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MODELING RELATIONS

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ABSTRACT. In the ordinary way of representing relations, the order of the relata plays a structural role, but in the states themselves such an order often does not seem to be intrinsically present. An alternative way to represent relations makes use of positions for the arguments. This is no problem for the love relation, but for relations like the adjacency relation and cyclic relations, different assignments of objects to the positions can give exactly the same states. This is a puzzling situation. The question is what is the internal structure of relations? Is the use of positions still justified, and if so, what is their ontological status? In this paper mathematical models for relations are developed that provide more insight into the structure of relations "out there" in the real world.

KEY WORDS: argument-places, mathematical models, metaphysics, relations, states of affairs, substitution

1. Introduction

When I say "Koos loves Marietje" do I signify the same state of affairs as I would if I said "Marietje is loved by Koos"? If you accept that states of affairs are "out there" in reality, you will probably say "yes". But then we have two ways to describe a single state of affairs. Which one is better? If we can't say, then we might ask if there is a way to express the state of affairs in a neutral way.

This is the problem Kit Fine takes up in his paper "Neutral Relations" [2]. Fine shows the inadequacy of what he calls the *standard view* on relations, according to which all relations hold of objects in a given order. In search of a better alternative, he first proposes a *positionalist view* on relations in which each relation comes with a set of positions. For example, for the amatory relation, the positions would be *Lover* and *Beloved*. Fine finds the positionalist view very natural and plausible, but he also regards it as problematic. One of Fine's objections is that on this view no relation can be strictly symmetric. A second alternative proposed by Fine is his *antipositionalist view*, which he claims combines the virtues of the standard and the positionalist view.

If Fine is right that the positionalist view is wrong, then this would be very disturbing, because it seems so natural and fundamental for our way of thinking. Therefore, I have three basic questions: (1) Can the positionalist view in some way be saved from the objections raised by Fine? (2) For what kind of relations are positional representations still adequate, and for what kind are they not? (3) What is the most natural way to look at relations?

In this paper I develop mathematical models for different views on relations that are not only of interest in themselves, but that also increase our insight into the adequacy of different conceptions of relations. In particular, I define *directional models* that agree with the standard view, *positional models* that agree with the positionalist view, and an elegant type of models, the *substitution models*, inspired by Fine's antipositionalist view. I prove that a natural subclass of the substitution models corresponds in a well-defined way to a subclass of the positional models. As a consequence, without any commitment to an ontology of positions, positional representations of relations are justified for a large class of relations, including all kinds of symmetrical relations.

The structure of the paper is as follows. Section 2 briefly discusses the views on relations as distinguished by Fine. Section 3 to 5 form the core of this paper. In Section 3 types of mathematical models for relations are developed, in Section 4 the relationship between the types is studied, and in Section 5 we take a closer look at the positional structure of models. In Section 6 we focus our attention on metaphysical aspects of the structure of relations. In Section 7 I finish with a recapitulation of the main argument, conclusions and suggestions for further inquiry.

A final note about the scope of this paper. The relations considered are relations "out there" in the real world. Occasionally I will use the term *"real" relations* to stress this point. If you like, you may regard the mathematical relations as a subclass of the "real" relations. However, in my arguments I will freely use what are called "relations" in set theory as tools.

2. Views on Relations

In his paper "Neutral Relations", Kit Fine presents three views on relations. He calls them the *standard view*, the *positionalist view* and the *antipositionalist view*. I will briefly describe here the views as presented by Fine, and criticize his objections against positionalism. In the description I will stay close to Fine's original formulations.

2.1. *Standard View*

According to the standard view, objects in a relation always come in a given order. For example, we may say that the relation *loves* holds of *a* and *b* (in that order) just in case *a* loves *b*. A consequence of this view is that each binary relation has a *converse* [2, p. 2]. For example, the converse of the relation *on top of* is the relation *beneath*. The consequence that each binary relation has a converse is a shortcoming of this view for the following reason.

Suppose a block *a* is on top of a another block *b*. Then we have a state of affairs *s* that may be described as the state of *a*'s being on top of *b*, but that may also be described as the state of *b*'s being beneath *a*. If *s* is a genuine relational complex, i.e. a state consisting of a relation in combination with its relata, then there must be a *single* relation that can be correctly said to figure in the complex in combination with the given relata. We have no reason to choose either *on top of* or *beneath* for this relation. Whatever this relation is, it cannot have a converse. Therefore the standard view is objectionable from a metaphysical perspective. [2, pp. 3–4]

If we consider relations as belonging to reality rather than to our representation of it, then the order of the arguments is to be attributed to our representations, not to the relation itself. It is an artifact of our language that it leads us to suppose that relations themselves must apply to arguments in a given order. [2, p. 6]

There is also a linguistic argument against the standard view. In a graphic language, the love predicate could be a heart-shaped body with a red and a black side. On the red side we write the name of the lover and on the black side the beloved one. The relation signified by the heart does not fit in with the standard view, since the sides of the heart are not ordered in a relevant sense. [2, pp. 6–7]

2.2. *Positionalist View*

The positionalist view assumes that each relation has a fixed number of positions or argument-places, which are *specific entities* that belong to the relation. For example, the love relation has the positions *Lover* and *Beloved*. [2, p. 10]

There is no intrinsic order to the positions. This makes the positionalist view a *neutral* or *unbiased* conception of relations. This does however not imply that on this view every relation is neutral in the sense that there is no meaningful notion of converse for it. For we may get biased relations like *loves* by imposing an order on positions from the "outside". [2, p. 11]

Fine has two objections to the positionalist view. The first objection is an ontological one. The positionalist is obliged to reify argument-places, and may have to include them among the "fundamental furniture of the universe". But according to Fine we are strongly inclined to think that there should be an account of relational facts without any reference to argument-places. The second objection concerns *strictly symmetric* relations, i.e. relations for which different assignments of the objects to the positions give identical states. For example, for the adjacency relation the state of a's being adjacent to b is the same as the state of b's being adjacent to a. But if a and b occupy distinct positions within a state, then switching the positions of a and b cannot yield the same state. A proposed way out to let objects in symmetric relations occupy the same position does not work; for cyclic relations the positions occupied must be distinct to distinguish certain states. Therefore, on the positionalist view no strictly symmetric relations are possible. [2, pp. 16–17]

2.3. *Antipositionalist View*

On the positionalist view, the *completion* of a relation is the state we get from assigning objects to the argument-places of the relation. In this case it is a single-valued operation. However, the antipositionalist has no argument-places to which objects can be assigned. He takes completion as a *multi-valued* operation, yielding a plurality of states for the different ways in which the relation might be completed by the objects. For example, the completion of the love relation by Don José and Carmen contains the state of Don José's loving Carmen and the state of Carmen's loving Don José. [2, p. 19]

To distinguish different states, the antipositionalist makes use of the idea that a state can be a completion of a relation *in the same manner* as another state. We say that a state s is a completion of a given relation $\Re$ by constituents $a_1, a_2, \ldots, a_m$ *in the same manner* as a state t is a completion of $\Re$ by constituents $b_1, b_2, \ldots, b_m$, if s can be obtained by simultaneously substituting $a_1, a_2, \ldots, a_m$ for $b_1, b_2, \ldots, b_m$ in t (and vice versa). We assume that if the a_i's are the same, then the corresponding b_i's are also the same (and vice versa). [2, p. 20]

So, for example, the state of Anthony's loving Cleopatra is a completion of the love relation by Anthony and Cleopatra in the same manner as the state of Abelard's loving Eloise is a completion of the love relation by Abelard and Eloise.

The antipositionalist view has certain advantages over the positionalist view: (1) It does not have the ontological problem of the positionalist view, for it has no argument-places, (2) it does not have problems with strictly symmetric relations, and (3) it can account for variably polyadic relations. [2, pp. 21–22]

The notion of *co-mannered completion*, i.e. the notion of completion *in the same manner*, should not be taken as a primitive. We defined the relation *in the same manner* in terms of substitution. Thus, we should see the notion of *co-mannered completion* as a special case of the more general notion of substitution. [2, pp. 25–28]

The antipositionalist can reconstruct the notion of position in terms of *co-positionality*. We say that *a* in *s* is *co-positional* to *b* in *t* if *s* results from *t* by a substitution in which *b* goes into *a* (and vice versa) [2, p. 29]. If the antipositionalist accepts the existence of strictly symmetric relations, he cannot satisfactorily reconstruct the positionalist's account of position, because if constituents occupy different positions, then interchanging constituents will give a different state [2, p. 32].

On the standard conception, a relation applies to its relata in an absolute manner; on the positionalist conception, a relation applies to its relata relative to the positions of the relation, but with an absolute notion of position; on the antipositionalist conception, we have the relative notion of co-positionality. The antipositionalist has stripped the concept of a relation to its core. [2, p. 32]

2.4. *Criticizing Fine's Objections to Positionalism*

The first objection Fine raised against the positionalist view is that a "full-blooded commitment to an ontology of positions" does not match our inclination to think that a position-free account of relations is possible. For me it is not clear whether we would *a priori* be strongly inclined to think that such an account is possible. But in any event, Fine comes with an elegant alternative account, the antipositionalist view.

