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SOME USEFUL 16-VALUED LOGICS: HOW A COMPUTER NETWORK SHOULD THINK

Dedicated to Nuel D. Belnap on the occasion of his 75th Birthday

ABSTRACT. In Belnap's useful 4-valued logic, the set $\mathbf{2} = \{T, F\}$ of classical truth values is generalized to the set $\mathbf{4} = \mathcal{P}(\mathbf{2}) = \{\varnothing, \{T\}, \{F\}, \{T, F\}\}$. In the present paper, we argue in favor of extending this process to the set $\mathbf{16} = \mathcal{P}(\mathbf{4})$ (and beyond). It turns out that this generalization is well-motivated and leads from the bilattice $FOUR_2$ with an information and a truth-and-falsity ordering to another algebraic structure, namely the trilattice $SIXTEEN_3$ with an information ordering together with a truth ordering *and* a (distinct) falsity ordering. Interestingly, the logics generated separately by the algebraic operations under the truth order and under the falsity order in $SIXTEEN_3$ coincide with the logic of $FOUR_2$, namely *first degree entailment*. This observation may be taken as a further indication of the significance of first degree entailment. In the present setting, however, it becomes rather natural to consider also logical systems in the language obtained by combining the vocabulary of the logic of the truth order and the falsity order. We semantically define the logics of the two orderings in the extended language and in both cases axiomatize a certain fragment comprising three unary operations: a negation, an involution, and their combination. We also suggest two other definitions of logics in the full language, including a *bi-consequence system*. In other words, in addition to presenting first degree entailment as a useful 16-valued logic, we define further useful 16-valued logics for reasoning about truth and (non-)falsity. We expect these logics to be an interesting and useful instrument in information processing, especially when we deal with a net of hierarchically interconnected computers. We also briefly discuss Arieli's and Avron's notion of a logical bilattice and state a number of open problems for future research.

KEY WORDS: 4-valued logic, 16-valued logic, bi-consequence logic, first degree entailment, generalized truth values, (logical) bilattices, trilattices, multilattices

1. Introduction: Generalized Valuations, Four-Valued Logic, and Bilattices

According to a strategy of semantic analysis elaborated by J. Michael Dunn, a sentence can be *rationally* considered to be not just true or just false, but also neither true nor false as well as both true and false.[1] This can be made explicit by developing a suitable valuation procedure that generalizes the notion of an ordinary, classical truth value function[2] by allowing “under-determined” and “over-determined” valuations. Let U be a “set of topics” such that $X_1 \subseteq U$ and $X_2 \subseteq U$. In [14, pp. 121–132], Dunn introduces so-called “aboutness valuations” that ascribe to each sentence a “proposition surrogate”, namely a pair (X_1, X_2), where X_1 represents topics the sentence gives definite information about, and X_2 represents topics the negation of the sentence gives definite information about (cf. [18, p. 36]). A proposition surrogate need be neither disjoint ($X_1 \cap X_2 = \varnothing$) nor exhaustive ($X_1 \cup X_2 = U$), thus making “truth value gaps” and “truth value gluts” possible.

The same idea is realized in [16] by interpreting a valuation not as a function but just as a *relation* connecting sentences of the language in use with elements from **2**. Such a valuation relates to each sentence either the value “true” (T) or “false” (F), or *neither* of these values (partial function), or *both* of them (non-functional relation). Dunn provides the non-standard valuations with an intuitive motivation in terms of abstract (epistemic) “situations” that may well be incomplete or inconsistent (see [16, pp. 155–157]).

An equivalent although somehow more “ontological” way of grasping this point is considering a valuation still a function, however, not from sentences to *elements* of the set **2**, but from sentences to *subsets* of this set. This latter interpretation can be found in [16] as well (cf. also [19, p. 7]). In what follows we take this method of generalizing the notion of a classical truth function as paradigmatic. We call a truth value function conceived in this way a *generalized truth value function*.[3]

In [9, 10] (also reproduced in [3, §81]) Nuel D. Belnap takes Dunn’s idea a step further by generalizing not just the notion of a truth value *function* but the very notion of a *truth value* itself. Namely, Belnap explicitly regards the empty and the “overcomplete” subsets of the set $\{F, T\}$ as new truth values. Using a highly heuristic interpretation of a truth value as information that “has been told to a computer” he arrives at a “useful four-valued logic” of “how a computer should think” with the following four *generalized truth values*:

$\mathbf{N} = \{\}$ – none (“told neither falsity nor truth”);

$\mathbf{F} = \{F\}$ – “plain” falsehood (“told only falsity”);

$\mathbf{T} = \{T\}$ – “plain” truth (“told only truth”);

$\mathbf{B} = \{F, T\}$ – both falsehood and truth (“told both falsity and truth”).

Belnap’s four-valued logic has found numerous applications in various fields such as the theory of deductive databases, distributed logic programming, and other areas. Inspired by applications of logic in computer

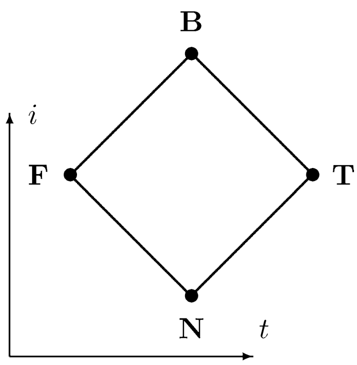

Figure 1. Bilattice $FOUR_2$.

science and AI, M. Ginsberg [31, 32] introduced the notion of a *bilattice* and pointed out that Belnap's four truth values form the smallest non-trivial bilattice.[4] Roughly speaking, a bilattice is a non-empty set with *two* partial orderings, each constituting its own lattice on this set. The bilattice based on the set $\mathbf{4} = \{\mathbf{N}, \mathbf{F}, \mathbf{T}, \mathbf{B}\}$, which we will call $FOUR_2$ (the subscript '2' stands for '*bi*'), is presented by a double Hasse diagram in Figure 1.

This diagram is placed into a two-dimensional coordinate plane, where the horizontal axis stands for a truth order ($\leq_t$) and the vertical axis stands for an information order ($\leq_i$).[5] These orderings represent an increase in truth and information, respectively, i.e., $x \leq_t y$ means that y is "at least as true" as x, and $x \leq_i y$ means that y is "at least as informative" as x.

From a logical point of view, the truth ordering is most important. It is a "logical order" that determines the properties of logical connectives as well as the relation of entailment defined on $FOUR_2$. Namely, the lattice operations of meet and join under this order are just logical conjunction and disjunction. The inversion of $\leq_t$ represents a certain kind of negation. As to entailment, it can be defined in the following way. Let v^4 (a 4-valuation) be a map from the set of propositional variables into **4**, and let this valuation be extended to compound formulas as usual, in accordance with the lattice operations under $\leq_t$ (cf. Definition 3.8). Then we have:

DEFINITION 1.1. $A \models^4 B$ iff $\forall v^4(v^4(A) \leq_t v^4(B))$.

This relation can be axiomatized by the consequence system $\mathbf{E}_{fde}$ of "tautological entailments" from [2, §15.2] (also called *first degree entailment*).

2. Taking Generalization Seriously

There is an interesting question concerning Dunn's and Belnap's four-valued semantics, namely: Why should we stop the "generalization procedure" just at the four-valued stage and not proceed further to considering, say, combinations of **T** and **B**, **N** and **B**, etc.? If we can get a "useful four valued logic" by taking the power-set of **2**, why should one not consider a *sixteen-valued logic* based on $\mathcal{P}(\mathbf{4})$, the power-set of the set of generalized truth values obtained previously? Would such a logic be "useful"? This question has been raised from time to time in research seminars and during conference debates, but it remains still chiefly a kind of a "standard question":[6] we are not aware of any comprehensive and systematic discussion of this question in print.[7] Yet, we believe that it is an interesting and important question.

The "oral tradition" supplies the question with one typical reply, arguing to the effect that any combination of Belnap's four truth values would be in a sense *superfluous*. The argument usually goes as follows. Consider, e.g., the combination $\mathbf{TB}(=\{\{T\}, \{F, T\}\})$ of **T** and **B**. This new truth value would then mean "true and true-and-false". But a repetition of truths gives us no new information (is superfluous)! Thus, the meaning of **TB**, it is claimed, collapses just into "true-and-false", and in this way we simply obtain **B**. An analogues argument reduces **FB** to **B**, and it is not difficult to argue in a similar way that **FT** is, in fact, also **B**. Further, a combination of **N** with any other truth value seems to be superfluous as well, for unifying the empty set with any other set gives just this latter set. As a consequence one might conclude that any attempt to continue generalizing truth values beyond the four values introduced by Belnap should fail due to a collapse of any new truth value into one of the initial four.

