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MODALITY, QUANTIFICATION, AND MANY VLACH-OPERATORS

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ABSTRACT. Consider two standard quantified modal languages **A** and **P** whose vocabularies comprise the identity predicate and the existence predicate, each endowed with a standard S5 Kripke semantics where the models have a distinguished actual world, which differ only in that the quantifiers of **A** are actualist while those of **P** are possibilist. Is it possible to enrich these languages in the same manner, in a non-trivial way, so that the two resulting languages are equally expressive—i.e., so that for each sentence of one language there is a sentence of the other language such that given any model, the former sentence is true at the actual world of the model iff the latter is? Forbes (1989) shows that this can be done by adding to both languages a pair of sentential operators called *Vlach-operators*, and imposing a syntactic restriction on their occurrences in formulas. As Forbes himself recognizes, this restriction is somewhat artificial. The first result I establish in this paper is that one gets sameness of expressivity by introducing infinitely many distinct pairs of indexed Vlach-operators. I then study the effect of adding to our enriched modal languages a rigid actuality operator. Finally, I discuss another means of enriching both languages which makes them expressively equivalent, one that exploits devices introduced in Peacocke (1978). Forbes himself mentions that option but does not prove that the resulting languages are equally expressive. I do, and I also compare the Peacockian and the Vlachian methods. In due course, I introduce an alternative notion of expressivity and I compare the Peacockian and the Vlachian languages in terms of that other notion.

KEY WORDS: modality, quantification, actualism, possibilism, Vlach-operators

1. Introduction

Consider two standard quantified modal languages **A** and **P** whose vocabularies comprise the identity predicate and the existence predicate, each endowed with a standard S5 Kripke semantics where the models have a distinguished actual world, which differ only in that the quantifiers of **A** are actualist while those of **P** are possibilist. Is it possible to enrich these languages in the same manner, in a non-trivial way (in particular, without enriching the actualist language with possibilist quantifiers), so that the two resulting languages are equally expressive—i.e. so that for each sentence of one language there is a sentence of the other language such that given any model, the former sentence is true at the actual world of the model iff the latter is? We know from Forbes (1989) that this can be done by adding to both languages a pair of sentential operators called *Vlach-operators* and imposing a syntactic restriction on their occurrences in formulas.

Yet, as Forbes himself recognizes, this restriction is somewhat artificial, and if we drop it the resulting languages fail to be equally expressive. The first result I establish in this paper is that one get sameness of expressivity by introducing infinitely many distinct pairs of indexed Vlach-operators without any artificial restriction. I then study the effect of adding to our enriched modal languages a rigid actuality operator. Finally, I discuss another means of enriching both languages which makes them expressively equivalent, one that exploits devices introduced in Peacocke (1978). Forbes himself mentions that option but does not prove that the resulting languages are equally expressive. I do, and I also compare the Peacockian and the Vlachian methods. In due course, I introduce an alternative notion of expressivity and I compare the Peacockian and the Vlachian languages in terms of that other notion.

The distinction between possibilist and actualist quantification mentioned above is a distinction between two kinds of formal semantics for expressions belonging to certain formal languages. This purely mathematical distinction is supposed to model a distinction between two types of universal and existential quantification which may be expressed by natural language phrases like ‘for all’ and ‘there is,’ which philosophers call possibilist and actualist, respectively.

Let me call *actualist* one who takes it that the only basic, primitive concepts of existential and universal quantification are those of actualist existential and universal quantification. An actualist may reject possibilist existential and universal quantification as unintelligible. But she may also take them to be legitimate. Let me call such an actualist a *liberal actualist*. A liberal actualist just holds the following view: there are meaningful sentences containing possibilist quantifiers, and given any such sentence S, there is a sentence S' which does not (which may or may not contain actualist quantifiers) and which is such that what S says consists in nothing more than what S' says. (A liberal actualist may wish to have at hand a general recipe for analyzing any meaningful sentence containing possibilist quantifiers into a sentence which does not, but she need not: she may be happy with case by case translations.)

The Vlach-operators may have no counterparts in English or in any other natural language. Yet my view is that one can legitimately add such operators to languages which contain modal expressions like 'possibly' and 'necessarily' and quantifiers which range over individuals, be they possibilist or actualist. On the other hand, I take it that the use of the Vlach-operators does not commit one to quantification over, or reference to, mere *possibilia*, more precisely to merely possible worlds. Forbes holds that the function of Peacocke's indexing devices is merely to effect scope-indication (see the discussion on pp. 90 ff), and I wish to hold the same for the Vlach-operators.[1] If I am right on all this, then non-liberal actualists are wrong, and the present study will obviously be useful to liberal actualists.

The plan of the paper is the following. Section 2 is devoted to some basic definitions. In Section 3 I present expressivity results about **A**, **P** and the languages obtained from them by adding the Vlach-operators, which can be found in Forbes (1989). In Section 4 I show that enriching both **A** and **P** with infinitely many distinct pairs of indexed Vlach-operators yields two languages with equal expressive power. In Section 5 I study the effect of adding to our enriched modal languages a rigid actuality operator and I introduce and exploit an alternative notion of expressivity which is specially relevant in that context. And finally, in Section 6 I turn to the Peacockian devices.

2. Basic Definitions

The languages we are dealing with in this paper divide into two classes: the *actualist* languages and the *possibilist* languages. Every actualist language is an extension of the basic actualist language **A**, and every possibilist language an extension of the basic possibilist language **P**. Each actualist language is obtained by enriching **A** with certain sentential operators, and each possiblist language is obtained by enriching **P** in the same way.

