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THE LOGIC AND MEANING OF PLURALS. PART II

ABSTRACT. In this sequel to "The logic and meaning of plurals. Part I", I continue to present an account of logic and language that acknowledges limitations of singular constructions of natural languages and recognizes plural constructions as their peers. To this end, I present a non-reductive account of plural constructions that results from the conception of plurals as devices for talking about the many. In this paper, I give an informal semantics of plurals, formulate a formal characterization of truth for the regimented languages that results from augmenting elementary languages with refinements of basic plural constructions of natural languages, and account for the logic of plural constructions by characterizing the logic of those regimented languages.

KEY WORDS: class, irreducibility of plurals, logic, model theory, natural language, non-axiomatizability of logic, plural, regimentation of plurals, second-order logic, semantics, set, singular, the many, the one

In this sequel to "The logic and meaning of plurals. Part I" (Yi (LMP I)), I continue to present an account of logic and language that acknowledges limitations of singular constructions of natural languages and recognizes plural constructions as their peers. The account is based on the conception of plural constructions as devices for talking about the *many*. On this conception, plural constructions (or *plurals*) belong to categories comparable to those of singular constructions (or *singulars*), and have a special semantic function. Plurals are, by and large, devices for talking about *many things* (as such), whereas their singular cousins are devices for talking about *one thing* ('at a time'). So my approach to plurals departs radically from traditional approaches, which rest on the view that plurals are more or less devices for abbreviating singulars. In Yi (LMP I), I have prepared for the presentation in this paper of a non-reductive account of plurals by arguing that plurals are not reducible to singulars, and constructing refinements of plurals by regimenting them as Frege has regimented singulars. In this paper, I present a non-reductive account of plurals that results from the conception of plurals as devices for talking about the many. In its first section, §4, I present the regimented languages, *plural languages*, that result from augmenting elementary languages with the refinements of plural constructions of natural languages presented in the last section, §3, of Yi (LMP I). In §5, I give an informal semantics of plurals, and present a formal characterization of truth for those regimented languages. In §§6–7, I account for the logic of plurals by characterizing the logic of the regimented languages. I give a model-theoretic characterization of the logic in §6, and present a partial, but substantial, axiomatization of the logic in §7. In §8, I conclude by highlighting the main ideas that drive the account of logic and meaning of plurals presented in the previous sections of the two papers.

4. Plural Languages: Expressions and Syntax

The last section, §3, of Yi (LMP I) regiments basic plural constructions of natural languages and presents their refinements: plural predicates, plural variables and quantifiers, and the term connective "**@**". These expressions can complement the elementary language refinements of basic singular constructions of natural languages to yield straightforward paraphrases of basic plural constructions of natural languages. So we can give natural paraphrases of these constructions, as well as their singular cousins, into the regimented languages that result from adding the refinements of plurals to elementary languages. To characterize the logic and meaning of plurals, it is useful to articulate those languages.

The regimented languages that result from augmenting elementary languages with some or all of the refinements of plurals are first-order extensions of elementary languages. Call them (*first-order*) *plural languages*. The plural languages with all the additional expressions can be taken to have primitive expressions of five kinds:

[A1] Primitive terms

[i] Singular constants: "c" (or "Carol"), "c_i" (or "Cicero"), etc.
[ii] Singular variables: "x", "y", "x_1", etc.
[iii] **Plural variables**: "xs", "ys", "xs_1", etc.

[A2] **Term connective**: "**@**" (or "**and**")
[B] Predicates

[i] Singular predicates

logical: "=" (or "is-identical-with")
non-logical: "C" (or "is-a-child"), "ϵ" (or "is-a-member-of"), "$\leq$" (or "is-a-part-of"), etc.

[ii] **Plural predicates**

logical: "**H**" (or "**is-one-of**")
non-logical: "$\mathbf{C_o}$" (or "**cooperate**"), "**L**" (or "**lift**"), "**W**" (or "**write**"),etc.

[C] Sentential connectives: "$\sim$" (or "it-is-not-the-case-that"), and "$\wedge$" (or "and")

[D] Quantifiers

The singular existential quantifier "$\exists$" (or "there-is-something ... such-that")

The **plural existential quantifier "$\boldsymbol{\Sigma}$"** (or "**there-are-some-things** ... **such-that**")

We can take plural languages to have only two primitive sentential connectives (namely, "$\sim$" and "$\wedge$") and introduce other truth-functional connectives (e.g., "$\vee$", "$\rightarrow$", or "$\leftrightarrow$") in the usual way. Similarly, we can take only the existential quantifiers as primitives and introduce the universal quantifiers "$\forall$" (the singular) and "$\boldsymbol{\Pi}$" (the plural) via definitions (see *Def.* 1 below).

For the purpose of succinct formulation of the logic and semantics of plurals, it is useful to focus on plural languages that do not contain the term connective "**@**". Call such plural languages *meager plural languages*, and the others *plenary plural languages*. Plenary plural languages can be considered definitional extensions of meager plural languages. We can define the connective "**@**" using the other logical expressions of plural languages: plural variables and quantifiers, and the logical predicate "**H**" (see *Def.* 4 below). Although we can consider extensions of elementary languages that do not have all of these basic logical expressions (e.g., those without "**H**"), I shall not discuss them. So by "plural language", I shall understand, for simplicity of exposition, only the languages that contain all the basic logical expressions. So the minimal plural languages (that I shall discuss) are the meager plural languages that result from adding just those expressions to elementary languages. The other plural languages are either plenary plural languages or plural languages with non-logical plural predicates. (I focus on plural languages without plural constants, because I think that their counterparts in natural languages, plural proper names, are rarely, if ever, found.[1] But it is straightforward to consider plural languages *with* plural constants, and extend to those languages the accounts of plural languages presented below.)

Let me now formulate the syntax of plural languages. To do so, it is useful to use metavariables for their expressions. I use lower-case Greek letters as metavariables of plural languages: "ϕ" and "ψ" for sentences (open or closed); "π" for predicates, and "π^n" for *n*-place predicates; "τ" and "μ" for terms of any kind; "ς" and "σ" for singular terms; and "υ" for singular variables, "ω" for plural variables, and "ν" for variables of any kind.[2] I also use the results of adding numerical subscripts to any of these as metavariables of the same kind. (And I use logical expressions of object languages, as usual, as names of themselves in metalanguage expressions for object language expressions.)

Plural languages have two kinds of *terms*: singular terms and plural terms. The primitive singular terms of a plural language are its singular variables and constants, and the primitive plural terms thereof its plural variables. In addition to primitive terms, *plenary* plural languages have complex terms formed by "**@**":[3]

> [τ**@**μ] is a term of a *plenary* plural language $\mathscr{L}$, if τ and μ are terms of $\mathscr{L}$.

All such complex terms are plural terms. So "[j**@**c]", "[j**@**j]", "[j**@***x*]", "[*xs***@**j]", "[*xs***@***ys*]", and "[*x***@**[*xs***@**j]]" are complex plural terms of plenary plural languages that contain "j" and "c".

Atomic sentences (or *predications*) of plural languages result from predicates combining with one or more terms. Predicates of plural languages have a fixed and finite number (greater than zero) of argument places, and each of the argument places is either singular or neutral. Say that a term τ *is suitable for* an argument place of a predicate (or the argument place *admits* τ), if one of the following conditions holds:

[i] τ is singular (whether the argument place is singular or neutral).
[ii] τ is plural and the argument place is neutral.

Then

> $\pi(\tau_1, \tau_2, \ldots, \tau_n)$ is a sentence of a plural language $\mathscr{L}$, if π is an *n*-place predicate of $\mathscr{L}$ and $\tau_1, \tau_2, \ldots, \tau_n$ terms of $\mathscr{L}$ suitable for the 1st, 2nd, ..., *n*-th argument places, respectively, of π.

I say that the predication $\pi(\tau_1, \tau_2, \ldots, \tau_n)$ is *formed* by the predicate π. And I say that the predication $\pi(\tau_1, \tau_2, \ldots, \tau_n)$ is a *singular predication*, if π is a singular predicate, and that it is a *plural predication*, if π is a plural predicate *and* at least one of the terms $\tau_1, \tau_2, \ldots, \tau_n$ is plural.[4] *Complex sentences* of plural languages are formed by sentential connectives or quantifiers:

> $\sim\phi$ is a sentence of a plural language $\mathscr{L}$, if φ is a sentence of $\mathscr{L}$.
> $[\phi \wedge \psi]$ is a sentence of a plural language $\mathscr{L}$, if both φ and ψ are sentences of $\mathscr{L}$.
> $\exists\upsilon\phi$ is a sentence of a plural language $\mathscr{L}$, if υ is a singular variable and φ a sentence of $\mathscr{L}$.

$\mathbf{\Sigma}\omega\phi$ is a sentence of a plural language $\mathscr{L}$, if ω is a plural variable and ϕ a sentence of $\mathscr{L}$.

Sentences formed by "∃" are called *singular existential quantifications*, and those formed by "**Σ**" *plural existential quantifications*.

We can define the universal quantifications, sentences formed by the universal quantifiers "∀" and "**Π**", as follows:

Def. 1 (Universal Quantifiers):

$\forall\upsilon\phi \equiv_{df} \sim\exists\upsilon\sim\phi.$
$\mathbf{\Pi}\omega\phi \equiv_{df} \sim\mathbf{\Sigma}\omega\sim\phi.$

So the universal quantifiers can be introduced into plural languages as defined expressions.

The predicate for plural 'identity' (viz. sameness) can also be introduced via definition:

Def. 2 (Sameness):

$\tau \approx \mu \equiv_{df} \forall\upsilon[\upsilon\mathbf{H}\tau \leftrightarrow \upsilon\mathbf{H}\mu]$, where υ does not occur in τ or μ.[5]

And neutral expansions and singular reducts can be introduced via contextual definitions:

Def. 3 (Neutral Expansion and Singular Reduct):

[a] $\{\phi\}^N(\upsilon/\tau) \equiv_{df} \forall\upsilon[\upsilon\mathbf{H}\tau \rightarrow \phi]$, where υ does not occur in τ.
[b] $\pi^{N(i)}(\tau_1, \ldots, \tau_i, \ldots, \tau_n) \equiv_{df} \{\pi(\tau_1, \ldots, \tau_{i-1}, \upsilon, \tau_{i+1}, \ldots, \tau_n)\}^N(\upsilon/\tau_i)$, where υ occurs in none of $\tau_1, \ldots, \tau_{i-1}, \tau_i, \tau_{i+1}, \ldots, \tau_n$.
[c] $\pi^{S(i)}(\tau_1, \ldots, \tau_{i-1}, \varsigma, \tau_{i+1}, \ldots, \tau_n) \equiv_{df} \exists\upsilon[\upsilon = \varsigma \wedge \pi(\tau_1, \ldots, \tau_{i-1}, \upsilon, \tau_{i+1}, \ldots, \tau_n)]$, where υ occurs in none of $\tau_1, \ldots, \tau_{i-1}, \varsigma, \tau_{i+1}, \ldots, \tau_n$.
[d] $\tau \sqsubseteq \mu \equiv_{df} \mathbf{H}^{N(1)}(\tau, \mu)$.

Note that I define neutral expansions of *predicates* indirectly in *Def.* 3. Clause [a] defines the neutral expansion of *sentences* on *given variables*, and clause [b] appeals to clause [a] to define the neutral expansions of predicates. Although we can define neutral expansions of predicates directly as in Yi (LMP I, §3), I take this detour because it is useful to have the notion of neutral expansion applicable to sentences. In plural languages, as in elementary languages, complex sentences (e.g., "∼H(x)" or "[C(x) ∧ H(x)]") work as surrogates of complex predicates because the languages have no complex predicates. So to clarify the logic of neutral expansions of complex predicates of natural languages (e.g., "is unhappy" or "is a happy child"), we need to consider neutral expansions of plural language sentences.[6] ([d] introduces another logical predicate, "$\sqsubseteq$". The predicate, introduced as the neutral expansion of "**H**", is the plural language counterpart of "*to* be some of" or "*to* be among".)

We can give a contextual definition of the term connective "**@**" in meager plural languages. To do so, it suffices to characterize its use in complex terms that occur in atomic sentences. We can characterize it as follows:[7]

Def. 4 (The Term Connective "**@**"):

$\pi^n(\tau_1, \ldots, \tau_{i-1}, [\mu_1 @ \mu_2], \tau_{i+1}, \ldots, \tau_n) \equiv_{df} \mathbf{\Sigma}\omega[\forall\upsilon(\upsilon\mathbf{H}\omega \leftrightarrow [\upsilon\mathbf{H}\mu_1 \vee \upsilon\mathbf{H}\mu_2]) \wedge \pi^n(\tau_1, \ldots, \tau_{i-1}, \omega, \tau_{i+1}, \ldots, \tau_n)]$, where ω occurs in none of $\mu_1, \mu_2, \tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots,$ and τ_n, and υ in neither μ_1 nor μ_2.

So we can regard plenary plural languages as definitional extensions of meager plural languages.

Definite descriptions can also be introduced via contextual definitions. We can characterize the use of singular definite descriptions (e.g., "$(\iota x)C(x)$") in atomic sentences in the usual way:

[a] $\pi^n(\tau_1, \ldots, \tau_{i-1}, (\iota\upsilon)\phi, \tau_{i+1}, \ldots, \tau_n) \equiv_{df} \exists\upsilon_1[\forall\upsilon(\upsilon = \upsilon_1 \leftrightarrow \phi) \wedge \pi^n(\tau_1, \ldots, \tau_{i-1}, \upsilon_1, \tau_{i+1}, \ldots, \tau_n)]$, where υ_1 occurs free in none of $\upsilon, \tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_n$, and ϕ.[8]

Similarly, we can introduce plural definite descriptions (e.g., "(***I****xs*) **write**(*xs*, *p*)") via contextual definition:

[b] $\pi^n(\tau_1, \ldots, \tau_{i-1}, (\boldsymbol{I}\omega)\phi, \tau_{i+1}, \ldots, \tau_n) \equiv_{df} \mathbf{\Sigma}\omega_1[\mathbf{\Pi}\omega(\omega \approx \omega_1 \leftrightarrow \phi) \wedge \pi^n(\tau_1, \ldots, \tau_{i-1}, \omega_1, \tau_{i+1}, \ldots, \tau_n)]$, where ω_1 occurs free in none of $\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_n, \omega$, and ϕ.

The singular definite description "$(\iota x)C(x)$", for example, amounts to the definite description "the child", which can be taken to result from applying the definite article "the" to the predicate phrase "is a child". Similarly, the plural definite description "(***I****xs*)**W**(*xs*, *p*)", for example, amounts to the English phrase "those who write *PM*", which can be taken to result from combining the plural predicate phrase "write *PM*" with a definite article. So [b] yields the result that "Those who write *PM* cooperate" amounts to "There are some things that are the same things as any things that write *PM*, and they cooperate." Now, I think that natural languages have plural definite descriptions of another kind, such as "the children (in this room)" or "the residents of London". "The residents of London", for example, cannot be taken to result from combining the

English counterpart of the plural description operator "$\boldsymbol{I}$" with the predicate phrase "reside in London"; analyzing it this way would yield the wrong result that "The residents of London cooperate" is false simply because London has more than two residents (if this holds, there are no things that are the same things as *any* things that reside in London). We can accommodate such definite descriptions by introducing another definite description operator, "ι", which combines only with neutral expansions (e.g., "C^N" or "$R^{N(1)}(xs, l)$"):

[c] $\pi^n(\tau_1, \ldots, \tau_{i-1}, (\boldsymbol{\iota}\omega)\{\phi\}^N(\upsilon/\omega), \tau_{i+1}, \ldots, \tau_n) \equiv_{df} \boldsymbol{\Sigma}\omega_1[\forall\upsilon(\upsilon\mathbf{H}\omega_1 \leftrightarrow \phi) \wedge \pi^n(\tau_1, \ldots, \tau_{i-1}, \omega_1, \tau_{i+1}, \ldots, \tau_n)]$, where ω does not occur free in ϕ, and ω_1 occurs free in none of $\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_n$, and ϕ.

We can then paraphrase "the residents of London cooperate" by "$\mathbf{C}_0((\boldsymbol{\iota}xs)R^{N(1)}(xs, l))$", which amounts to "There are some things such that something is one of them if and only if it resides in London, and they cooperate."[9] And we may abbreviate $(\boldsymbol{\iota}\omega)\{\phi\}^N(\upsilon/\omega)$, where ω does not occur free in ϕ, by $<\upsilon: \phi>$. To sum up, definite descriptions of three kinds can be defined as follows:

Def. 5 (Definite Descriptions):

[a] $\pi^n(\tau_1, \ldots, \tau_{i-1}, (\iota\upsilon)\phi, \tau_{i+1}, \ldots, \tau_n) \equiv_{df} \exists\upsilon_1[\forall\upsilon(\upsilon = \upsilon_1 \leftrightarrow \phi) \wedge \pi^n(\tau_1, \ldots, \tau_{i-1}, \upsilon_1, \tau_{i+1}, \ldots, \tau_n)]$, where υ_1 occurs free in none of $\upsilon, \tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_n$, and ϕ.

[b] $\pi^n(\tau_1, \ldots, \tau_{i-1}, (\boldsymbol{I}\omega)\phi, \tau_{i+1}, \ldots, \tau_n) \equiv_{df} \boldsymbol{\Sigma}\omega_1[\boldsymbol{\Pi}\omega(\omega \approx \omega_1 \leftrightarrow \phi) \wedge \pi^n(\tau_1, \ldots, \tau_{i-1}, \omega_1, \tau_{i+1}, \ldots, \tau_n)]$, where ω_1 occurs free in none of $\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_n, \omega$, and ϕ.

[c] $\pi^n(\tau_1, \ldots, \tau_{i-1}, (\boldsymbol{\iota}\omega)\{\phi\}^N(\upsilon/\omega), \tau_{i+1}, \ldots, \tau_n) \equiv_{df} \boldsymbol{\Sigma}\omega_1[\forall\upsilon(\upsilon\mathbf{H}\omega_1 \leftrightarrow \phi) \wedge \pi^n(\tau_1, \ldots, \tau_{i-1}, \omega_1, \tau_{i+1}, \ldots, \tau_n)]$, where ω does not occur free in ϕ, and ω_1 occurs free in none of $\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_n$, and ϕ.[10]

[d] $<\upsilon: \phi> \equiv_{df} (\boldsymbol{\iota}\omega)\{\phi\}^N(\upsilon/\omega)$, where ω does not occur free in ϕ.

And we can define (pure) numerical predicates in plural languages. To do so, it is useful to use restricted quantifiers defined as follows:

Def. 6 (Restricted Quantifiers):

$(\exists\upsilon\mathbf{H}\tau)\phi \equiv_{df} \exists\upsilon[\upsilon\mathbf{H}\tau \wedge \phi]$.

$(\forall\upsilon\mathbf{H}\tau)\phi \equiv_{df} \forall\upsilon[\upsilon\mathbf{H}\tau \rightarrow \phi]$.

$(\boldsymbol{\Sigma}\omega\sqsubseteq\tau)\phi \equiv_{df} \boldsymbol{\Sigma}\omega[\omega \sqsubseteq \tau \wedge \phi]$.