However, a more careful examination shows that such a conclusion is not justified. First, recall that the proper interpretation of **T** is not simply "true" but "true-*only*" (and analogously for falsehood). And the combination of "true-only" and "true-and-false", which we get in the new truth value **TB**, is not so trivial and, in any case, is not so easily reducible to "true-and-false" as the above argument seems to suggest. Second, one may notice that this argument works only under the implicit interpretation of the comma between elements in new truth values as set-theoretical union and the identification of a set x with the singleton $\{x\}$. Only then one would be able to conduct the suggested manipulation: $\{\{T\}, \{F, T\}\} = \{\{T\} \cup \{F, T\}\} = \{T, F, T\} = \{F, T\}$, which is obviously incorrect. $\{\{T\}, \{F, T\}\}$ is, of course, distinct from $\mathbf{B} = \{T\} \cup \{F, T\}$, and therefore, it would be more natural to consider the generalized truth value $\{\{T\},\{F,T\}\}$

an independent value in its own right. Similarly, $\{\varnothing, \{F, T\}\}$ is not the same as $\{F, T\}$, etc.

Thus, there are good reasons for taking the above mentioned generalizing procedure seriously and considering a *second-order generalization*, which results from taking the power-set of {**N**, **F**, **T**, **B**}.[8] In this way we obtain the following set **16** of *sixteen* generalized truth values (denotations for most values are obvious, and **A** stands for "all"):

1. $\mathbf{N} = \varnothing$	9. $\mathbf{FT} = \{\{F\}, \{T\}\}$
2. $\boldsymbol{N} = \{\varnothing\}$	10. $\mathbf{FB} = \{\{F\}, \{F, T\}\}$
3. $\boldsymbol{F} = \{\{F\}\}$	11. $\mathbf{TB} = \{\{T\}, \{F, T\}\}$
4. $\boldsymbol{T} = \{\{T\}\}$	12. $\mathbf{NFT} = \{\varnothing, \{F\}, \{T\}\}$
5. $\boldsymbol{B} = \{\{F, T\}\}$	13. $\mathbf{NFB} = \{\varnothing, \{F\}, \{F, T\}\}$
6. $\mathbf{NF} = \{\varnothing, \{F\}\}$	14. $\mathbf{NTB} = \{\varnothing, \{T\}, \{F, T\}\}$
7. $\mathbf{NT} = \{\varnothing, \{T\}\}$	15. $\mathbf{FTB} = \{\{F\}, \{T\}, \{F, T\}\}$
8. $\mathbf{NB} = \{\varnothing, \{F, T\}\}$	16. $\mathbf{A} = \{\varnothing, \{T\}, \{F\}, \{F, T\}\}$.

That **16** makes perfect sense can also be shown by employing Belnap's interpretation of truth values as information that could be told to a computer. One of the main motivations for Belnap's interpretation is the obvious observation that a computer can receive information from various (maybe independent) sources. Now a situation is possible, when one source informs a computer that a sentence is true-only, while another informant supplies (perhaps without being aware of this) inconsistent data, namely that the sentence is both true and false: a clear case for **TB**. And if a computer has been simultaneously "told" that a sentence is true-only (informant 1), false-only (informant 2), both-true-and-false (informant 3) and neither-true-nor-false (informant 4), then **A** appears to be not a "madness" but just an adequate device for representing this situation. Dunn and Hardegree [20, p. 277, our emphasis] seem to argue along the same lines, when they remark that "there can be states of information that are inconsistent, incomplete, *or both*". Obviously, an information state that is both inconsistent *and* incomplete cannot be represented by any single value from **4**, whereas **16** offers the value **NB** to account for this situation.

In fact, Belnap's interpretation suits perfectly well when applied to *a single* computer. In addition, it presupposes that this computer receives information from *classical* sources, i.e., from sources which can operate exclusively with classical truth values (T, F). But what if the computer's informants behave nonclassically? Moreover, nowadays it is quite rare for a computer to stay completely isolated and not being connected (even from time to time) to other computers. And it appears that Belnap's interpretation cannot adequately be applied when we deal not just with one com-

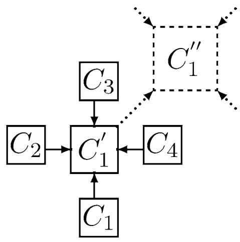

Figure 2. A computer network.

puter but with several interconnected computers, i.e., a *computer network* (conceived hierarchically).

Indeed, consider four Belnap computers[9] (C_1, C_2, C_3, C_4) connected to some *central computer* (C_1'), a server, to which they are supposed to supply information (Figure 2). It is fairly clear that the logic of the server itself (so, the network as a whole) cannot remain four-valued any more, and to get an adequate tool for handling this situation we have to involve **16**. Incidentally, it is interesting to observe that if we wish to extend our network and connect our server to some "higher" computer (C_1''), then generalized truth values of the *third order* (the set $\mathcal{P}(\mathbf{16})$) come into question (and so on), which motivates further generalization of the present construction (see the last section of this paper).

Note that in **16**, each of Belnap's original four truth values from **4** underwent an important transformation by being promoted to the single element of some "higher set". We mark this by putting the corresponding truth values in **16** into italics. Obviously, $\mathbf{N} \in \boldsymbol{N}$, $\mathbf{F} \in \boldsymbol{F}$, $\mathbf{T} \in \boldsymbol{T}$ and $\mathbf{B} \in \boldsymbol{B}$. It is also interesting to observe the difference between **FT** and Belnap's **B**. We will discuss this difference and generally an intuitive motivation for "generalized truth values of higher order" in more detail later.

3. The Trilattice *SIXTEEN*$_3$

The rest of the paper is devoted to investigating **16** and elucidating some of its logical and algebraic properties. To meet this goal, we introduce one central notion that will be of great importance for the whole further analysis.

DEFINITION 3.1. An *n-dimensional multilattice* (or simply *n-lattice*) is a structure $\mathcal{M}_n = (S, \leq_1, \ldots, \leq_n)$ such that S is a non-empty set and $\leq_1, \ldots, \leq_n$ are partial orders defined on S such that $(S, \leq_1), \ldots, (S, \leq_n)$ are all distinct lattices.

The notion of a *multilattice*[10] (cf. [43]) is a generalization of the notion of a bilattice along the lines proposed in [40], where a generalized truth value space of constructive logics has been presented in the form of a *trilattice*. The trilattice in [40][11] is a direct extension of a bilattice structure with a *third* partial order ($\leq_c$), which represents there an increase in *constructivity*. However, elements of a truth value space may well incorporate some other property (or properties), and then some additional partial orders may be needed. Note that we do not require that $\leq_1, \dots, \leq_n$ should be *the only* (and all) partial orders that exist for the given set. That is, a basic set of generalized truth values may give rise to a 2-lattice, 3-lattice, 4-lattice, etc., depending on the purposes of the analysis and, of course, on how many partial orders exist on it.

Following Fitting and others (see, e.g., [27]), we call an n-lattice *complete* iff all the lattices that constitute this multilattice are complete, *interlaced* iff each pair of meet and join is monotone (i.e., order preserving) with respect to each partial order of the multilattice, and *distributive* iff all $2(2n^2 - n)$ distributive laws connecting its meets and joins hold.

Consider any two distinct ordering relations defined on some non-empty set. We say that these relations are *mutually independent* with respect to these definitions (or are defined *mutually independently*) iff they are not inversions of each other and the only common terms that are used in both definitions, except of metalogical connectives and quantifiers, are the usual set theoretical terms (for the intuitive motivation for the idea of "mutual independence" see [40, pp. 782–783]). The following definition introduces an important class of multilattices allowing to reasonably reduce the amount of partial orders in a multilattice to relations that are in a certain sense "non-trivial" (or "interesting").

DEFINITION 3.2. A multilattice is called *proper* iff all its partial orders can be defined mutually independently.