A is a standard first-order modal language whose vocabulary comprises individual constants, the identity predicate = and the existence predicate E. We assume that it has $\wedge$ (conjunction) and $\neg$ (negation) as sole truth-functional connectives, $\forall$ (universal quantification) as sole quantifier, and $\Box$ (necessity) as sole modal operator. **P** differs from **A** only in that its universal quantifier is Π and not $\forall$. Standard abbreviations, notational conventions and vocabulary will be used throughout the paper. We will take $\exists$ as short for $\neg\forall\neg$, Σ as short for $\neg\Pi\neg$, and $\Diamond$ as short for $\neg\Box\neg$.

Each language, actualist or possibilist, is interpreted by means of models I simply call *models*. A model is any quadruple $\langle \alpha, W, D, I \rangle$, where W (worlds) is a non-empty set, α (the actual world) is in W, D (domain) is a function which takes any world w into the set D_w (the world's domain) such that $\bigcup_{w \in W} D_w$ is not empty, and I (interpretation) a function which takes any constant into a member of $\bigcup_{w \in W} D_w$ (the reference of that constant) and any n-place predicate distinct from $=$ and world into a set of n-tuples taken from $\bigcup_{w \in W} D_w$ (the extension of that predicate at that world)–with the requirement that for every world w, $I(E, w) = D_w$. An *assignment* to the variables in model $\langle \alpha, W, D, I \rangle$ is a function which assigns to each variable an element of $\bigcup_{w \in W} D_w$. Where x is a variable, two assignments ρ and μ are *x-alternatives* iff ρ and μ take the same values for all variables, except possibly for x.

The truth-predicate $\vDash$ for **A** is defined in the following way. Let $M = \langle \alpha, W, D, I \rangle$ be a model, w a world of that model, ρ an assignment to the variables in M, and A an **A**-formula. We define '$(M, w) \vDash_\rho A$' recursively as follows ($\rho I(t)$ is used for $\rho(t)$ if t is a variable, and for $I(t)$ if t is a constant; I will follow that notational convention in the rest of the paper):

- $(M, w) \vDash_\rho F t_1 ... t_n$ iff $\langle \rho I(t_1), ..., \rho I(t_n) \rangle \in I(F, w)$ for F any n-ary predicate distinct from =;
- $(M, w) \vDash_\rho t_1 = t_2$ iff $\rho I(t_1) = \rho I(t_2)$;
- $(M, w) \vDash_\rho \neg A$ iff $(M, w) \nvDash_\rho A$;
- $(M, w) \vDash_\rho A \wedge B$ iff $(M, w) \vDash_\rho A$ and $(M, w) \vDash_\rho B$;
- $(M, w) \vDash_\rho \forall x A$ iff $(M, w) \vDash_\mu A$ for every x-alternative μ of ρ such that $\mu(x) \in D_w$;
- $(M, w) \vDash_\rho \Box A$ iff for every $v \in W$, $(M, v) \vDash_\rho A$.[2]

Given a model $M = \langle \alpha, W, D, I \rangle$, an **A**-sentence (i.e. a closed **A**-formula) A is *true at world* w in M iff $(M, w) \vDash_\rho A$ for every (or equivalently, for some) assignment ρ to the variables in M; and it is *true* in M iff it is true at α in M.

The truth-predicate $\Vdash$ for **P** is defined in exactly the same way, except that the clause for $\forall$ is replaced by:

- $(M, w) \Vdash_\rho \Pi x A$ iff $(M, w) \Vdash_\mu A$ for every x-alternative μ of ρ.

Truth-at-a-world-in-a-model and truth-in-a-model for **P**-sentences are defined in the same way as above.

For each of the other languages we are going to deal with in this paper, a (unique) notion of truth-at-a-world-in-a-model will be defined, and in terms of that notion, truth-in-a-model for sentences of that language will be defined in the same way as for **A** and **P**, by reference to the actual world of the model.

Let $\mathcal{L}$ and $\mathcal{L}'$ be any two languages of interest to us. We adopt the following definitions:

- For A an $\mathcal{L}'$-sentence and B an $\mathcal{L}$-sentence: A is $\mathcal{L}'/\mathcal{L}$ *expressible* by B iff A and B are true in exactly the same models;
- For A an $\mathcal{L}'$-sentence: A is *expressible* in $\mathcal{L}$ iff there is an $\mathcal{L}$-sentence B such that A is $\mathcal{L}'/\mathcal{L}$ expressible by B;
- $\mathcal{L}$ is *at least as expressive* as $\mathcal{L}'$ iff every $\mathcal{L}'$-sentence is expressible in $\mathcal{L}$;
- $\mathcal{L}$ is *more expressive* than $\mathcal{L}'$ iff $\mathcal{L}$ is at least as expressive as $\mathcal{L}'$ but not *vice versa*;
- $\mathcal{L}$ and $\mathcal{L}'$ are *equally expressive* iff each is at least as expressive as the other.

I will use '$\mathcal{L} \succ \mathcal{L}'$' for '$\mathcal{L}$ is more expressive than $\mathcal{L}'$' and '$\mathcal{L} \approx \mathcal{L}'$' for '$\mathcal{L}$ and $\mathcal{L}'$ are equally expressive,' and I will follow the standard convention of writing expressions like '$\mathcal{L} \approx \mathcal{L}' \succ \mathcal{L}''$...' to mean '$\mathcal{L} \approx \mathcal{L}'$, and $\mathcal{L}' \succ \mathcal{L}''$, and ...'. Alternative expressivity concepts will be introduced in Section 5.

3. Adding The Vlach-Operators ↑ and ↓: Forbes (1989)

It can be shown that[3]:

PROPOSITION 1. $\mathbf{P} \succ \mathbf{A}$.