$(\boldsymbol{\Pi}\omega\sqsubseteq\tau)\phi \equiv_{df} \boldsymbol{\Pi}\omega[\omega \sqsubseteq \tau \rightarrow \phi]$.

We can then define numerical predicates as follows:

Def. 7 (Numerical Predicates):

[a] $\mathbf{One}(\tau) \equiv_{df} \exists \upsilon \upsilon \approx \tau$,[11] where υ does not occur in τ.[12]
[b] $\mathbf{N}^*(\tau) \equiv_{df} \mathbf{\Sigma}\omega\exists\upsilon[\tau \approx [\omega@\upsilon] \wedge \sim\upsilon\mathbf{H}\omega \wedge \mathbf{N}(\omega)]$, where υ and ω do not occur in τ.
[c] $\mathbf{Two}(\tau) \equiv_{df} \mathbf{One}^*(\tau)$; $\mathbf{Three}(\tau) \equiv_{df} \mathbf{Two}^*(\tau)$; etc.
[d] $\mathbf{Many}(\tau) \equiv_{df} (\exists\upsilon\mathbf{H}\tau)(\exists\upsilon_1\mathbf{H}\tau)\upsilon \neq \upsilon_1$, where υ and υ_1 do not occur in τ, and υ is not υ_1.[13]

And it is useful to define plural definite descriptions of a special kind as follows:

Def. 8 (Ancestry):

$<\phi>^{\upsilon/\tau}_{\nu} \equiv_{df} <\upsilon : \mathbf{\Pi}\omega[(\tau \sqsubseteq \omega \wedge (\forall\nu\mathbf{H}\omega)\forall\upsilon_1 [\phi(\nu/\upsilon_1) \rightarrow \upsilon_1\mathbf{H}\omega]) \rightarrow \upsilon\mathbf{H}\omega]>$, where υ does not occur in τ, υ is not υ_1, υ_1 does not occur free in ϕ, and υ_1 is substitutable for ν in ϕ.

To see what the defined expressions amount to, consider the case in which ϕ amounts to a two-place predicate that indicates the relation of *being a parent of*[14] and τ a term that refers to John. Then $<\phi>^{\upsilon/\tau}_{\nu}$ abbreviates the plural definite description tantamount to "John and his ancestors".

Let me complete this section by comparing the plural languages that I *discuss* in this paper to clarify the logic and semantics of plurals with the language (or languages) that I *use* for that purpose. The plural languages presented above are first-order plural languages. They have no higher-order variables, quantifiers, or predicates. They are, so to speak, *horizontal* extensions of elementary languages that contain refinements of basic plural constructions of natural languages, whereas the usual (i.e., singular) higher-order languages are their *vertical* extensions that contain expressions built on elementary language predicates.[15] So I concentrate on those plural languages to clarify the logic and semantics of basic plural constructions of natural languages. (And I usually omit "first-order" to talk about the languages, unless it is necessary to contrast them explicitly with higher-order languages.) But we can consider their higher-order extensions, languages that result from augmenting first-order plural languages with higher-order expressions built on their predicates. We can obtain, for example, their second-order extensions that contain second-order variables, quantifiers, or predicates. It is straightforward to formulate such plural languages as regimented languages that are to first-order plural languages what the usual higher-order languages are to elementary languages. And we can give the semantics of those languages, as well as their first-order fragments, using the resources that I develop in the next section. Now, I think that we need to use higher-order plural languages to give the proper semantics of even first-order languages, including elementary languages. So the characterizations of truth and logic for first-order plural languages that I present in the next two sections can be seen to be formulated in higher-order plural languages. Although I present those characterizations in a mixed language, English augmented with expressions of regimented plural languages of a higher-order, my remarks on the expressions used in the characterizations should make it clear how to regiment the languages with the additional expressions.

5. Plural Languages: Semantics

To give a proper semantics of plurals, it is crucial to draw a sharp distinction between terms and predicates. There is a clear distinction between singular terms and predicates. It is wrong to assimilate the singular terms "he", "the author of *Academica*", and "Cicero", for example, to related predicate phrases, such as "is identical with him", "is an author of *Academica*", and "is Cicero". Similarly, it is wrong to assimilate plural terms to predicates related to them. We must clearly distinguish the plural terms "they", "John and Carol", "the authors of *PM*", and "those who write *PM*", for example, from related predicate phrases, such as "is one of them", "is John or Carol", "is an author of *PM*", and "is one of those who write *PM*". Predicates (or predicate phrases) are *predicable* expressions with argument places that admit other expressions. In particular, first-order predicates have argument places that admit non-predicable expressions. The predicate "*to* be a child", for example, admits "Carol" into its argument place to form the sentence "Carol is a child", and the predicate "*to* love" (or "*to* be") admits "John" and "Carol" in its first and second argument places, respectively, to form the sentence "John loves Carol" (or "John is Carol"). Singular terms, by contrast, are non-predicable expressions. "John", for example, cannot combine with "Carol" to form a sentence, although they can help to yield "John loves Carol" and "John is Carol", for example, by filling the argument places of predicates. It is the same with plural terms. "John and Carol", for example, cannot combine with "Carol" to form a sentence. It helps to yield "*John and Carol* lift Bob" or "Carol is one of *John and*

Carol", but it does so by filling argument places of predicates (e.g., the underlined) as "John" helps to yield "*John* loves Carol" or "Carol is *John.*"

Accordingly, there is an important, if not the basic,[16] semantic description of predicates that is not appropriate for non-predicable expressions. The predicate "*to* be a child" *is true of* (or *designates* or, in general, *is satisfied by*)[17] John–given that he is a child. But it is not appropriate to say that the singular term "John" is true of John, although it *refers to* him. To see the difference between the two semantic relations, *designation* and *reference*, consider the usual characterization of the truth or falsity of atomic sentences, such as "John is a child":

> "John is a child" is true if and only if its predicate "*to* be a child" *is true of* (or *designates*) what the term "John" *refers to*.

The characterization presupposes a disparity between the two relations. It assumes that "John" cannot refer to something while referring to something else as well, but not that "*to* be a child" cannot be true of something while being true of something else as well. Surely, it would be wrong to assume this. Expressions that *are true of* something can in general be true of something else as well (while being used in the very same sense). The predicate "*to* be a child", for example, can be true of John, and also of Carol. An expression that *refers to* something, by contrast, cannot refer to something else (unless it is used in a different context or in a different sense), as is rightly assumed in the above characterization. Now, the semantic function of designating (or its genus, being satisfied) suits only predicable expressions. In giving recursive characterizations of truth, it is necessary to invoke the function only for expressions used as predicates that *form* atomic sentences, and predicable expressions are precisely those expressions that can be used as predicates that form some atomic sentences.[18]

Non-predicable expressions are not subject to the designation or satisfaction relation. They are subject to the reference relation. The semantic function of referring suits singular terms. Typical singular terms (e.g., "John") refer to something (e.g., John).[19] Similarly, I say, referring suits plural terms as well. If so, *what* do they refer to? The plural term "John and Carol", for example, does not refer to John. Nor does it refer to Carol, or to anything else. So it does not refer to anything whatsoever. But this does not mean that the plural term does not refer at all. Although it does not refer to any *one thing*, it still refers to *some things that are more than one*. That is, it refers to: John and Carol, who are not some one thing, but two things (viz. two humans). A plural term may refer to some things without referring to any one of them, just as a piano may be lifted by some children without being lifted by any one of them. That *is* what typical plural terms do.[20]

This is the natural conclusion to draw on the semantics of plural terms. But most of those who agree that we must account for plurals without paraphrasing them away would still resist the conclusion. To do so, they might hold that "John and Carol", for example, refers to a composite object that in a way comprehends both John and Carol, such as a set or class, a group or collection, a sum or aggregate, or a 'totality' or 'plurality'. But this view yields the wrong result that "John and Carol lift Bob" (sentence [5]), for example, logically implies "There is something that lifts Bob" (sentence [6a]).[21] Some might be prepared to tolerate this result. It is not plausible at all, however, to hold that "John and Carol are two children" must likewise logically imply "There is something that is two children."[22] And if "Genie", for example, is the singular term that refers to the composite object that "Russell and Whitehead" refers to, then "Genie is one of *Russell and Whitehead*" and "Genie is one of *Genie*" (sentences [19] and [20]) must have the same truth value; but the former is false while the latter is true.[23]

Some might assimilate plural terms to predicates and hold that the plural term "John and Carol", for example, refers to John and to Carol, just as the predicate "*to* be John or Carol" (or "*to* be one of John and Carol") is true of John and of Carol.[24] There is no denying that one may, if one wishes, use the word "refer" in this way. One who thinks, as I do, that it refers (or, let us say, refers$_1$) to John and Carol (but not to John alone) can introduce a defined expression, "*to* refer$_2$", for its derivative semantic function (one can say that a term refers$_2$ to something if and only if it is one of the things that the term refers$_1$ to). But it is wrong to conclude from this that there is nothing more to the semantics of the plural term than referring$_2$. This is not the function that we use to characterize the truth of plural predications, such as "*John and Carol* <u>are</u> <u>two</u> <u>children</u>", "*John and Carol* <u>lift</u> Bob", or "*John and Carol* <u>cooperate</u>." The first of these, for example, is true if and only if its predicate "*to* be two children" is true of what the term "John and Carol" **refers** to (i.e., *John and Carol*), just as the singular predication "*John* <u>is</u> <u>a</u> <u>child</u>" is true if and only if its predicate "*to* be a child" is true of what the term "John" refers to (i.e., *John*). To characterize truth in this way, it is necessary to use "**refers**" to indicate the basic semantic function of plural terms: referring$_1$. Using it to indicate their derivative function, referring$_2$, yields the wrong result. The plural predicate "*to* be two children" is *not* true of John, whom the term $refers_2$ to (John is not two children, but only one child).[25]

Why would one who declines to paraphrase plurals away be still inclined to give their semantics by assimilating them to singular terms or predicates (e.g., "the set {John, Carol}" or "*to* be John or Carol")? Those who do so, I think, have failed to free themselves from the grip of the bias against plurals. While embracing plurals as they are in non-semantic object languages, they have yet to reject the idea that languages with no plural semantic predicates must be rich enough for giving the semantics of plurals. But it is hard to see how they can retain this idea, given that the proper reason for rejecting the bias is the poverty of singular languages. One must use plurals to give natural semantic descriptions of plurals, such as the following:

[a] The term "John and Carol" <u>refers</u> <u>to</u> *John and Carol.*
[b] The predicate "*to* be two children" <u>is</u> <u>true</u> <u>of</u> *any things* that *are two children*.

In these statements, the underlined predicates are used as plural predicates (their second argument places admit plural terms, e.g., "John and Carol"). And the plural constructions in metalanguages, like those in object languages, must also be considered devices for talking about many things (as such), such as John and Carol, who are two humans. The italicized plural term in sentence [a] is used to talk about two humans (as such); and [a] states that the plural term mentioned in it refers to those same humans (as such). The semantic plural predication [a] is as irreducible to singular constructions as are comparable non-semantic plural predications, such as "*John and Carol* lift Bob" or "Bob <u>is</u> <u>lifted</u> <u>by</u> *John and Carol*." We cannot do without plurals in metalanguages, either.

Using plurals, we can state the truth conditions of plural predications and quantifications in the natural way:

[c] The plural predication "*John and Carol* <u>are</u> <u>two</u> <u>children</u>", for example, is true if and only if its predicate (i.e., the underlined) is true of *the things* that the plural term "John and Carol" refers to.
[d] The plural quantification "Some things <u>are</u> <u>two</u> <u>children</u>", for example, is true if and only if *there are some things* that the predicate "*to* be two children" is true of.

And we can extend the Tarski-style characterization of truth for elementary languages in terms of reference and satisfaction to obtain a Tarski-style characterization of truth for (first-order) plural languages.

We can do so without resorting to set theory by formulating the characterization in *higher-order plural languages*, higher-order extensions of first-order plural languages.[26] But Tarski-style characterizations of truth in terms of reference and satisfaction have serious limitations. We cannot generalize them to give characterizations of truth for *languages with higher-order predications* (e.g., plenary second-order languages or meager third-order languages). Nor can we generalize it to characterize the *logic* of even first-order languages, including elementary languages.[27] The reason is that the Tarski-style characterizations invoke what I think is only a derivative semantic function of predicates: designating or being satisfied. By invoking their primary semantic function, we can formulate improvements of the Tarski-style characterizations that can be straightforwardly turned into characterizations of logic for plural languages as well as characterizations of truth for higher-order languages.[28] To give such characterizations of plural language truths, it is necessary to discuss the primary semantic function of predicates: *indicating* attributes.[29]

Predicable expressions (e.g., predicates), I think, relate to predicable entities (or *attributes*), such as properties or relations, as non-predicable expressions (e.g., singular terms) relate to non-predicable entities (or objects). The predicate "*to* be a human", for example, relates to the property of *being a human*, as the singular term "John", for example, relates to the object, *John*, that the term refers to. If such a relation holds between a predicable expression and an attribute, say that the expression *indicates* the attribute. So, for example, "*to* be red" (or "*to* be a human") indicates the property of *being red* (or *being a human*), and "*to* admire" (or "*to* live in") the relation of *admiring* (or *living in*). In general, one-place predicates indicate properties, and multi-place predicates relations. For properties are predicable entities with one argument place, and relations those with more than one. We can then see that designating or being satisfied is a derivative semantic function of predicates. The predicate "*to* be a human", for example, is satisfied by Bush, because the property that it indicates (viz. being a human) is instantiated by him; the predicate "*to* live in" is satisfied by Bush and Washington in that order, because the relation that it indicates (viz. living in) holds for the human and the city in that order (that is, it relates Bush to Washington).

Now, the above account applies to plural predicates as well as to their singular cousins. Plural predicates, too, indicate attributes. The one-place predicate "*to* be two children", for example, indicates *being two children*. This is a property (i.e., an attribute with one argument place) that has an argument place of a special kind. Its argument place must be one that admits what a plural term refers to, because it corresponds to the plural argument place of the predicate "*to* be two children". And the plural term "John and Carol", for example, refers to some things that are more than one: John and Carol (as such). So the argument place of the property of *being two children* must admit those many things, namely, the two humans (as such). Call such a special argument place of predicable entities, one that admits many things (as such), a *plural argument place*. Accordingly, say that an attribute is *plural*, if it has at least one plural argument place. We can then see that plural predicates indicate plural attributes. For example, "*to* cooperate" (like "*to* be two children") indicates a plural property, namely, *cooperating* (its only argument place is plural); "*to* lift" and "*to* write" two-place plural relations whose first argument places are plural, *lifting* and *writing*; and "*to* be one of" a two-place relation whose second argument place is plural, *being one of*.

Attributes indicated by singular predicates, such as "*to* be a child" or "*to* be identical with" (or their elementary language siblings), by contrast, have no plural argument place. The argument places of singular predicates admit only singular terms, and no singular term refers to more than one thing (as such). So attributes indicated by singular predicates have no plural argument place. Their argument places do not admit, for example, John and Carol (as such), although they admit any one of them (for any one of them is one thing). Say that an argument place of an attribute is *singular*, if it does not admit many things (as such); and that an attribute itself is *singular*, if it has only singular argument places.[30] Then singular predicates, we have seen, indicate singular attributes.

Note that when I say that a property, for example, has an argument place that admits some things, I do not mean that the property is instantiated by those things. The predicate "*to* be a child" is not true of London, but its argument place admits "London"; the predicate combines with "London" to form a sentence, although the sentence is not true. Similarly, London does not instantiate the property of *being a child*, but it can still fill its argument place. Although the property does not combine with London to form a fact, its complement, namely, *not being a child*,[31] combines with London to form a fact, namely, the fact that London is not a child. So the argument place of *being a child*, like that of its complement, must be one that admits London. Similarly, some things that do not (as such) instantiate a plural property may still fill its argument place. For example, London and Chicago (as two cities) do not instantiate the property of *being two children*, but can fill its argument place. The two cities (as such) instantiate its complement, namely, *not being two children*;[32] "Chicago and London are not two children" is true.

But a property cannot be instantiated by some things, unless they can fill its argument place. As assumed above, London cannot instantiate *not being a child*, unless it can fill the argument place of the property; nor can London and Chicago (as such) instantiate *not being two children*, unless they can fill its argument place. So any property instantiated by many things (as such) is a plural property. Note that the converse of this does not hold. Some of the plural properties are not instantiated by any two or more things (as such).[33] But most of them are. *Being two children* is instantiated by John and Carol (as such), and its complement by London and Chicago. Call such properties *plurally instantiated properties*. Then we can see that the existence of plurally instantiated properties is a straightforward consequence of the existence of plural properties: if the argument place of a property admits some things (as such), the same things must instantiate either the property itself or its complement.[34]

Surely, the account of attributes sketched above deviates radically from the standard conception of reality. It is usually taken for granted that a property can be instantiated by each one of many things, and that a relation can in some sense relate many things. But it is widely assumed that there is no property instantiated by many things (as such). This thesis, as we have seen, presupposes the thesis that there is no plural attribute. These two theses are entrenched in the standard conception of reality,[35] and they, I think, lie under the longstanding bias against plurals. Those who cannot conceive alternatives to the theses would argue that there could be no genuine plural predicates (and, thus, no genuine plural terms) because such predicates would be ones that indicate no attributes. So they would conclude that the apparent plural constructions of natural languages must be considered devices for abbreviating singular constructions that involve no plural predicates. But this stilted view of plurals must be rejected, as we have seen, because there are robust logical relations pertaining to plurals that one cannot accommodate by paraphrasing them away. So I propose to reject the two theses of the standard conception that lead to the view, and to accept a liberal conception of reality that acknowledges the existence of plural attributes as well as their singular cousins. Call this the *plural conception of reality*.[36]

With this conception of reality in hand, we can complete the natural account of the logic and meaning of plurals. The conception admits attributes that plural predicates can relate to, as well as those that their singular cousins can. So in the case of plural predicates, too, designation or satisfaction can be seen to be a derivative semantic relation that results from the primary relation of indication. For example, the one-place predicate "*to* be two children" is true of John and Carol (as such), because the plural property that it indicates, *being two children*, is instantiated by those children (as such); the two-place predicate "*to* write" is satisfied by two humans (as such) and a book (e.g., *PM*) in that order, because the relation that it indicates, *writing*, relates those humans (as such) to that book.

By invoking the primary semantic function of predicates, we can give characterizations of truth that improve on Tarski-style characterizations. To give such characterizations of truth for (first-order) plural languages, it is necessary to *use* resources of higher-order plural languages.

First, we need to use a second-order predicate for the *indication relation* between predicates of a plural language and attributes.[37] Second, we need to use plural predicates for the *reference relations* for terms of plural languages. We can take the reference predicates for *meager* plural languages to be singular, because all their constants are singular terms.[38] But we still need to account for the semantics of plural variables, and it is necessary to use plural predicates comparable to the reference predicates to state what are the things that a plural variable (e.g., "*xs*") is assigned (by a given assignment function of the variables of the language in question). So for even meager plural languages, we need to use plural predicates for the *denotation* relations, i.e., relations akin reference relations that pertain to variables as well as constant terms. Third, it is necessary to use plural variables and quantifiers to invoke the *assignment functions* suitable for plural variables. To give a recursive characterization of truth for a plural language, it is necessary to consider the semantics of its variables. To this end, we need to formulate a generalized notion of *assignment function* applicable to plural variables. We can do so by considering relations similar to the reference relation for a plenary plural language, which has constant plural terms (e.g., "[j@c]"). The reference relation for such a language is a plural relation that satisfies the following condition:

[T0] If there are some things, *xs*, some things *ys*, and something *z* such that $\boldsymbol{S}(z, xs)$ and $\boldsymbol{S}(z, ys)$, then $xs \approx ys$.