Let us return for a while to the bilattice $FOUR_2$ and formally define its ordering relations ($\leq_i$ and $\leq_t$). The definition of $\leq_i$ is very simple: for any $x \in \mathbf{4}$ we just put $x \leq_i y$ iff $x \subseteq y$.[12] For $\leq_t$ the situation is more intricate. For each element of **4** we first define its "truth part" and its "falsity part" as follows:

$$x^t := \{z \in x \mid z = T\}; \qquad x^f := \{z \in x \mid z = F\}.$$

Then we have: $x \leq_t y$ iff $x^t \subseteq y^t$ and $y^f \subseteq x^f$.[13] This definition clearly shows that within $FOUR_2$, $\leq_t$ is in fact not just a *truth* order but rather a "*truth-and-falsity* order": by ordering the truth values we have to take into account not only the "truth-content" of each value but also its "falsity-content". An increase in truth-content automatically means here a decrease in falsity content (cf. [26, p. 480]). In other words, $\leq_t$ in $FOUR_2$ seems to presuppose that falsehood *by itself* is less true than truth, and thus one may suspect that truth and falsity in $FOUR_2$ are not entirely autonomous notions.

Now, when we turn to the algebraic structure of **16**, it appears that it is here possible to discriminate between an increase in truth and a decrease in falsity and thus to define a truth order and a (non-)falsity order as distinct and in effect *mutually independent* relations. To do so we have to redefine the sets x^t and x^f, and to explicitly introduce their complements:

$$x^t := \{y \in x \mid T \in y\}; \qquad x^{-t} := \{y \in x \mid T \notin y\};$$

and analogously for falsity:[14]

$$x^f := \{y \in x \mid F \in y\}; \qquad x^{-f} := \{y \in x \mid F \notin y\}.$$

Then we define:

DEFINITION 3.3. For every x, y in **16**:

(1) $x \leq_i y$ iff $x \subseteq y$;
(2) $x \leq_t y$ iff $x^t \subseteq y^t$ and $y^{-t} \subseteq x^{-t}$;
(3) $x \leq_f y$ iff $x^f \subseteq y^f$ and $y^{-f} \subseteq x^{-f}$.

In this way the algebraic structure of **16** is that of a proper 3-lattice, a *trilattice* with *three* mutually independent partial orders that represent increase in information, truth and falsehood. This trilattice, which we for obvious reason call $SIXTEEN_3$, is presented by a triple Hasse diagram in Figure 3 (cf. Figure 5 in [40]). One can clearly observe in this diagram all three partial orderings. **A** and **N** are, respectively, the lattice top and bottom relative to $\leq_i$, **TB** and **NF** relative to $\leq_t$, and **FB** and **NT** relative to $\leq_f$. In accordance with the underlying interpretation, **A** and **N** are then the most and the least *informative* elements of **16**, **TB** and **NF** are the most and the least *true* of its elements, and **FB** and **NT** are the most and the least *false* elements. Note that the f-axis in Figure 3 is drawn a bit approximately and gives only a rough idea about the third "dimension" of $SIXTEEN_3$. Another projection of $SIXTEEN_3$ (on the plain t–f) is represented in Figure 4 below.

It is interesting to observe that $SIXTEEN_3$ has altogether exactly *eight* distinct partial orders that are not inversions of each other. In addition to the relations introduced in Definition 3.3 there are: a "truth-only" order (with ***T*** and **NFB** as top and bottom); a "falsity-only" order (with ***F*** and **NTB**);

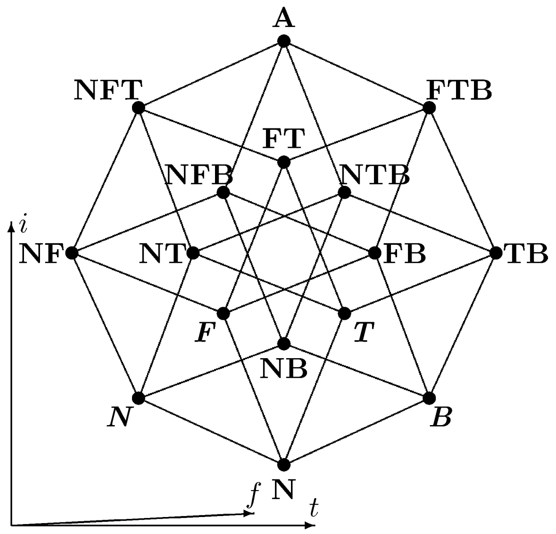

Figure 3. Trilattice *SIXTEEN*$_3$ (projection i–t).

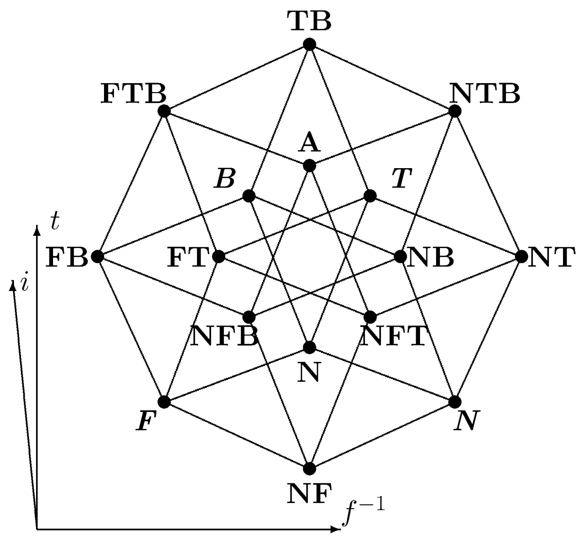

Figure 4. Trilattice *SIXTEEN*$_3$ (projection t–f).

a "truth-only and-falsity-only" order (with **FT** and **NB**); a "both-only" order (with ***B*** and **NFT**); a "none-only" order (with ***N*** and **FTB**). But these five additional relations are all not independent and in fact "derivative" of the truth order and falsity order from Definition 3.3 in the sense that any of them can be defined through a certain combination of the functions $(\cdot)^t$, $(\cdot)^{-t}$, $(\cdot)^f$ and $(\cdot)^{-f}$ used in the definitions of $\leq_t$ and $\leq_f$.

One might consider the generalized truth values ***B***, ***N***, ***F***, and ***T*** form **16** as *analogues* of the truth values **B**, **N**, **F** and **T** from **4**.[15] However, some of these truth values from **16** and **4** behave quite differently under one and the same ordering relations in $SIXTEEN_3$ and in $FOUR_2$. For example, within $FOUR_2$, **B** is more and **N** is less informative than **F** and **T**, but in $SIXTEEN_3$, we see that ***B***, ***N***, ***F***, and ***T*** are situated on the same informational level. This latter fact might seem to violate some basic intuitive motivations. Recall that according to Belnap's interpretation **B** carries more information than **T**, because **B** stands for "both true and false", whereas **T** stands for "only true" (and analogously for other truth values).

However, it turns out that the behavior of ***B***, ***N***, ***F***, and ***T*** in $SIXTEEN_3$ is much more natural than one might think on the face of it. First, note again that analogous values from **4** and from **16** are *not* the same, e.g., $\mathbf{T} = \{T\}$ but $\boldsymbol{T} = \{\{T\}\}$, etc. Moreover, we remark that under $\leq_i$ generalized truth values should be ordered exclusively by the amount of elements in the corresponding sets: the more elements a set has, the more informative it is. But in **16** the truth values ***B***, ***N***, ***F***, and ***T*** are *all* singletons (in contrast to **N**, **F**, **T** and **B** in **4**), and hence they all are equally informative, precisely as $SIXTEEN_3$ suggests. Intuitively this means that – as one may easily observe – *any* of Belnap's initial truth values, and not only **B**, may be viewed as saying (explicitly or implicitly) *something* both about truth and falsehood, either in a positive or in a negative mode. To make this point explicit, the elements of **4** need a slightly different reading. To justify this reading, we remark that, e.g., **T** – "truth-only" – means actually nothing more than "true and *not false*". But from a purely quantitative point of view a negative piece of information is exactly of *the same* cash value as a positive one. In this way we arrive at the following reinterpretation (or maybe more precise interpretation) of Belnap's four truth values:

N – "a sentence is not false and not true", [*non-falsehood*, *non-truth*];
F – "a sentence is false and not true", [*falsehood*, *non-truth*];
T – "a sentence is not false and true", [*non-falsehood*, *truth*];
B – "a sentence is false and true", [*falsehood*, *truth*].