For one thing, **A** is not at least as expressive as **P**: e.g. the **P**-sentence $\Sigma x \neg Ex$ is not expressible in **A**. And for another thing, **P** is at least as expressive as **A**. For take any **A**-sentence A and replace in it each subformula $\forall xB$ by $\Pi x(Ex \supset B)$. The result is a **P**-sentence, and it can be shown that it is true in a model iff A is.

Is it possible to enrich both **A** and **P** in a non-trivial way (i.e. without enriching **A** with possibilist quantifiers), with the same vocabulary, in such a way that the resulting languages are equally expressive? Forbes (1989) gives a positive answer.

The tools used by Forbes are the *Vlach-operators* ↑ and ↓.[4] These are sentential operators: putting any one of them in front of a formula results in a formula. In a nutshell, the job of ↑ is to store the current world of evaluation, and the job of ↓ is to make the currently stored world the world of evaluation. Suppose we want to express the proposition that there are two worlds such that all the non-existents of one exist in the other. This can be done in **P** enriched with the Vlach-operators as follows: $\Diamond \uparrow \Diamond \Pi x (\neg Ex \supset \downarrow Ex)$. That formula says: there is a world w (here ↑ stores w) and a world v such that the following holds at v: given any possible object, if that object fails to exist, then (here $\downarrow$ makes the stored world the world of evaluation) it is true at w that this object exists–which is equivalent to what we wanted to say.

Let $\mathbf{A}_v$ and $\mathbf{P}_v$ be, respectively, $\mathbf{A}$ and $\mathbf{P}$ enriched with the Vlach-operators.

The truth-predicate $\vDash$ for $\mathbf{A}_v$ is defined in the following way. We define '$(M, v, w) \vDash_\rho A$', for $M = \langle \alpha, W, D, I \rangle$ a model, v (the store world) and w (the world of evaluation) worlds of that model, ρ an assignment to the variables in M, and A an $\mathbf{A_v}$-formula, recursively as follows:

- $(M, v, w) \vDash_\rho Ft_1 ... t_n$ iff $\langle \rho I(t_1), ..., \rho I(t_n) \rangle \in I(F, w)$ for F any n-ary predicate distinct from =;
- $(M, v, w) \vDash_\rho t_1 = t_2$ iff $\rho I(t_1) = \rho I(t_2)$;
- $(M, v, w) \vDash_\rho \neg A$ iff $(M, v, w) \nvDash_\rho A$;
- $(M, v, w) \vDash_\rho A \wedge B$ iff $(M, v, w) \vDash_\rho A$ and $(M, v, w) \vDash_\rho B$;
- $(M, v, w) \vDash_\rho \forall x A$ iff $(M, v, w) \vDash_\mu A$ for every x-alternative μ of ρ such that $\mu(x) \in D_v$;
- $(M, v, w) \vDash_\rho \Box A$ iff for every $w' \in W$, $(M, v, w') \vDash_\rho A$;
- $(M, v, w) \vDash_\rho \uparrow A$ iff $(M, w, w) \vDash_\rho A$;
- $(M, v, w) \vDash_\rho \downarrow A$ iff $(M, v, v) \vDash_\rho A$.

The only special clauses are the ones for the Vlach-operators. The truth-predicate $\Vdash$ for $\mathbf{P}_v$ is defined in exactly the same way, except for the clause for quantification of course, which has to be modified in the obvious way.

The job of the Vlach-operators is to store and then retrieve. So any formula of our enriched languages the evaluation of which involves storing without retrieving or *vice versa* is, as far as the nature of the Vlach-operators is concerned, a deviant one. Say that in a formula, an occurrence o of $\uparrow$ *binds* an occurrence o' of $\downarrow$ iff (a) o' is within the scope of o, and (b) there is no occurrence o'' of $\uparrow$ within the scope of o and having o' within its scope. Let us then say that a formula is *nice* iff in that formula, any occurrence of $\uparrow$ binds an occurrence of $\downarrow$, and any occurrence of $\downarrow$ is bound by an occurrence of $\uparrow$. In the course of evaluating a nice formula, no useless storing occurs, and retrieving is always of a world which has previously been stored. We define the sentences of our two enriched languages by requiring that they be nice. That is to say, a *sentence* of any of the two languages is defined as a formula of that language which is both closed and nice.

An $\mathbf{A}_v$-sentence (resp. a $\mathbf{P}_v$-sentence) A is then said to be *true at a world* w in model M iff $(M, w, w) \vDash_\rho A$ (resp. $(M, w, w) \Vdash_\rho A$) for every (or equivalently, for some) assignment ρ to the variables in M. *Truth-in-a-model* for sentences of both languages is defined by reference to the actual world of the model.

Let us now turn to comparisons of expressive power.

Take any **P**-sentence A and replace in it each sub-formula ΠxB by $\uparrow \Box\forall x \downarrow B$. The result is an $\mathbf{A}_v$-sentence, and it is true in a model iff A is. On the other hand, the $\mathbf{A}_v$-sentence $\Diamond \uparrow \Diamond\forall x \downarrow \neg Ex$, which says that there are two worlds such that all the existents of one fail to exist in the other, is not expressible in **P**. So we have:

PROPOSITION 2. $\mathbf{A}_v \succ \mathbf{P}$.

One may think that the two enriched languages are equally expressive. But in fact,

PROPOSITION 3. $\mathbf{P}_v \succ \mathbf{A}_v$.