According to Fine the antipositionalist can reconstruct positions, but I find his solution not very satisfactory. Fine claims [2, p. 29]: "Positions can then be taken to be the abstracts of constituents in relational complexes with respect to the relation *co-positionality*." The way I understand this is that we would get for certain states of cyclic relations just one position. Another peculiarity of Fine's reconstruction of positions is that certain relations can have more positions than the maximum number of arguments an instance can have. Consider the love relation and the state *s* of Narcissus's loving Narcissus. By definition of co-positionality, Narcissus in *s* is only co-positional to other objects in states where they love themselves. This would give us three positions for the love relation instead of two. So, positions reconstructed in this way are in general not very similar to the positions of the positionalist. I find it somewhat confusing to call the reconstructed entities *positions*. Perhaps it would be better to call them *roles*.

Fine's second objection against the positionalist view is that, for strictly symmetric relations it is contradictory to assume both (1) that distinct objects occupy different positions, and (2) that position is preserved under substitution [2, pp. 31–32]. However, in my view, this does *not* mean that positional representations are dubious. I even think that a positionalist might concede the argument of Fine, but respond that the positions of a relation only have a *mediating* role. Assigning objects to positions *yields* states, but I see no reason to assume that the objects *occupy* these positions within the states.

Finally, Fine argues that if we accept strictly symmetric relations, then the antipositionalist cannot give a satisfactory reconstruction of the positionalist's account of positions [2, p. 32, note 22]. In his argument for this claim, Fine uses the supposition that according to a positionalist objects must occupy positions within the states. But if you drop this supposition—and as I argued, we have good reasons for doing so—then a satisfactory reconstruction of "normal" positions, as seen by the positionalist, is possible for a large class of relations including all kind of symmetric relations, as I will show in Section 4.

In response to this criticism, Fine has said [private communication, October 30, 2005] that in his paper "Neutral Relations" he was, for simplicity, ignoring the fact that substitution is properly done on occurrences, as is made clear in [1]. If we use the notion of what it is for one occurrence of an individual to be co-positional with another occurrence, then we can avoid the difficulty over there being too many positions. If positions are something to be occupied, then we cannot properly distinguish different positions within a cyclic relation.

Further, Fine has granted that he has no objection to a "thin" notion of position (one which is not occupied) as such. But he does not think it is basic; exemplification or completion through thin positions must be understood in other terms. He remarked that he thinks his paper "Neutral Relations" was not clear on how both of his objections to positionalism are to positionalism as a *basic* account of what relations are.

In conclusion, I would say that these considerations give hope for a non-basic form of positionalism, but they still leave us with questions with respect to the adequacy of positional representations.

3. Framing Relations

We will define different types of *frames* to model the *logical space* of relations. The frames will all be of the form $\langle S, O, \ldots \rangle$, where S is a nonempty set of states, and O a nonempty set of objects. We may extend the frames to *models* of the form $\langle S, O, \ldots, H \rangle$, where H is a subset of S representing the states that obtain.

3.1. *Directional Frames*

We start with frames in which the order of the relata is relevant. Predicates like '__ loves __' can perfectly be expressed in these frames.

DEFINITION 3.1. A *directional frame* is a quadruple $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$, where S is a nonempty set of states, O is a nonempty set of objects, α is an ordinal number, and Γ is a function from O^α to S.

We call the cardinality of α the degree of the frame. We denote it as $\text{degree}_\mathcal{F}$.

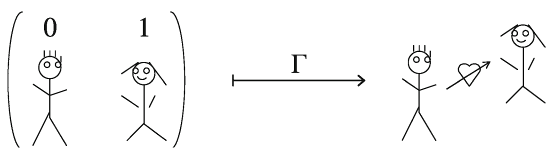
We call $\mathcal{F}$ a *full frame* if $\text{im}\ \Gamma = S$.

Γ is the function that sends the sequence of objects $f \in O^\alpha$ to the state that is the completion of the modeled relation with f.

Note that we allow the degree of a frame to be infinite. We do not want to exclude upfront that some "real" relations might have an infinite number of relata per state. But also if all "real" relations are of finite degree, then it may still be useful to consider in our analysis frames of infinite degree, because it may highlight how certain properties depend on the degree.

Note further that we do not make use of typed domains for the objects. So, the models are not accurate for a relation like *drinks*, because "Mo drinks beer" corresponds in a natural way to a state, but "beer drinks Mo" does not. However, such refinements can easily be incorporated into the models.

EXAMPLE 3.2. For the relation *loves* we can make a directional frame $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$ with S states of loving, O people, $\alpha = \{0, 1\}$, and Γ depicted as:



⊣

Because the arguments in directional frames are ordered, binary directional frames have converses. More generally, all directional frames have *permutations*:

DEFINITION 3.3. A directional frame $\mathcal{F} = \langle S, O, \alpha, \Gamma \rangle$ is a *permutation of* $\mathcal{F}' = \langle S', O', \alpha', \Gamma' \rangle$ if $S = S'$, $O = O'$, $\alpha = \alpha'$, and there is a bijection $\pi: \alpha \to \alpha$ such that for each $f \in O^\alpha$, $\Gamma(f) = \Gamma'(f \circ \pi)$.

We say that $\mathcal{F}$ has *strict symmetry* if there is a bijection $\pi: \alpha \to \alpha$ with $\pi \neq \text{id}_\alpha$ such that for each $f \in O^\alpha$, $\Gamma(f) = \Gamma(f \circ \pi)$.

In our definition of directional frames we have chosen the arguments to be well-ordered. However, it is not obvious that this is the most appropriate choice for the infinite case. Perhaps we should also allow other linear orderings.

In Section 2.1 I mentioned Fine's objection against the standard view that, as a consequence of that view, each binary relation has a converse [2, p. 2]. This objection may also be expressed as a shortcoming of directional frames.

If there is a single underlying relation, then we would like to give a neutral representation for it. The directional frames obviously fail in this respect. We could of course take the class of permutations of a directional frame as a neutral representation, but, as we will show in this paper, there are simpler alternatives.

REMARK 3.4. Fine thinks that there are both neutral and biased relations [2, p. 1]. Therefore he might find for a certain class of relations directional models adequate. A different view on relations has been proposed by Timothy Williamson [5]. For Williamson all relations are neutral, and he will probably see no reason (apart from conventional ones) to prefer for any relation any specific directional model as a representation.

3.2. *Positional Frames*

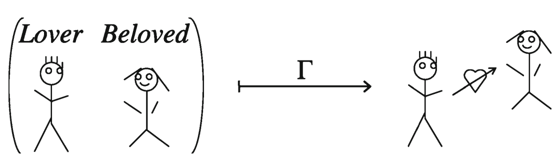
Instead of letting the order of the objects play a constitutive role for relations, we can assign objects to orderless *positions*:

DEFINITION 3.5. A *positional frame* is a quadruple $\mathcal{F} = \langle S, O, P, \Gamma \rangle$, where S is a nonempty set of states, O is a nonempty set of objects, P is a set of positions, and Γ is a function from O^P to S.

We call the cardinality of P the *degree of* the frame. We denote it as $\text{degree}_{\mathcal{F}}$.

We call $\mathcal{F}$ a *full frame* if $\text{im}\ \Gamma = S$.

EXAMPLE 3.6. For the love relation we can make a positional frame $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ with S states of loving, O people, $P = \{\textit{Lover}, \textit{Beloved}\}$, and Γ depicted as:



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Analogous to permutations of directional frames we can define *positional variants* of positional frames:

DEFINITION 3.7. A positional frame $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ is a *positional variant of* $\mathcal{F}' = \langle S', O', P', \Gamma' \rangle$ if $S = S'$, $O = O'$, and there is a bijection π: $P' \to P$ such that for each $f \in O^P$, $\Gamma(f) = \Gamma'(f \circ \pi)$.

For directional frames we discussed the problem with converses. This problem does not occur for positional frames, since there is no intrinsic order in the positions. In Section 2.2 I mentioned two objections of Fine against the positionalist view [2, p. 16] whose impact for positional frames needs to be considered. The first objection is an ontological one, namely that positions do not belong to the "fundamental furniture of the universe". The second objection concerns strictly symmetric relations. Different assignments to positions may give identical completions.

The first objection has no force against positional frames as models for the logical space of relations if we do not have to take the identity of positions in the frames as basic. By presenting alternative frames and by analyzing their relationship with positional frames we will show that positions may be defined in other terms. Also the second objection I do not consider as a disqualification of positional frames. What Fine convincingly showed is that it would be wrong to assume that the positions correspond one-to-one to some kind of entities in the complexes of relations and that these entities are occupied by the constituents of the relation. But in the positional frames the positions are not part of the states nor is it said that objects occupy positions. The positions in the positional frames only have a kind of mediating function. Assigning objects to them *yields* states. In this paper I will defend that positional frames are most appropriate for representing a large class of relations.