Let us emphasize that this reading for elements of **4** could be explicated only within the higher-order construction **16**. Generally speaking, the key point of the semantic approach by Dunn and Belnap consists in taking the power set of some *basic set* of truth values and thereby obtaining a new set of generalized truth values, which in turn should provide information *concerning this basic set*, more specifically, information about the assignment of elements of the given base to a sentence. Thus, $\sigma(x) := \{x\}$ may naturally be viewed as an operation of "informatization". It creates a "piece of information" that refers to some "reality of a (one-step) lower order": the truth value $\{x\}$ is supposed to supply information just about x.[16]

It is now clearer what the difference between **N** and ***N*** consists in. The only feature of **N** is that it presents *no information* at all (relative to the truth values from the corresponding base). That is, in **4** the value **N** just gives *no* information concerning the classical truth values and in **16** concerning Belnap's truth values. But ***N*** in **16** is more expressive. Namely, it provides specific information saying that a sentence has been assigned Belnap's value **N**, which can be articulated by the *metastatement* that a sentence is neither classically true nor classically false.

The reader may also observe the difference between **FT** and **B**. Recall that Belnap's **B** is often interpreted as representing the idea of *paraconsistency*, the view that nontrivial contradictions can well exist. However, this interpretation makes sense only under some implicit linguistic convention, namely the assumption that truth and falsehood are, in effect, contradictory notions. But the real, *logical* contradiction to truth, that does not depend on any assumption, is just *non*-truth, and a logical contradiction to falsehood is *non*-falsehood. Thus, **FT** – saying "false and not-true *as well as* not-false and true" – is not only more informative, but seems to express the idea of a (nontrivial) contradictory truth value much better than **B** does.

Moreover, it appears that the second-order value **A** is inconsistent in an even stronger sense, stating that a sentence with this value is *not only* true-and-false, but also true and not-false, false and not-true, and neither-true-nor-false. Such a sentence takes all the values available at the level of first-order values, and it becomes clear that this idea of strengthening the notion of inconsistency can be extended to higher levels. If we define $\mathcal{P}_1(\mathbf{2}) := \mathcal{P}(\mathbf{2})$, and $\mathcal{P}_n(\mathbf{2}) := \mathcal{P}(\mathcal{P}_{n-1}(\mathbf{2}))$ for $n > 1$, a sentence taking the value $\{x \mid x \in \mathcal{P}_n(\mathbf{2})\}$ is the maximal inconsistency of order $n + 1$, while **B** can be defined as the inconsistency of order 0.

It is also instructive to notice that **N**, **F**, **T** and **B** in $FOUR_2$ and ***B***, ***N***, ***F***, and ***T*** in $SIXTEEN_3$ are ordered differently under $\leq_t$ as well. Within $FOUR_2$, **T** is "more true" than **B**, and **N** is more true than **F**, whereas **N** and **B** are of the same truth level. But in $SIXTEEN_3$, ***B*** comes out more true than ***N***, while ***T*** and ***B*** as well as ***N*** and ***F*** are (pairwise) "equally true". This situation is entirely clear. As it was noted above, in $SIXTEEN_3$ (but not in $FOUR_2$) $\leq_t$ is a *pure truth relation*: it orders the truth values by exclusively taking into account what they say about *truth*, leaving any information about falsehood without attention.

Clearly, meets and joints exist in $SIXTEEN_3$ for all three partial orders. We will use $\sqcap$ and $\sqcup$ with the appropriate subscripts for these operations under the corresponding ordering relations. *SIXTEEN*$_3$ emerges then as the structure $(\mathbf{16}, \sqcap_i, \sqcup_i, \sqcap_t, \sqcup_t, \sqcap_f, \sqcup_f)$.

It was already mentioned above that in *FOUR*$_2$ the partial order $\leq_t$ is often called (and, in fact, identified with) a "logical order", for it is supposed to determine the properties of logical connectives and the entailment relation. But as we have seen, within *FOUR*$_2$ actually *two* ordering relations – one for truth and one for falsehood – are merged into one logical order. However, when proceeding one step higher to *SIXTEEN*$_3$, it turns out that there an increase in truth does not necessarily mean a decrease in falsehood (and *vice versa*) any more. Hence, within *SIXTEEN*$_3$ the logical order explicitly splits into two distinct relations: the truth order $\leq_t$ and the falsity order $\leq_f$. To display both orders in a precise way, we present in Figure 4 another projection of *SIXTEEN*$_3$ (namely the projection on the plain $t - f$). Here the falsity order is inverted, because for defining central logical notions we will be interested in decreasing (rather than increasing) falsehood.

Some important properties of $\sqcap_t$ and $\sqcup_t$, as well as $\sqcap_f$ and $\sqcup_f$ are summarized in the following, directly checkable proposition:

PROPOSITION 3.4. *For any x, y in SIXTEEN*$_3$*:*

(1) $\mathbf{T} \in x \sqcap_t y \Leftrightarrow \mathbf{T} \in x$ *and* $\mathbf{T} \in y$;
$\mathbf{B} \in x \sqcap_t y \Leftrightarrow \mathbf{B} \in x$ *and* $\mathbf{B} \in y$;
$\mathbf{F} \in x \sqcap_t y \Leftrightarrow \mathbf{F} \in x$ *or* $\mathbf{F} \in y$;
$\mathbf{N} \in x \sqcap_t y \Leftrightarrow \mathbf{N} \in x$ *or* $\mathbf{N} \in y$;

(2) $\mathbf{T} \in x \sqcup_t y \Leftrightarrow \mathbf{T} \in x$ *or* $\mathbf{T} \in y$;
$\mathbf{B} \in x \sqcup_t y \Leftrightarrow \mathbf{B} \in x$ *or* $\mathbf{B} \in y$;
$\mathbf{F} \in x \sqcup_t y \Leftrightarrow \mathbf{F} \in x$ *and* $\mathbf{F} \in y$;
$\mathbf{N} \in x \sqcup_t y \Leftrightarrow \mathbf{N} \in x$ *and* $\mathbf{N} \in y$.

(3) $\mathbf{T} \in x \sqcup_f y \Leftrightarrow \mathbf{T} \in x$ *and* $\mathbf{T} \in y$;
$\mathbf{N} \in x \sqcup_f y \Leftrightarrow \mathbf{N} \in x$ *and* $\mathbf{N} \in y$;
$\mathbf{F} \in x \sqcup_f y \Leftrightarrow \mathbf{F} \in x$ *or* $\mathbf{F} \in y$;
$\mathbf{B} \in x \sqcup_f y \Leftrightarrow \mathbf{B} \in x$ *or* $\mathbf{B} \in y$;

(4) $\mathbf{T} \in x \sqcap_f y \Leftrightarrow \mathbf{T} \in x$ *or* $\mathbf{T} \in y$;
$\mathbf{N} \in x \sqcap_f y \Leftrightarrow \mathbf{N} \in x$ *or* $\mathbf{N} \in y$;
$\mathbf{F} \in x \sqcap_f y \Leftrightarrow \mathbf{F} \in x$ *and* $\mathbf{F} \in y$;
$\mathbf{B} \in x \sqcap_f y \Leftrightarrow \mathbf{B} \in x$ *and* $\mathbf{B} \in y$.

In bilattices, a logical negation is usually defined as an operation that inverts the truth order only, leaving the information order unchanged. Fitting [22] considers also an operation of *conflation* that inverts $\leq_i$, while $\leq_t$ remains as it is. For trilattices this point has been generalized in [40], where several unary operations have been introduced under the general label of *inversion*. The idea is that an inversion can invert some partial orders in a trilattice, possibly leaving the other(s) without change. We have then the following definition:

DEFINITION 3.5. Let $\mathcal{T}$ be a trilattice. Then we can introduce the following unary operations on $\mathcal{T}$ with the following properties:

(1) t-inversion($-_t$):
  (a) $a \leq_t b \Rightarrow -_t b \leq_t -_t a$;
  (b) $a \leq_f b \Rightarrow -_t a \leq_f -_t b$;
  (c) $a \leq_i b \Rightarrow -_t a \leq_i -_t b$;
  (d) $-_t -_t a = a$.

(2) f-inversion($-_f$):
  (a) $a \leq_t b \Rightarrow -_f a \leq_t -_f b$;
  (b) $a \leq_f b \Rightarrow -_f b \leq_f -_f a$;
  (c) $a \leq_i b \Rightarrow -_f a \leq_i -_f b$;
  (d) $-_f -_f a = a$.