For on one hand, replacing each sub-formula $\forall xB$ in an $\mathbf{A}_v$-sentence by $\Pi x(Ex \supset B)$ results in a $\mathbf{P}_v$-sentence which is true in exactly the same modal models as the original sentence. And on the other hand, the $\mathbf{P}_v$-sentence $\Diamond \uparrow \Diamond\Pi x(\neg Ex \supset\downarrow Ex)$, which says that there are two worlds such that all the non-existents of one exist in the other, is not expressible in $\mathbf{A}_v$.[5]

Thus, adding the Vlach-operators to both basic languages the way we did does not give us what we were looking for, namely two languages with the same expressive power. But there is a means of achieving sameness of expressivity, the one proposed by Forbes I previously mentioned, which consists in restricting the formation rules corresponding to the Vlach-operators. For the restricted actualist language, which we shall call $\mathbf{A}_v^{®}$, there is a unique formation rule for the two operators, which says that if A is a formula, then so is $\uparrow \Box\forall x \downarrow A$. For the possibilist language $\mathbf{P}_v^{®}$, the rule is of course that if A is a formula, then so is $\uparrow \Box\Pi x \downarrow A$. It can then be shown that:

PROPOSITION 4. $\mathbf{A}_v^{®} \approx \mathbf{P}_v^{®}$.

Interestingly, the new possibilist language and the basic one are equally expressive: introducing the Vlach-operators in **P** with the proposed restriction has no effect. So, $\mathbf{A}_v^{®}$'s expressive power exactly matches **P**'s.

This final result is interesting, and gives us what we wanted. But, as Forbes himself recognizes, the proposed syntactic restrictions are somewhat artificial. In fact, there are perfectly meaningful things one can express by means of sentences containing the Vlach-operators which violate the restrictions. The $\mathbf{P}_v$-sentence $\Diamond \uparrow \Diamond\Pi x(\neg Ex \supset\downarrow Ex)$ we previously met is a case in point. As I will show in the next section, there is a more natural way of achieving matching of expressive power.

4. Many Vlach-operators

The idea is to enrich both languages with infinitely many pairs of operators $\langle \uparrow^1, \downarrow_1 \rangle$, $\langle \uparrow^2, \downarrow_2 \rangle$, ... which semantically behave like the pair $\langle \uparrow, \downarrow \rangle$, and which I will also call *Vlach-operators*. In a nutshell, $\uparrow^n$ has the effect of storing the current world of evaluation at a certain place in a list, and $\downarrow_n$ that of putting the world which is currently at that place in the position of world of evaluation–each pair $\langle \uparrow^n, \downarrow_n \rangle$ having its own designated place in the list in which to store and from which to retrieve.[6]

Let us thus enrich both languages **P** and **A** with an "up" Vlach-operator $\uparrow^n$ and a "down" Vlach-operator $\downarrow_n$ for each natural number n. The formation rules for formulas corresponding to these operators are unrestricted, and the resulting languages are called $\mathbf{P}_V$ and $\mathbf{A}_V$, respectively.

Let us define a *store list* of a model as an ω-tuple of worlds of that model. Where s is a store list, w a world and n a natural number, $s^{n \rightarrow w}$ is the store list which results from replacing the nth item in s by w, and $s(n)$ is the nth item in s. Where w is a world, $[w]$ is the store list whose members are all w.

We define '$(M, s, w) \vDash_\rho A$,' for $M = \langle \alpha, W, D, I \rangle$ a model, s (the store list) and w (the world of evaluation) a store list and a world of that model, respectively, ρ an assignment to the variables in M, and A an $\mathbf{A}_V$-formula, recursively as follows:

- $(M, s, w) \vDash_\rho F t_1 ... t_n$ iff $\langle \rho I(t_1), ..., \rho I(t_n) \rangle \in I(F, w)$ for F any n-ary predicate distinct from =;
- $(M, s, w) \vDash_\rho t_1 = t_2$ iff $\rho I(t_1) = \rho I(t_2)$;
- $(M, s, w) \vDash_\rho \neg A$ iff $(M, s, w) \nvDash_\rho A$;
- $(M, s, w) \vDash_\rho A \wedge B$ iff $(M, s, w) \vDash_\rho A$ and $(M, s, w) \vDash_\rho B$;
- $(M, s, w) \vDash_\rho \forall x A$ iff $(M, s, w) \vDash_\mu A$ for every x-alternative μ of ρ such that $\mu(x) \in D_v$;
- $(M, s, w) \vDash_\rho \Box A$ iff for every $w' \in W$, $(M, s, w') \vDash_\rho A$;
- $(M, s, w) \vDash_\rho \uparrow^n A$ iff $(M, s^{n \rightarrow w}, w) \vDash_\rho A$;
- $(M, s, w) \vDash_\rho \downarrow_n A$ iff $(M, s, s(n)) \vDash_\rho A$.

The only special clauses are the ones for the Vlach-operators. The truth-predicate $\Vdash$ for $\mathbf{P}_V$ is defined in exactly the same way, except for the clause for quantification which has to be modified in the obvious way.

We define the notion of an occurrence of $\uparrow^n$ *binding* an occurrence of $\downarrow_n$ in the same way as we did for $\uparrow$ and $\downarrow$, and we define a *nice* formula of any of our two new languages as a formula of that language in which, for every natural number n, any occurrence of $\uparrow^n$ binds an occurrence of $\downarrow_n$, and any occurrence of $\downarrow_n$ is bound by an occurrence of $\uparrow^n$. A *sentence* of any of the two languages is defined as a formula of that language which is both closed and nice.

An $\mathbf{A}_V$-sentence (resp. a $\mathbf{P}_V$-sentence) A is said to be *true at a world w* in model M iff $(M, [w], w) \vDash_\rho A$ (resp. $(M, [w], w) \Vdash_\rho A$) for every (or equivalently, for some) assignment ρ to the variables in M. *Truth-in-a-model* is, as before, defined by reference to the actual world of the model.

Let us now turn to the main result of this section.