(In [T0], "$\boldsymbol{S}$" is used as a second-order variable for two-place plural relations.[39]) Relations that satisfy this condition can be considered one-place functions of a special kind.[40] Call such relations (*one-many*) *plural functions*. (If [T0] holds, it is useful to abbreviate "$\boldsymbol{S}(z, xs)$", for example, as "$\boldsymbol{S}(z) \approx xs$" or "$xs \approx \boldsymbol{S}(z)$".) And we can invoke one-many plural functions that pertain to the variables of plural languages to give the semantics of those variables. To do so, it is necessary to use second-order variables and quantifiers that range over such functions. Finally, it is necessary, for that reason, to use second-order predicates as the *satisfaction predicates* for plural languages to define the truth predicates for the languages.

Now, let me formulate a characterization of truth for (first-order) plural languages. To do so, it is sufficient to characterize the truths of meager plural languages, because plenary plural languages can be considered their definitional extensions (see *Def*. 4 in §4).

Let $\mathcal{L}$ be a meager plural language. Say that a one-many plural function $\boldsymbol{S}$ is an *assignment function* (or *assignment*) for $\mathcal{L}$, if the following conditions hold:

[T1] There is something x such that $x \approx \boldsymbol{S}(\nu)$, if ν is a singular variable of $\mathcal{L}$.
[T2] There are some things xs such that $xs \approx \boldsymbol{S}(\nu)$, if ν is a plural variable of $\mathcal{L}$.

Then we can define what a term τ (of $\mathcal{L}$) *denotes on* an assignment $\boldsymbol{S}$ for $\mathcal{L}$ (in symbols, $\tau^{\boldsymbol{S}}$) as follows:

[T3] $\tau^{\boldsymbol{S}} \approx x$, if τ is a singular constant of $\mathcal{L}$ that refers to x.[41]
[T4] $\tau^{\boldsymbol{S}} \approx \boldsymbol{S}(\tau)$, if τ is a variable of $\mathcal{L}$.

And say that an assignment $\boldsymbol{R}$ (for $\mathcal{L}$) *is a variant of* an assignment $\boldsymbol{S}$ (for $\mathcal{L}$) *on* a variable ν *for* some things xs (in symbols, $\boldsymbol{R} \cong^{\nu}_{xs} \boldsymbol{S}$), if the following conditions hold:

[T5] There is something x such that $x \approx xs$ (i.e., **One**(xs)), if ν is singular.
[T6] $\boldsymbol{R}(\nu_1) \approx xs$, if $\nu_1 = \nu$.
[T7] $\boldsymbol{R}(\nu_1) \approx \boldsymbol{S}(\nu_1)$, if $\nu_1 \neq \nu$.

And say that an assignment $\boldsymbol{S}$ of $\mathcal{L}$ *satisfies* a sentence ϕ of $\mathcal{L}$ (in symbols, $\boldsymbol{S} \models^{\mathcal{L}} \phi$), if it is so determined by the following conditions:

[T8] $\boldsymbol{S} \models^{\mathcal{L}} \varsigma = \sigma$, if $\varsigma^{\boldsymbol{S}} = \sigma^{\boldsymbol{S}}$.
[T9] $\boldsymbol{S} \models^{\mathcal{L}} \varsigma \mathbf{H} \tau$, if $\varsigma^{\boldsymbol{S}}$ **is one of** $\tau^{\boldsymbol{S}}$.
[T10] $\boldsymbol{S} \models^{\mathcal{L}} \pi^n(\tau_1, \tau_2, \ldots, \tau_n)$, if τ_1, τ_2, ..., and τ_n are suitable for the 1st, 2nd, ..., and n-th argument places of π^n, respectively, and π^n is a non-logical predicate that indicates ***an*** n-place attribute $\boldsymbol{P}$ such that $\boldsymbol{P}$ is instantiated by $\tau_1^{\boldsymbol{S}}$, $\tau_2^{\boldsymbol{S}}$, ..., $\tau_n^{\boldsymbol{S}}$ in that order.[42]
[T11] $\boldsymbol{S} \models^{\mathcal{L}} [\phi \wedge \psi]$, *if* $\boldsymbol{S} \models^{\mathcal{L}} \phi$ and $\boldsymbol{S} \models^{\mathcal{L}} \psi$.
[T12] $\boldsymbol{S} \models^{\mathcal{L}} \sim\phi$, if it is not the case that $\boldsymbol{S} \models^{\mathcal{L}} \phi$.
[T13] $\boldsymbol{S} \models^{\mathcal{L}} \exists \upsilon \phi$, if there is something x and ***there is*** an assignment $\boldsymbol{R}$ for $\mathcal{L}$ such that $\boldsymbol{R} \cong^{\upsilon}_{xs} \boldsymbol{S}$ and that $\boldsymbol{R} \models^{\mathcal{L}} \phi$.
[T14] $\boldsymbol{S} \models^{\mathcal{L}} \boldsymbol{\Sigma} \omega \phi$, if there are some things xs and ***there is*** an assignment $\boldsymbol{R}$ for $\mathcal{L}$ such that $\boldsymbol{R} \cong^{\omega}_{xs} \boldsymbol{S}$ and that $\boldsymbol{R} \models^{\mathcal{L}} \phi$.

Then the truths of $\mathscr{L}$ can be characterized in the usual way:

[T15] A closed sentence ϕ of $\mathscr{L}$ *is true* if and only if ***any*** assignment for $\mathscr{L}$ satisfies ϕ.

This completes the characterization of truth for first-order plural languages. Note that it is straightforward to generalize it for higher-order languages.[43] We can characterize the truths of a second-order plural language in a third-order language. To give the semantics of monadic second-order variables, for example, we can use third-order variables to invoke second-order functions from those variables to first-order properties; and we can use a third-order predicate for the indication relation that pertains to second-order predicates. Moreover, we can obtain characterizations of logic by making straightforward modifications to the characterizations of truth. In the next section, I give such a characterization logic for first-order plural languages to clarify the logic of natural language plurals.

Now, it is useful to compare the above characterization with a Tarski-style characterization that is equivalent to it. We can get a Tarski-style characterization by replacing [T10] with the following:[44]

[T10*] $\boldsymbol{S} \models^{\mathscr{L}} \pi^n(\tau_1, \tau_2, \ldots, \tau_n)$, if τ_1, τ_2, ..., and τ_n are suitable for the 1st, 2nd, ..., and n-th argument places of π^n, respectively, and π^n is a non-logical predicate that **is satisfied by** $\tau_1^{\boldsymbol{S}}$, $\tau_2^{\boldsymbol{S}}$, ..., $\tau_n^{\boldsymbol{S}}$ in that order.

We can replace [T10] with this, because they concern first-order predications, which involve no predicates except those that *form* the predications. But this assumption fails in higher-order predications. Second-order predications, for example, may have first-order predicates that fill the argument places of second-order predicates. To characterize the satisfaction condition of second-order predications, it is necessary to appeal to the primary semantic function of first-order predicates. So one cannot generalize the Tarski-style characterization for languages with second-order predications, such as plenary second-order languages or third-order languages.

6. Plural Logic: Model Theory

We can characterize the logic of plural constructions by characterizing the logic of plural languages. And we can give a model-theoretic characterization of the logic of these languages by modifying the characterization of their truths given in §5. I call the system of logic that results from the characterization *plural logic*. Plural logic is a conservative extension of elementary logic to plural languages. But it diverges from elementary logic in an important way. Plural logic is not compact and, thus, not axiomatizable. This is not an artefact of my characterization. A system of logic that does justice to plurals, as we shall see, cannot be axiomatizable.

I use higher-order plural languages in characterizing the logic of first-order plural languages. This is necessary to align the characterization of logic to the characterization of truth given in §5. Moreover, I think that higher-order plural languages are optimal languages to use to characterize the logic of even elementary languages.

It is usual to characterize elementary logic *in* elementary languages sufficient for stating set theory. The usual characterization of the logic proceeds by specifying a set as the *domain* of its quantifiers, and a set of a certain kind that is related to the domain as the *interpretation* of the non-logical expressions of the language (the interpretation is a 'function' that assigns members of the domain to its singular constants, and certain sets over the domain to its predicates). But this leads to serious limitations of the resulting characterization of logic. A quantifier of an elementary language may range over all the objects that there are (including any sets that there are), and they do not form a set; similarly, the things each of which satisfies a one-place predicate of an elementary language (e.g., "is identical with itself" or "is not a member of itself") may not form a set. So one cannot directly use the usual characterization of elementary logic to get the result that all the *logical truths* of elementary languages are *true*.[45] To specify some things that an elementary language quantifier may range over, however, it is not necessary to make a detour through a set that in a sense comprehends them. We can use plural languages to specify those things without invoking any one thing that comprehends them. And it is not necessary to have a set, a so-called interpretation, that relates sets of a certain kind (e.g., sets of ordered pairs) to predicates of an elementary language. The sets assigned to predicates are used as substitutes of attributes (properties or relations) in the usual characterization of elementary logic. So we can assign attributes, rather than their set-theoretic substitutes, to predicates. Using higher-order languages, we can invoke functions that assign attributes to predicates without making a detour through their set-theoretic substitutes. So it is straightforward to use higher-order plural languages to characterize the logic of elementary languages, and to use the characterization to show that all the logical truths of elementary languages are true.

Similarly, we can use higher-order plural languages to characterize the logic of first-order plural languages.[46] Let me now present a natural characterization of the logic that results from making straightforward modifications to the characterization of truth given in §5.

To do so, I define two semantic predicates (or open sentences) that pertain to (first-order) plural languages:

ϕ *is a model-theoretic truth* of $\mathscr{L}$ (in symbols, $\models^{\mathscr{L}} \phi$)
ϕ *is a model-theoretic consequence in* $\mathscr{L}$ *of* Γ (in symbols, $\Gamma \models^{\mathscr{L}} \phi$)

where "$\mathscr{L}$" is for plural languages, "ϕ" for closed sentences of the languages, and "Γ" for sets of closed sentences of the languages. The predicates are so defined that they yield an adequate and full characterization of the logic of plural languages. That is, the following are to hold:

$\models^{\mathscr{L}} \phi$ if and only if ϕ is a logical truth of $\mathscr{L}$.
$\Gamma \models^{\mathscr{L}} \phi$ if and only if ϕ is a logical consequence in $\mathscr{L}$ of the sentences in Γ.[47]

In defining these semantic predicates, too, we may focus on *meager* plural languages.

Let $\mathscr{L}$ be a meager plural language. To give a 'domain' of the quantifiers of $\mathscr{L}$, it suffices to specify some one or more things that the quantifiers may range over. So let *ds* be some things.[48] Then the following holds:

[L1] There is something x such that x **is one of** *ds*.[49]

An interpretation for $\mathscr{L}$ over those things (i.e., *ds*) specifies how the non-logical expressions of $\mathscr{L}$ relate to them. A singular constant of $\mathscr{L}$ is to be assigned to one of the things, and a non-logical predicate of $\mathscr{L}$ to an attribute over them. So it is useful to distinguish two kinds of interpretations: those that concern singular constants (*c*-interpretations) and those that concern non-logical predicates (*p*-interpretations). The former can be taken to be first-order functions, and the latter second-order functions. Now, say that a one-place function $\boldsymbol{I}$ is a *c-interpretation for* $\mathscr{L}$ *over ds*, if the following holds:

[L2] $\exists x[x \approx \boldsymbol{I}(\tau) \wedge x\mathbf{H}ds]$, if τ is a singular constant of $\mathscr{L}$.[50]

And say that a second-order function $\boldsymbol{J}$ is a *p-interpretation for* $\mathscr{L}$ *over ds*, if the following holds:[51]

[L3] If π is a non-logical n-place predicate of $\mathscr{L}$, then $\boldsymbol{J}(\pi)$ is an n-place plural attribute that satisfies the following conditions:

[1] $xs_i \sqsubseteq ds$, if $1 \leq i \leq n$ and $\boldsymbol{J}(\pi)$ is instantiated by $xs_1, xs_2, \ldots, xs_n$ in that order.

[2] $\textbf{One}(xs_i)$, if the i-th argument place of π is singular and $\boldsymbol{J}(\pi)$ is instantiated by $xs_1, xs_2, \ldots, xs_n$ in that order.

It is straightforward to modify the definition of *assignment* in §5 to obtain the relativized notion of *assignment over* some things (e.g., *ds*). Say that a (one-many) plural function $\boldsymbol{S}$ is an *assignment for* $\mathscr{L}$ *over ds*, if the following conditions hold:

[L4] $\exists x[x \approx \boldsymbol{S}(\nu) \wedge x\mathbf{H}ds]$, if ν is a singular variable of $\mathscr{L}$.
[L5] $\boldsymbol{\Sigma} xs[xs \approx \boldsymbol{S}(\nu) \wedge xs \sqsubseteq ds]$, if ν is a plural variable of $\mathscr{L}$.

And say that an assignment $\boldsymbol{R}$ (for $\mathscr{L}$ over *ds*) *is a variant of* an assignment $\boldsymbol{S}$ (of $\mathscr{L}$ over *ds*) *on* a variable ν of $\mathscr{L}$ *for* some things *xs* among *ds* (in symbols, $\boldsymbol{R} \cong^{\nu}_{xs} \boldsymbol{S}$), if the conditions [T5]–[T7] in §5 hold.

Now, let $\boldsymbol{I}$ be a *c*-interpretation, $\boldsymbol{J}$ a *p*-interpretation, and $\boldsymbol{S}$ an assignment for $\mathscr{L}$ over *ds*. Then we can define what a term τ (of $\mathscr{L}$) *denotes at* $\boldsymbol{S}$ *on* $\boldsymbol{I}$ *over ds* (in symbols, $[\![\tau^{\boldsymbol{S}}]\!]^{ds}_{\boldsymbol{I}}$) as follows:

[L6] $[\![\tau^{\boldsymbol{S}}]\!]^{ds}_{\boldsymbol{I}} \approx \boldsymbol{I}(\tau)$, if τ is a singular constant of $\mathscr{L}$.
[L7] $[\![\tau^{\boldsymbol{S}}]\!]^{ds}_{\boldsymbol{I}} \approx \boldsymbol{S}(\tau)$, if τ is a variable of $\mathscr{L}$.

And say that $\boldsymbol{S}$ *satisfies* a sentence ϕ (of $\mathscr{L}$) *on* $\boldsymbol{I}$ *and* $\boldsymbol{J}$ *over ds* (in symbols, $\boldsymbol{S} \models^{ds}_{\boldsymbol{I},\boldsymbol{J}} \phi$), if it is so determined by the following conditions:

[L8] $\boldsymbol{S} \models^{ds}_{\boldsymbol{I},\boldsymbol{J}} \varsigma = \sigma$, if $[\![\varsigma^{\boldsymbol{S}}]\!]^{ds}_{\boldsymbol{I}} = [\![\sigma^{\boldsymbol{S}}]\!]^{ds}_{\boldsymbol{I}}$.
[L9] $\boldsymbol{S} \models^{ds}_{\boldsymbol{I},\boldsymbol{J}} \varsigma\mathbf{H}\tau$, if $[\![\varsigma^{\boldsymbol{S}}]\!]^{ds}_{\boldsymbol{I}}$ **is one of** $[\![\tau^{\boldsymbol{S}}]\!]^{ds}_{\boldsymbol{I}}$.
[L10] $\boldsymbol{S} \models^{ds}_{\boldsymbol{I},\boldsymbol{J}} \pi^n(\tau_1, \tau_2, \ldots, \tau_n)$, if π^n is a non-logical predicate and $\boldsymbol{J}(\pi^n)$ is instantiated by $[\![\tau_1^{\boldsymbol{S}}]\!]^{ds}_{\boldsymbol{I}}, [\![\tau_2^{\boldsymbol{S}}]\!]^{ds}_{\boldsymbol{I}}, \ldots, [\![\tau_n^{\boldsymbol{S}}]\!]^{ds}_{\boldsymbol{I}}$ in that order.
[L11] $\boldsymbol{S} \models^{ds}_{\boldsymbol{I},\boldsymbol{J}} [\phi \wedge \psi]$, if $\boldsymbol{S} \models^{ds}_{\boldsymbol{I},\boldsymbol{J}} \phi$ and $\boldsymbol{S} \models^{ds}_{\boldsymbol{I},\boldsymbol{J}} \psi$.
[L12] $\boldsymbol{S} \models^{ds}_{\boldsymbol{I},\boldsymbol{J}} \sim\phi$, if it is not the case that $\boldsymbol{S} \models^{ds}_{\boldsymbol{I},\boldsymbol{J}} \phi$.
[L13] $\boldsymbol{S} \models^{ds}_{\boldsymbol{I},\boldsymbol{J}} \exists\upsilon\phi$, if there is something x that **is one of** *ds* and ***there is*** an assignment $\boldsymbol{R}$ for $\mathscr{L}$ over *ds* such that $\boldsymbol{R} \cong^{\nu}_{x} \boldsymbol{S}$ and that $\boldsymbol{R} \models^{ds}_{\boldsymbol{I},\boldsymbol{J}} \phi$.
[L14] $\boldsymbol{S} \models^{ds}_{\boldsymbol{I},\boldsymbol{J}} \boldsymbol{\Sigma}\omega\phi$, if there are some things *xs* that **are some of** *ds* and ***there is*** an assignment $\boldsymbol{R}$ for $\mathscr{L}$ over *ds* such that $\boldsymbol{R} \cong^{\omega}_{xs} \boldsymbol{S}$ and that $\boldsymbol{R} \models^{ds}_{\boldsymbol{I},\boldsymbol{J}} \phi$.

Then we can complete the characterization of the logic of $\mathscr{L}$ in the usual fashion:

[L15] Let ϕ be a closed sentence of $\mathscr{L}$, and Γ a set of closed sentences of $\mathscr{L}$. Then

[a] $\boldsymbol{I}$ *with* $\boldsymbol{J}$ *satisfies* ϕ *over ds in* $\mathscr{L}$ (in symbols, $ds \models^{\mathscr{L}}_{\boldsymbol{I},\boldsymbol{J}} \phi$) if and only if ***any*** assignment for $\mathscr{L}$ over *ds* satisfies ϕ on $\boldsymbol{I}$ and $\boldsymbol{J}$ over *ds*.

[b] ***I*** *with* ***J*** *satisfies* Γ *over ds in* $\mathcal{L}$ (in symbols, *ds* $\models^{\mathcal{L}}_{\boldsymbol{I},\boldsymbol{J}}$ Γ) if and only if ***I*** with ***J*** satisfies every sentence in Γ over *ds* in $\mathcal{L}$.