(3) i-inversion($-_i$):
  (a) $a \leq_t b \Rightarrow -_i a \leq_t -_i b$;
  (b) $a \leq_f b \Rightarrow -_i a \leq_f -_i b$;
  (c) $a \leq_i b \Rightarrow -_i b \leq_i -_i a$;
  (d) $-_i -_i a = a$.

(4) tf-inversion($-_{tf}$):
  (a) $a \leq_t b \Rightarrow -_{tf} b \leq_t -_{tf} a$;
  (b) $a \leq_f b \Rightarrow -_{tf} b \leq_f -_{tf} a$;
  (c) $a \leq_i b \Rightarrow -_{tf} a \leq_i -_{tf} b$;
  (d) $-_{tf} -_{tf} a = a$.

(5) ti-inversion($-_{ti}$):
  (a) $a \leq_t b \Rightarrow -_{ti} b \leq_t -_{ti} a$;
  (b) $a \leq_f b \Rightarrow -_{ti} a \leq_f -_{ti} b$;
  (c) $a \leq_i b \Rightarrow -_{ti} b \leq_i -_{ti} a$;
  (d) $-_{ti} -_{ti} a = a$.

(6) fi-inversion($-_{fi}$):
  (a) $a \leq_t b \Rightarrow -_{if} a \leq_t -_{if} b$;
  (b) $a \leq_f b \Rightarrow -_{if} b \leq_f -_{if} a$;
  (c) $a \leq_i b \Rightarrow -_{if} b \leq_i -_{if} a$;
  (d) $-_{if} -_{if} a = a$.

(7) tfi-inversion($-_{tfi}$):
  (a) $a \leq_t b \Rightarrow -_{tif} b \leq_t -_{tif} a$;
  (b) $a \leq_f b \Rightarrow -_{tif} b \leq_f -_{tif} a$;
  (c) $a \leq_i b \Rightarrow -_{tif} b \leq_i -_{tif} a$;
  (d) $-_{tif} -_{tif} a = a$.

In *SIXTEEN*$_3$ all seven inversion operations can be defined as shown in Table I. A routine calculation over this table immediately gives us the following proposition:

PROPOSITION 3.6. *For any x in SIXTEEN$_3$:*

(1) $-_t -_f x = -_f -_t x = -_{tf} x$;
(2) $-_t -_i x = -_i -_t x = -_{ti} x$;
(3) $-_f -_i x = -_i -_f x = -_{fi} x$;
(4) $-_t -_f -_i x = -_f -_t -_i x = -_t -_i -_f x = -_f -_i -_t x =$
$-_i -_t -_f x = -_i -_f -_t x = -_{tf} -_i x = -_{ti} -_f x =$
$-_{fi} -_t x = -_i -_{tf} x = -_f -_{ti} x = -_t -_{fi} x = -_{tfi} x$.

Our main concern will naturally be focused on t-inversion, f-inversion and tf-inversion as the most obvious candidates for representing an object-language negation. The following proposition highlights some key features of these operations that will be employed in the further analysis.

PROPOSITION 3.7. *For any x in SIXTEEN$_3$:*

(1) $\mathbf{T} \in -_t x \Leftrightarrow \mathbf{N} \in x$;	(2) $\mathbf{T} \in -_f x \Leftrightarrow \mathbf{B} \in x$;	(3) $\mathbf{T} \in -_{tf} x \Leftrightarrow \mathbf{F} \in x$;
$\mathbf{N} \in -_t x \Leftrightarrow \mathbf{T} \in x$;	$\mathbf{B} \in -_f x \Leftrightarrow \mathbf{T} \in x$;	$\mathbf{B} \in -_{tf} x \Leftrightarrow \mathbf{N} \in x$;
$\mathbf{F} \in -_t x \Leftrightarrow \mathbf{B} \in x$;	$\mathbf{F} \in -_f x \Leftrightarrow \mathbf{N} \in x$;	$\mathbf{F} \in -_{tf} x \Leftrightarrow \mathbf{T} \in x$;
$\mathbf{B} \in -_t x \Leftrightarrow \mathbf{F} \in x$;	$\mathbf{N} \in -_f x \Leftrightarrow \mathbf{F} \in x$;	$\mathbf{N} \in -_{tf} x \Leftrightarrow \mathbf{B} \in x$.

TABLE I

Inversions in *SIXTEEN*$_3$.

a	$-_t a$	$-_f a$	$-_i a$	$-_{tf} a$	$-_{ti} a$	$-_{fi} a$	$-_{tfi} a$
N	**N**	**N**	**A**	**N**	**A**	**A**	**A**
N	***T***	***F***	**NFT**	***B***	**NTB**	**NFB**	**FTB**
F	***B***	***N***	**NFB**	***T***	**FTB**	**NFT**	**NTB**
T	***N***	***B***	**NTB**	***F***	**NFT**	**FTB**	**NFB**
B	***F***	***T***	**FTB**	***N***	**NFB**	**NTB**	**NFT**
NF	**TB**	**NF**	**NF**	**TB**	**TB**	**NF**	**TB**
NT	**NT**	**FB**	**NT**	**FB**	**NT**	**FB**	**FB**
FT	**NB**	**NB**	**NB**	**FT**	**FT**	**FT**	**NB**
NB	**FT**	**FT**	**FT**	**NB**	**NB**	**NB**	**FT**
FB	**FB**	**NT**	**FB**	**NT**	**FB**	**NT**	**NT**
TB	**NF**	**TB**	**TB**	**NF**	**NF**	**TB**	**NF**
NFT	**NTB**	**NFB**	***N***	**FTB**	***T***	***F***	***B***
NFB	**FTB**	**NFT**	***F***	**NTB**	***B***	***N***	***T***
NTB	**NFT**	**FTB**	***T***	**NFB**	**N**	***B***	***F***
FTB	**NFB**	**NTB**	***B***	**NFT**	***F***	***T***	***N***
A	**A**	**A**	**N**	**A**	**N**	**N**	**N**

Note that $\sqcap_t$, $\sqcup_t$, and $-_t$ are now not the only algebraic operations that naturally correspond to logical conjunction, disjunction, and negation; $\sqcup_f$, $\sqcap_f$, and $-_f$ (or even $-_{tf}$) may play this role as well. And taking into account the fact that $x \sqcap_t y \neq x \sqcup_f y$, $x \sqcup_t y \neq x \sqcap_f y$ and $-_t x \neq -_f x$, we can state that both logical orders bring into existence "parallel" and, in fact, *distinct* logical connectives.

Thus, it seems rather natural to explore the possibility of a unified approach to all of these operations within a joint logical framework. To determine such a framework syntactically, we consider (in the most general case) the language $\mathcal{L}_{tf}$ that comprises $\wedge_t$, $\vee_t$, $\sim_t$, $\wedge_f$, $\vee_f$, and $\sim_f$ as propositional connectives. As to the semantics, let v^{16} (a 16-valuation) be a map from the set of propositional variables into **16**, and let us define:

DEFINITION 3.8. For any A and B:

(1) $v^{16}(A \wedge_t B) = v^{16}(A) \sqcap_t v^{16}(B)$; (4) $v^{16}(A \wedge_f B) = v^{16}(A) \sqcup_f v^{16}(B)$;
(2) $v^{16}(A \vee_t B) = v^{16}(A) \sqcup_t v^{16}(B)$; (5) $v^{16}(A \vee_f B) = v^{16}(A) \sqcap_f v^{16}(B)$;
(3) $v^{16}(\sim_t A) = -_t v^{16}(A)$; (6) $v^{16}(\sim_f A) = -_f v^{16}(A)$.

This definition naturally extends a 16-valuation v^{16} to a valuation of compound formulas, thereby enabling an evaluation of arbitrary formulas from $\mathcal{L}_{tf}$.[17] In this way $SIXTEEN_3$ allows a nontrivial coexistence of pairs of different (although analogous) logical connectives without collapsing them into each other. It may be helpful to think of $\wedge_t$, $\vee_t$, $\sim_t$ in terms of the *presence of truth* and to treat $\wedge_f$, $\vee_f$, $\sim_f$ as essentially highlighting the *absence of falsity*.