Define the translation function $^\circlearrowright$ from the formulas of $\mathbf{A}_V$ to the formulas of $\mathbf{P}_V$ recursively as follows:

- $A^\circlearrowright$ is A for A atomic;
- $[\neg A]^\circlearrowright$ is $\neg A^\circlearrowright$;
- $[A \wedge B]^\circlearrowright$ is $A^\circlearrowright \wedge B^\circlearrowright$;
- $[\Box A]^\circlearrowright$ is $\Box A^\circlearrowright$;
- $[\forall x A]^\circlearrowright$ is $\Pi x(Ex \supset A^\circlearrowright)$;
- $[\uparrow^n A]^\circlearrowright$ is $\uparrow^n A^\circlearrowright$;
- $[\downarrow_n A]^\circlearrowright$ is $\downarrow_n A^\circlearrowright$.

The translation takes any $\mathbf{A}_V$-sentence into a $\mathbf{P}_V$-sentence. It is easy to show that given any $\mathbf{A}_V$-sentence A, A is true at a world in a model iff A is true at that world in that model.

An occurrence of $\downarrow_n$ in a formula which is bound by no occurrence of $\uparrow^n$ will be said to be *free* in that formula. Define now the translation function $^\circlearrowleft$ from the formulas of $\mathbf{P}_V$ to the formulas of $\mathbf{A}_V$ as follows:

- $A^\circlearrowleft$ is A for A atomic;
- $[\neg A]^\circlearrowleft$ is $\neg A^\circlearrowleft$;
- $[A \wedge B]^\circlearrowleft$ is $A^\circlearrowleft \wedge B^\circlearrowleft$;
- $[\Box A]^\circlearrowleft$ is $\Box A^\circlearrowleft$;
- $[\Pi x A]^\circlearrowleft$ is $\uparrow^n \Box \forall x \downarrow_n A^\circlearrowleft$, where n is the first natural number m such that $A^\circlearrowleft$ contains no free occurrence of $\downarrow_m$;
- $[\uparrow^n A]^\circlearrowleft$ is $\uparrow^n A^\circlearrowleft$;
- $[\downarrow_n A]^\circlearrowleft$ is $\downarrow_n A^\circlearrowleft$.

The translation takes any $\mathbf{P}_V$-sentence into an $\mathbf{A}_V$-sentence. We have then:

LEMMA *Take any model $M = \langle \alpha, W, D, I \rangle$, any storing function s, any world w and any assignment ρ. Then given any $\mathbf{P}_V$-formula A, $(M, s, w) \Vdash_\rho A$ iff $(M, s, w) \vDash_\rho A^\circlearrowleft$.*

Proof. By induction on the length of the formulas. The only non-trivial step is the one involving universal quantification. Let A be any $\mathbf{P}_V$-formula of the form $\Pi x B$. $A^\circlearrowleft$ is thus $\uparrow^n \Box \forall x \downarrow_n B^\circlearrowleft$, where n is the first natural number m such that $B^{\cup}$ contains no free occurrence of $\downarrow_m$. The following six propositions are equivalent:

1. $(M, s, w) \vDash_\rho \uparrow^n \forall x \downarrow_n B^{\cup}$
2. for every world v and every x-alternative μ of ρ such that $\mu(x) \in D_v$, $(M, s^{n \to w}, w) \vDash_\mu B^{\cup}$
3. for every x-alternative μ of ρ, $(M, s^{n \to w}, w) \vDash_\mu B^{\cup}$
4. for every x-alternative μ of ρ, $(M, s, w) \vDash_\mu B^{\cup}$
5. for every x-alternative μ of ρ, $(M, s, w) \Vdash_\mu B$
6. $(M, s, w) \Vdash_\rho \Pi x B$.

1 $\Leftrightarrow$ 2 by the truth-clauses for $\uparrow^n$, $\Box$, $\forall$ and $\downarrow_n$; 2 $\Leftrightarrow$ 3 trivially; 3 $\Leftrightarrow$ 4 because $B^{\cup}$ contains no free occurrence of $\downarrow_n$; 4 $\Leftrightarrow$ 5 by induction hypothesis; and finally 5 $\Leftrightarrow$ 6 by the truth-clause for Π. It follows that $(M, s, w) \vDash_\rho A^{\cup}$ iff $(M, s, w) \Vdash_\rho A$. $\Box$

As a consequence:

THEOREM 1. $\mathbf{P}_V \approx \mathbf{A}_V$.

Thus, we have found a natural way of enriching both basic languages in the same manner so that the resulting languages have the same expressive power.

5. Actually

Let us enrich $\mathbf{A}_V$ and $\mathbf{P}_V$ with a (unary sentential) actuality operator @, call the new languages $\mathbf{A}_{V@}$ and $\mathbf{P}_{V@}$, respectively, and define the sentences of the resulting languages accordingly. Then supplement the definition of the truth-predicate for $\mathbf{A}_V$ by adding the following truth-clause:

- $(M, s, w) \vDash_\rho @A$ iff $(M, s, \alpha)_\rho A$,

and similarly for the possibilist language.

We then have:

THEOREM 2. $\mathbf{P}_{V@} \approx \mathbf{A}_{V@} \approx \mathbf{P}_V \approx \mathbf{A}_V$.