[c] ϕ *is a model-theoretic consequence of* Γ *in* $\mathcal{L}$ (in symbols, $\Gamma \models^{\mathcal{L}} \phi$) if and only if ***any*** *c*-interpretation ***I*** and *p*-interpretation ***J*** for $\mathcal{L}$ over some things, *ds*, are such that if ***I*** with ***J*** satisfies Γ over *ds* in $\mathcal{L}$, then ***I*** with ***J*** satisfies ϕ over *ds* in $\mathcal{L}$.

[d] ϕ *is a model-theoretic truth of* $\mathcal{L}$ (in symbols, $\models^{\mathcal{L}} \phi$) if and only if $\Lambda \models^{\mathcal{L}} \phi$.[52]

Given this characterization of the logic of plural languages, it is straightforward to show that logical truths of plural languages are true. To see this, we may focus on meager plural languages. So let $\mathcal{L}$ be a meager plural language, and *ds* all the objects (in the world). Then let ***I*** be a *c*-interpretation for $\mathcal{L}$ over *ds* that assigns to any singular constant of $\mathcal{L}$ the object that it refers to, and ***J*** a *p*-interpretation for $\mathcal{L}$ over *ds* that assigns to any non-logical predicate of $\mathcal{L}$ the attribute that it indicates. Then a sentence ϕ of $\mathcal{L}$ is true, if ***I*** with ***J*** satisfies ϕ over *ds* in $\mathcal{L}$. So any sentence of $\mathcal{L}$ is true, if it is a model-theoretic truth of $\mathcal{L}$. (Similarly, it is straightforward to show that truths are closed under logical consequence.)

Let me complete this section by discussing two important features of plural logic. First, plural logic is a conservative extension of elementary logic. That is, it agrees with elementary logic on sentences under their common jurisdiction. Second, plural logic is non-compact. That is, there is a sentence that is logically implied by an infinite number of sentences, but not by any finite number of sentences among them. And a direct consequence of this is that plural logic is not axiomatizable.

To show that plural logic is a conservative extension of elementary logic, we need to use a characterization of elementary logic. We can obtain a model-theoretic characterization of elementary logic by simplifying the above characterization of plural logic. Dropping clauses pertaining to expressions not available in elementary languages[53] yields the definition of the predicate "$\models_e$" on elementary languages that satisfies the following condition:

> Let $\mathcal{L}$* be an elementary language, ϕ a sentence of $\mathcal{L}$*, and Γ a set of sentences of $\mathcal{L}$*. Then ϕ is a model-theoretic consequence$_e$ in $\mathcal{L}$* of Γ (in short, $\Gamma \models^{\mathcal{L}*}_e \phi$) if and only if ϕ is a logical consequence of the sentences in Γ in $\mathcal{L}$* on elementary logic (and $\Lambda \models^{\mathcal{L}*}_e \phi$ if and only if ϕ is a logical truth in $\mathcal{L}$* on elementary logic).

Then we can prove the following:

> *Conservativeness of Plural Logic*: Let $\mathscr{L}$ be a plural language, $\mathscr{L}^*$ an elementary language, ϕ a closed sentence of both $\mathscr{L}$ and $\mathscr{L}^*$, and Γ a set of closed sentences of both $\mathscr{L}$ and $\mathscr{L}^*$. Then $\Gamma \models^{\mathscr{L}} \phi$ if and only if $\Gamma \models_e^{\mathscr{L}^*} \phi$. (So ϕ is a logical consequence of the sentences in Γ on plural logic if and only if ϕ is a logical consequence of the sentences on elementary logic.)

So elementary language sentences are not logical truths on plural logic unless they are logical truths on elementary logic. So "$\exists x[\mathrm{r}\epsilon x \wedge \mathrm{w}\epsilon x]$", "$\mathrm{r} \neq \mathrm{w} \rightarrow \exists x[\mathrm{r}\epsilon x \wedge \mathrm{w}\epsilon x]$", and "$[\mathrm{C(j, b)} \wedge \mathrm{C(c, b)}] \rightarrow \exists x[\mathrm{j} \leq x \wedge \mathrm{c} \leq x]$", for example, are not logical truths on plural logic.

Using the conservativeness of plural logic, we can confirm that plural quantifications, such as "There are some critics who admire only one another" (sentence [13]), do not logically imply singular quantifications over composite objects. Consider the following sentences:[54]

[16a] $[\mathrm{C_r(e)} \wedge \mathrm{C_r(t)}] \wedge \mathrm{e} \neq \mathrm{t} \wedge [\forall z(\mathrm{A(e,} z) \rightarrow z = \mathrm{t}) \wedge \forall z(\mathrm{A(t,} z) \rightarrow z = \mathrm{e})]$.
[15a] $\forall y[y\mathbf{H}[\mathrm{e@t}] \rightarrow [\mathrm{C_r}(y) \wedge \forall z(\mathrm{A}(y, z) \rightarrow y \neq z \wedge z\mathbf{H}[\mathrm{e@t}])]]$.
[13a] $\mathbf{\Sigma} xs \forall y[y\mathbf{H}xs \rightarrow [\mathrm{C_r}(y) \wedge \forall z(\mathrm{A}(y, z) \rightarrow y \neq z \wedge z\mathbf{H}xs)]]$.
[14a] $\exists x[\exists y y\epsilon x \wedge \forall y(y\epsilon x \rightarrow [\mathrm{C_r}(y) \wedge \forall z(\mathrm{A}(y, z) \rightarrow y \neq z \wedge z\epsilon x)])]$.

[16a] logically implies [15a],[55] and [15a] does [13a][56] (on plural logic). So [13a] does not logically imply [14a]. Otherwise, [16a] must logically imply [14a]; but [16a] does not do so on plural logic, by conservativeness, because it does not do so on elementary logic. Similarly, plural logic yields the result that the following plural constructions do not logically imply the existence of any composites, such as sets, classes, aggregates, or 'pluralities':

John and Carol are children.
Some humans admire one another.
There are some critics who admire only one another. (sentence [13])
There are some philosophers who write a book.
There are twelve apostles.

This, to be sure, does not mean that there are no composite objects. It means that to use plurals to assert [13], for example, is not to commit oneself to the existence of sets, classes, or the like. One can commit oneself to the existence of composites by asserting, for example, "There are some sets that are not members of themselves" or There are some classes that are the non-self-membered classes." These sentences logically imply the existence of sets or classes (they logically imply "There is a set" or "There is a class"). But they do not logically imply the existence of any composites *of* sets or classes. The first of them is logically equivalent to "There is a set that is not a member of itself", and the second to "There is a non-self-membered class."[57]

Another important feature of plural logic is that it is not compact. To see this, consider the following:

[32] $\mathbf{\Sigma} xs(\forall y\mathbf{H}xs)(\exists z\mathbf{H}xs)\mathrm{A}(y, z)$.[58]

Now, let Γ^* be the infinite set {"$\mathrm{A}(\mathrm{c}_n, \mathrm{c}_{n+1})$": n is a natural number}. We can then see that [32] is logically implied by the sentences in Γ^*, but not by any finite number of sentences among them ([32] is a model-theoretic consequence of Γ^*, but not of any finite subset of Γ^*).[59]

An immediate corollary of the non-compactness of plural logic is that it is not axiomatizable. In particular, there is no adequate axiomatic system of logic that captures the logical relation between the sentences in Γ^* and [32]. And we can use this logical relation to obtain another interesting result: [32] is not expressible in elementary languages. Because all the sentences in Γ^* are elementary language sentences, any elementary language sentence logically implied by them must be logically implied by a finite number of sentences among them (for elementary logic is compact). So, no sentence in elementary languages can be logically equivalent to [32]. Similarly, we can get Kaplan's result: [13] is not expressible in elementary languages. Consider an infinite sequence of sentences of the form "$\mathrm{C}(\mathrm{c}_n) \wedge \mathrm{c}_n \neq \mathrm{c}_{n+1} \wedge \forall z[\mathrm{A}(\mathrm{c}_n, z) \rightarrow z = \mathrm{c}_{n+1}]$." [13a], the plural language paraphrase of [13], is logically implied by these elementary language sentences, but not by any finite number of sentences among them. So [13a] has no logical equivalent in elementary languages; that is, [13] cannot be paraphrased into elementary languages.[60]

7. PLURAL LOGIC: PARTIAL AXIOMATIZATION

We cannot give a complete axiomatization of plural logic, as we have seen, but we can formulate strong axiomatic systems that yield good approximations to plural logic. This is what I aim to do in this section. I present a system of logic, S_{PL}, that captures most of the logical relations pertaining to plurals that we usually appeal to. In particular, all the logical relations that I have invoked to argue that plurals are not reducible to singulars can be captured by the system.[61]

Let $\mathscr{L}$ be a plural language. Say that a sentence ϕ of $\mathscr{L}$ is a *closure* of a sentence ψ of $\mathscr{L}$, if one of the following conditions holds:

[a] $\phi = \psi$.
[b] $\phi = Q_1 \nu_1 Q_2 \nu_2 \ldots Q_n \nu_n \psi$, where $Q_1, Q_2, \ldots, Q_n$ are singular or plural universal quantifiers and $\nu_1, \nu_2, \ldots, \nu_n$ variables suitable for $Q_1, Q_2, \ldots, Q_n$, respectively.[62]

Then the axioms of System S_{PL} for $\mathscr{L}$ are the closures of instances of the following:[63]

Group A

Ax. 1. Truth-functional tautologies.
Ax. 2. $\phi(\upsilon/\varsigma) \rightarrow \exists \upsilon \phi$, where ς is substitutable for υ in ϕ.
Ax. 3. $\forall \upsilon[\phi \rightarrow \psi] \rightarrow [\forall \upsilon \phi \rightarrow \forall \upsilon \psi]$.
Ax. 4. $\phi \rightarrow \forall \upsilon \phi$, where υ does not occur free in ϕ.[64]
Ax. 5. $\varsigma = \sigma \rightarrow [\pi^n(\tau_1, \ldots, \tau_{i-1}, \varsigma, \tau_{i+1}, \ldots, \tau_n) \rightarrow \pi^n(\tau_1, \ldots, \tau_{i-1}, \sigma, \tau_{i+1}, \ldots, \tau_n)]$.

Group B

Ax. 6. $\phi(\omega/\tau) \rightarrow \mathbf{\Sigma} \omega \phi$, where τ is substitutable for ω in ϕ.
Ax. 7. $\mathbf{\Pi} \omega[\phi \rightarrow \psi] \rightarrow [\mathbf{\Pi} \omega \phi \rightarrow \mathbf{\Pi} \omega \psi]$.
Ax. 8. $\phi \rightarrow \mathbf{\Pi} \omega \phi$, where ω does not occur free in ϕ.

Group C

Ax. 9. $\exists \upsilon \upsilon \mathbf{H} \tau$.
Ax. 10. $\varsigma \mathbf{H} \sigma \rightarrow \varsigma = \sigma$.
Ax. 11. $\varsigma \mathbf{H}[\tau @ \mu] \leftrightarrow [\varsigma \mathbf{H} \tau \vee \varsigma \mathbf{H} \mu]$.[65]
Ax. 12. $\mu \approx \mu_1 \rightarrow [\pi^n(\tau_1, \ldots, \tau_{i-1}, \mu, \tau_{i+1}, \ldots, \tau_n) \rightarrow \pi^n(\tau_1, \ldots, \tau_{i-1}, \mu_1, \tau_{i+1}, \ldots, \tau_n)]$.
Ax. 13. $\exists \upsilon \phi \rightarrow \mathbf{\Sigma} \omega \forall \upsilon[\upsilon \mathbf{H} \omega \leftrightarrow \phi]$, where ω does not occur free in ϕ.

S_{PL} has only one rule of inference:

Modus Ponens: from ϕ and $[\phi \rightarrow \psi]$, we may infer ψ.

So deducibility and provability can be defined as follows:

Deducibility in S_{PL}: Let ϕ be a sentence of $\mathscr{L}$, and Γ a set of sentences of $\mathscr{L}$. Then ϕ *is deducible from* Γ in S_{PL} for $\mathscr{L}$ (in symbols, $\Gamma \vdash^{\mathscr{L}} \phi$) if and only if it can be so determined by the following conditions:

[a] $\Gamma \vdash^{\mathscr{L}} \phi$, if ϕ is an axiom of S_{PL} for $\mathscr{L}$.
[b] $\Gamma \vdash^{\mathscr{L}} \phi$, if ϕ is in Γ.

[c] $\Gamma \vdash^{\mathscr{L}} \phi$, if there is a sentence ψ of $\mathscr{L}$ such that $\Gamma \vdash^{\mathscr{L}} [\psi \rightarrow \phi]$ and that $\Gamma \vdash^{\mathscr{L}} \psi$.

Provability in S_{PL}: Let ϕ be a sentence of $\mathscr{L}$. Then ϕ *is provable* in S_{PL} for $\mathscr{L}$ (in symbols, $\vdash^{\mathscr{L}} \phi$) if and only if $\Lambda \vdash^{\mathscr{L}} \phi$.

Now, we can show in the usual fashion that S_{PL} is a sound system for plural logic:

Soundness of S_{PL}: Let ϕ be a closed sentence of $\mathscr{L}$, and Γ a set of closed sentences of $\mathscr{L}$. Then the following hold:

[a] If $\Gamma \vdash^{\mathscr{L}} \phi$, then $\Gamma \models^{\mathscr{L}} \phi$. (So ϕ is a logical consequence of the sentences in Γ on plural logic, if ϕ is deducible from Γ in S_{PL}.)
[b] If $\vdash^{\mathscr{L}} \phi$, then $\models^{\mathscr{L}} \phi$. (So ϕ is a logical truth on plural logic, if ϕ is provable in S_{PL}.)

But S_{PL} is not complete with regard to plural logic. Γ^* and [32], specified above (see §6), yield a counterexample to the converse of [a]. The converse of [b] does not hold, either. We can show that the following is a logical truth of plural languages that contain the one-place plural predicate "**D**", but not a theorem of S_{PL} for the languages:

[33] $\mathbf{\Sigma} xs \mathbf{\Pi} ys([\mathbf{D}(ys) \wedge \mathbf{\Pi} zs(\forall x \mathbf{H} zs)([\mathbf{D}(zs) \wedge x\mathbf{H}ys] \rightarrow zs \approx ys)] \rightarrow \exists x \forall y(y = x \leftrightarrow [y\mathbf{H}xs \wedge y\mathbf{H}ys]))$.[66]

Let me now discuss individual axioms of S_{PL}. The axioms of Group A are essentially those of elementary logic.[67] To discuss axioms of Groups B and C, it is useful to consider them together with some of their straightforward consequences:

Th. 1. $\exists \upsilon \phi(\omega/\upsilon) \rightarrow \mathbf{\Sigma} \omega \phi$, where υ does not occur free in ϕ and is substitutable for ω in ϕ.

Th. 2.

[a] $\{\phi\}^N(\upsilon/\tau) \rightarrow \exists \upsilon \phi$, where υ does not occur in τ.
[b] $\mathbf{\Sigma} \omega \{\phi\}^N(\upsilon/\omega) \leftrightarrow \exists \upsilon \phi$, where ω does not occur free in ϕ.
[c] $\mathbf{\Sigma} \omega \pi^{N(i)}(\tau_1, \ldots, \tau_{i-1}, \omega, \tau_{i+1}, \ldots, \tau_n) \leftrightarrow \exists \upsilon \pi(\tau_1, \ldots, \tau_{i-1}, \upsilon, \tau_{i+1}, \ldots, \tau_n)$, where ω and υ occur in none of $\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_n$.
[d] $\{\phi \wedge \psi\}^N(\upsilon/\tau) \leftrightarrow [\{\phi\}^N(\upsilon/\tau) \wedge \{\psi\}^N(\upsilon/\tau)]$, where υ does not occur in τ or μ.

Th. 3.

[a] $\varsigma \mathbf{H} \varsigma$.
[b] $\varsigma \mathbf{H} \sigma \leftrightarrow \varsigma = \sigma$.

[c] **Many**$(\tau) \leftrightarrow \sim$**One**$(\tau)$.

Th. 4.

[a] $\{\phi\}^{N}(\upsilon/\varsigma) \leftrightarrow \phi(\upsilon/\varsigma)$, where ς is substitutable for υ in ϕ.
[b] $\pi^{N(i)}(\tau_1, \ldots, \tau_{i-1}, \varsigma, \tau_{i+1}, \ldots, \tau_n) \leftrightarrow \pi(\tau_1, \ldots, \tau_{i-1}, \varsigma, \tau_{i+1}, \ldots, \tau_n)$, where ς occurs in none of $\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_n$.
[c] $\exists\upsilon\pi^{N(i)}(\tau_1, \ldots, \tau_{i-1}, \upsilon, \tau_{i+1}, \ldots, \tau_n) \leftrightarrow \exists\upsilon\pi(\tau_1, \ldots, \tau_{i-1}, \upsilon, \tau_{i+1}, \ldots, \tau_n)$, where υ occurs in none of $\tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_n$.

Th. 5. $\{\phi\}^{\mathrm{N}}(\upsilon/[\tau@\mu]) \leftrightarrow [\{\phi\}^{\mathrm{N}}(\upsilon/\tau) \wedge \{\phi\}^{\mathrm{N}}(\upsilon/\mu)]$, where υ does not occur in τ or μ.

Th. 6.

[a] $\varsigma\mathbf{H}[\sigma@\mu] \leftrightarrow [\varsigma = \sigma \vee \varsigma\mathbf{H}\mu]$.
[b] $\varsigma\mathbf{H}[\mu@\sigma] \leftrightarrow [\varsigma\mathbf{H}\mu \vee \varsigma = \sigma]$.
[c] $\varsigma\mathbf{H}[\sigma_1@\sigma_2] \leftrightarrow [\varsigma = \sigma_1 \vee \varsigma = \sigma_2]$.

Th. 7. $\varsigma = \varsigma$.

Th. 8. $\tau \approx \tau$.

Th. 9. $\varsigma = \sigma \leftrightarrow \varsigma \approx \sigma$.

Th. 10.

[a] $\tau \approx \mu \rightarrow [\phi(\omega/\tau) \leftrightarrow \phi(\omega/\mu)]$, where τ and μ are subtitutable for ω in ϕ.
[b] $\phi(\omega/\tau) \leftrightarrow \mathbf{\Sigma}\omega(\omega \approx \tau \wedge \phi)$, where τ is subtitutable for ω in ϕ, and ω does not occur in τ.
[c] $\pi^{n}(\tau_1, \ldots, \tau_{i-1}, [\mu_1@\mu_2], \tau_{i+1}, \ldots, \tau_n) \equiv \mathbf{\Sigma}\omega[\forall\upsilon(\upsilon\mathbf{H}\omega \leftrightarrow [\upsilon\mathbf{H}\mu_1 \wedge \upsilon\mathbf{H}\mu_2]) \vee \pi^{n}(\tau_1, \ldots, \tau_{i-1}, \omega, \tau_{i+1}, \ldots, \tau_n)]$, where ω occurs in none of $\mu_1, \mu_2, \tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_n$, and υ in neither μ_1 nor μ_2.

Th. 11. $\mathbf{\Sigma}\omega\forall\upsilon[\upsilon\mathbf{H}\omega \leftrightarrow \phi] \leftrightarrow \exists\upsilon\phi$, where ω does not occur free in ϕ.