Incidentally, one may notice that $SIXTEEN_3$ in a way "improves" some perhaps disputable aspects of $FOUR_2$. It has been observed that the account of the truth functions applied to the "nonstandard" truth values (**N** and **B**) in $FOUR_2$ looks a bit "puzzling" or even "odd" (see [3, pp. 516–518]). Indeed, intuitively it is not so evident why we should get $\mathbf{N} \wedge \mathbf{B} = \mathbf{F}$ and $\mathbf{N} \vee \mathbf{B} = \mathbf{T}$. $SIXTEEN_3$ offers in fact a quite different account of gaps and gluts. Their conjunction, for example, can produce again either a gap or a glut, depending on whether we wish to stress the presence of truth (using $\wedge_t$) or the absence of falsity (using $\wedge_f$), cf. Note 7.

Following Belnap [3, p. 518], we may now notice that at this point we have a nice algebraic structure, but we still do not have a *logic*. To get a full-fledged logic, a mere lattice of truth values is not enough (no matter how beautiful it is) – this lattice has also to be equipped with a suitable entailment relation. The canonical way to do so is to define entailment just through the logical order as it is done, e.g., in $FOUR_2$ by using $\leq_t$ (see Definition 1.1).

But in $SIXTEEN_3$ – as it was noted above – we actually have *two* distinct logical orders (one for truth and one for falsity), and it would be hardly justifiable to prefer the truth order over the (non-)falsity order as "the most proper" representative of the notion of "logical inference". This means that we get at least three options: to consider the logic of the truth order (only), to deal with the logic of the falsity order (only), and to define a logic based on both orderings.

4. THE LOGIC OF THE TRUTH ORDER

Belnap [3, p. 518] thinks of a logic as "rules for generating and evaluating inferences". Let us first concentrate on the latter task. Typically a (semantic) definition of an entailment relation provides a method for checking the validity of any inference. And we can use the *truth order* of $SIXTEEN_3$ to obtain such a definition (for arbitrary formulas $A, B \in \mathcal{L}_{tf}$) in a straightforward way.

DEFINITION 4.1. $A \models_t^{16} B$ iff $\forall v^{16}(v^{16}(A) \leq_t v^{16}(B))$.[18]

This definition gives a precise semantic characterization of the logic that corresponds to the order $\leq_t$ in $SIXTEEN_3$. That is, the (semantically defined) logic $(\mathcal{L}_{tf}, \models^{16}_t)$ is the set of all statements $A \models^{16}_t B$ with $A, B \in \mathcal{L}_{tf}$ such that for every 16-valuation v^{16}, $v^{16}(A) \leq_t v^{16}(B)$. Paraphrasing Belnap [3, p. 518], we might state that having an argument involving any inferential connections in the language $\mathcal{L}_{tf}$, we can now unambiguously decide whether "it is a good one" from the standpoint of the truth order.

To obtain rules for *generating* valid inferences (in a systematic way), we have to formalize the logic $(\mathcal{L}_{tf}, \models^{16}_t)$, i.e., characterize it syntactically by means of a suitable deductive system. We shall approach this task using the apparatus of first degree consequence. We shall introduce complete systems for some important fragments of the language $\mathcal{L}_{tf}$, leaving the syntactic formalization of the whole logic $(\mathcal{L}_{tf}, \models^{16}_t)$ to future work.

4.1. *The Language $\mathcal{L}_t$ and the System* $\mathbf{FDE}^t_t$

Let us first investigate a logic that is generated solely by algebraic operations determined by the truth order. To do so, we consider the language $\mathcal{L}_t$ with $\wedge_t$, $\vee_t$ and $\sim_t$ as propositional connectives. Having a 16-valuation v^{16}, we can use Definition 3.8(1)–(3) and Propositions 3.4(1), (2) and 3.7(1) for evaluating any formula of the language $\mathcal{L}_t$.

LEMMA 4.2. *In SIXTEEN*$_3$ *the following clauses are all equivalent* (*for any* $A, B \in \mathcal{L}_t$):

(a) $\forall v^{16}(\mathbf{F} \in v^{16}(B) \Rightarrow \mathbf{F} \in v^{16}(A))$, (b) $\forall v^{16}(\mathbf{N} \in v^{16}(B) \Rightarrow \mathbf{N} \in v^{16}(A))$,
(c) $\forall v^{16}(\mathbf{T} \in v^{16}(A) \Rightarrow \mathbf{T} \in v^{16}(B))$, (d) $\forall v^{16}(\mathbf{B} \in v^{16}(A) \Rightarrow \mathbf{B} \in v^{16}(B))$.

Proof. This is a generalization (and, in fact, an *extension*) of Dunn's analogous result for *FOUR*$_2$ (see, e.g., the proof of Proposition 4 in [19, p. 10]).

(a) $\Rightarrow$ (b) First, we define for any valuation v^{16} a *t-counterpart* valuation $v^{16\prime}$ as follows:

$$\mathbf{T} \in v^{16\prime}(p) \Leftrightarrow \mathbf{B} \in v^{16}(p); \qquad \mathbf{B} \in v^{16\prime}(p) \Leftrightarrow \mathbf{T} \in v^{16}(p);$$
$$\mathbf{F} \in v^{16\prime}(p) \Leftrightarrow \mathbf{N} \in v^{16}(p); \qquad \mathbf{N} \in v^{16\prime}(p) \Leftrightarrow \mathbf{F} \in v^{16}(p).$$

An easy induction extends $v^{16\prime}$ to any formula of $\mathcal{L}_t$.

Now, let $\forall v^{16}(\mathbf{F} \in v^{16}(B) \Rightarrow \mathbf{F} \in v^{16}(A))$. Assume $\exists v^{16}(\mathbf{N} \in v^{16}(B)$ and $\mathbf{N} \notin v^{16}(A))$. Then $\mathbf{F} \in v^{16\prime}(B)$ and $\mathbf{F} \notin v^{16\prime}(A)$. A contradiction.

(b) $\Rightarrow$ (c) For any valuation v^{16} we define a *t-dual* valuation v^{16*} as follows:

$$\mathbf{T} \in v^{16*}(p) \Leftrightarrow \mathbf{N} \notin v^{16}(p); \qquad \mathbf{B} \in v^{16*}(p) \Leftrightarrow \mathbf{F} \notin v^{16}(p);$$
$$\mathbf{F} \in v^{16*}(p) \Leftrightarrow \mathbf{B} \notin v^{16}(p); \qquad \mathbf{N} \in v^{16*}(p) \Leftrightarrow \mathbf{T} \notin v^{16}(p);$$

and show by induction that it can be extended to any formula of $\mathcal{L}_t$.

Let $\forall v^{16}(\mathbf{N} \in v^{16}(B) \Rightarrow \mathbf{N} \in v^{16}(A))$. Assume $\exists v^{16}(\mathbf{T} \in v^{16}(A)$ and $\mathbf{T} \notin v^{16}(B))$. Then $\mathbf{N} \notin v^{16*}(A)$ and $\mathbf{N} \in v^{16*}(B)$. A contradiction.

(c) $\Rightarrow$ (d) Let a counterpart valuation be defined as above, and let $\forall v^{16}(\mathbf{T} \in v^{16}(A) \Rightarrow \mathbf{T} \in v^{16}(B))$. Assume $\exists v^{16}(\mathbf{B} \in v^{16}(A)$ and $\mathbf{B} \notin v^{16}(B))$. Then $\mathbf{T} \in v^{16\prime}(A)$ and $\mathbf{T} \notin v^{16\prime}(B)$. A contradiction.

(d) $\Rightarrow$ (a) Let a dual valuation be defined as above. Let $\forall v^{16}$ $(\mathbf{B} \in v^{16}(A) \Rightarrow \mathbf{B} \in v^{16}(B))$. Assume $\exists v^{16}(\mathbf{F} \in v^{16}(B)$ and $\mathbf{F} \notin v^{16}(A))$. Then $\mathbf{B} \notin v^{16*}(B)$ and $\mathbf{B} \in v^{16*}(A)$. A contradiction. □

LEMMA 4.3. *For any* $A, B \in \mathcal{L}_t$: $A \models_t^{16} B$ *iff* $\forall v^{16}(\boldsymbol{T} \in v^{16}(A) \Rightarrow \boldsymbol{T} \in v^{16}(B))$.