Proof. Theorem 1 says that $\mathbf{A}_V$ and $\mathbf{P}_V$ are equally expressive. As for the rest, for one thing it is easy to show that $\mathbf{A}_{V@}$ and $\mathbf{P}_{V@}$ are equally expressive: add the clauses '$[@A]^{\cup}$ is $@A^{\cup}$' and '$[@A]^{\cup}$ is $@A^{\cup}$' to the translation schemes of the previous section. In order to establish Theorem 2, and given that every $\mathbf{A}_V$-sentence is an $\mathbf{A}_{V@}$-sentence, it is then sufficient to show that $\mathbf{A}_V$ is at least as expressive as $\mathbf{A}_{V@}$. Let A be an $\mathbf{A}_{V@}$-sentence. If A does not contain @, then A is an $\mathbf{A}_V$-sentence and so is expressible in $\mathbf{A}_V$ Suppose A does contain @, and consider the formula $\uparrow^n B$, where n is the first natural number m such that A contains neither $\uparrow^m$ nor $\downarrow_m$, and B is the result of replacing in A all occurrences of @ by $\downarrow_n$. Then $\uparrow^n B$ is an $\mathbf{A}_V$-sentence, and A is true in a model iff $\uparrow^n B$ is. □

This result is remarkable, since $\mathbf{A}_{v@}$ is more expressive than $\mathbf{A}_v$, and $\mathbf{P}_{v@}$ than $\mathbf{P}_v$.[7]

In the first section of this paper, I defined various expressivity concepts by reference to the actual world. But one could so by reference to all worlds. Where $\mathcal{L}$ and $\mathcal{L}'$ are any two languages dealt with in this paper, A is an $\mathcal{L}'$-sentence and B an $\mathcal{L}$-sentence, say that A is $\mathcal{L}'/\mathcal{L}$ *expressible** by B iff for every model and every world of that model, A and B have the same truth-value at that world in that model. Then say that an $\mathcal{L}'$-sentence is *expressible** in $\mathcal{L}$ iff there is an $\mathcal{L}$-sentence such that the former is $\mathcal{L}'/\mathcal{L}$ expressible* by the latter, and define 'at least as expressive*', 'more expressive*' and 'equally expressive*' accordingly. (Following the notational convention for expressivity, I will use $\succ *$ for superior expressivity* and $\approx *$ for sameness of expressivity*.) To say that A is $\mathcal{L}'/\mathcal{L}$ expressible by B is to say that A and B say the same thing, in some sense; and to say that A is $\mathcal{L}'/\mathcal{L}$ expressible* by B is to say that A and B say the same thing, in some other (as we shall see, for some languages stronger) sense.

Theorem 2 fails if we replace $\approx$ by $\approx *$. As one can check from the proofs of Theorem 1 and Theorem 2, $\mathbf{A}_V$ and $\mathbf{P}_V$ are equally expressive*, as well as $\mathbf{A}_{V@}$ and $\mathbf{P}_{V@}$. But the simple formula $@Et$ (t a given constant), which is a sentence of both $\mathbf{A}_{V@}$ and $\mathbf{P}_{V@}$, is expressible* neither in $\mathbf{A}_V$ nor in $\mathbf{P}_V$. To sum up:

THEOREM 3. $\mathbf{P}_{V@} \approx * \mathbf{A}_{V@} \succ * \mathbf{P}_V \approx * \mathbf{A}_V$.

Say that a sentence of any of the languages we met so far is *weakly valid* iff the sentence is true in every model, and that it is *strongly valid* iff for every model, it is true at every world in that model. Weak and strong validity are coextensive for the languages we met so far which do not contain operator @, but not for the other languages. For instance, the formula $Et \supset @Et$ (t a given constant), which is both an $\mathbf{A}_{V@}$-sentence and a $\mathbf{P}_{V@}$-sentence, is weakly but not strongly valid.[8]

Unstarred expressibility goes hand in hand with weak validity, and starred expressibility with strong validity, in the following sense. Let $\mathcal{L}$

and $\mathcal{L}'$ be any two languages taken from those introduced so far, A an $\mathcal{L}'$-sentence and B an $\mathcal{L}$-sentence. Then: if A is $\mathcal{L}'/\mathcal{L}$ expressible by B, then A is weakly valid iff B is, and if A is $\mathcal{L}'/\mathcal{L}$ expressible* by B, then A is strongly valid iff B is. Of course, if A is $\mathcal{L}'/\mathcal{L}$ expressible* by B, then A is weakly valid iff B is. But there are cases where A is $\mathcal{L}'/\mathcal{L}$ expressible by B, and A is strongly valid but not B: take $\mathcal{L}' = \mathbf{A}_V$, $\mathcal{L} = \mathbf{A}_{V@}$, $A = \uparrow^1 (Et \supset\downarrow_1 Et)$, and $B = Et \supset @Et$ (t a given constant).

6. Peacocke's Devices

Adding infinitely many pairs of Vlach-operators to both **A** and **P** yields two languages which are equally expressive, and the same holds as well if in addition we enrich both languages with @. There is an alternative way of getting the same result mentioned by Forbes himself (see pp. 87 ff). It consists in enriching both **A** and **P** with @ and infinitely many indexed modal and actuality operators of the sort introduced in Peacocke (1978, pp. 485–6). Forbes claims, without proof, that the resulting languages are equally expressive. In this last section I establish the claim, and more generally I compare the languages dealt with in the previous section and the Peacockian languages in terms of both expressivity and expressivity*.

Let us enrich both **A** and **P** with @ and infinitely many indexed necessity operators $\Box_1$, $\Box_2$, ... and infinitely many indexed actuality operators $@_1$, $@_2$, ..., and call the resulting languages $\mathbf{A}_P$ and $\mathbf{P}_P$, respectively. These are the Peacockian languages discussed by Forbes. The truth-predicates for these new languages are just like the ones for $\mathbf{A}_{V@}$ and $\mathbf{P}_{V@}$: they are 5-place predicates with places for models, store lists, worlds, assignments to the variables and formulas. The truth-clauses for the new operators in $\mathbf{A}_P$ are the following:

- $(M, s, w) \vDash_\rho \Box_n A$ iff for every $w' \in W$, $(M, s^{n \to w'}, w') \vDash_\rho A$;
- $(M, s, w) \vDash_\rho @_n A$ iff $(M, s, s(n)) \vDash_\rho A$,

and similarly for the case of $\mathbf{P}_P$. The remaining truth-clauses are unchanged.