It is possible that certain positions play individually absolutely no role for the states assigned by the function Γ. We call such positions *dummy positions*:

DEFINITION 3.8. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a positional frame. We call $p \in P$ a *dummy position* if for each $f, g \in O^P$,[1]

$$f =_{P-\{p\}} g \Rightarrow \Gamma(f) = \Gamma(g).$$

Perhaps surprisingly, dummy positions cannot always be dropped all simultaneously as the next example shows.

EXAMPLE 3.9. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a positional frame with O an infinite set of objects, S the set of subsets of O modulo a finite difference, i.e.

$$S = \{\widehat{A} \mid A \subseteq O\}$$

with $\widehat{A} = \{A' \subseteq O \mid A \,\Delta\, A' \text{ is finite}\}$, P an infinite set of positions, and Γ defined by $\Gamma(f) = \widehat{\text{im} f}$.[2] Then each $p \in P$ is a dummy position, but not for every f and g, $\Gamma(f) = \Gamma(g)$. ⊣

To handle *variadic* relations, i.e. relations with a variable number of relata, we could define a *variadic positional frame* as a quadruple $\langle S, O, P, \Gamma \rangle$, where Γ a function from V to S with V a set of partial functions from P to O. In this paper we will not discuss this type of frames.

3.3. *Substitution Frames*

In this section, we present a type of frames, the *substitution frames*, that agrees with the antipositionalist view.[3]

DEFINITION 3.10. A *substitution frame* is a triple $\mathcal{F} = \langle S, O, \Sigma \rangle$, where S is a nonempty set of states, O is a nonempty set of objects, and Σ is a function from $S \times O^O$ to S such that

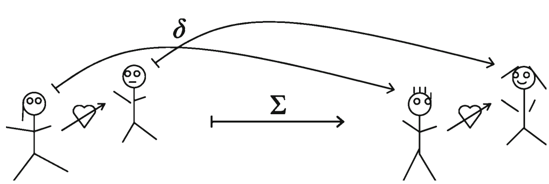
1. $\Sigma(s, \text{id}_O) = s$,
2. $\Sigma(s, \delta' \circ \delta) = \Sigma(\Sigma(s, \delta), \delta')$.

For convenience, we will often write $s \cdot_{\mathcal{F}} \delta$ or $s \cdot \delta$ for $\Sigma(s, \delta)$. Further, we will also often write $f \cdot g$ for $g \circ f$. With this notation, Σ is such that for all $s \in S$ and for all $\delta, \delta' \in O^O$, $s \cdot \text{id}_O = s$, and $s \cdot (\delta \cdot \delta') = (s \cdot \delta) \cdot \delta'$.

The two conditions on Σ agree with how we understand substitution. The intended interpretation of $s \cdot \delta$ is the state we get when we simultaneously substitute in s for each object a the object $\delta(a)$.

REMARK 3.11. The two conditions on Σ say that Σ is a *right action of the monoid O^O on S*. In terms of category theory we could alternatively have defined a substitution frame as a triple $\langle S, O, \Sigma^* \rangle$, where S is a nonempty set of states, O is a nonempty set of objects, and Σ^* is a functor from the monoid O^O to the monoid S^S.

EXAMPLE 3.12. For the love relation we can make a substitution frame $\mathcal{F} = \langle S, O, \Sigma \rangle$ with S states of loving, O people, and Σ depicted as:



For the state s of Hans's loving Riëtte, and δ: $O \to O$ with Hans $\mapsto$ Jan and Riëtte $\mapsto$ Jos, $s \cdot \delta$ is the state of Jan's loving Jos. With Σ defined in this way, the two conditions imposed on Σ in Definition 3.10 are clearly fulfilled. Take for example δ' with Jan $\mapsto$ Henk and Jos $\mapsto$ Lieke, then $s \cdot (\delta \cdot \delta')$ is the same state as $(s \cdot \delta) \cdot \delta'$, namely the state of Henk's loving Lieke. ⊣

Substitution frames can also accommodate variadic relations (i.e. variably polyadic relations), since states not connected by substitutions can be united into a single frame. For example, for the variadic relation *is surrounded by* we can make a substitution frame as follows:

EXAMPLE 3.13. Define $\mathcal{F} = \langle S, O, \Sigma \rangle$ with O a set of objects, S states of objects being surrounded by a variable number of other objects, and Σ such that for the state of a_1's being surrounded by $a_2, a_3, \ldots, a_m$ a substitution may result in the state of b_1's being surrounded by $b_2, b_3, \ldots, b_m$, where we do not exclude that in a state certain objects may occur more than once. ⊣

We now define the objects or relata of a state. The idea is to define them as the objects for which it can make a difference for the resulting state which objects are substituted for them.[4] We have, however, to be a bit cautious in our formulation:

DEFINITION 3.14. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame. We call $A \subseteq O$ an *object-domain of* $s \in S$ if for every δ, δ': $O \to O$,

$$\delta =_A \delta' \Rightarrow s \cdot \delta = s \cdot \delta'.$$

We define the *core of s* as:

$$\mathrm{Core}_{\mathcal{F}}(s) = \bigcap \{A \mid A \text{ is an object-domain of } s\}.$$

If $\mathrm{Core}_{\mathcal{F}}(s)$ is an object-domain, then we call this set the *objects of s*. We denote this set as $\mathrm{Ob}_{\mathcal{F}}(s)$. If $\mathrm{Core}_{\mathcal{F}}(s)$ is not an object-domain, then we leave $\mathrm{Ob}_{\mathcal{F}}(s)$ undefined.

We will often write $\mathrm{Core}(s)$ and $\mathrm{Ob}(s)$ for $\mathrm{Core}_{\mathcal{F}}(s)$ and $\mathrm{Ob}_{\mathcal{F}}(s)$.

LEMMA 3.15. *Let* $\mathcal{F} = \langle S, O, \Sigma \rangle$ *be a substitution frame. For every* $s \in S$, *the object-domains of s form a (possibly non-proper) filter on* O.[5]

Proof. To prove that the object-domains of s are closed under finite intersection, let A and A' be object-domains of s. Let $\delta, \delta' \colon O \to O$ be such that $\delta =_{A \cap A'} \delta'$. Define

$$\delta''(a) = \begin{cases} \delta(a) & \text{if } a \in A - A', \\ \delta(a) = \delta'(a) & \text{if } a \in A \cap A', \\ \delta'(a) & \text{if } a \in A' - A. \end{cases}$$

Then $\delta'' =_A \delta$ and $\delta'' =_{A'} \delta'$. So, $s \cdot \delta = s \cdot \delta'' = s \cdot \delta'$. Thus, $A \cap A'$ is an object-domain of s.

It is trivial that the object-domains of s are upward closed.

Since an object-domain may be empty, we may have a non-proper filter. ⊣

The next example, which is related to Example 3.9, shows that not every core is an object-domain.

EXAMPLE 3.16. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame with O an infinite set, S the set of subsets of O modulo a finite difference, i.e.

$$S = \{\widehat{A} \mid A \subseteq O\}$$

with $\widehat{A} = \{A' \subseteq O \mid A \,\Delta\, A' \text{ is finite}\}$, and Σ defined by[6]

$$\widehat{A} \cdot \delta = \widehat{\delta[A]}.$$

Σ is well-defined, since for any $A, B \subseteq O$, if $\widehat{A} = \widehat{B}$, then $\widehat{\delta[A]} = \widehat{\delta[B]}$. Further, $\mathcal{F}$ is a substitution frame, since

1. $\widehat{A} \cdot \mathrm{id}_O = \widehat{\mathrm{id}_O[A]} = \widehat{A}$,
2. $\widehat{A} \cdot (\delta \cdot \delta') = \widehat{(\delta \cdot \delta')[A]} = \widehat{\delta'[\delta[A]]} = \widehat{\delta[A]} \cdot \delta' = (\widehat{A} \cdot \delta) \cdot \delta'$.

It is not difficult to see that for any $A \subseteq O$ the core of $\widehat{A}$ is the empty set, but if A is infinite, then the empty set is not an object-domain of $\widehat{A}$. ⊣

The next lemma gives a characterization of the core of a state in terms of substitutions of one object at a time.

LEMMA 3.17. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame, for every $s \in S$,[7]

$$\mathrm{Core}(s) = \{a \mid \exists b \in O\,(s \cdot \mathrm{id}_O[a \mapsto b] \neq s)\}.$$

Proof. Consider $a \in \mathrm{Core}(s)$. Then $O - \{a\}$ is not an object-domain. Therefore, for some δ_1, δ_2 we have $\delta_1 =_{O-\{a\}} \delta_2$ and $s \cdot \delta_1 \neq s \cdot \delta_2$. We may choose $b \in O$ with $b \neq a$. Then, for $\delta_0 = \mathrm{id}_O[a \mapsto b]$ we have $\delta_0 \cdot \delta_1 = \delta_0 \cdot \delta_2$, and thus

$$(s \cdot \delta_0) \cdot \delta_1 = s \cdot (\delta_0 \cdot \delta_1) = s \cdot (\delta_0 \cdot \delta_2) = (s \cdot \delta_0) \cdot \delta_2.$$

So, because $s \cdot \delta_1 \neq s \cdot \delta_2$, we see that $s \cdot \delta_0 \neq s$.

