpha \vee \beta|\neg\alpha) > d(\alpha \to \beta|\alpha)$. We wish to first show that $\min_{\sqsubseteq^-_\alpha}[\neg\beta] = \min_\sqsubseteq[\neg\beta \wedge \alpha]$ from which the desired result will easily follow. First the trivial case: if $\vdash \alpha \vee \beta$, then clearly $[\neg\beta] \subseteq [\alpha]$ whereby $\min_{\sqsubseteq^-_\alpha}[\neg\beta] = \min_\sqsubseteq[\neg\beta \wedge \alpha]$. Now the non-trivial case: assume that $\nvdash \alpha \vee \beta$, i.e., $[\neg\alpha \wedge \neg\beta] \neq \emptyset$. Let $\omega' \in \min_\sqsubseteq[\neg\beta \wedge \neg\alpha]$. Now, it is easily noted that $\nvdash \alpha \to \beta$, i.e., $[\alpha \wedge \neg\beta] \neq \emptyset$. Let $\omega \in \min_\sqsubseteq[\neg\beta \wedge \alpha]$. Hence, by the properties of conditional degree of beliefs, there is a chain of worlds in $[\neg\alpha]$ ending in ω' whose length is greater than any chain of worlds in $[\alpha]$ ending in ω. Hence we have $\omega \sqsubset^-_\alpha \omega'$. Therefore, from $\mathcal{OP}_\alpha[\alpha]$ we get $\min_{\sqsubseteq_\alpha^-}[\neg\beta] = \min_{\sqsubseteq}[\neg\beta \wedge \alpha]$. From Eq. G2 we get $(\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\to\beta}^-$. Hence we see that $-$ satisfies Eq. 13. Similarly we can show that $-$ satisfies Eqs. 12 and 14.

(Right to Left). Let $-$ be an AGM-rational contraction function which satisfies Eqs. 12, 13 and 14 when contracting two arbitrary beliefs α and β. We need to show that $-$ is a lexicographic contraction function, i.e., it satisfies $\mathcal{OP}_\alpha[\alpha]$, $\mathcal{OP}_\alpha[\neg\alpha]$ and **LC3**.

Lemma 5 states that any AGM contraction function that satisfies Eq. 13 also satisfies Eq. 2. Now from Lemma 1 we know that any AGM contraction function that satisfies Eq. 2 also satisfies $\mathcal{OP}_\alpha[\alpha]$. Hence we see that $-$ satisfies $\mathcal{OP}_\alpha[\alpha]$. Similarly from Lemmas 6 and 2 we can conclude that $-$ satisfies $\mathcal{OP}_\alpha[\neg\alpha]$. Hence it will be sufficient to show that $-$ satisfies **LC3**.

Case 1: $\vdash \alpha$. We have shown that $-$ satisfies $\mathcal{OP}_\alpha[\alpha]$. Hence, we have $\sqsubseteq_\alpha^- = \sqsubseteq$.

Case 2: $\nvdash \alpha$. Consider two worlds ω and ω' such that ω is a model of α and ω' is a model of $\neg\alpha$. Let n denote the minimum length of a complete chain of worlds in $[\alpha]$ that end in ω and m denote the minimum length of a complete chain of worlds in $[\neg\alpha]$ that end in ω'. Since $\mathcal{L}$ is finitely generated there exists a sentence β in the language such that the only models of $\neg\beta$ are ω and ω'. There are four possible subcases that arise.

Subcase (2a): Suppose $n = 0$ whereby, $n \leq m$. Then we have $\omega \in \min_{\sqsubseteq}[\alpha]$, i.e., $\omega \in \min_{\sqsubseteq}(\Omega)$. Therefore $\omega \in \min_{\sqsubseteq_\alpha^-}(\Omega)$. This gives $\omega \sqsubseteq_\alpha^- \omega'$ when $n = 0$. However suppose $m = 0$, i.e., $m \leq n$. Then $\omega' \in \min_{\sqsubseteq}[\neg\alpha]$. Since $-$ is an AGM contraction function it satisfies Eq. G2 and hence $\omega' \in \min_{\sqsubseteq_\alpha^-}(\Omega)$. Therefore $\omega' \sqsubseteq_\alpha^- \omega$.

Subcase (2b): Suppose $n, m \neq 0$ and $n < m$. Since $n, m \geq 0$, and $[\neg\beta] = \{\omega, \omega'\}$, we have $\beta \in \mathcal{K}_\alpha^-$. By definition of conditional degree of belief, we have that $d(\beta|\alpha) = n$ and $d(\beta|\neg\alpha) = m$. Therefore $d(\alpha \vee \beta|\neg\alpha) > d(\alpha \to \beta|\alpha)$. Since the contraction function satisfies Eq. 13 we have $(\mathcal{K}_\alpha^-)_\beta^- = \mathcal{K}_\alpha^- \cap \mathcal{K}_{\alpha\to\beta}^-$. From Eq. G2 and from $\beta \in \mathcal{K}_\alpha^-$ we get $\min_{\sqsubseteq_\alpha^-}[\neg\beta] = \min_{\sqsubseteq}[\neg\beta \wedge \alpha]$, whereby, $\omega \sqsubset_\alpha^- \omega'$.

Subcase (2c): Suppose $n, m \neq 0$ and $n > m$. With similar reasoning as in case b above, together with the fact that $-$ satisfies Eq. 12, we get $\omega' \sqsubset_\alpha^- \omega$.

Subcase (2d): Consider $n, m \neq 0$ and $n = m$. Since $-$ satisfies Eq. 14, following in similar lines as presented in case b, we can conclude that $\omega \approx_\alpha^- \omega'$.

Thus $-$ satisfies **LC3** and hence we see that $-$ is a lexicographic contraction function. □

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