Say that in a formula of our Peacockian languages, an occurrence o of $\Box_n$ *binds* an occurrence o' of $@_n$ iff (a) o' is within the scope of o, and (b) there is no occurrence o'' of $\Box_n$ within the scope of o and having o' within its scope. Let us then say that a formula of these languages is *nice* iff in that formula, for every natural number n, any occurrence of $\Box_n$ binds an occurrence of $@_n$, and any occurrence of $@_n$ is bound by an occurrence of $\Box_n$. (I use the term 'nice' here again for a reason that already is, or will be, obvious to the reader.) In the course of evaluating a nice formula, no useless storing occurs, and retrieving is always of a world which has previously been stored, so nice formulas are not deviant as far as the function of the indexed operators is concerned. A *sentence* of any of the two languages is defined as a formula of that language which is both closed and nice. *Truth-at-a-world-in-a-model* and *truth-in-a-model* for sentences of these two languages are defined as before.

Two Vlachian languages defined in terms of those we already met will be useful. Say that an occurrence of an up-arrow in a Vlachian formula is *□-free* iff in that formula, it is not within the scope of any occurrence of □. We define language $\mathbf{A}^{-}_{V@}$ by stipulating that its formulas are those of $\mathbf{A}_{V@}$, and that its sentences are the $\mathbf{A}_{V@}$-sentences in which there are no □-free occurrences of up-arrows. We define the language $\mathbf{P}^{-}_{V@}$ in a similar way.

The first result I wish to establish is that $\mathbf{A}_{P}$ and $\mathbf{A}^{-}_{V@}$ are equally expressive*. Half of this is rather obvious. Take any $\mathbf{A}_{P}$-sentence and replace, for each natural number n such that the formula contains an operator indexed by n, each occurrence of $\Box_n$ by $\Box\uparrow^{n}$ and each occurrence of $@_n$ by $\downarrow_n$. Then the result is an $\mathbf{A}^{-}_{V@}$-sentence, and it is clear that for every model and world of that model, it is true at that world in that model iff the original sentence is. Thus $\mathbf{A}^{-}_{V@}$ is at least as expressive* as $\mathbf{A}_{P}$.

The other half is a little bit more difficult to prove. Say that an $\mathbf{A}^{-}_{V@}$-sentence is *regular* iff in that sentence, every occurrence of an up-arrow immediately follows an occurrence of □. Let us first establish that for every $\mathbf{A}^{-}_{V@}$-sentence A, there is a regular $\mathbf{A}^{-}_{V@}$-sentence $\rho(A)$ such that for every model and every world of that model, the two sentences have the same truth-value at that world in that model. First, two preliminary definitions:

- The *immediate content* of an occurrence of □ in an $\mathbf{A}^{-}_{V@}$-formula is the longest formula it has in its scope;
- An occurrence of □ in an $\mathbf{A}^{-}_{V@}$-formula is *pregnant* iff its immediate content contains □-free occurrences of up-arrows.

Let then A be an $\mathbf{A}^{-}_{V@}$-sentence, and let k be the number of pregnant occurrences of □ in it. If $k = 0$, then we let $\rho(A)$ be A itself. If $k \geq 1$, then let o_1, ..., o_k be these occurrences in order of appearance reading from left to right, and let M be the greatest natural number m such that A contains an occurrence of $\uparrow^{M}$. Then for each $1 \leq i \leq k$, we replace o_i's immediate content B by $\uparrow^{M+i} C$, where C is the result of deleting all the □-free occurrences of up-arrows in B and of replacing all the occurrences of down-arrows they bind by $\downarrow_{M+i}$. The resulting formula is an $\mathbf{A}^{-}_{V@}$-sentence, and we let $\rho(A)$ be identical to it. One can show that for every model and every world of that model, A and $\rho(A)$ have the same truth-value at that world in that model.

It is now easy to see that $\mathbf{A}_P$ is at least as expressive* as $\mathbf{A}^-_{V@}$. For take any $\mathbf{A}^-_{V@}$-sentence A. If A contains no Vlach-operator, then A is an $\mathbf{A}_P$-sentence, and so trivially, A is expressible* in $\mathbf{A}_P$. Suppose A does contain Vlach-operators. Then replace, for each natural number n such that A contains an operator indexed by n, each sequence $\Box \uparrow^n$ by $\Box_n$ and each occurrence of $\downarrow_n$ by $@_n$. The result is an $\mathbf{A}_P$-sentence, and for every model and world of that model, it is true at that world in that model iff A is.

So, as previously announced, $\mathbf{A}_P$ and $\mathbf{A}^-_{V@}$ are equally expressive*. An almost identical argument establishes that $\mathbf{P}_P$ and $\mathbf{P}^-_{V@}$ are equally expressive* too. So, $\mathbf{A}_{V@}$ is more expressive* than $\mathbf{A}_P$ ($\uparrow^1 \Diamond\forall x \downarrow_1 \neg Ex$ is not expressible* in $\mathbf{A}^-_{V@}$) and $\mathbf{P}_{V@}$ than $\mathbf{P}_P$ ($\uparrow^1 \Diamond\Pi x(Ex \supset\downarrow_1 \neg Ex)$ is not expressible* in $\mathbf{P}^-_{V@}$). Moreover, $\mathbf{P}_P$ is more expressive* than $\mathbf{A}_P$. For on one hand, one can use the proof that $\mathbf{P}_{V@}$ is at least as expressive* as $\mathbf{A}_{V@}$ to show that $\mathbf{P}^-_{V@}$ is at least as expressive* as $\mathbf{A}^-_{V@}$. And on the other hand, the $\mathbf{P}^-_{V@}$-sentence $\Sigma x\neg Ex$ is not expressible* in $\mathbf{A}^-_{V@}$. To sum up:

THEOREM 4. $\mathbf{P}_{V@} \approx^* \mathbf{A}_{V@} \succ^* \mathbf{P}_P \succ^* \mathbf{A}_P$.