The inclusion in the other direction is obvious. ⊣

In the next lemma and examples we investigate the relationship between object-domains, cores, and objects of different states.

LEMMA 3.18. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame. For every $s \in S$, if A is an object-domain of s, then $\delta[A]$ is an object-domain of $s \cdot \delta$.

Proof. Let δ', δ'' be functions from O to O. If $\delta' =_{\delta[A]} \delta''$, then $\delta \cdot \delta' =_A \delta \cdot \delta''$. So, if A is an object-domain of s, then $s \cdot \delta \cdot \delta' = s \cdot \delta \cdot \delta''$. ⊣

Thus, by the lemma, if $\mathrm{Ob}(s)$ is defined, then $\mathrm{Core}(s \cdot \delta) \subseteq \delta[\mathrm{Ob}(s)]$. The next example shows that not always $\mathrm{Core}(s \cdot \delta) \subseteq \delta[\mathrm{Core}(s)]$.

EXAMPLE 3.19. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame with O an infinite set,

$$S = \{(A, \widehat{B}) \mid A, B \subseteq O\}$$

with $\widehat{B} = \{B' \subseteq O \mid B \,\Delta\, B' \text{ is finite}\}$, and Σ defined by

$$(A, \widehat{B}) \cdot \delta = (\delta[A] \cup C, \widehat{\delta[B]}\ )$$

with $C = \{a \mid \delta(b) = a \text{ for infinitely many } b \in B\}$.

It is not difficult to verify that Σ is well-defined, that $\mathcal{F}$ is indeed a substitution frame, and that $\mathrm{Core}(A, \widehat{B}) = A$. So, for $s_0 = (\emptyset, \widehat{O})$, $\mathrm{Core}(s_0) = \emptyset$, but for any constant function $c_a \colon D \to D$ with value a, $\mathrm{Core}(s_0 \cdot c_a) = a$. ⊣

Sometimes $\mathrm{Ob}(s \cdot \delta) \neq \delta[\mathrm{Ob}(s)]$, as is shown in the next example.

EXAMPLE 3.20. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame with $S = \mathcal{P}(\mathrm{O})$, and Σ defined by

$$s \cdot \delta = \begin{cases} \delta[s] & \text{if } \delta \text{ is injective on } s, \\ \emptyset & \text{otherwise.} \end{cases}$$

It is easy to verify that $\mathcal{F}$ is indeed a substitution frame, and that for all $s \in S$, we have $\text{Ob}(s) = s$. But, we also see that for any function $\delta\colon O \to O$ that is not injective on s, $\text{Ob}(s \cdot \delta) = \emptyset$.

If $\text{Ob}(s)$ is undefined, then not necessarily $\text{Ob}(s \cdot \delta)$ is undefined as well. This follows easily from Example 3.16. Also, if $\text{Ob}(s)$ is defined, then not necessarily $\text{Ob}(s \cdot \delta)$ is defined as well:

EXAMPLE 3.21. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame with O an infinite set,

$$S = \{(A, 0) \mid A \subseteq O\} \cup \{(\widehat{A}, 1) \mid A \subseteq O\}$$

with $\widehat{A} = \{A' \subseteq O \mid A \,\Delta\, A' \text{ is finite}\}$, and Σ defined by

$$s \cdot \delta = \begin{cases} (\delta[A], 0) \text{ if } s = (A, 0) \text{ and } \delta \text{ is injective on } s, \\ (\widehat{\delta[A]}, 1) \text{ if } s = (A, 0) \text{ and } \delta \text{ is not injective on } s, \text{or } s = (\widehat{A}, 1). \end{cases}$$

It is not difficult to verify that Σ is well-defined, and that $\mathcal{F}$ is indeed a substitution frame. Further, for all $s = (A, 0)$, $\text{Ob}(s)$ is defined, but for any function $\delta\colon O \to O$ that is not injective on s, $\text{Ob}(s \cdot \delta)$ is not defined. ⊣

For each substitution frame we can define its degree as a cardinal number:

DEFINITION 3.22. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame. For a state s in S, we define the *degree of s* as:

$$\text{degree}(s) = \text{glb}\,\{|A| \mid A \text{ is an object-domain of } s\}.$$

The *degree of* $\mathcal{F}$ we define as:

$$\text{degree}_{\mathcal{F}} = \text{lub}\,\{\text{degree}(s) \mid s \in S\}.$$

Here $|A|$ denotes as usual the cardinality of A, "glb" denotes the greatest lower bound, and "lub" denotes the least upper bound. Note that the degree of s and the degree of $\mathcal{F}$ always exist and are indeed cardinal numbers.

If the degree of a frame is infinite, then either (1) for all states s the set $\text{Ob}(s)$ is finite, but the size of the sets $\text{Ob}(s)$ is unbounded, or (2) there is some state s for which $\text{Ob}(s)$ is infinite or not defined. In the last case all object-domains of s are obviously infinite sets.

For any substitution frame $\mathcal{F} = \langle S, O, \Sigma \rangle$, $\text{degree}_{\mathcal{F}} \leq |O|$. If $\mathcal{F}$ is of finite degree and $\text{degree}_{\mathcal{F}} < |O|$, then the simultaneous substitution of many objects of a state can be defined in terms of substitutions of one object at a time. (cf. [2, p. 26, note 15]).

We may ask ourselves whether substitution frames are perhaps not too limited. With substitution frames it is not possible to frame relations for which $\Re abc$ and $\Re cba$ represent the same state iff a and b are *not* equal. But could a “real” relation like this exist? Actually, I don't think this is very likely. However, if such relations exist, then it might be worth considering frames based on *injective* substitution. We will not discuss such frames in this paper.

4. Corresponding Frames

In this section, we present the main results of this paper. We show how intimately related positional frames are with substitution frames. Metaphysically the results are of interest because they give a justification for using positional models for a large class of relations without any commitment to an ontology of positions. In other words, the use of positional frames for such relations does not force us to accept positions as fundamental entities, because we can treat positions as “light” products of our own mind. In Section 6 these metaphysical aspects will be discussed in more detail.

DEFINITION 4.1. A substitution frame $\mathcal{F} = \langle S, O, \Sigma \rangle$ and a positional frame $\mathcal{G} = \langle S', O', P, \Gamma \rangle$ *correspond* if

1. $S = S' = \operatorname{im} \Gamma$,
2. $O = O'$,
3. $\Gamma(f) \cdot_{\mathcal{F}} \delta = \Gamma(f \cdot \delta)$.

Note that the first condition implies that $\mathcal{G}$ is a full frame. The last condition states that for any δ: $O \to O$ the following diagram commutes:

$$\begin{array}{ccc} O^P & \xrightarrow{\ \tilde{\delta}\ } & O^P \\ \Big\downarrow{\scriptstyle \Gamma} & & \Big\downarrow{\scriptstyle \Gamma} \\ S & \xrightarrow[\ \Sigma_\delta\ ]{} & S \end{array}$$

where $\widetilde{\delta}$ is defined by $\widetilde{\delta}(f) = f \cdot \delta$, and Σ_δ is defined by $\Sigma_\delta(s) = s \cdot_{\mathcal{F}} \delta$.

Further, we say that a substitution model $\mathcal{M} = \langle S, O, \Sigma, H \rangle$ and a positional model $\mathcal{N} = \langle S', O', P, \Gamma, H' \rangle$ *correspond* if their frames correspond and $H = H'$.

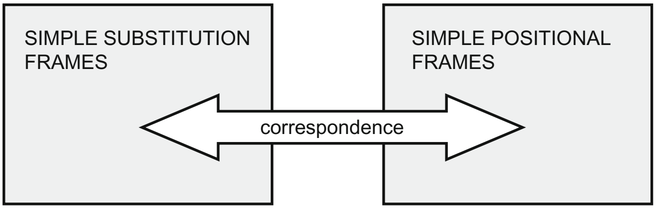


Figure 1. Relationship between simple frames

Not every substitution frame corresponds to a positional frame, but we will show in Theorems 4.3 and 4.6 that the *simple* substitution frames correspond to the *simple* positional frames (see Figure 1). Further, we will show (1) that for each simple substitution frame of finite degree the corresponding positional frame of the same degree is unique, modulo positional variants, and (2) that for each simple positional frame the corresponding substitution frame is always unique.

DEFINITION 4.2. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame. We call $\mathcal{F}$ a *simple substitution frame* if there is a state s_0 such that

$$S = \{s_0 \cdot \delta \mid \delta\colon O \to O\}.$$

We call s_0 an *initial state*.

THEOREM 4.3. A substitution frame $\mathcal{F}$ corresponds to some positional frame $\mathcal{G}$ of the same degree iff $\mathcal{F}$ is a simple substitution frame.

Further, if $\mathrm{degree}_{\mathcal{F}}$ is finite, then $\mathcal{G}$ is unique, modulo positional variants.