Th. 12.

[a] $\mathbf{\Sigma}\omega\forall\upsilon\upsilon\mathbf{H}\omega$.
[b] $\mathbf{\Sigma}\omega\mathbf{\Pi}\omega_1\omega_1 \sqsubseteq \omega$.

Th. 13.

[a] $\upsilon\mathbf{H}<\upsilon\colon \phi> \leftrightarrow \phi$.

[b] $\boldsymbol{\Sigma}\omega\omega \approx <\upsilon\colon \phi> \leftrightarrow \exists\upsilon\phi$, where ω does not occur free in φ.
[c] $<\upsilon\colon \phi> \approx <\upsilon\colon \psi> \leftrightarrow [\exists\upsilon\phi \wedge \forall\upsilon(\phi \leftrightarrow \psi)]$.

Th. 14.

[a] $\Sigma\omega\omega \approx <\phi>_{\nu}^{\upsilon/\tau}$, where υ does not occur in τ, and ω does not occur free in ν, τ, or φ.
[b] $\tau \sqsubseteq <\phi>_{\nu}^{\upsilon/\tau}$, where υ does not occur in τ.
[c] $(\forall\upsilon\mathbf{H}<\phi>_{\nu}^{\upsilon/\tau})\forall\upsilon_1[\phi(\nu/\upsilon_1) \rightarrow \upsilon_1\mathbf{H}<\phi>_{\nu}^{\upsilon/\tau}]$, where υ does not occur in τ, υ is not υ_1, υ_1 does not occur free in φ, and υ_1 is substitutable for ν in φ.

Axioms of Group B are plural cousins of *Axs.* 2–4, and characterize the logic of plural quantifications. The role of *Axs.* 7 & 8 lies mostly in helping to draw consequences of the other axioms. Their singular cousins, *Axs.* 3 & 4, together with truth-functional tautologies, yield the following metatheorem:

∃-*Elimination*: Let υ be a singular variable that does not occur free in ψ or Γ.[68] Then if $\Gamma \vdash^{\mathscr{L}} [\phi \rightarrow \psi]$, then $\Gamma \vdash^{\mathscr{L}} [\exists\upsilon\phi \rightarrow \psi]$.

Similarly, *Axs.* 7 & 8 yield the following:

Σ-*Elimination*: Let ω be a plural variable that does not occur free in ψ or Γ. Then if $\Gamma \vdash^{\mathscr{L}} [\phi \rightarrow \psi]$, then $\Gamma \vdash^{\mathscr{L}} [\Sigma\omega\phi \rightarrow \psi]$.

And the two metatheorems yield the following:

Universal Generalization: Let ψ be a closure of φ, and Γ a set of closed sentences of $\mathscr{L}$. Then if $\Gamma \vdash^{\mathscr{L}} \phi$, then $\Gamma \vdash^{\mathscr{L}} \psi$.

So the metatheorems justify derivative rules of inference that we can use to draw consequences of the other axioms.

Ax. 6 has the following instance:

$$\mathrm{C}^N([\mathrm{j}@\mathrm{c}]) \wedge \mathbf{L}([\mathrm{j}@\mathrm{c}], \mathrm{b}) \rightarrow \boldsymbol{\Sigma} xs\left[\mathrm{C}^N(xs) \wedge \mathbf{L}(xs, \mathrm{b})\right].$$

This is a straightforward paraphrase of the conditional of which "**John and Carol** are *children*" and "**There are some** *children who lift Bob*" (sentences [3] and [4]) are the antecedent and consequent, respectively. So we can use plural logic to explain that [3] logically implies [4], as we can use elementary logic to explain that "John is a child who is healthy" logically implies "There is a child who is healthy." Similarly, we can use plural logic to explain that plural predications, such as the following, logically imply their plural existential generalizations:[69]

John and Carol *are children*.
John and Carol are *children who are healthy*.
Russell, Moore, and Whitehead are *famous humans who are three philosophers*.
Russell, Whitehead, Ezra, and Thomas are *people who admire (only) one another*.
Bill, Boris, Ted, Dick, and Ken are *politicians who cooperate*.

Note that the following are also instances of *Ax.* 6 (for "τ" is a meta-variable for *any* terms):

$$\mathrm{P}^N(\mathrm{c}_i) \rightarrow \mathbf{\Sigma} xs \mathrm{P}^N(xs).$$
$$\left[\mathrm{C}^N(\mathrm{j}) \wedge \mathrm{H}^N(\mathrm{j})\right] \rightarrow \mathbf{\Sigma} xs\left[\mathrm{C}^N(xs) \wedge \mathrm{H}^N(xs)\right].$$
$$\left[\mathrm{C}^N(\mathrm{j}) \wedge \mathbf{L}(\mathrm{j},\mathrm{b})\right] \rightarrow \mathbf{\Sigma} xs\left[\mathrm{C}^N(xs) \wedge \mathbf{L}(xs,\mathrm{b})\right].$$

So we can use plural logic to show that the following are also logical truths:[70]

If Cicero is a philosopher, then there are some philosophers.
If John is a child who is healthy, then some children are healthy.
If John is a child who lifts Bob, then some children lift Bob.

Some might object that these are not logical truths because the plural quantification "There are some philosophers", for example, implies the existence of *at least two* philosophers. But it is wrong to hold this. The plural quantification is logically implied by "Cicero and Tully are philosophers", which is logically implied by "Cicero is a philosopher, and Tully is a philosopher."[71] Clearly, the last sentence can be true whether or not Cicero is Tully. So the quantification cannot logically imply "There are at least two philosophers." The plural quantifier phrase "there are some" must be taken to be interchangeable with "there are some one or more."

Given ∃-*Elimination*, *Ax.* 6 yields *Th.* 1. So singular existential quantifications logically imply the corresponding plural existential quantifications. Surely, the converse does not hold. "$\mathbf{\Sigma} xs\mathbf{L}(xs, \mathrm{b})$" does not logically imply "$\exists x\mathbf{L}(x, \mathrm{b})$."[72] But some plural quantifications are logically equivalent to their singular cousins. "$\mathbf{\Sigma} xs\mathrm{C}^N(xs)$", for example, is logically equivalent to "$\exists x\mathrm{C}(x)$", which is logically equivalent to "$\exists x\mathrm{C}^N(x)$". Similarly, the following three existential quantifications are logically equivalent to each other:

[8a] $\mathbf{\Sigma} xs[\mathrm{C}^N(xs) \wedge \mathrm{H}^N(xs)]$.

[2b] $\exists x[\mathrm{C}^N(x) \wedge \mathrm{H}^N(x)]$.
[2a] $\exists x[\mathrm{C}(x) \wedge \mathrm{H}(x)]$.

We can show the equivalences in S_{PL}. To do so, however, we need to appeal to axioms of Group C because the equivalences rest on the logic of neutral expansions.

Group C pertains to the logical predicate "**H**". *Ax.* 9,[73] given the definition of neutral expansion, yields *Th.* 2.[74] This yields the result that "$\boldsymbol{\Sigma} xs\mathrm{C}^N(xs)$" and "$\exists x\mathrm{C}(x)$" are logically equivalent. *Axs.* 9 & 10 yield *Th.* 3,[75] and this yields *Th.* 4. So "$\mathrm{C}^N(x)$" and "$\mathrm{C}(x)$" are derivable from each other, which yields the equivalence between "$\exists x\mathrm{C}^N(x)$" and "$\exists x\mathrm{C}(x)$." Similarly, *Ths.* 2 & 4 yield the result that [8a], [2a], and [2b] are derivable from one another.[76]

Notice that [8a] is the straightforward paraphrase of "Some children are healthy" (sentence [8]) into plural languages whereas [2a] is the straightforward paraphrase of its singular cousin "A child is healthy" (sentence [2]). So plural logic can explain the logical equivalence between [8] and [2], and justify the practice of paraphrasing [8] by [2a] (see Yi (LMP I, §3.3.2)). Similarly, the logic can explain why some special plural predications can be paraphrased by singular conjunctions. Consider, for example, "John and Carol are children" (sentence [7]). Its straightforward paraphrase is the plural predication "$\mathrm{C}^N([\mathrm{j}@\mathrm{c}])$" (sentence [7a]). But it is usual to paraphrase [7] by "$[\mathrm{C}(\mathrm{j})\wedge\mathrm{C}(\mathrm{c})]$" (sentence [9a]), the elementary language sibling of the singular cousin of [7]: "John is a child and Carol is a child" (sentence [9]). Using the distributivity of neutral expansions over plural terms formed by "**@**", we can justify the usual practice. *Th.* 5, which states the distributivity, results from *Ax.* 9, and *Th.* 5, together with *Th.* 4, yields the equivalence between [9a] and [7a].[77] This explains that [7] is logically equivalent to [9].

We can also explain the logical equivalence between, e.g., "Cicero is one of John and Tully" and "Cicero is either John or Tully." Their equivalence is captured by *Th.* 6, which is essentially a result of *Ax.* 11.[78] *Th.* 6 captures other similar logical equivalences, such as the equivalence between "Jack is one of the boys and the girls" and "Jack is either one of the boys or one of the girls." We can also explain that "Jack is one of the boys" is logically equivalent to "Jack is a boy", but this requires *Ax.* 13.[79]

We can prove in S_{PL} both the principle of reflexivity of identity and its plural cousin. *Th.* 7, which formulates the principle of reflexivity of identity, is an immediate corollary of *Th.* 3, which results from *Ax.* 9. Note that *Th.* 8 (i.e., $\tau \approx \tau$) is the plural cousin of *Th. 7*, because the sameness predicate "$\approx$" is the plural cousin of the identity predicate "=". Its instances include paraphrases of the following sentences:

Russell and Whitehead *are* Russell and Whitehead.
Bill, Hillary, and Chelsea *are* Bill, Hillary, and Chelsea.
They *are* themselves.[80]

We can see that these, like "Russell is Russell", are logical truths.[81] And we can show this in S_{PL}. Given the definition of "$\approx$" (see *Def.* 2), *Th.* 8 results from axioms of Group A.[82]

Let me turn to the last two axioms, *Axs.* 12 & 13. *Ax.* 12 is the plural cousin of *Ax.* 5. It encapsulates the plural cousin of substitutivity of identity: if some things are the same things as some things and the former are so-and-so, then the latter must also be so-and-so. So we can use *Ax.* 12 to show that the following are logical truths:

If Russell and Whitehead are the authors of *PM*, and Russell and Whitehead cooperate, then the authors of *PM* cooperate.
If the children who lift Bob are John and Carol, then they play together if and only if John and Carol play together.
If something is one of Bill and Hillary if and only if it is a parent of Chelsea, then Chelsea's parents live in Washington only if Bill and Hillary live in Washington.
If there are some things that are John and Carol, then they lift a piano just in case John and Carol lift it.

Note that *Ax.* 12, like *Ax.* 5, applies only to atomic sentences. But we can use it to derive its generalization that applies to all sentences (see *Th.* 10 [a]).[83]

To see the content of *Ax.* 13, consider one of its instances:

[34] $\exists x[\mathrm{C}(x) \wedge \mathrm{H}(x)] \rightarrow \mathbf{\Sigma} zs \forall x(x\mathbf{H}zs \leftrightarrow [\mathrm{C}(x) \wedge \mathrm{H}(x)])$.

The antecedent of [34] (i.e., [2a]) is the usual paraphrase of [2], "A child is healthy", and its consequent is a straightforward paraphrase of the following:

[35] There are some things such that something is one of them if and only if it is a healthy child.

So *Ax.* 13 yields the result that [2] logically implies [35]. Note that the converse holds as well. We can show this, too, in S_{PL}. *Ax.* 9 yields the converse of *Ax.* 13. So *Th.* 11 results from *Axs.* 9 & 13. This explains the logical equivalence between [2] and [35].[84]

Ax. 13 yields the principle that there are some things that include everything (see *Th.* 12).[85] And it is pivotal to characterizing the logic of plural definite descriptions. The axiom, given *Def.* 5, yields *Th.* 13.[86] The instances of [a]–[b] include the following:

$$\mathrm{j}\mathbf{H}<x: [\mathrm{C}(x) \wedge \mathrm{H}(x)]> \leftrightarrow [\mathrm{C}(\mathrm{j}) \wedge \mathrm{H}(\mathrm{j})].$$
$$\mathbf{\Sigma} ys\ ys \approx <x: [\mathrm{C}(x) \wedge \mathrm{H}(x)]> \leftrightarrow \exists x[\mathrm{C}(x) \wedge \mathrm{H}(x)].$$

So we can show that "John is one of *the* happy children (in the world)" and "There are *the* happy children (in the world)" are logically equivalent to "John is a happy child" and "There is a happy child", respectively. Similarly, we can show that "Genie is one of *the* non-self-membered sets" is logically equivalent to "Genie is a non-self-membered set", and "There are *the* non-self-membered sets" to "There is a non-self-membered set."[87] [a] and [b] have a consequence, [c], that it is useful to compare with Frege's infamous Law V. Although this is inconsistent as Russell (1902) has shown, its counterpart in plural languages is correct except for the case in which the open sentences in question are satisfied by no object.[88]

Th. 14 is a corollary of *Th.* 13. So we can use plural logic to explain that, roughly, "If there are some philosophers, then there are the things that are either the philosophers or their ancestors" is a logical truth.[89]

Now, *Ax.* 13 has an intriguing consequence that lies under Cantor's Theorem. We cannot prove the theorem without assuming proper axioms of set theory, because it concerns the existence of sets.[90] But there is a logical principle underlying Cantor's theorem, and others of its sort, that does not pertain to sets, classes, or the like.

To have a grasp of the underlying principle, consider sets with two or more members, such as the doubleton {Russell, Whitehead}. This set has more subsets than it has members. So there is no way to assign one of its members to each one of its subsets without assigning the same member to two or more of the subsets. This holds, to be sure, whether or not the doubleton has a power set. And it holds even if we ignore the empty set: we cannot assign one of the members of the doubleton to each one of its *non-empty* subsets without assigning the same member more than once. This still concerns sets as well as the two humans, Russell and Whitehead, who are the members of the doubleton. We can see, however, that there is a parallel fact that concerns only the two humans. Suppose that you assign one of them to *any things that are some of them*.[91] That is, you are to assign one of them to *Russell* (because he is some of them), one of them to *Whitehead* (because he is also some of them), and one of them to the two humans, *Russell and Whitehead* (because they are also some of themselves). To do so, you must assign Russell or else Whitehead more than once. One of the two must be assigned to some things among them while being assigned to some other things among them as well.

Now, the situation is the same *as long as* the things in question are more than one.[92] In plural languages, we can state this without invoking any sets, classes, or the like:

Th. 15. $[\mathbf{M}\textsc{any}(\tau) \wedge (\mathbf{\Pi}\omega{\sqsubseteq}\tau)(\exists \upsilon \mathbf{H}\tau)\psi] \rightarrow (\mathbf{\Sigma}\omega{\sqsubseteq}\tau)(\mathbf{\Sigma}\omega_1{\sqsubseteq}\tau)(\exists \upsilon \mathbf{H}\tau)[\sim\omega \approx \omega_1 \wedge \psi \wedge \psi\,(\omega/\omega_1)]$, where ω_1 does not occur free in ψ and is substitutable for ω in ψ, ω is not ω_1, and ω, ω_1, and υ do not occur in τ.

It is useful to compare this with a principle that can be taken to concern the special cases in which the things in question are all of the things in the world:

Th. 16. $[\exists \upsilon \exists \upsilon_1 \upsilon \neq \upsilon_1 \wedge \mathbf{\Pi}\omega \exists \upsilon \psi] \rightarrow \mathbf{\Sigma}\omega\mathbf{\Sigma}\omega_1 \exists \upsilon[\sim\omega \approx \omega_1 \wedge \psi \wedge \psi\,(\omega/\omega_1)]$, where ω_1 does not occur free in ψ and is substitutable for ω in ψ, ω is not ω_1, and υ is not υ_1.

This formulates the principle that if there are at least two things (in the world), then there is no 'one'-to-one function that assigns to any things whatsoever some one thing–that is, if any things whatsoever (taken together) have some one thing assigned to them, then there must be something that is assigned to some things while being assigned to some other things as well. *Th.* 15 formulates a more general principle that applies even to those cases in which the things in question are not all of the things in the world: if there are some things that are many, then there is no 'one'-to-one function that assigns to any things *that are some of those things* something that *is one of them*. Using *Ax.* 13, we can show that *Ths.* 15 & 16 are logical principles.[93]

8. Concluding Remarks

The accounts of plurals presented above are based on the conception of plurals as devices for talking about the many. On this conception, plurals belong to categories on a par with those to which their singular cousins belong, and have a special semantic function. They are, by and large, devices for talking about many things (as such), whereas singulars are devices for talking about one thing ('at a time'). A typical plural term refers to (or denotes) many things, whereas a typical singular term refers to (or denotes) one thing; and a plural predicate indicates a plural attribute, whereas a singular predicate indicates a singular attribute. Those who hold this conception do not find it surprising that some plurals have no singular equivalents. This confirms the potency of plurals as devices with a separate function: we can indeed use them to say things that we cannot say using only their singular cousins. What is somewhat surprising is that some plurals do have singular equivalents. This calls for explanation. And we can explain it, as we have seen, by analyzing the logic of plurals.

It is wrong to infer from the potency of plurals the existence of special objects (e.g., 'plural objects') that we cannot talk about using singular terms. Surely, there can be new things to say without talking about new objects. Consider limitations of languages that contain no predicates except one-place predicates. Using such languages, we cannot say some things that we can say using sentences that contain two-place predicates, such as "Everyone who *draws* a circle *draws* a figure."[94] This is because there are special attributes (i.e., relations) that predicates of the languages cannot relate to. Similarly, there are special attributes that predicates of singular languages cannot relate to. Singular languages have no predicates that indicate *plural attributes*. So we cannot use the languages to attribute a plural property (e.g., cooperating) to many things (e.g., the authors of *PM*) without separating them.

Acknowledging plural attributes requires a radical departure from traditional conceptions of reality that date back to Aristotle. One of their central tenets is the thesis that there can be no plural attributes. I think that this thesis lies under the prevalent bias against plurals. One cannot recognize genuine plural predicates without acknowledging plural attributes, and this leaves those committed to the thesis no choice but to try to accommodate natural language plurals as mere abbreviation devices. But the conception of plurals as abbreviation devices is ruled out by the logic of plurals, which clarifies the potency of plurals. So we must reject traditional conceptions of reality and accept a liberal conception that acknowledges plural attributes.

The conception of plurals as devices for talking of the many and the liberal conception of reality complement, and call for, each other. And they yield natural accounts of the logic and meaning of plurals. The account of logic based on them surpasses contemporary Fregean accounts in its scope. This extension of the scope of logic results from extending the range of languages that logic can directly relate to.

Underlying the view of language that makes room for this is a perspective on reality that locates in the world what plurals can relate to. Ruminations over plurals, I think, point to a broader framework for understanding logic, language, and reality that can replace the contemporary Fregean framework as this replaced its Aristotelian ancestor.