Proof. In view of the previous lemma, our claim is equivalent with: For any $A, B \in \mathcal{L}_t$: $A \models_t^{16} B$ iff

$$(*) \qquad \begin{aligned} &\forall v^{16}\forall y(((\exists x \in \mathbf{16})\ y \in x \ \&\ T \in y) \\ &\quad \Rightarrow (y \in v^{16}(A) \Rightarrow y \in v^{16}(B))) \\ &\quad \&\ \forall v^{16}\forall y(((\exists x \in \mathbf{16})\ y \in x \ \&\ T \notin y) \Rightarrow (y \in v^{16}(B) \\ &\quad \Rightarrow y \in v^{16}(A))). \end{aligned}$$

($\Rightarrow$) If $A \models_t^{16} B$, by definition, $\forall v^{16}(v^{16}(A) \leq_t v^{16}(B))$, and thus (i) $v^{16}(A)^t \subseteq v^{16}(B)^t$ and (ii) $v^{16}(B)^{-t} \subseteq v^{16}(A)^{-t}$. By (i), if $T \in y \in v^{16}(A)$, then $y \in v^{16}(B)$. By (ii), if $T \notin y \in v^{16}(B)$, then $y \in v^{16}(A)$.

($\Leftarrow$) Suppose (∗) holds. We must show that (1) $v^{16}(A)^t \subseteq v^{16}(B)^t$ and (2) $v^{16}(B)^{-t} \subseteq v^{16}(A)^{-t}$. Ad (1): Let $y \in v^{16}(A)^t$. Then $T \in y$ and $y \in v^{16}(A)$. Suppose $y \notin v^{16}(B)^t$. Then $y \notin v^{16}(B)$ or $T \notin y$. But since $T \in y$, we have $y \notin v^{16}(B)$. Since $y \in v^{16}(A)$, by (∗), $y \in v^{16}(B)$, a contradiction. Ad (2): Let $y \in v^{16}(B)^{-t}$. Then $T \notin y$ and $y \in v^{16}(B)$. Suppose $y \notin v^{16}(A)^{-t}$. Then $y \notin v^{16}(A)$ or $T \in y$. But since $T \notin y$, we have $y \notin v^{16}(A)$. By (∗), $y \notin v^{16}(B)$, a contradiction. □

Now we can determine the logic that corresponds to the entailment relation introduced by Definition 4.1, when $A, B \in \mathcal{L}_t$. For formulas built up from $\wedge_t$, $\vee_t$ and $\sim_t$ this relation can be axiomatized by a (first degree) consequence system, which we call $\mathbf{FDE}_t^t$. The superscript indicates the type of language used, and the subscript explicates the kind of consequence. The system is thus a pair $(\mathcal{L}_t, \vdash_t)$, where $\vdash_t$ is a binary relation (consequence) on the language $\mathcal{L}_t$ satisfying the following postulates (axiom schemes and rules of inference):

a_t1. $A \wedge_t B \vdash_t A$
a_t2. $A \wedge_t B \vdash_t B$
a_t3. $A \vdash_t A \vee_t B$
a_t4. $B \vdash_t A \vee_t B$

$a_t5.\ A \wedge_t (B \vee_t C) \vdash_t (A \wedge_t B) \vee_t C$
$a_t6.\ A \vdash_t \sim_t\sim_t A$
$a_t7.\ \sim_t\sim_t A \vdash_t A$

$r_t1.\ A \vdash_t B, B \vdash_t C / A \vdash_t C$
$r_t2.\ A \vdash_t B, A \vdash_t C / A \vdash_t B \wedge_t C$
$r_t3.\ A \vdash_t C, B \vdash_t C / A \vee_t B \vdash_t C$
$r_t4.\ A \vdash_t B / \sim_t B \vdash_t \sim_t A.$

Note that postulates of $\mathbf{FDE}^t_t$ are direct analogues to the postulates of the *first degree entailment* system $\mathbf{E}_{fde}$ from [2, p. 158] (Dunn in [17] dubs it $\mathbf{R}_{fde}$, emphasizing the fact that it is simultaneously a fragment of both relevance logics **E** and **R**). In our view, this observation reinforces the claim that the logic of first degree entailment is a significant logical system that can be arrived at using different well-motivated approaches.

First, we prove the consistency of $\mathbf{FDE}^t_t$ relative to $\models^{16}_t$.

THEOREM 4.4. *For any* $A, B \in \mathcal{L}_t$*: If* $A \vdash_t B$*, then* $A \models^{16}_t B$.

Proof. Taking into account Lemma 4.3, it suffices to prove that (1) if $A \vdash_t B$ is an axiom of $\mathbf{FDE}^t_t$, then $\forall v^{16}(\mathbf{T} \in v^{16}(A) \Rightarrow \mathbf{T} \in v^{16}(B))$, and (2) all the rules of $\mathbf{FDE}^t_t$ preserve this property. This is mainly a routine check (employing Propositions 3.4(1), (2) and 3.7(1)) and can be safely left to the reader, except of r_t4 which, in addition, needs Lemma 4.2 for its justification. Assume $A \models^{16}_t B$, i.e., $\forall v^{16}(\mathbf{T} \in v^{16}(A) \Rightarrow \mathbf{T} \in v^{16}(B))$ (Lemma 4.3). Then by Lemma 4.2 $\forall v^{16}(\mathbf{N} \in v^{16}(B) \Rightarrow \mathbf{N} \in v^{16}(A))$. Suppose $\sim_t B \not\models^{16}_t \sim_t A$, i.e., $\exists v^{16}(\mathbf{T} \in v^{16}(\sim_t B)$ and $\mathbf{T} \notin v^{16}(\sim_t A))$. Then $\exists v^{16}(\mathbf{N} \in v^{16}(B)$ and $\mathbf{N} \notin v^{16}(A))$ (Proposition 3.7), a contradiction. □

To prove completeness we have to construct a suitable canonical model. Let a *theory* be a set of sentences closed under $\vdash_t$ (i.e., for every theory α, if $A \in \alpha$ and $A \vdash_t B$, then $B \in \alpha$) and $\wedge_t$ (if $A \in \alpha$ and $B \in \alpha$, then $A \wedge_t B \in \alpha$). A theory α is *prime* iff the following holds: if $A \vee_t B \in \alpha$, then $A \in \alpha$ or $B \in \alpha$. The following fact about prime theories is very well known (Lindenbaum's Lemma, a proof is given, e.g., in [19, p. 13]):

LEMMA 4.5. *For any* A *and* $B \in \mathcal{L}_t$*, if* $A \nvdash_t B$*, then there exists a prime theory* α *such that* $A \in \alpha$ *and* $B \notin \alpha$.

We next consider *ordered pairs* of prime theories. For any ordered pair of prime theories $\mathcal{T} = \langle \alpha_1, \alpha_2 \rangle$ we define the canonical 16-valuation $v^{16}_{\mathcal{T}}$ as follows:

$$\mathbf{N} \in v^{16}_{\mathcal{T}}(p) \text{ iff } \sim_t p \in \alpha_1; \qquad \mathbf{F} \in v^{16}_{\mathcal{T}}(p) \text{ iff } \sim_t p \in \alpha_2;$$
$$\mathbf{T} \in v^{16}_{\mathcal{T}}(p) \text{ iff } p \in \alpha_1; \qquad \mathbf{B} \in v^{16}_{\mathcal{T}}(p) \text{ iff } p \in \alpha_2.$$

Now we can show that the canonical 16-valuation so defined is naturally extended to any formula of the language:

LEMMA 4.6. *Let $v_{\mathcal{T}}^{16}$ be defined as above. Then for any $A \in \mathcal{L}_t$:*

$$\mathbf{N} \in v_{\mathcal{T}}^{16}(A) \quad \textit{iff} \quad \sim_t A \in \alpha_1; \qquad \mathbf{F} \in v_{\mathcal{T}}^{16}(A) \quad \textit{iff} \quad \sim_t A \in \alpha_2;$$
$$\mathbf{T} \in v_{\mathcal{T}}^{16}(A) \quad \textit{iff} \quad A \in \alpha_1; \qquad \mathbf{B} \in v_{\mathcal{T}}^{16}(A) \quad \textit{iff} \quad A \in \alpha_2.$$

Proof. This is a usual induction on the construction of formulas. We show only the case with negation leaving other cases to the reader.