Proof. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a substitution frame, and let $\mathcal{G} = \langle S, O, P, \Gamma \rangle$ be a corresponding positional frame of the same degree. Let $f_0\colon P \to O$ be an injection. Such a function exists, since $|P| = \mathrm{degree}_{\mathcal{G}} = \mathrm{degree}_{\mathcal{F}} \leq |O|$. By the injectivity of f_0, there is for each $f \in O^P$ a $\delta \in O^O$ such that $f = f_0 \cdot \delta$. Thus, by condition 1 and 3 of the definition of corresponding frames,

$$S = \mathrm{im}\,\Gamma = \{\Gamma(f_0 \cdot \delta) \mid \delta\colon O \to O\} = \{\Gamma(f_0) \cdot_{\mathcal{F}} \delta \mid \delta\colon O \to O\}.$$

So, $\mathcal{F}$ is a simple substitution frame with $\Gamma(f_0)$ as an initial state.

Conversely, let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a simple substitution frame. We construct a corresponding positional frame $\mathcal{G} = \langle S, O, P, \Gamma \rangle$ as follows:

1. Choose an initial state $s_0 \in S$.
2. Choose an object-domain A of s_0 with $|A| = \mathrm{degree}_{\mathcal{F}}(s_0)$.
3. Define $P = A$.

4. Let f be an arbitrary element of O^P. Let f^+ extend f to $O \to O$. Define $\Gamma(f) = s_0 \cdot_{\mathcal{F}} f^+$.

Note that Γ is well-defined, since all extensions of f are identical on P and P is an object-domain of s_0.

We now show that the conditions of the definition of corresponding frames are fulfilled and that $\mathcal{G}$ has the same degree as $\mathcal{F}$.

Condition 1 follows from the following observations. Let s be an arbitrary state in S. Because s_0 is an initial state, for some δ we have $s = s_0 \cdot_{\mathcal{F}} \delta$. Let f in O^P be the restriction of δ to P. Then $\Gamma(f) = s_0 \cdot_{\mathcal{F}} \delta = s$, which proves that $S \subseteq \operatorname{im}\Gamma$. Conversely, from the definition of Γ it follows immediately that $\operatorname{im}\Gamma \subseteq S$.

Condition 2 is trivially fulfilled, and condition 3 follows from

$$\Gamma(f) \cdot_{\mathcal{F}} \delta = (s_0 \cdot_{\mathcal{F}} f^+) \cdot_{\mathcal{F}} \delta = s_0 \cdot_{\mathcal{F}} (f \cdot \delta)^+ = \Gamma(f \cdot \delta).$$

This proves that $\mathcal{F}$ corresponds to $\mathcal{G}$.

Since s_0 is an initial state, for any s in S, $\text{degree}(s) \leq \text{degree}(s_0)$. Therefore, $\text{degree}_{\mathcal{F}} = \text{degree}(s_0) = |P| = \text{degree}_{\mathcal{G}}$.

To prove the uniqueness claim of the theorem, assume $\text{degree}_{\mathcal{F}}$ is finite. Let $\mathcal{G}' = \langle S, O, P', \Gamma' \rangle$ be another corresponding frame of the same degree. Consider again the injection $f_0 \in O^P$. Because the frames have the same, finite degree, there is an injection $f_0' \in O^{P'}$ such that $\Gamma'(f_0') = \Gamma(f_0)$. So there is a bijection $\pi \colon P' \to P$ such that $f_0' = \pi \cdot f_0$. Further, for each $f \in O^P$ there is a $\delta \colon O \to O$ such that $f = f_0 \cdot \delta$. Therefore,

$$\begin{aligned}\Gamma(f) &= \Gamma(f_0 \cdot \delta) = \Gamma(f_0) \cdot_{\mathcal{F}} \delta = \Gamma'(f_0') \cdot_{\mathcal{F}} \delta = \Gamma'(f_0' \cdot \delta) = \Gamma'(\pi \cdot f_0 \cdot \delta) \\ &= \Gamma'(\pi \cdot f).\end{aligned}$$

Thus, we showed that $\mathcal{G}$ and $\mathcal{G}'$ are positional variants. $\dashv$

If $\text{degree}_{\mathcal{F}}$ is infinite, then the corresponding positional frames with the same degree are not always unique, modulo positional variants. A trivial cause are dummy positions (see Definition 3.8), but dummy positions are not the only obstacle, as the next example shows.

EXAMPLE 4.4. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ with $O = \omega$, the set of natural numbers, $S = \{s \colon \omega \to (\omega \cup \{\infty\}) \mid \exists i\, (s(i) = \infty)\}$, and Σ defined by

$$(s \cdot_{\mathcal{F}} \delta)(i) = \sum_{\delta(j)=i} s(j)$$

with $i + \infty = \infty + i = \infty + \infty = \infty$. Note that S can be regarded as a set of multisets. It is easy to check that $\mathcal{F}$ is a simple substitution frame.

A peculiar property of $\mathcal{F}$ is that it has initial states s_0, s'_0 such that for any δ: $O \to O$ with $s_0 \cdot_{\mathcal{F}} \delta = s'_0$, δ is not injective on $\mathrm{Ob}(s_0)$, namely

$s_0 = [0^\infty, 1, 2, 3, \ldots]$, i.e. $s_0(0) = \infty$, and for $i \geq 1, s_0(i) = 1$;
$s'_0 = [0^\infty, 1^\infty, 2, 3, \ldots]$, i.e. $s'_0(0) = s'_0(1) = \infty$, and for $i \geq 2, s'_0(i) = 1$.

We exploit this peculiarity to define two dissimilar positional frames:

$$\mathcal{G} = \langle S, O, \omega, \Gamma \rangle \text{ with } \Gamma(f) = s_0 \cdot_{\mathcal{F}} f;$$
$$\mathcal{G}' = \langle S, O, \omega, \Gamma' \rangle \text{ with } \Gamma'(f) = s'_0 \cdot_{\mathcal{F}} f.$$

By the proof of Theorem 4.3 we see that $\mathcal{G}$ and $\mathcal{G}'$ correspond to $\mathcal{F}$. Clearly, $\mathcal{G}$ and $\mathcal{G}'$ have no dummy positions. To see that $\mathcal{G}$ and $\mathcal{G}'$ are not positional variants, let π: $\omega \to \omega$ be such that $\Gamma'(\mathrm{id}_\omega) = \Gamma(\mathrm{id}_\omega \cdot \pi)$. Then $s'_0 = s_0 \cdot_{\mathcal{F}} \pi$, and so π cannot be bijective.

It is an open question what in this and similar cases could be regarded as the "most natural" corresponding frame. ⊣

Theorem 4.3 shows that for simple substitution frames of finite degree, we can reconstruct indirectly the notion of position in a satisfactory way. To characterize the positional frames that correspond to substitution frames, we need the following definition:

DEFINITION 4.5. We say that a positional frame $\mathcal{G} = \langle S, O, P, \Gamma \rangle$ *respects substitution* if for every δ: $O \to O$,

$$\Gamma(f) = \Gamma(g) \Rightarrow \Gamma(f \cdot \delta) = \Gamma(g \cdot \delta).$$

Now we state the counterpart of Theorem 4.3:

THEOREM 4.6. A positional frame $\mathcal{G}$ corresponds to some substitution frame iff $\mathcal{G}$ is a full frame that respects substitution.

Further, the corresponding substitution frame is unique.

Proof. Let $\mathcal{G} = \langle S, O, P, \Gamma \rangle$ be a positional frame and let $\mathcal{F} = \langle S, O, \Sigma \rangle$ be a corresponding substitution frame. Assume $\Gamma(f) = \Gamma(g)$. Then for every δ: $O \to O$,

$$\Gamma(f \cdot \delta) = \Gamma(f) \cdot_{\mathcal{F}} \delta = \Gamma(g) \cdot_{\mathcal{F}} \delta = \Gamma(g \cdot \delta).$$

So, $\mathcal{G}$ respects substitution. Further, by condition 1 of the definition of corresponding frames, $\mathcal{G}$ is a full frame.

Conversely, let $\mathcal{G} = \langle S, O, P, \Gamma \rangle$ be a full, substitution-respecting positional frame. Let $\mathcal{F} = \langle S, O, \Sigma \rangle$ with Σ defined by

$$\Gamma(f) \cdot_{\mathcal{F}} \delta = \Gamma(f \cdot \delta).$$

Since $\mathcal{G}$ respects substitution, $\Gamma(f) = \Gamma(g) \Rightarrow \Gamma(f \cdot \delta) = \Gamma(g \cdot \delta)$. Therefore Σ is well-defined.

It is easy to see that $\mathcal{F}$ is a substitution frame:

1. $\Gamma(f) \cdot_{\mathcal{F}} \mathrm{id}_O = \Gamma(f \cdot \mathrm{id}_O) = \Gamma(f)$,
2. $\Gamma(f) \cdot_{\mathcal{F}} (\delta \cdot \delta') = \Gamma(f \cdot \delta \cdot \delta') = \Gamma(f \cdot \delta) \cdot_{\mathcal{F}} \delta' = (\Gamma(f) \cdot_{\mathcal{F}} \delta) \cdot_{\mathcal{F}} \delta'$.

The frames $\mathcal{F}$ and $\mathcal{G}$ correspond, since the conditions of the definition of corresponding frames are trivially fulfilled.