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APPENDIX 1: REMARKS ON BOOLOS'S SEMANTICS OF SECOND-ORDER LANGUAGES

Meager second-order languages, like elementary languages, have no second-order predications. For second-order variables are variables that can replace *first-order* predicates. Boolos (1985a, pp. 335–7) takes advantage of this limitation of those languages to extend Tarski's characterization of truth for elementary languages to *singular meager monadic second-order languages*, languages that can be obtained from elementary languages by adding just monadic second-order variables and quantifiers (call them *B-languages*).[95] His idea is to specify the semantics of those variables by considering relations similar to the *designation relation* for first-order predicates. Say that relations similar to the designation relation except that they pertain to the variables of a B-language are *D-relations* for the language. Then we can give recursive definitions of satisfaction predicates for such languages by clauses that include the following:

[B1] Let *S* be an assignment function for the first-order variables of a B-language, and *R* a D-relation for the language. Then *R with S satisfies* the predication X(v), if *R* relates the variable X to *S*(v) (in short, *R*(X, *S*(v))).[96]

(Boolos's own characterization of truth is based on a variant of [B1] that does away with the *dyadic* second-order variable "*R*" and "*S*".[97]) But one cannot generalize [B1] (or its variants) for *second-order predications*, which have first-order predicates or second-order variables in the argument positions. One cannot use languages acceptable to Boolos (or his followers)[98] to characterize the satisfaction condition of second-order predications involving first-order predicates by appealing to what these predicates designate; nor can one use those languages to characterize the satisfaction condition of predications involving second-order variables by appealing to a given D-relation for these variables. So one cannot extend Boolos-style characterizations of truth to languages with higher-order predications, including plenary second-order languages and meager third-order languages.[99]

There is a close, if not exact, parallel to be drawn between *meager* (first-order) plural languages and B-languages. So it is straightforward to give a Boolos-style characterization of truth for those plural languages in singular languages acceptable to Boolos. Say that a singular relation *R* that pertains to the plural variables of a meager plural language is a *D-relation* for the language, *if* every one of the variables is related by *R* to at least one object.[100] Then we can give recursive definitions of satisfaction predicates for such languages by clauses that include the following:

[B2] Let *S* be an assignment function for the first-order variables of a meager (first-order) plural language, and *R* a D-relation for the language. Then *R with S satisfies* the plural predication ν**H**ω, if *R* relates the plural variable ω to *S*(ν) (in short, *R*(ω, *S*(ν))).

But Boolos's followers cannot generalize this clause to the usual plural predications, those formed by non-logical plural predicates, such as "**cooperate**(*xs*)" (in symbols, "$\mathbf{C_o}$(*xs*)"). So one cannot extend the Boolos-style characterization of truth based on [B2] to languages with non-logical plural predicates.[101]

It is worthwhile to note that Boolo's characterization of truth, unlike Tarski-style characterizations, can be turned into a characterization of logic.[102] Rayo and Uzquiano (1999) present a Boolos-style characterization of logic for B-languages.[103] They apply Boolos's idea of using D-relations for second-order variables to first-order predicates (e.g., "ϵ") in specifying the interpretation of the predicates in a model. So their characterization of logic makes essential use of the lack of second-order predications in those languages, and inherits the limitations of Boolos's characterization of truth. The characterization of logic cannot be extended to languages with second-order predications. And it cannot be turned to a characterization of logic for (first-order) plural languages with non-logical plural predicates, although it is straightforward to turn it into one for (first-order) plural languages without those predicates.

Defenders of Boolos might reply that there is a sharp line to draw between B-languages and languages with second-order predications. They might take Boolos to show that the former are legitimate languages, and argue that second-order predications cannot be considered legitimate unless they can be reduced to constructions available in B-languages. But this is not a view that Boolos can hold. The three-place satisfaction predicate "... with --- satisfies ***" that is defined recursively by clauses including [B2] is a second-order predicate. One might avoid using it as a primitive predicate by turning the recursive definition into an explicit definition in the usual way. To do so, however, it is necessary to replace the satisfaction predicate with a *third-order* variable (and use quantifiers that bind the variable). So Boolos must accept the legitimacy of second-order predications.

Rayo and Uzquiano (1999) must do so as well, because they use a second-order satisfaction predicate to characterize the logic of B-languages. They acknowledge this, and continue: "we would be forced to resort to an even higher-order satisfaction predicate in order to give a semantics of a [plenary second-order] language" (*ibid*., p. 322). But, as I have argued, they cannot give a characterization of truth or logic for plenary second-order languages within the confines of languages that they, who follow Boolos, can accept.

Those who understand second-order quantifiers as invoking attributes do not have much difficulty in embracing languages with higher-order predications. They can take second-order predicates to indicate second-order attributes (i.e., attributes of attributes), and third-order quantifiers to range over those attributes; and similarly for third-order predicates and quantifiers, and so on. And they can give natural characterizations of truth and logic for plenary second-order language as well as for their meager fragments.[104] Moreover, one can use languages of orders higher than the second to give characterizations of truth or logic that extend Boolos-style characterizations to plenary second-order languages.[105] But Boolos and his followers cannot accept such characterizations, because they have difficulties in accepting languages with third-order quantifications. The difficulties stem not from any intrinsic disparity between second-order quantifications and their third-order cousins, but from limitations of their approach to higher-order languages.

To meet Quine's charge that higher-order logics (e.g., second-order logic) are not systems of logic in the proper sense, but set theories "in sheep's clothing" (1970, p. 66), Boolos attempts to derive second-order logic, the logic of second-order languages, from the logic of English plural constructions. To do so, he in effect presents a scheme for rendering B-language sentences into meager plural languages. The scheme renders monadic second-order variables (e.g., "X") to plural variables (e.g., "*xs*"), predications formed by monadic second-order variables (e.g., "$X(y)$") to plural predications formed by the logical predicate "**H**" (e.g., "$y\mathbf{H}xs$"), and second-order existential quantifications (e.g., "$\exists X \forall y {\sim} X(y)$") to certain disjunctions involving plural quantifications (e.g., "$[\forall y {\sim} y \neq y \vee \mathbf{\Sigma} xs \forall y {\sim} y\mathbf{H}xs]$").[106] One can use the scheme to derive the logic of B-languages from the logic of plurals. But B-languages are modest higher-order languages whose logic comprises only a small fragment of the entire second-order logic (or its cousins of higher orders). So one might attempt to apply Boolos's approach to those richer languages. But there are serious difficulties in doing so. In particular, those who attempt to extend Boolos's scheme of 'translation' to languages with second-order predications meet obvious difficulties They might tackle plenary second-order languages by attempting to relate primitive second-order predicates to non-logical plural predicates.[107] But they are helpless in dealing with third-order variables and quantifiers. These expressions do not have even remote analogues in (first-order) plural languages.[108]

I think that this shows serious limitations of Boolos's approach to higher-order languages and logic. Those who take the approach cannot accept third-order languages as legitimate languages. And we can see that this is what bars them from giving characterizations of truth or logic for even plenary second-order languages, which they accept as legitimate languages.

Defenders of Boolos might respond by drawing a sharp line between second-order languages and their third-order cousins. They might hold that constructions beyond those found in second-order languages cannot be made legitimate, and that third-order logic, unlike second-order logic, is at best a set theory (or class theory) in disguise. I do not think that one can justify Boolos's approach to draw such a stark contrast between the two cases. For those who take the approach meet serious difficulties in handling even some second-order constructions. They meet difficulties in dealing with non-monadic second-order quantifications, as is well-known. Boolos renders dyadic second-order quantifications, for example, into B-languages by taking them to be quantifications over ordered pairs, namely, sets of a certain kind. This yields the wrong result that "$\forall x \forall y \exists z\ z = \langle x, y \rangle$" is a logical truth, because "$\forall x \forall y \exists R(R(x, y) \leftrightarrow x = x)$" is a logical truth on second-order logic. Staunch advocates of Boolos's approach might conclude that the logic of non-monadic second-order quantifications turns out not to be a system of logic in the proper sense. But they cannot make a similar response to another problem.

Boolos's scheme for rendering B-language sentences to plural languages cannot be extended to languages with second-order predicates. We cannot render second-order quantifications into plural languages simply by replacing second-order quantifiers with plural quantifiers. For we can find a conspicuous disparity between monadic second-order quantifiers and plural quantifiers even in comparing B-languages and meager plural languages: "$\exists X \forall y \sim X(y)$" is a logical truth, but its plural language analogue "$\mathbf{\Sigma} xs \forall y \sim y \mathbf{H} xs$" is a logical falsity. Boolos gets around this disparity by noting that B-languages have no second-order predications. By exploiting this limitation of the languages, he succeeds in giving a scheme for rendering their sentences into plural languages in a way that preserves their logic. To see how his scheme works for second-order quantifications, consider the following sentences:

[a] $\exists X \forall y \sim X(y)$.
[b] $\exists X[\underline{\forall y \sim X(y)} \wedge \forall y \sim X(y)] \vee \exists X[\underline{\exists y X(y)} \wedge \forall y \sim X(y)]$.
[c] $\forall y \sim \overline{y \neq y} \vee \mathbf{\Sigma} xs \forall y \sim y \mathbf{H} xs$.

[a] and [b] are logically equivalent. So Boolos renders the second-order existential quantification [a] into plural languages by in effect rendering [b]. By replacing the restricted second-quantifier construction "$\exists X[\underline{\exists y X(y)} \wedge ...]$" in its right disjunct into the plural quantifier construction "$\mathbf{\Sigma} xs$ ...", we can get "$\mathbf{\Sigma} xs \forall y \sim y \mathbf{H} xs$." This helps to render [b] into plural languages, because its left disjunct is logically equivalent to the elementary language sentence "$\forall y \sim y \neq y$." We can obtain this from "$\forall y \sim X(y)$" in [a] by replacing "$X(y)$" with "$y \neq y$." This is the crucial maneuver behind Boolos's scheme, and it works for all second-order quantifications in B-languages, because no second-order variables in those quantifications occur in argument positions of second-order predications. But this condition does not hold for languages with second-order predicates. Consider, for example, "$\exists X \mathbb{C}(X)$", where "$\mathbb{C}$" is a second-order predicate tantamount to the plural predicate "$\mathbf{C_o}$". This sentence is logically equivalent to the following disjunction:

[d] $\exists X[\underline{\forall y \sim X(y)} \wedge \mathbb{C}(X)] \vee \exists X[\underline{\exists y X(y)} \wedge \mathbb{C}(X)]$.

And we can render the right disjunct of [d] as "$\mathbf{\Sigma} xs \mathbf{C_o}(xs)$." But this does not help to obtain a suitable rendering of [d] or, for that matter, "$\exists X \mathbb{C}(X)$", because we cannot find an elementary language equivalent of the left disjunct of [d] by eliminating its second-order variable and quantifier.[109] So Boolos's approach provides no way to explain how the second-order predicates that he uses to give a semantics of his favorite second-order languages can be understood.

APPENDIX 2: THE LOGIC UNDERLYING CANTOR'S THEOREM

We can prove the following principles using *Ax.* 13:

> *Th.* 15. $[\mathbf{MANY}(\tau) \wedge (\mathbf{\Pi}\omega \sqsubseteq \tau)(\exists \upsilon \mathbf{H} \tau)\psi] \rightarrow (\mathbf{\Sigma}\omega \sqsubseteq \tau)(\mathbf{\Sigma}\omega_1 \sqsubseteq \tau)(\exists \upsilon \mathbf{H} \tau)[\sim \omega \approx \omega_1 \wedge \psi \wedge \psi(\omega/\omega_1)]$, where ω_1 does not occur free in ψ and is substitutable for ω in ψ, ω is not ω_1, and ω, ω_1, and υ do not occur in τ.

> *Th.* 16. $[\exists \upsilon \exists \upsilon_1 \upsilon \neq \upsilon_1 \wedge \mathbf{\Pi}\omega \exists \upsilon \psi] \rightarrow \mathbf{\Sigma}\omega \mathbf{\Sigma}\omega_1 \exists \upsilon [\sim \omega \approx \omega_1 \wedge \psi \wedge \psi(\omega/\omega_1)]$, where ω_1 does not occur free in ψ and is substitutable for ω in ψ, ω is not ω_1, and υ is not υ_1.

Here is a proof of *Th.* 15 in S_{PL}:

> *Proof of Th.* 15. Assume that $\mathbf{MANY}(\tau)$ and $(\mathbf{\Pi}\omega \sqsubseteq \tau)(\exists \upsilon \mathbf{H} \tau)\psi$ hold. Then we may assume that $\exists \upsilon (\mathbf{\Sigma}\omega \sqsubseteq \tau)[\upsilon \mathbf{H} \tau \wedge \psi \wedge \sim \upsilon \mathbf{H} \omega]$ holds.[110] This implies $\mathbf{\Sigma}\omega \forall \upsilon (\upsilon \mathbf{H} \omega \leftrightarrow (\mathbf{\Sigma}\omega \sqsubseteq \tau)[\upsilon \mathbf{H} \tau \wedge \psi \wedge \sim \upsilon \mathbf{H} \omega])$ by *Ax. 13*. So assume that the following holds:
>
> [*] $\forall \upsilon (\upsilon \mathbf{H} \omega_1 \leftrightarrow (\mathbf{\Sigma}\omega \sqsubseteq \tau)[\psi \wedge \upsilon \mathbf{H} \tau \wedge \sim \upsilon \mathbf{H} \omega])$, where ω_1 does not occur in ψ, τ, or ω.

Then $\omega_1 \sqsubseteq \tau$ holds. This implies $(\exists\upsilon\mathbf{H}\tau)\psi(\omega/\omega_1)$. So assume $[\upsilon_0\mathbf{H}\tau \wedge \psi(\omega/\omega_1)(\upsilon/\upsilon_0)]$, where υ_0 does not occur in τ or ψ. This implies $\upsilon_0\mathbf{H}\omega_1$.[111] So $(\mathbf{\Sigma}\omega\sqsubseteq\tau)[\upsilon_0\mathbf{H}\tau \wedge \psi(\upsilon/\upsilon_0) \wedge {\sim}\upsilon_0\mathbf{H}\omega]$ holds by [*]. And ${\sim}\upsilon_0\mathbf{H}\omega$ implies ${\sim}\omega \approx \omega_1$. So $(\mathbf{\Sigma}\omega\sqsubseteq\tau)[{\sim}\omega \approx \omega_1 \wedge \psi(\upsilon/\upsilon_0) \wedge \psi(\omega/\omega_1)(\upsilon/\upsilon_0)]$ must hold. This, given $\upsilon_0\mathbf{H}\tau$ and $\omega_1 \sqsubseteq \tau$, implies the consequent of *Th.* 15. □

We can prove *Th.* 16 in the same way. But it is useful to see that it is a corollary of *Th.* 15. We can derive it from *Th.* 15 by setting τ as "$<x: x = x>$".[112]

We can see that *Th.* 15 is the logical basis of Cantor's Theorem. To do so, it is necessary to formulate the plural version of the subset axiom:

[a] $\forall\alpha\mathbf{\Pi}xs\exists\beta\beta = \{y{\in}\alpha: y\mathbf{H}xs\}$ (that is, $\forall\alpha\mathbf{\Pi}xs\exists\beta\forall y[y{\in}\beta \leftrightarrow y{\in}\alpha \wedge y\mathbf{H}xs]$)

where "α" and "β" are restricted singular variables for sets. This principle, unlike the usual subset axiom, does not imply the existence of an empty set,[113] but we can add a principle that states the existence of an empty set:

[b] $\exists\alpha\forall\beta{\sim}\beta{\in}\alpha$.

Using these two principles, it is straightforward to obtain Cantor's Theorem from *Th.* 15.[114] Note, however, that *Th.* 15 has a much wider range of application than Cantor's Theorem. This does not apply to the cases in which the things in question (e.g., the objects that are not members of themselves) do not form a set.[115] *Th.* 15 has no such restriction.

APPENDIX 3: LIST OF NUMBERED SENTENCES[116]

[2] A child is healthy (i.e., there is a child who is healthy).
[2a] $\exists x[C(x) \wedge H(x)]$.
[2b]* $\exists x[C^N(x) \wedge H^N(x)]$.
[3] John and Carol are children who lift Bob.
[4] Some children lift Bob.
[5] John and Carol lift Bob.
[6a] There is something that lifts Bob.
[7] John and Carol are children.
[7a] $C^N([j@c])$.
[8] Some children are healthy.
[8a] $\mathbf{\Sigma}xs[C^N(xs) \wedge H^N(xs)]$.
[9] John is a child and Carol is a child.
[9a] $[C(j) \wedge C(c)]$.
[13] There are some critics who admire only one another.
[13a]* $\mathbf{\Sigma}xs\forall y[y\mathbf{H}xs \rightarrow [C_r(y) \wedge \forall z(A(y, z) \rightarrow y \neq z \wedge z\mathbf{H}xs)]]$.
[14] $\exists X[\exists yX(y) \wedge \forall y(X(y) \rightarrow [C_r(y) \wedge \forall z(A(y, z) \rightarrow y \neq z \wedge X(z))])]$.
[14a] $\exists x[\exists y y\epsilon x \wedge \forall y(y\epsilon x \rightarrow [C_r(y) \wedge \forall z(A(y, z) \rightarrow y \neq z \wedge z\epsilon x)])]$.
[15] Ezra and Thomas are critics who admire only each other.
[15a] $\forall y[y\mathbf{H}[e@t] \rightarrow (C_r(y) \wedge \forall z[A(y, z) \rightarrow y \neq z \wedge z\mathbf{H}[e@t]])]$.
[16] Ezra is a critic, Thomas is a critic, Ezra is not identical with Thomas, Ezra admires only Thomas, and Thomas admires only Ezra.
[16a] $[C_r(e) \wedge C_r(t)] \wedge e \neq t \wedge [\forall z(A(e, z) \rightarrow z = t) \wedge \forall z(A(t, z) \rightarrow z = e)]$.
[19] Genie is one of Russell and Whitehead.

[20] Genie is one of Genie.
[32]* $\mathbf{\Sigma}xs(\forall y\mathbf{H}xs)(\exists z\mathbf{H}xs)\mathrm{A}(y, z)$.
[33]* $\mathbf{\Sigma}xs\mathbf{\Pi}ys([\mathbf{D}(ys) \wedge \mathbf{\Pi}zs(\forall x\mathbf{H}zs)([\mathbf{D}(zs) \wedge x\mathbf{H}ys] \rightarrow zs \approx ys)] \rightarrow \exists x\forall y(y = x \leftrightarrow [y\mathbf{H}xs \wedge y\mathbf{H}ys]))$.
[34]* $\exists x[\mathrm{C}(x) \wedge \mathrm{H}(x)] \rightarrow \mathbf{\Sigma}zs\forall x(x\mathbf{H}zs \leftrightarrow [\mathrm{C}(x) \wedge \mathrm{H}(x)])$.
[35]* There are some things such that something is one of them if and only if it is a healthy child.

NOTES

[1] And in any case, it is useful to compare plural languages with no singular or plural constants with elementary languages with no singular constants. Incidentally, Simons (1982, p. 165) gives “Benelux” as an example of plural proper name. This is objectionable, because “Benelux”, it seems, refers to *the customs union* of three countries (i.e., Belgium, the Netherlands, and Luxembourg) rather than the three countries themselves. If it is still used as a plural noun (so that we say, e.g., “Benelux *were* formed in 1947”), it might count as a plural proper name of a natural language. But one might still object to introducing its counterpart in regimented languages as a plural constant. (It seems that it was not unusual to use “The United States of America” as a plural noun a century or so ago (see Garner (1998, p. 669)), but I do not think that this makes it necessary to introduce a plural constant amounting to the noun as used in that way.) Despite these reservations about Simons’s example, I see no reason why speakers of a language can choose to use “Benelux” (or another word) unambiguously as a plural proper name that refers to the three Benelux countries rather than a union thereof.