Notice that $\mathbf{P}_P$ and $\mathbf{P}_V$ are incomparable as far as expressivity* is concerned, in the sense that neither is $\mathbf{P}_P$ at least as expressive* as $\mathbf{P}_V$, nor *vice versa*. For on one hand, the formula $\uparrow^1 \Diamond\Pi x(Ex \supset\downarrow_1 \neg Ex)$ we just met is a $\mathbf{P}_V$-sentence, and it is not expressible* in $\mathbf{P}_P$. And on the other hand, the $\mathbf{P}_P$-sentence $@Et$ is not expressible* in $\mathbf{P}_V$. All the same, $\mathbf{A}_P$ and $\mathbf{A}_V$ are incomparable in this sense.

Things are nicer with expressivity:

THEOREM 5. $\mathbf{P}_{V@} \approx \mathbf{A}_{V@} \approx \mathbf{P}_V \approx \mathbf{A}_V \approx \mathbf{P}_P \approx \mathbf{A}_P$.

Proof. That $\mathbf{P}_{V@} \approx \mathbf{A}_{V@} \approx \mathbf{P}_V \approx \mathbf{A}_V$ is given by Theorem 2. In order to get Theorem 5, it is then sufficient to show that (A) $\mathbf{A}_{V@} \approx \mathbf{A}_P$ and (B) $\mathbf{P}_P \approx \mathbf{A}_P$. (A) By Theorem 4, $\mathbf{A}_{V@}$ is at least as expressive* as $\mathbf{A}_P$. So $\mathbf{A}_{V@}$ is at least as expressive as $\mathbf{A}_P$. Conversely, let A be an $\mathbf{A}_{V@}$-sentence. If A is an $\mathbf{A}^-_{V@}$-sentence, we know that A is expressible*, and so expressible, in $\mathbf{A}_P$. Suppose that A is not an $\mathbf{A}^-_{V@}$-sentence. This means that A has $\Box$-free occurrences of up-arrows. Delete them all and replace all the occurrences of down-arrows they bind by $@$. The result is an $\mathbf{A}_P$-sentence, and it is true in a model iff A is. So $\mathbf{A}_P$ is at least as expressive as $\mathbf{A}_{V@}$. (B) By Theorem 4, $\mathbf{P}_P$ is at least as expressive* as $\mathbf{A}_P$. So $\mathbf{P}_P$ is at least as expressive as $\mathbf{A}_P$. For the converse, let us make a detour via the Vlachian languages. Let A be a $\mathbf{P}^-_{V@}$-sentence. Suppose A contains no occurrence of Π not within the scope of an occurrence of $\Box$. Then its translation $A^{\cup}$ (see Section 4) is an $\mathbf{A}^-_{V@}$-sentence, and we know that A

and $A^{\cup}$ are true in exactly the same models. Suppose now that A contains occurrences of Π which are not within the scope of some occurrences of $\Box$. Then $A^{\cup}$ has $\Box$-free occurrences of up-arrows. Delete them all and replace the corresponding down-arrows they bind by @. The result is an $A^{-}_{V@}$-sentence, and it is true in a model iff A is. So $\mathbf{A}^{-}_{V@}$ is at least as expressive as $\mathbf{P}^{-}_{V@}$. Consequently, $\mathbf{A}_{P}$ is at least as expressive as $\mathbf{P}_{P}$. (Thus the claim made by Forbes I mentioned at the beginning of this section is established.) $\Box$

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Notes

[1] A proper defense of that view would require an extensive discussion which cannot be undertaken here. Of course, the view is compatible with my belief that there is an appropriate formal world-semantics for the Vlach-operators (to wit, the one I will introduce below): as a matter of general fact, accepting a *formal* world-semantics for languages containing certain expressions (modal operators, relevant implication connectives, etc.) does not commit one to mere *possibilia*.

[2] By these truth-clauses, the existence predicate is redundant in **A**.

[3] The results presented in this section can be found in Forbes (1989), chapters I and II.

[4] For references on the Vlach-operators, see Forbes (1989), p. 27, fn. 10.

[5] Forbes gives $\Diamond \uparrow \Box(A \supset \Pi x(Fx \supset \downarrow Fx))$ as an example of a $\mathbf{P}_{v}$-sentence which is not expressible in $\mathbf{A}_{v}$ (footnote 15, page 30). But this is wrong, for that sentence is equivalent to the **P**-sentence $\Diamond\Pi x(\Diamond(A \wedge Fx) \supset Fx)$, and as we saw every **P**-sentence is expressible in $\mathbf{A}_{v}$. Replacing Π by Σ in Forbes' sentence gives something not expressible in $\mathbf{A}_{v}$ (for a suitable choice of A and F).

[6] Forbes mentions the idea of using arbitrarily many pairs of Vlach-operators in order to increase expressive power, but he prefers to use the Peacockian devices instead (pp. 87 ff). See Section 6 on the Peacockian method.

[7] See Forbes (1989), p. 28, fn. 11.

[8] There is an issue as to which notion of validity, if any, correctly represents the concept of logical truth, which I will not address here. See Humberstone (2004, pp. 22–23) for a short discussion and further references on the topic.

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