The uniqueness of $\mathcal{F}$ follows immediately from the fact that for any substitution frame $\mathcal{F}'$ that corresponds to $\mathcal{G}$ we must have $\Gamma(f) \cdot_{\mathcal{F}'} \delta = \Gamma(f \cdot \delta)$ by condition 3 of the definition of corresponding frames. $\dashv$

REMARK 4.7. For some substitution-respecting frames $\Gamma(f) = \Gamma(g)$ does not imply $\mathrm{im} f = \mathrm{im}\, g$, not even if the frames have no dummy positions. Take for example a frame $\mathcal{G} = \langle S, O, P, \Gamma \rangle$ with $|O| \geq 2$, $P = \{p_1, p_2\}$, and Γ such that

$$\Gamma(f) = \Gamma(g) \Leftrightarrow (f = g \text{ or } (f(p_1) = f(p_2) \text{ and } g(p_1) = g(p_2))).$$

Note that in the substitution frame corresponding to $\mathcal{G}$ the set of objects of one of the states is empty. So, in particular for some f, $\mathrm{Ob}(\Gamma(f)) \subsetneq \mathrm{im} f$. It is an open question whether some "real" relations have frames with such properties.

DEFINITION 4.8. Let $\mathcal{G} = \langle S, O, P, \Gamma \rangle$ be a positional frame. We call $\mathcal{G}$ a *simple positional frame* if

1. $|O| \geq |P|$,
2. $\mathcal{G}$ is a full frame,
3. $\mathcal{G}$ respects substitution.

From Theorem 4.3 and Theorem 4.6 it follows immediately that:

COROLLARY 4.9. Every simple substitution frame corresponds to a simple positional frame and vice versa.

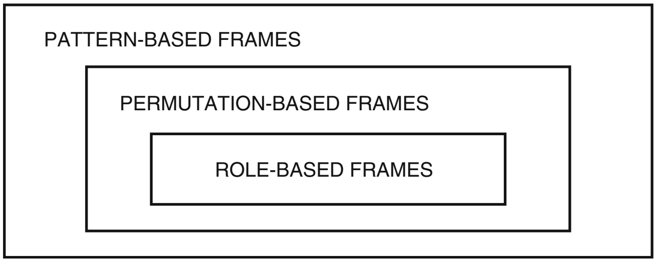


Figure 2. Subtypes of positional frames

5. Positional Structure

The frames developed in Section 3 might be too general or too limited for "real" relations. For example, it might be that relations can only have certain limited forms of strict symmetry. Because a positionalist view on relations is so natural for our way of thinking, we will investigate to what extent certain subtypes of the positional frames are adequate for "real" relations. In our analysis the notion of *positional structure* will play a central role.

DEFINITION 5.1. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a positional frame. We define the *positional structure* $E_{\mathcal{F}}$ as:

$$E_{\mathcal{F}} = \{(f, g) \mid \Gamma(f) = \Gamma(g)\}.$$

Note that $E_{\mathcal{F}}$ is an equivalence relation. Insight in the positional structure for metaphysically meaningful relations might provide a better understanding of the essence of relations. We will define three subtypes of the positional frames in terms of structures that involve only their positions (see Figure 2). We follow a bottom-up approach, starting with the role-based frames.

5.1. *Role-Based Frames*

Positions can fulfill certain roles. In the positional frame for the amatory relation one position fulfills the role of *Lover* and the other position the role of *Beloved*. In this case positions and roles coincide. However, there is no compelling reason why there would always be a one-to-one correspondence between positions and roles. On the contrary, it is natural to say that in the positional frame for the adjacency relation the two positions fulfill the same role.

DEFINITION 5.2. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a positional frame. Let p, p' be elements of P. We say that *p' fulfills the same role as p* if for some bijection $\pi: P \to P$,

$$p' = \pi(p) \ \ \& \ \ \forall f \in O^P (f \ E_{\mathcal{F}} \ (f \circ \pi)).$$

We define the *role of p* as:

$$\mathsf{Role}(p) = \{p' \in P \mid p' \text{ fulfills the same role as } p\},$$

and the *roles of $\mathcal{F}$* as:

$$\mathsf{Roles}_{\mathcal{F}} = \{\mathsf{Role}(p) \mid p \in P\}.$$

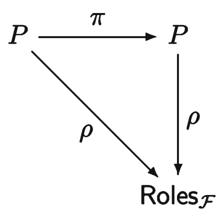
Note that the relation *fulfills the same role as* is an equivalence relation.

We now define a type of positional frames for which changing the positions of the objects does not change the corresponding state as long as the roles of objects are kept invariant.

DEFINITION 5.3. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a positional frame, and let $E_{\mathcal{F}}$ be its positional structure. We call $\mathcal{F}$ a *role-based frame* if

$$E_{\mathcal{F}} = \left\{ (f, f \circ \pi) \mid f \in O^P \ \ \& \ \ \pi \text{ is a role-preserving permutation} \right\},$$

where $\pi: P \to P$ *is a role-preserving permutation* if π is a bijection for which the following diagram commutes:



with $\rho: p \mapsto \mathsf{Role}(p)$.

Note that the role-preserving permutations form a group.

We defined roles within the context of individual frames. In this respect they differ from *thematic roles* which apply to arguments of different predicates. For example, thematic roles like *agent*, *patient*, and *location* are used to classify arguments of natural language predicates. We will not study such kind of global roles in this paper.

Certain strictly symmetric relations can be modeled by role-based models. For example, for the adjacency relation we can define a frame $\mathcal{F}$ with two positions, say *Next* and *Nixt*. If $\mathcal{F}$ is strictly symmetric, then $\mathcal{F}$

is clearly a role-based frame with one role. However, in general for circular relations no role-based frames are possible. We can see this by rephrasing an argument of Fine [2, p. 17, note 10] in terms of roles:

EXAMPLE 5.4. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a role-based frame with a, b, and c three different objects in O, $P = \{p_1, p_2, p_3\}$, and[8]

$$\Gamma abc = \Gamma bca = \Gamma cab.$$

Then the frame has just one role. But then also the state Γacb is necessarily identical to Γabc. Similar observations can of course be made for frames with more positions. Therefore, frames for "genuine" n-ary circular relations with $n \geq 3$ are not role-based. ⊣

5.2. *Permutation-Based Frames*

A natural generalization of the role-based frames are the *permutation-based frames*:

DEFINITION 5.5. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a positional frame, and let $E_{\mathcal{F}}$ be its positional structure. We define the *permutation group of* $\mathcal{F}$ as:

$$\mathsf{Perm}_{\mathcal{F}} = \left\{ \pi \in P^P \mid \pi \text{ is a bijection } \ \& \ \forall f \in O^P (f \, E_{\mathcal{F}} \, (f \circ \pi)) \right\}.$$

We call $\mathcal{F}$ a *permutation-based frame* if

$$E_{\mathcal{F}} = \{ (f, f \circ \pi) \mid f \in O^P \ \& \ \pi \in \mathsf{Perm}_{\mathcal{F}} \}.$$

Note that $\mathsf{Perm}_{\mathcal{F}}$ is indeed a group.

Circular relations can adequately be framed by permutation-based frames. For example, a ternary circular relation can be framed by a permutation-based frame $\mathcal{F}$ with $\mathsf{Perm}_{\mathcal{F}}$ generated by

$$\begin{pmatrix} p_1 & p_2 & p_3 \\ p_2 & p_3 & p_1 \end{pmatrix}.$$

Unfortunately, also this type of frames has shortcomings, in particular for certain relations of degree three and higher. Consider a relation $\Re$ in which $\Re abc$ represents the state that a loves b and b loves c. Then $\Re aba$ represents the same state as $\Re bab$, but aba is not a permutation of bab. This means that no permutation-based frame for this relation is possible. From this we can infer a more general conclusion, namely that the class of permutation-based frames is not closed under identification of positions.

5.3. *Pattern-Based Frames*

We now define a class of positional frames which turns out to be identical to the class of substitution-respecting frames. These frames are of special interest, since, as we showed in Theorem 4.6, the positional frames that correspond to a substitution frame are precisely the full, substitution-respecting frames.

DEFINITION 5.6. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a positional frame, and let $E_{\mathcal{F}}$ be its positional structure. We define the *pattern of* $\mathcal{F}$ as:

$$\mathsf{Pattern}_{\mathcal{F}} = \{(\sigma, \sigma') \in Q^P \times Q^P \mid \forall h \in O^Q\, ((h \circ \sigma)\ E_{\mathcal{F}}\ (h \circ \sigma'))\}$$

with $Q = 2 \times P$.

We call $\mathcal{F}$ a *pattern-based frame* if

$$E_{\mathcal{F}} = \{(h \circ \sigma, h \circ \sigma') \mid h \in O^{2 \times P}\ \&\ (\sigma, \sigma') \in \mathsf{Pattern}_{\mathcal{F}}\}.$$

Note that $\mathsf{Pattern}_{\mathcal{F}}$ is an equivalence relation.