[2] Note that although plural variables (e.g., “*xs*”) result from attaching the italicized “*s*” to singular variables, the metavariables for the former (e.g., “ω”) do not contain the italicized “*s*”.

[3] Meager plural languages have no complex terms.

[4] The predication $\pi(\tau_1, \tau_2, \ldots, \tau_n)$ is neither a singular predication nor a plural predication, on this terminology, if π is a plural predicate while all of the terms $\tau_1, \tau_2, \ldots, \tau_n$ are singular. In any case, such predications are not purely singular constructions.

[5] The definition of *occurrence* is straightforward. Note that singular variables (e.g., “*x*”) do not occur in plural variables (e.g., “*xs*”) despite their typographical affinities.

[6] Or we may add complex predicates (e.g., “$\lambda x{\sim}\mathbf{H}(x)$” or “$\lambda x[\mathrm{C}(x) \wedge \mathbf{H}(x)]$”) to regimented languages while defining neutral expansions of predicates directly as in Yi (LMP I, §3).

[7] Alternatively, we can regard plural terms formed by “**@**” as plural definite descriptions. See *Def.* 5 and note 10 below.

[8] The definition of *free occurrence* of variables in plural language expressions (e.g., terms or sentences) is straightforward.

[9] $(\boldsymbol{I}\omega)\phi$ cannot be taken to abbreviate $(\boldsymbol{\iota}\omega)\{\phi\}^N(\upsilon/\omega)$. For “those who write *PM*” is not interchangeable with “those who [each] write *PM*”. But $(\boldsymbol{\iota}\omega)\{\phi\}^N(\upsilon/\omega)$ is interchangeable with $(\boldsymbol{I}\omega)\forall\upsilon[\upsilon\mathbf{H}\omega \leftrightarrow \phi]$, where ω does not occur free in ϕ. So “the children” (or “$(\boldsymbol{\iota}xs)\mathrm{C}^N(xs)$”), for example, can be taken to abbreviate “the things such that something is one of them if and only if it is a child” (or “$(\boldsymbol{I}xs)\forall y[y\mathbf{H}xs \leftrightarrow \mathrm{C}(y)]$”).

[10] Replacing ϕ in the definiens of [c] with $[\upsilon\mathbf{H}\tau \vee \upsilon\mathbf{H}\mu]$ yields that of *Def.* 4. So $[\tau\,\mathbf{@}\,\mu]$ can be taken to abbreviate $(\boldsymbol{\iota}\omega)[\upsilon\mathbf{H}\tau \vee \upsilon\mathbf{H}\mu]^N(\upsilon/\omega)$, where υ and ω do not occur in τ or μ.

[11] $\varsigma \approx \tau$ is logically equivalent to $\forall\upsilon(\upsilon = \varsigma \leftrightarrow \upsilon\mathbf{H}\tau)$, where υ does not occur in ς or τ (see *Def.* 2 in §4 and *Th.* 3 in §7). So $\exists\upsilon\upsilon \approx \tau$ is logically equivalent to $\exists\upsilon\forall\upsilon_1(\upsilon_1 = \upsilon \leftrightarrow \upsilon_1\mathbf{H}\tau)$, where υ_1 does not occur in υ or τ.

[12] We can define the numerical predicate corresponding to zero as follows:

$\textbf{Z}\textsc{ero}(\tau) \equiv_{\text{df}} \sim\exists\upsilon\upsilon\mathbf{H}\tau$,

where υ does not occur in τ. But we cannot apply the definition [b] of successor to define "**O**NE" as "**Z**ERO*", because "$\boldsymbol{\Sigma}xs$**Z**ERO(xs)" is logically false (see *Ax. 9* in §7).

[13] I think that natural numbers are properties indicated by the numerical predicates corresponding to them. See Yi (1995, ch. 4), (1998) and (1999). Notice that the definitions of the numerical predicates involve only logical expressions. This might be taken to render some support to Frege's logicism (see, e.g., Frege 1884), if in a different framework. To defend logicism, however, it is necessary to reduce arithmetical truths to logical truths using the definitions. I do not think that arithmetical truths can be reduced to logical truths even in (first- or higher-order) plural languages. I leave it for another occasion to present my views on the nature of number and of arithmetic.

[14] That is, ϕ is a sentence that results from the variables υ and ν filling the first and second argument places, respectively, of the predicate.

[15] See Yi (LMP I, note 7 and §2.1), where I make remarks on the contrast between first-order plural languages and the usual higher-order languages.

[16] I think that the basic semantic function of predicates is to indicate properties or relations. The function of designating (or being satisfied) can be seen to derive from this. See below.

[17] I use "is true of" and "designates" interchangeably for a relation that pertains to one-place predicates, while using "is satisfied by" for a broader relation that pertains to any predicate.

[18] Note that not all predicates in atomic sentences (or predications) are expressions that *form* the sentences. In second-order predications, where first-order predicates occur in the argument positions of second-order predicates, the first-order predicates do not form the predications. We cannot characterize the semantics of such predications by invoking the entities that *satisfy* the first-order predicates. To do so, we need to appeal to what I think is the basic semantic function of predicates: *indicating* properties or relations. See the discussion of this semantic function below.

[19] I add the qualification "typical" because natural languages have vacuous singular terms (e.g., "Pegasus") that do not refer to anything. Note, however, that plural languages, like elementary languages, are assumed to have no vacuous terms.

[20] And a plural that refers to some things cannot refer to any other things, as a singular term that refers to something cannot refer to anything else. The qualification "typical" is necessary, because [1] natural languages have vacuous plural terms (e.g., "Pegasus and Hamlet"), and [2] some plural terms, on my view, refer to some one thing. "Cicero and Tully" is a plural term but refers to Cicero, i.e., Tully, i.e., Cicero and Tully, who are not many things but just one thing (because Cicero *is* Tully).

[21] Most of the sentences discussed in this paper are those discussed in Yi (LMP I). To refer to such sentences, this paper retains the numerals used to refer to them in Yi (LMP I). See Appendix 3 for the list of numbered sentences discussed in this paper.

[22] Some might object that "*is* two children" is ungrammatical. We can bypass this objection by considering "*John and Carol*, and Chelsea are three children", which does not logically imply "There is something such that it and Chelsea are three children."

[23] Recall note 25 in Yi (LMP I, §2.2) and the discussion that begins with the paragraph it is attached to. The discussion applies to the view that a non-degenerate plural term refers to a composite object (e.g., Genie).

[24] See Simons (1982), who holds that "when an expression designates A and B and C . . . , where these are individuals, this is to say no more than that it designates A and designates B and designates C . . ." (1982, p. 166).

[25] One can characterize "refer_1" from "refer_2", because "John and Carol", for example, refers_1 to the *things* each one of which the term refers_2 to. To do so, however, it is necessary to use plurals in the metalanguage. And it is necessary to invoke the reference_1 relation to give the truth condition of plural predications.

[26] To do so, it is necessary to obtain a generalized notion of *assignment function* applicable to plural variables. We can consider relations of a special kind, *plural relations*, similar to the reference relation that pertains to constant plural terms, e.g., "John and Carol" (as well as constant singular terms), and regard those relations as functions of a special kind, (*one-many*) *plural functions*. By using second-order variables and quantifiers that range over such special functions or relations, we can characterize the truths of first-order plural languages without invoking the set-theoretic substitutes, such as sets of ordered pairs. See below on plural relations, plural functions, and assignments to plural variables, and the last paragraph of §4 for the Tarski-style characterization mentioned above.

[27] Boolos (1985a, pp. 335–7) extends Tarski's characterization of truth for elementary languages to *singular meager monadic second-order languages*, languages that can be obtained from elementary languages by adding just monadic second-order variables and quantifiers. And Rayo and Uzquiano (1999) modify Boolos's characterization of truth to give a characterization of logic for those languages. But their characterizations of truth and logic cannot be extended to languages with second-order predications. See Appendix 1.

[28] The improvement of the Tarski-style characterization of truth for first-order plural language that I present below is also formulated in *higher-order* plural languages, and does without invoking set-theoretic substitutes of relations or functions.

[29] I use "attribute" broadly as a term that applies to both properties and relations. For this use of the word, see Kim (1998, p. 6).

[30] Note that all higher-order predicates and variables of orders higher than the second, on this definition, are also singular predicates. But it is useful to classify them further on the basis of whether or not they can be found in singular higher-order languages. Second-order variables can be classified into singular and plural ones in the way that first-order predicates are.

[31] Or *being something that is not a child*.

[32] Or *being some things that are not two children*.

[33] Consider, e.g., [a] *being not the same things as themselves*, and [b] *being one child*. The former is a non-instantiated plural property, and the latter a plural property not instantiated by any two or more things (as such).

[34] Similarly, an argument place of a two-place relation (e.g., *writing*) may admit some things (e.g., Chicago and London) that the relation does not relate to any things at all; but a relation cannot form a fact by combining with some things (as such), unless they (as such) can fill its argument place. And if a relation is plural, either the relation itself or its complement can combine with many things (as such) to form a fact.

[35] The two theses are equivalent, as we have seen in the previous paragraph. I think that they are central components of the standard conception of reality, and call them the *Principle of Singular Instantiation* and the *Principle of Singularity*, respectively, in Yi (1999, p. 167ff).

[36] In Yi (1999), I elaborates on the account of attribute sketched above and develops the plural conception of reality. See also Yi (1998, esp. pp. 104–8).

[37] To invoke attributes indicated by predicates, it is necessary to use higher-order languages even in characterizing the truths of elementary languages. One might avoid

this by invoking set-theoretic substitutes of attributes, but this makes the resulting characterizations of truth inapplicable to languages used to talk about objects that do not form a set (or class), e.g., all the objects, including any sets (or classes) that there are. (The relation indicated by, e.g., the identity predicate in such a language cannot be represented by a set (or class).)

38 This is because the plural languages that I focus on have no plural constants. But we can consider meager plural languages with primitive plural constants, and the reference predicates for such languages must be plural predicates.

39 Note that I use italics for higher-order expressions, and boldface for expressions that are not available in singular languages. "***S***" is a dyadic second-order variable that can replace two-place predicates whose second-argument place is plural. Such a variable, used in the metalanguage that I use to give a semantics of first-order plural languages, ranges over (first-order) plural attributes whose second-argument place is plural.

40 The condition can be seen to generalize the usual condition to be satisfied by singular functions.

41 If the language in question has primitive plural constants, it is necessary to replace [T3] with the following:

[T3*] $\tau^{\boldsymbol{S}} \approx xs$, if τ is a constant that **refers to** *xs*.

[T3] follows from this together with the following condition on the reference of singular constants:

[T1*] **One**(*xs*), if τ is a *singular* constant that **refers to** *xs*.

42 I use boldface italics for "***a***" in [T10], and for "***there is***", and "***any***" in [T13]–[T15], below, to indicate that the quantifiers are to be replaced in regimented languages by higher-order quantifiers ranging over plural functions or relations.

43 Note that the characterization includes one for elementary languages, which are the singular fragments of first-order plural languages. Similarly, characterizations for higher-order plural languages include those for the usual higher-order languages as their singular fragments.

44 The two characterizations are equivalent, because [T10] and [T10*] are equivalent.

45 Similarly, the characterization does not directly yield the result that logical consequences of truths are true. I add "directly" because there are devious ways of getting the results, which should be quite straightforward. See, e.g., Boolos (1985b, p. 340).

46 The characterization of plural logic yields a characterization of elementary logic in higher-order plural languages, because plural languages are extensions of elementary languages. See below. In Yi (1995, pp. 52–56), I formulate the logic of plural languages in an elementary language (sufficient for stating set theory) by invoking set-theoretic substitutes of attributes, etc. The elementary language formulation of the logic of plural languages is to the characterization of the logic given below what the usual formulation of elementary logic is to the characterization thereof in higher-order languages.

47 The definite description "the sentences in Γ" is vacuous, if Γ is the empty set. The right side of the biconditional in such cases is meant to be equivalent to "ϕ is a logical truth of $\mathscr{L}$."

48 "*ds*" is used as a plural variable in the metalanguage.

49 The plural universal generalization of [L1], i.e., "If there are some things, *ds*, then there is something *x* such that *x* **is one of** *ds*", is a logical truth (see *Ax.* 9 in §7).

50 "***I***" is a second-order variable for first-order plural functions. Because all constants of $\mathscr{L}$ are singular, we may replace it with a variable for singular first-order functions.

(Then we can replace "≈" in [L2] with "="). Given any singular function, however, there is an *equivalent* plural function (i.e., a plural function that yields the same value for the same argument).

[51] "***J***" is a third-order variable for second-order functions whose values are plural attributes.

[52] Λ is the empty set.

[53] I.e., [L5], [L9], and [L14].

[54] The four sentences are paraphrases of "Ezra is a critic, Thomas is a critic, Ezra is not identical with Thomas, Ezra admires only Thomas, and Thomas admires only Ezra" (sentence [16]), "Ezra and Thomas are critics who admire only each other" (sentence [15]), sentence [13], and "There is something every member of which is a critic and admires only its other members", respectively.

[55] "*y***H**[e**@**t]" is logically equivalent to "[*y***H**e ∨ *y***H**t]" by *Def.* 4, and the latter to "[*y* = e ∨ *y* = t]" by [L6] and [L9] (see also *Ax*. 5 and *Th*. 6 in §7).

[56] By [L14] (see also *Ax*. 6 in §7).

[57] See *Th*. 2 and *Th*. 13.

[58] [32] is the straightforward paraphrase of "There are some things each one of which admires one of them." See *Def.* 6 for the restricted quantifiers used in [32].

[59] The use of constants in the above example is not essential. Instead of Γ*, consider the set that contains "$\exists x_0 \exists x_1 A(x_0, x_1)$", "$\forall x_0 \forall x_1[A(x_0, x_1) \rightarrow \exists x_2 A(x_1, x_2)]$", "$\forall x_0 \forall x_1 \forall x_2[A(x_0, x_1) \wedge A(x_1, x_2) \rightarrow \exists x_3 A(x_2, x_3)]$", etc. This example shows that the logic of plural languages that contain a non-logical 2-place predicate is non-compact.

[60] The proof given above of the non-axiomatizability of plural logic does not yield the result that one cannot give an axiomatic characterization of the logical truths of plural languages. Kaplan's proof of the non-expressibility of [13] in effect yields this stronger result. The proof assumes that [13] can be paraphrased by the second-order sentence [14], the negation of which can be seen to have the same structure as the second-order induction principle, and invokes the result that this principle helps to give a complete characterization of arithmetical truths (see Yi (LMP I, note 27)). I think that the assumption is controversial. By considering the straightforward paraphrases of sentences similar to [13] into plural languages, however, we can show that the logical truths of even meager plural languages cannot be characterized by an axiomatic system. For we can see that the negation of the plural language paraphrase of "Some non-zero natural numbers are successors only of one another" helps to give a complete characterization of arithmetical truths. To show this, however, we need to make some assumptions about the semantics of plural quantifiers, assumptions parallel to those made in the proof that the second-order induction principle helps to yield a complete characterization of arithmetical truths. The assumptions, which I think are correct, might be challenged by those who are skeptical about the idea that logic can fail to be axiomatizable. I think that we can meet the challenges by appealing to the strong intuitions that we have about the natural language counterparts of the examples used to prove the non-compactness of plural logic. But I still think that the Kaplan-style proof, though correct, is less convincing to the skeptics than the proof via non-compactness. In Yi (preprint), I elaborate on the view that the logic of plurals must be non-compact and non-axiomatizable.

[61] I have sometimes appealed to the fact that a given sentence is *not* a logical truth or that a given sentence does *not* logically imply another. We can use elementary logic, together with the conservativeness of plural logic, to explain those logical relations, because the sentences in question in those cases are elementary language sentences.

[62] I say that a variable ν is suitable for a quantifier *Q*, if *Q* and ν are both singular or both plural.

[63] I say that a sentence ψ is an instance of, e.g., *Ax.* 4, if there is a sentence φ and a singular variable υ that does not occur free in φ such that ψ is [φ → ∀υφ].

[64] The definitions of "ν is substitutable for μ in φ" and "ν occurs free in φ" for $\mathscr{L}$ are straightforward (e.g., ν occurs free in a term μ if ν occurs in μ). φ (ν/τ), where ν is plural if τ is, is the sentence that results from φ by substituting τ for ν in φ wherever ν occurs free in φ.

[65] In plenary plural languages, *Ax.* 11 is independent of the other axioms. We can do without *Ax.* 11 in meager plural languages, where it abbreviates the following (see *Def.* 4):

$$\mathbf{\Sigma}\omega[\forall\upsilon(\upsilon\mathbf{H}\omega \leftrightarrow [\upsilon\mathbf{H}\tau \vee \upsilon\mathbf{H}\mu]) \wedge \varsigma\mathbf{H}\omega] \leftrightarrow [\varsigma\mathbf{H}\tau \vee \varsigma\mathbf{H}\mu],$$

where neither υ nor ω occurs in τ or μ. We can derive this from the other axioms (e.g., *Ax.* 13). For the purpose of axiomatization, however, it is simpler to remove *Def.* 4 and derive the equivalences warranted by the definition from *Ax.* 11 (see *Th.* 10 below).

[66] [33] is an instance of the plural cousin of the global Axiom of Choice in set theory:

Plural Choice: $\mathbf{\Sigma}\omega_1\mathbf{\Pi}\omega([\phi \wedge \mathbf{\Pi}\omega_2(\forall\upsilon\mathbf{H}\omega_2)([\phi(\omega/\omega_2) \wedge \upsilon\mathbf{H}\omega] \rightarrow \omega_2 \approx \omega)] \rightarrow \exists\upsilon\forall\upsilon_1(\upsilon_1 = \upsilon \leftrightarrow [\upsilon_1\mathbf{H}\omega_1 \wedge \upsilon_1\mathbf{H}\omega]))$, where ω_1 does not occur free in φ or ω, ω_2 does not occur free in φ and is substitutable for ω in φ, and υ_1 is not υ.

All the instances of *Plural Choice* must strike one as logical truths (see, e.g., Lewis (1991, p. 71f)). They lie under instances of the (non-global) Axiom of Choice (*not*: the well-ordering axiom), although its instances are not logical truths of plural languages (they concern the existence of sets). I leave further discussion of this issue for another occasion.

[67] They are essentially the axioms of Enderton's axiomatization of elementary logic, but they include some plural language sentences that are not available in elementary languages (e.g., "**L**(*xs*, b) → **L**(*xs*, b)"). See Enderton (1972, p. 104f). Enderton's axioms include instances of ς = ς, but they are derivable in S_{PL} given *Axs.* 5, 9, &10. See *Th.* 7.

[68] I say that υ occurs free in Γ, if υ occurs free in any sentence in Γ.