Not every pattern-based frame is permutation-based, as the next example shows:

EXAMPLE 5.7. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a positional frame with $P = \{p_1, p_2, p_3\}$, and Γabc being the state of a's loving b and b's loving c. $\mathcal{F}$ has a special symmetry, namely $\Gamma aba = \Gamma bab$. Hence, $\mathcal{F}$ is clearly not permutation-based. It is straightforward to verify that $\mathcal{F}$ is pattern-based with $\mathsf{Pattern}_{\mathcal{F}}$ consisting of all pairs (σ, σ) with $\sigma \in (2 \times P)^P$, and all pairs

$$(\begin{pmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_1 \end{pmatrix}, \begin{pmatrix} p_1 & p_2 & p_3 \\ q_2 & q_1 & q_2 \end{pmatrix})$$

with $q_1, q_2 \in 2 \times P$. ⊣

The next theorem is of metaphysical interest since it implies that the positional structure $E_{\mathcal{F}}$ of a substitution-respecting frame of finite degree is determined by a finite subset of $E_{\mathcal{F}}$. Also from an epistemological point of view this is of interest, since it means that in principle we can learn the positional structure of such frames by a finite number of substitutions.

THEOREM 5.8. A positional frame is pattern-based iff it respects substitution.

Proof. Let $\mathcal{F} = \langle S, O, P, \Gamma \rangle$ be a positional frame with positional structure $E_{\mathcal{F}}$. Assume that $\mathcal{F}$ is pattern-based. Further, assume that $f\, E_{\mathcal{F}}\, g$. Then for some $(\sigma, \sigma') \in \mathsf{Pattern}_{\mathcal{F}}$ and some h we have $f = h \circ \sigma$ and $g = h \circ \sigma'$. So for every δ: $O \to O$:

$$\delta \circ f = \delta \circ h \circ \sigma \ E_{\mathcal{F}} \ \delta \circ h \circ \sigma' = \delta \circ g.$$

So, $\mathcal{F}$ respects substitution.

Conversely, assume that $\mathcal{F}$ respects substitution. Obviously

$$E_{\mathcal{F}} \supseteq \{(h \circ \sigma, h \circ \sigma') \mid h \in O^{2\times P} \ \& \ (\sigma, \sigma') \in \mathsf{Pattern}_{\mathcal{F}}\}.$$

To prove the reverse inclusion, assume that $f E_{\mathcal{F}} g$. We have an injection j: $\mathsf{im} f \cup \mathsf{im}\, g \to 2 \times P$. Choose a function h: $2 \times P \to O$ such that $h \circ j$ is the identical embedding emb of $\mathsf{im} f \cup \mathsf{im}\, g$ in O. Define $\sigma = j \circ f$ and $\sigma' = j \circ g$. Then

$$h \circ \sigma = h \circ j \circ f = \mathsf{emb} \circ f = f.$$

Similarly, $h \circ \sigma' = g$. So, it is sufficient to show that $(\sigma, \sigma') \in \mathsf{Pattern}_{\mathcal{F}}$. But this follows from the fact that for every h': $2 \times P \to O$:

$$h' \circ \sigma = h' \circ j \circ f \ \ E_{\mathcal{F}} \ \ h' \circ j \circ g = h' \circ \sigma'.$$

⊣

Note that the permutation-based frames also respect substitution. Thus, it follows from the theorem and Example 5.7 that the class of permutation-based frames is a proper subclass of the pattern-based frames.

We are not going to discuss operations on frames and models, but I like to mention that a nice property of the class of pattern-based models is that they are closed under operations like identification of positions, conjunction, and disjunction. For more details about operations on models and relations, see [3].

6. Metaphysical Interpretations

In this section, we first look at metaphysical principles for relations. Then we translate technical results of the previous sections to metaphysical claims. In particular, we consider justification for positional representations and epistemological aspects of relations.

6.1. *Metaphysical Principles For Relations*

Let me start by formulating a number of metaphysical principles concerning states of relations. We postulate three sorts of entities: states, objects, and substitutions.

Constituents Principle:

CP-1 Every state of a relation has exactly one set of objects.

Substitution Principles:

SP-1 The objects of any state can simultaneously be substituted by other objects.

SP-2 Any substitution of objects in any state yields exactly one state of the same relation.

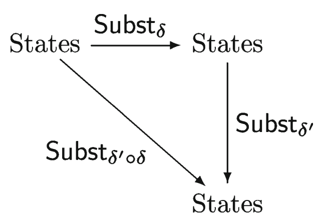
SP-3 An object a belongs to a state iff for some substitution it makes a difference for the resulting state which object is substituted for a.

SP-4 Substitution is a monoidal action on states, i.e.

(a) for any state s, the identity substitution yields the same state, i.e.

$$\mathsf{Subst}_{\mathsf{id}}(s) = s,$$

(b) the following diagram commutes:



with Subst_δ representing a substitution of objects.

Factuality Principle:

FP-1 Every state either obtains or does not obtain.

What supporting arguments can be given for these principles? It seems hard to give any conclusive arguments, since I have not given any clear definition of what relations are. Nevertheless, for several views on relations these principles might be acceptable. Let me briefly comment on each principle separately.

The constituents principle by itself does not say much. Therefore I consider it in combination with the other principles. The first substitution principle is perhaps too restrictive. It might be argued that within the context of a relation a state can have two kinds of objects: those that can be substituted by other objects, and those that fulfill a kind of background role for the state. For example, for the relation $\mathfrak{R}$ in which $\mathfrak{R}ab$ is the state that there is a flight from *a* to *b* via Amsterdam, Amsterdam could be regarded as a fixed object for the states of this relation.

We might ask whether the second substitution principle is perhaps not too strong and that it would be better to replace it with a weaker principle, namely:

SP-2′ Any substitution of objects in any state yields *no more than* one state of the same relation.

This weaker principle might be a better choice if you consider the states of a relation as *possible* states of affairs. That Bin Laden loves Bush is possible, but that $1 = 2$ is clearly not possible. What is also impossible, I think, is that I am identical to Mo, my daughter. Such examples make clear that not every substitution yields a possible state of affairs. However, if you regard states of relations as propositions or if you are willing to accept *impossible* states of affairs, then the stronger principle seems preferable.

To a certain extent it is a matter of definition whether you accept substitution principle SP-3. One might entertain the view that constituents of constituents are constituents of the same state. So, on that view, for the state of Gitte's loving Mo the heart of Gitte is also a constituent of this state. A way to effectively deal with this view would be to refine the constituents principle and to enrich our models with a dependency ordering on O, the set of objects.

I think that our intuitive understanding of the identity substitution and of composing substitutions is fully in accordance with the fourth substitution principle.

Note that in the principles we only talk about substituting objects and not about substituting individual occurrences of an object by possibly different objects. The reason for not considering a more refined substitution mechanism is that otherwise we would have to make clear what exactly occurrences are. It might perhaps be possible to do this, but it would be an extra complication.

Another point is that we can ask ourselves if there is not a more primitive operation in which substitution could be expressed. Fine [2, p. 27] discusses the question whether substitution should be understood in terms of a structural operation. However, he considers the notion of substitution of a lower logical type. But even if Fine would be wrong in this respect, this would not make the substitution principles less credible.

With respect to the factuality principle, note that the principle does not say that there are states that do not obtain. So, it is also true for hard actualists, who hold that the only states of affairs that exist are those that obtain.

The considerations above make it plausible that any relation can be modeled by a substitution model. However, this does not mean that such models are always completely satisfactory. In some cases the model might leave out essential aspects of the relation, like the (relative) order of relata in biased relations. Or consider the variadic relation *standing in a line*, where we allow an object to have multiple occurrences. Then how could we get from a line of length n to a line of length $n-1$? The state transition graph $\langle S, E\rangle$ with S the states of the relation, and E the set of pairs (s, s') such that s' can be obtained from s by substitution contains isolated islands of states. To get a more satisfactory model, we perhaps need to consider *subtraction* of objects from a state as a complementary basic operation. This will be a topic for further inquiry.

6.2. *Justifying Positional Representations*

I do not claim to have obtained a complete understanding of the essence of relations. I cannot even give a satisfactory definition. In this section, we limit ourselves to *simple* relations:

DEFINITION 6.1. We call a relation $\Re$ a *simple relation* if

1. $\Re$ satisfies the constituents principle, the substitution principles, and the factuality principle;
2. $\Re$ has an initial state, i.e. a state from which any state of the relation can be obtained by substitution.

We define the *degree of* $\Re$ as the maximum number of objects per state.

Claim 1. Any simple relation can be modeled by a unique simple substitution model.

Note that we do not claim that the substitution model models all aspects of the relation.

DEFINITION 6.2. Given a simple relation $\Re$. Then we call a positional model $\mathcal{M}$ a *natural model for* $\Re$ if $\mathcal{M}$ corresponds to the substitution model for $\Re$.

By Theorem 4.3 we have the following result:

Claim 2. Any simple relation of finite degree has a natural positional model of the same degree that is unique, modulo positional variants.

The last claim gives a justification for using positional representations for a large class of relations. I consider this claim as a main result of this paper. It is a modest result, but it is also very fundamental. Perhaps it looks completely trivial, but I do not think it is. The fact that we continually use positional representations for relations is in itself no evidence for their validity.