[69] Their plural existential generalizations are:

There are some *children*.
There are some *children who are healthy*.
There are some *famous humans who are three philosophers*.
There are some *people who admire (only) one another*.
There are some *politicians who cooperate*.

[70] To show that using *Ax.* 6, it is necessary to use neutral expansions to paraphrase the sentence "John is a child who is healthy", for example, by "$[C^N(j) \wedge H^N(j)]$." But this is logically equivalent to the usual paraphrase "[C(j) ∧ H(j)]." See *Th.* 4.

[71] Some might insist that "Cicero is a philosopher, and Tully is a philosopher" does not logically imply "*Cicero and Tully* are philosophers" because this cannot be true unless "Cicero" and "Tully" refer to two different things. On their reading of the plural terms, however, "*Cicero and Tully* are not two different people, but one and the same person" would be false no matter what.

[72] Similarly, "$\mathbf{\Sigma}xs[C^N(xs) \wedge \mathbf{L}(xs, b)]$" does not logically imply "$\exists x[C^N(x) \wedge \mathbf{L}(x, b)]$" (nor does it logically imply "$\exists x[C(x) \wedge \mathbf{L}(x, b)]$").

[73] An instance of *Ax.* 9 is "$\exists xx\mathbf{H}ys$", whose closure (i.e., "$\mathbf{\Pi}ys\exists xx\mathbf{H}ys$") can be considered the plural language paraphrase of "If there are some things, there is something that is one of them." So *Ax.* 9 can be taken to formulate a necessary condition for the existence of some things: there be something that is one of them. It is, moreover, a logically necessary condition.

[74] *Th.* 2 [a] is derivable from $\exists\upsilon\upsilon\mathbf{H}\tau$, and *Th.* 2 [b] from *Th.* 2 [a] via **Σ**-*Elimination*. *Th.* 2 [d] does not require *Ax.* 9; it results directly from the definition of neutral expansion on elementary logic.

[75] *Th.* 3 [a] (i.e., $\varsigma\mathbf{H}\varsigma$) is derivable from *Axs.* 9–10 (and *Ax.* 5), and *Th.* 3 [b] from *Th.* 3 [a] and *Ax.* 10.

[76] To show that [8a] and [2a] are equivalent, we need to appeal to *Th.* 2 [d], which states the distributivity of neutral expansion over conjunction. We can use *Th.* 2 [d] to get generalizations of *Th.* 2 [b]–[c] that apply to conjunctions of neutral expansions. But we cannot generalize them for all sentences that involve no predicates except neutral expansions. "$\mathbf{\Sigma} xs[\sim C^N(xs) \wedge \sim H^N(xs)]$" does not logically imply "$\exists x[\sim C^N(x) \wedge \sim H^N(x)]$." Suppose that there is only one non-child (e.g., Bill) and only one thing that is not healthy (e.g., John), and that the former is not the latter. Then the plural quantification would be true (on account of the two things) while the singular quantification being false.

[77] *Th.* 5 alone yields the equivalence between [7a] and "$[C^N(j) \wedge C^N(c)]$." *Th.* 4 [a] yields the equivalence between "$[C^N(j) \wedge C^N(c)]$" and [9a].

[78] We can derive *Th.* 6 from *Ax.* 11 and *Th.* 3, which we have seen to result from *Ax.* 9.

[79] See *Th.* 13 [a]. Note, however, that "Jack is one of those who carry Bob upstairs" and "Jack carries Bob upstairs" are not logically equivalent (nor are "Russell is one of those who wrote *PM*" and "Russell wrote *PM*").

[80] This sentence occurs in, e.g., "There are some things such that they *are* themselves."

[81] But "Chelsea's parents are Chelsea's parents" and "The authors of *PM* are the authors of *PM*" are not logical truths; they logically imply the existence of Chelsea's parents or authors of *PM*. Their paraphrases are not instances of *Th.* 8, because definite descriptions are introduced into plural languages only as contextually defined expressions.

[82] Given *Th.* 8, *Th.* 9 results from *Ax.* 10 and *Ax.* 5. (Note that *Ax.* 5 is derivable from *Ax.* 13 and *Th.* 9. But this does not mean that *Ax.* 5 is eliminable given *Ax.* 13.)

[83] *Th.* 10 [a], given *Th.* 8 and *Ax.* 6, yields *Th.* 10 [b], and *Th.* 10 [c] results from *Th.* 10 [b] and *Ax.* 11. This justifies defining plural terms formed by the connective "**@**" as in *Def.* 4 in meager plural languages.

[84] Similarly, *Th.* 11 yields the logical equivalence between the following:

There is a philosopher who writes something.
There are some things such that something is one of them if and only if it is a philosopher who writes something.

These can be paraphrased "$\exists x[P(x) \wedge \exists y\mathbf{W}(x, y)]$" and "$\mathbf{\Sigma} zs\forall x(x\mathbf{H}zs \leftrightarrow [P(x) \wedge \exists y\mathbf{W}(x, y)])$." Notice, however, that the following is not an instance of *Ax.* 13:

$$\mathbf{\Sigma} xs\left[P^N(xs) \wedge \exists y\mathbf{W}(xs, y)\right] \rightarrow \mathbf{\Sigma} zs\forall x(x\mathbf{H}zs \rightarrow [P(x) \wedge \exists y\mathbf{W}(x, y)]).$$

This is not a logical truth; its antecedent, which paraphrases "There are some philosophers who write something", does not logically imply "$\exists x[P(x) \wedge \exists y\mathbf{W}(x, y)]$." (The English sentence is sometimes used interchangeably with "There are some philosophers who [each] write something", which can be paraphrased by "$\exists x[P(x) \wedge \exists y\mathbf{W}(x, y)]$." This is logically equivalent to "$\mathbf{\Sigma} zs\forall x(x\mathbf{H}zs \leftrightarrow [P(x) \wedge \exists y\mathbf{W}(x, y)])$.")

[85] For "$\exists xx = x \rightarrow \mathbf{\Sigma} zs\forall x(x\mathbf{H}zs \leftrightarrow x = x)$" is an instance of *Ax. 13* whose antecedent is a logical truth of elementary logic while its consequent is equivalent to "$\mathbf{\Sigma} zs\forall xx\mathbf{H}zs$" in S_{PL}.

[86] On *Def.* 5, $\upsilon\mathbf{H}\langle\upsilon: \phi\rangle$ abbreviates $\mathbf{\Sigma}\omega[\forall\upsilon(\upsilon\mathbf{H}\omega \leftrightarrow \phi) \wedge \upsilon\mathbf{H}\omega]$, where ω does not occur free in ϕ. This, given *Ax.* 13, yields *Th.* 13 [a]. Similarly, *Th.* 13 [b] results from *Ax.* 13, because their consequents are equivalent in S_{PL}. On *Def.* 5, $\mathbf{\Sigma}\omega\omega \approx \langle\upsilon: \phi\rangle$ (where ω does not occur free in ϕ) abbreviates $\mathbf{\Sigma}\omega\mathbf{\Sigma}\omega_1[\forall\upsilon(\upsilon\mathbf{H}\omega_1 \leftrightarrow \phi) \wedge \omega \approx \omega_1]$

(where ω_1 does not occur free in ϕ or ω). This is equivalent to $\mathbf{\Sigma}\omega\forall\upsilon[\upsilon\mathbf{H}\omega \leftrightarrow \phi]$ in S_{PL}. *Th.* 13 [c] results from [a]–[b].

[87] We can replace *Axs.* 9 & 13 in S_{PL} with *Th.* 13 [b], which can be taken to state a necessary and sufficient condition for there to be *the things of a certain kind*: there are *the* things each one of whom is so-and-so if and only if there is at least one thing that is so-and-so. This, as we have seen, is a logical principle.

[88] Frege's notion of *class* (or *course-of-value*) is motivated by the informal talk of the *extension* of predicate (or 'concept'). See, e.g., Frege (1884, §68), who understands the extension of a predicate (or of a 'concept') to be an object. But I think that many of those who talk of the extension of, e.g., a (singular) one-place predicate might simply mean: the things that satisfy the predicate. This interpretation helps to give a better justification of most of the principles commonly accepted about extensions, because instances of [c] are logical truths. But a one-place predicate that is not true of any object has no 'extension'. And more importantly, one cannot characterize what a (one-place) plural predicate is true of by its 'extension'. A plural predicate that is true of an object if and only if another plural predicate is true of the object might not be true of some things (as such) that the other predicate is true of.

[89] To obtain a logical truth, we need to replace "ancestor" in the sentence with its analysis in terms of "parent". See the discussion of *Def.* 7 in §4.

[90] Cantor's Theorem, on the usual formulation, states that there is no one-to-one correspondence between a set and its power set. The theorem presupposes the power set axiom, which states that every set has a power set (i.e., a set whose members are its subsets), and its proof rests on the subset axiom.

[91] Recall that "any (or some) things that are some of them" is used interchangeably with "any (or some) one or more things that are some of them", which is equivalent to "any (or some) one or more things each of which is one of them" or "any (or some) one or more things that they include." So Russell, for example, is some of Russell and Whitehead (i.e., they include him).

[92] If the things in question are just one, assigning the one thing to itself will do. See Appendix 2.

[93] We can derive both *Th.* 15 and *Th.* 16 from *Ax.* 13. See Appendix 2 for their proofs.

[94] We can see that this sentence is logically implied by "Every circle is a figure." But one cannot use monadic predicate logic, which relates only to the languages in question, to explain this, because the languages do not have an adequate paraphrase of the former sentence.

[95] Boolos (1985a, pp. 335–7) formulates the characterization for one such language, the one that extends the elementary language whose only non-logical expression is the membership predicate "ϵ". But it is straightforward to modify the characterization for other such languages.

[96] Here "*R*" and "*S*" are used as (singular) second-order variables, and "ν" and "X" as metavariables for (singular) first- and second-order variables, respectively.

[97] Boolos replaces the dyadic second-order variable "*S*" with a first-order variable ranging over set-theoretics substitutes of functions (i.e., sequences), and the dyadic second-order variable "*R*" with a monadic second-order variable while invoking ordered pairs. So the languages that he uses to characterize the truths of B-languages are close to B-languages, but they must still reach beyond B-languages because the satisfaction predicates for the object languages are second-order predicates. This causes a problem for Boolos's approach to higher-order languages. See below.

[98] Those languages must be singular languages, although Boolos defends the legitimacy of B-languages against the Quinean charge that second-order logic is set

theory in disguise by relating B-language sentences to natural language plurals. In particular, they cannot accept languages with *both* plural expressions and higher-order expressions, such as second-order plural languages. I think that Boolos and his followers must accept the legitimacy of singular languages that reach somewhat beyond B-languages (e.g., plenary second-order languages), but that they meet serious difficulties in doing so. In any case, I think that they must fall short of languages with third-order quantifiers because their project is to defend the legitimacy of higher-order languages without embracing predicable entities (i.e., attributes). See below for more on these.

[99] Those who, unlike Boolos, use *plural* languages augmented with (singular monadic) second-order variables as metalanguages can cope with predications (in the object languages) that are formed by monadic *second-order predicates* by treating the predicates in effect as plural predicates, which can be true of some things (as such), such as the things that a monadic first-order predicate designates or those that a D-relation relates a monadic second-order variable to. But they still cannot deal with second-order predications formed by monadic *third-order variables*. To deal with them, it is necessary to invoke *second-order functions* that assign *plural properties* to such variables. (And those who do so would have no more reason to decline to take monadic second-order predicates to indicate plural properties.) Using *higher-order* plural languages, one can extend the Boolos-style characterization of truth to higher-order singular languages. Note, however, that this requires drawing a stark contrast between second-order variables and their third-order cousins. I do not think that one can justify the stark contrast by any disparity between the two kinds of variables. (The characterizations of the logic of higher-order singular languages that one can get by generalizing the characterization of truth as suggested above yield the result that the substitutivity of extensionally equivalent *second-order* variables in third-order predications holds as a logical principle while the substitutivity of extensionally equivalent *third-order* variables in fourth-order predications does not.)

[100] It is necessary to put this restriction on D-relations for plural variables, because "$\Sigma xs \forall y {\sim} y \mathbf{H} xs$" is a logical falsity. By contrast, its second-order analogue, "$\exists X \forall y {\sim} X(y)$", is a logical truth. So D-relations for second-order variables cannot be required to satisfy the same restriction. Note, however, that Boolos is somewhat indefinite about this point. He says that his satisfaction predicate for a meager second-order language "is true or false relative to an assignment of ... some (or *perhaps* no) ordered pairs of second-order variables and sets to the second-order variable R" (1985a, p. 336f; my italics). This makes it unclear whether his characterization is meant for second-order languages (where "$\exists X \forall y {\sim} X(y)$" is a logical truth) or for meager plural languages in a misleading notation (where "$\exists X \forall y {\sim} X(y)$" is a logical falsity because their 'second-order' variables and quantifiers are plural variables and quantifiers in disguise). I think that Boolos's aim in Boolos (1985a) is to give a characterization of truth for his fovorite *second-order* languages (viz. B-languages), although he defends the logic of the languages by translating their sentences in effect into meager plural languages in other closely related works (see Boolos (1984, 1985b)). See also Rayo and Uzquiano (1999, p. 320f), who explicitly allowed relations that relate some second-order variables to no object.

[101] Using higher-order *plural* languages, we can characterize the satisfaction condition of monadic plural predications, for example, as follows:

> *R* with *S* satisfies $\pi(\omega)$, if π is a non-logical one-place plural predicate that **is satisfied by** the **things** that *R* relates ω to, i.e., $\langle x: R(\omega, x)\rangle$ (as such).

(Here "**is satisfied by**" is a plural predicate.) So we can give a characterization of truth for (first-order) plural languages based on this clause in second-order plural languages. But one cannot turn the characterization, which we can see is equivalent to the Tarski-style characterization given in the last paragraph of §5, into a characterization of logic. To obtain an improvement of the characterization that can be turned into a characterization of logic, it is necessary to appeal to the *indication* relation between plural predicates and *plural* attributes (rather than the satisfaction relation). This requires embracing attributes (specifically, plural attributes).

102 Boolos, however, fails to formulate a characterization of logic connected to his characterization of truth, and admits that this gives rise to a "sense of loss" (1985a, p. 344).

103 See Rayo and Uzquiano (1999, p. 319). They also formulate the characterization only for one B-language, the one whose only non-logical expression is "ϵ", but it is straightforward to modify the characterization for other B-languages.

104 It is straightforward to turn the characterizations of truth and logic that I give for first-order plural languages in §§5–6 into those for singular second-order languages. The resulting characterizations for these languages are formulated in higher-order plural languages (and I think it is necessary to use these languages to give the most natural characterizations), but one can use singular higher-order languages to simulate those characterizations.

105 See note 99, where I suggest a way to give such characterizations in higher-order plural languages. One can simulate those characterizations in singular languages of orders one step higher than those plural languages.

106 The scheme is presented in Boolos (1984, p. 444) and (1985b, p. 341). I do not think that it can be extended to plenary second-order languages. See below.

107 But I think that the attempts would fail. See below.

108 Surely, relating singular third-order variables to plural second-order variables (i.e., variables that can replace plural predicates) would not serve their aim. Those who, like me, accept higher-order plural languages might explore relating the former to the latter as an exercise (as I have done in note 99), but they would not take Boolos's approach to justifying the legitimacy of second-order languages and logic in the first place.

109 Those who try to relate second-order predicates to plural predicates might consider making "$\mathbb{C}(X)$" to be false (on a D-relation) if "X" does not denote anything (on the D-relation). On this idea, the left disjunct of [a] would count as a logical falsity and, thus, its right disjunct would be considered logically equivalent to "$\exists X\mathbb{C}(X)$." (So the idea leads to a scheme that renders "$\exists X\mathbb{C}(X)$" simply as "$\boldsymbol{\Sigma} xs\mathbf{C_o}(xs)$.") But this conflict with second-order logic, on which the two sentences, "$\exists X\mathbb{C}(X)$" and "$\exists X[\exists yX(y) \wedge \mathbb{C}(X)]$", fail to be logically equivalent.

110 Both $(\boldsymbol{\Pi}\omega\sqsubseteq\tau)(\exists\upsilon\mathbf{H}\tau)\psi$ and $\tau \sqsubseteq \tau$ hold. So assume $[\upsilon_0\mathbf{H}\tau \wedge \psi(\omega/\tau)(\upsilon/\upsilon_0)]$, and $[\upsilon_1\mathbf{H}\tau \wedge \psi(\omega/\upsilon_0)(\upsilon/\upsilon_1)]$ (where υ_0 does not occur in ψ or τ, and υ_1 does not occur in ψ, τ, or υ_0). If $\upsilon_0 = \upsilon_1$ holds, τ and υ_0 witness the consequent of *Th.* 15 ($\sim\tau \approx \upsilon_0$ follows from $\mathbf{M\small{ANY}}(\tau)$ [see *Th.* 3 [c]]). So we may assume that $\upsilon_0 \neq \upsilon_1$ holds. This implies $\sim\upsilon_1\mathbf{H}\upsilon_0$. This, together with $\upsilon_0\mathbf{H}\tau$ and $[\psi(\omega/\upsilon_0)(\upsilon/\upsilon_1) \wedge \upsilon_1\mathbf{H}\tau]$, implies $\exists\upsilon(\boldsymbol{\Sigma}\omega\sqsubseteq\tau)[\upsilon\mathbf{H}\tau \wedge \psi \wedge \sim\upsilon\mathbf{H}\omega]$.

111 $\sim\upsilon_0\mathbf{H}\omega_1$ implies $(\boldsymbol{\Sigma}\omega\sqsubseteq\tau)[\upsilon_0\mathbf{H}\tau \wedge \psi(\upsilon/\upsilon_0) \wedge \sim\upsilon_0\mathbf{H}\omega]$, which implies $\upsilon_0\mathbf{H}\omega_1$ by [*].

112 Note that the assumption $\mathbf{M\small{ANY}}(\tau)$ is essential in *Th.* 15. Its negation implies both $(\boldsymbol{\Pi}\omega\sqsubseteq\tau)(\exists\upsilon\mathbf{H}\tau)\omega \approx \upsilon$ and $\sim(\boldsymbol{\Sigma}\omega\sqsubseteq\tau)(\boldsymbol{\Sigma}\omega_1\sqsubseteq\tau)(\exists\upsilon\mathbf{H}\tau)[\sim\omega \approx \omega_1 \wedge \omega \approx \upsilon \wedge \omega_1 \approx \upsilon]$ in S_{PL}. Similarly, the assumption $\exists\upsilon\exists\upsilon_1\upsilon \neq \upsilon_1$ in *Th.* 16 is essential.

113 It implies the existence of an empty set under the assumption that there is a non-universal set.

[114] *Th.* 15 does not apply to the cases in which the set in question is empty or has only one member, but we can use [b] to show that Cantor's theorem holds for those special cases.

[115] So we cannot apply Cantor's Theorem to show that, roughly, there are more properties than there are objects (or objects that are not members of themselves), because there is no set (or class) that comprehends all the objects (or all the objects that are not members of themselves).

[116] All the sentences except those marked by "*" are mentioned in Yi (LMP I).

[117] Only works directly referred to in this paper are listed below. See Yi (LMP I) for a more comprehensive list of works germane to the subject matter of this paper.

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