What is also worth noting is that for a simple relation of finite degree, a natural positional model (modulo positional variants) of the same degree has exactly the same information content as its substitution model.

I do not claim that the positional model for a simple relation is unique in an absolute sense. The model is defined in terms of our definition of correspondence between substitution models and positional models. Other definitions of correspondence might give other positional models. But the given definition strikes me as the most natural one. It seems plausible that if a relation can adequately be modeled by a substitution model and a positional model, then these models correspond.

For simple relations, substitution of objects in any state yields by definition exactly one state of the same relation. For the class of relations satisfying the weaker substitution principle that says that any substitution of objects in any state yields *no more than* one state of the same relation, another kind of models might be more appropriate, namely substitution models with a *partial* function Σ and positional models with a *partial* function Γ. For these kind of models, a theory very similar to the one given in this paper could be developed. In particular, we would get results similar to Claims 1 and 2.

Claim 2 says nothing about the ontological status of positions. But can we say something about the ontology of positions? Can we deny them a place in the "fundamental furniture of the universe"? I think we have no reason to grant them such an honorable place, but I also don't think that we have as yet a decisive argument why they cannot belong to this furniture. As long as you do not consider positions as things occupied by objects within the states, their fundamental existence seems hard to disprove. It is perhaps also not contradictory to claim that objects occupy a kind of *internal* positions within the states. For example, you could argue that the states of the adjacency relation have two internal positions, but that these positions have *no determinate identity*. The indeterminacy of the positions could be compared with the quantum-theoretic ontic indeterminacy of the electrons of a He atom (cf. [4]). For a cyclic relation the internal positions might be argued to be partially indiscernible like the unlabeled vertices of a regular mathematical polygon are (see Figure 3). Perhaps for arbitrary positional structures a similar defense can be given for a kind of internal (super)positions. But we do well to realize that the adequacy of positional models does not imply that the constituents of a state really need to occupy some kind of position. I think we can be

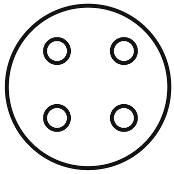


Figure 3. Could states have internal positions without identity?

perfectly happy with assigning to positions nothing more than the status of an innocent mental construction, *unless* we have reason to assume that there is a "real" relation that can only be adequately modeled by a positional model that does *not* respect substitution. Then we still have something to explain.

Although the two claims in this section are about relations, they could be generalized to any entity that satisfies the principles of the previous section. For an extreme nominalist who does not believe in relations, substitution might also still be a notion that makes sense, and he might find positional models useful. Substitution definitely has a wider scope of application than just states of affairs. I see no objection for applying substitution also to situations and propositions, and for using positional representations for them.

6.3. *Epistemological Aspects of Relations:*

Substitution models seem to be more primitive than positional models. So why would we use positional models? I think there are good reasons for this. The strength of positional models (and of directional models) is that they provide us a well-organized framework for all states, and the possibility to refer efficiently to each individual state. Natural languages like English obviously take advantage of this kind of representation. As far as I can see, almost any linguistic relational statement in English makes explicitly or implicitly use of positions.

With respect to determining the structure of simple relations we have the following result:

Claim 3. **For any natural positional model of finite degree, the positional structure can in principle be determined by a finite number of substitutions.**

The claim is a direct consequence of Theorem 4.6, Theorem 5.8, and the presupposition that in principle one can determine for any pair of substitutions whether they result in the same state. I don't know if this is true in practice. Also, I do not know to what extent it is needed for a practical understanding of a relation to know explicitly or tacitly its positional structure.

How do we "learn" relations? I have as yet no answer to this question. I have not investigated what kind of empirical research has been done on this subject, but it would be very interesting to find answers to the following questions:

Do we learn relations by substitution, by abstraction, by positional representations or via processes with a completely different logic? Do we learn complex relations by applying operations like conjunction to simple relations? In processing perception, do we use neutral rather than biased relations? How do small children learn relations, and how animals and other organisms? What is the role of language in learning relations? Do all natural languages use directional and positional representations of relations?

Answers to these questions might deepen our insight in fundamental aspects of the way we understand and represent the world. In addition, they might suggest new learning programs or new ways for learning Artificial Intelligence systems to "discover" and handle relations. For example, if we can implement a general notion of substitution in an AI system, then it might perhaps be possible to learn the system a variety of relations by examples.

7. Conclusions

My aim in this paper was to develop models for relations that would give a better understanding of the essence of relations. To what extent did we accomplish this goal? Let me recapitulate the main results. We developed mathematical models for the views on relations that Fine described in his paper "Neutral Relations". We proved that the *simple substitution frames* correspond in a natural way with the *simple positional frames*, and that if the frames have the same finite degree, then the correspondence is one-to-one, modulo positional variants. Further, I argued that the simple substitution models adequately model a large class of relations, which we called the *simple relations*.

The results can be interpreted in two directions. They provide support for the antipositionalist view. They show—with the proviso that adequate substitution models and adequate positional models for a relation correspond—that for simple relations of finite degree the primitive notion of substitution has the same expressive power as the use of positions, modulo positional variants. On the other hand, the positionalist can claim for the same class of relations that the results show that his use of positions is innocent.

One of the objections Fine raised against the positionalist view was that it could not handle strictly symmetric relations. But, as I argued in Section 2.4, this is only an argument against a positionalist who would claim that objects *occupy* positions in relational states. If a positionalist only claims a *mediating* role for positions, then this argument of Fine does not apply.

Do we have arguments to prefer one view over the other? A strong argument for the antipositionalist view is that it is based on the very general notion of substitution, a primitive kind of operation. I don't think that for a positional approach a similar claim can be made. The positionalist could point out that there are positional models that do not correspond to substitution models, but the converse is also true. Moreover, for positional models that do not correspond to substitution models it seems highly unlikely that they could adequately model “real” relations. These models probably have no metaphysical significance at all for relations. In defense of a positional approach, it may be claimed that positional representations are very natural and practical. But that does not mean that they are very basic. In fact it is not an argument for a positionalist view, but only for the use of a certain representation. I conclude that the results of this paper give extra support to the antipositionalist view, but that they also give a justification for the use of positional representations for relations.

The models and theory developed in this paper can have more use than their contribution in reaching the conclusion I drew in the previous paragraph. The models can be useful tools for analyzing relations, and also for empirical research on how we “learn” relations. With respect to learnability, I showed in Section 6.3 that the positional structure of simple relations of finite degree can in principle be learned by a finite number of substitutions. Also further theoretical study of substitution models might be helpful to deepen our understanding of the structure of relations. We already mentioned a promising approach to defining objects of states in substitution models in terms of ultrafilters.

I want to conclude with a remark about the research approach I have taken in this paper. I started with questions about the structure of “real” relations. To find answers, I developed mathematical models that highlighted certain aspects of relations. By playing with the models and studying properties of them, ideas for new models emerged and connections between them became clear. Subsequently, these insights could be translated back to characteristic properties of “real” relations. I think that this type of approach might be fruitful also for other areas of metaphysics. Unfortunately, in metaphysics there is still hardly any consensus about almost anything. I would say that ontological claims about our world have value only if they are accompanied by very strong arguments.

Developing and analyzing mathematical models or axiomatic systems might provide the right level of certainty in this respect. Doing physics without mathematics is hardly thinkable, but strangely enough, many seem to think that metaphysics can largely do without it. A reason to refrain from using rigorous mathematical methods for metaphysics might be that such methods are considered too difficult to handle for this discipline. This might be true in some cases, but if we want to take metaphysics seriously, I see no other way.

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The idea to develop mathematical models for the views on relations as presented by Kit Fine comes from Albert Visser. Comments from Kit Fine on my master's thesis about modeling relations gave impetus to writing this paper. Discussions with Rosja Mastop contributed to a better understanding of the subject. Questions from Vincent van Oostrom and his detailed reading of a semi-final version of this paper helped to get many parts in a better shape. I thank them very much for their support and contribution.

NOTES

[1] We say that $f =_X g$ if $f{\restriction}X = g{\restriction}X$, i.e. f restricted to X is equal to g restricted to X.

[2] $A \,\Delta\, A' = (A - A') \cup (A' - A)$, the symmetric difference of A and A'.

[3] For a better appreciation of the definition of substitution frames it might be useful to look at [3, pp. 23–25], where more frames are presented that reflect ideas of the antipositionalist view.

[4] This approach to define the objects of a state corresponds to a remark Fine made in [2, p. 26, note 15].

[5] As suggested by Albert Visser, it might be possible to give a general definition of the objects of s as the ultrafilters extending the collection of object-domains of s. The details of this approach are under development.

[6] $f[X] = \{f(x) \mid x \in X\}$, the image of X under f.

[7] $f[a \mapsto b]$ denotes the function defined by $f[a \mapsto b](x) = b$ if $x = a$; $f(x)$ otherwise.

[8] We use Γabc as an abbreviation for $\Gamma\left(\begin{pmatrix} p_1 & p_2 & p_3 \\ a & b & c \end{pmatrix}\right)$.

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