Let $A = \sim_t B$, and the lemma holds for B. Then we have:

$\mathbf{N} \in v_{\mathcal{T}}^{16}(\sim_t B) \Leftrightarrow \mathbf{T} \in v_{\mathcal{T}}^{16}(B)$ (Proposition 3.7) $\Leftrightarrow B \in \alpha_1$ (inductive assumption) $\Leftrightarrow \sim_t\sim_t B \in \alpha_1$ (a_t6).

$\mathbf{F} \in v_{\mathcal{T}}^{16}(\sim_t B) \Leftrightarrow \mathbf{B} \in v_{\mathcal{T}}^{16}(B)$ (Proposition 3.7) $\Leftrightarrow B \in \alpha_2$ (inductive assumption) $\Leftrightarrow \sim_t\sim_t B \in \alpha_2$ (a_t6).

$\mathbf{T} \in v_{\mathcal{T}}^{16}(\sim_t B) \Leftrightarrow \mathbf{N} \in v_{\mathcal{T}}^{16}(B)$ (Proposition 3.7) $\Leftrightarrow \sim_t B \in \alpha_1$ (inductive assumption).

$\mathbf{B} \in v_{\mathcal{T}}^{16}(\sim_t B) \Leftrightarrow \mathbf{F} \in v_{\mathcal{T}}^{16}(B)$ (Proposition 3.7) $\Leftrightarrow \sim_t B \in \alpha_2$ (inductive assumption). □

THEOREM 4.7. *For any $A, B \in \mathcal{L}_t$: If $A \models_t^{16} B$, then $A \vdash_t B$.*

Proof. Let $A \models_t^{16} B$. For the sake of contradiction assume $A \nvdash_t B$. Then, by Lemma 4.5, there exists a prime theory α such that $A \in \alpha$ and $B \notin \alpha$. Consider a pair of prime theories $\mathcal{T} = \langle \beta_1, \beta_2 \rangle$ such that $\alpha = \beta_1$, and β_2 is arbitrary. Then we have $\mathbf{T} \in v_{\mathcal{T}}^{16} A$ and $\mathbf{T} \notin v_{\mathcal{T}}^{16} B$. A contradiction (by Lemma 4.3). □

4.2. *The Language $\mathcal{L}_{tf}$ for $\leq_t$*

4.2.1. *Adding More Negations*

The system $\mathbf{FDE}_t^t$ is quite standard. It is formulated in a customary language with conjunction, disjunction and negation, and it is equipped with a suitable entailment relation corresponding to $\leq_t$ in *SIXTEEN*$_3$. But what kind of logical systems can we obtain by axiomatizing the relation $\models_t^{16}$ (introduced by Definition 4.1) when it is extended to richer fragments of the language $\mathcal{L}_{tf}$ and to the entire language? It turns out that in the presence of formulas with $\wedge_f$, $\vee_f$ and $\sim_f$, the previous Lemmas 4.2 and 4.3 require some essential modifications (restrictions).

LEMMA 4.8. *For any A, $B \in \mathcal{L}_{tf}$ in SIXTEEN$_3$ the clause* (a) *from Lemma* 4.2 *is equivalent to the clause* (c)*, and* (b) *is equivalent to* (d).

Proof. We prove here only the case (c) $\Rightarrow$ (a), because the other cases are analogous. For any 16-valuation v^{16} we define a *tf-dual* valuation $v^{16\#}$ as follows:

$$\mathbf{T} \in v^{16\#}(p) \Leftrightarrow \mathbf{F} \notin v^{16}(p); \qquad \mathbf{B} \in v^{16\#}(p) \Leftrightarrow \mathbf{N} \notin v^{16}(p);$$
$$\mathbf{F} \in v^{16\#}(p) \Leftrightarrow \mathbf{T} \notin v^{16}(p); \qquad \mathbf{N} \in v^{16\#}(p) \Leftrightarrow \mathbf{B} \notin v^{16}(p).$$

An induction by construction shows that these conditions also hold for any formula $A \in \mathcal{L}_{tf}$. Let $\forall v^{16}(\mathbf{T} \in v^{16}(A) \Rightarrow \mathbf{T} \in v^{16}(B))$ and assume there exists a 16-valuation v^{16} such that $\mathbf{F} \in v^{16}(B)$ and $\mathbf{F} \notin v^{16}(A)$. Then $\mathbf{T} \notin v^{16\#}(B)$ and $\mathbf{T} \in v^{16\#}(A)$, which contradicts the assumption. □

LEMMA 4.9. *For any $A, B \in \mathcal{L}_{tf}$:*

$$A \models_t^{16} B \quad \textit{iff} \quad \text{(a) } \forall v^{16}(\boldsymbol{T} \in v^{16}(A) \Rightarrow \boldsymbol{T} \in v^{16}(B)) \textit{ and}$$
$$\text{(b) } \forall v^{16}(\boldsymbol{B} \in v^{16}(A) \Rightarrow \boldsymbol{B} \in v^{16}(B)).$$

Proof. Mutatis mutandis, as in Lemma 4.3, taking into account that ($*$) from Lemma 4.3 is also equivalent to the right-hand side of the statement in the present lemma. □

The logical system that should correspond to the logic $(\mathcal{L}_{tf}, \models_t^{16})$ defined at the beginning of this section can be called $\mathbf{FDE}_t^{tf} = (\mathcal{L}_{tf}, \vdash_t)$. As a first step toward axiomatizing $\mathbf{FDE}_t^{tf}$, we enrich the language $\mathcal{L}_t$ by another negation operator and consider the language $\mathcal{L}_{t+\sim_f} := \{\wedge_t, \vee_t, \sim_t, \sim_f\}$. Now we can introduce the system $\mathbf{FDE}_t^{t+\sim_f} = (\mathcal{L}_{t+\sim_f}, \vdash_t)$, where $\vdash_t$ is the binary relation on $\mathcal{L}_{t+\sim_f}$ satisfying the axioms and rules of inference $a_t1 - a_t7$ and $r_1 - r_4$ stated above, as well as the following additional postulates for $\sim_f$:

$a_t8.$ $A \vdash_t \sim_f\sim_f A$
$a_t9.$ $\sim_f\sim_f A \vdash_t A$
$a_t10.$ $\sim_f\sim_t A \vdash_t \sim_t\sim_f A$
$r_t5.$ $A \vdash_t B / \sim_f A \vdash_t \sim_f B.$

It is not difficult to show (using Lemmas 4.8 and 4.9) that Theorem 4.4 holds for any formula of the language $\mathcal{L}_{t+\sim_f}$. That is, we have:

THEOREM 4.10. *For any $A, B \in \mathcal{L}_{t+\sim_f}$: If $A \vdash_t B$, then $A \models_t^{16} B$.*

Note that the following statements are theorems of $\mathbf{FDE}_t^{t+\sim_f}$:

$t_t1.$ $\sim_f(A \wedge_t B) \vdash_t \sim_f A \wedge_t \sim_f B$;
$t_t2.$ $\sim_f A \wedge_t \sim_f B \vdash_t \sim_f(A \wedge_t B)$;

$t_t3.\ {\sim_f}A \vee_t {\sim_f}B \vdash_t {\sim_f}(A \vee_t B)$;
$t_t4.\ {\sim_f}(A \vee_t B) \vdash_t {\sim_f}A \vee_t {\sim_f}B$;
$t_t5.\ {\sim_t}{\sim_f}A \vdash_t {\sim_f}{\sim_t}A$.

To prove completeness, we continue to deal with prime theories closed under $\vdash_t$. But now for any theory α, we define the set of formulas

$$\alpha^* := \{A \mid {\sim_f}A \in \alpha\}.$$

LEMMA 4.11. *Let α be a theory and let α^* be defined as above. Then:*

(1) *α^* is a theory;*
(2) *${\sim_f}A \in \alpha^*$ iff $A \in \alpha$;*
(3) *α^* is prime iff α is prime.*

Proof. (1) Assume $A \vdash_t B$ and $A \in \alpha^*$. Then, by r_t5 ${\sim_f}A \vdash_t {\sim_f}B$ and by definition of α^*, ${\sim_f}A \in \alpha$. Hence ${\sim_f}B \in \alpha$, and thus $B \in \alpha^*$. Next, assume $A \in \alpha^*$ and $B \in \alpha^*$. Then ${\sim_f}A \in \alpha$ and ${\sim_f}B \in \alpha$. Hence, ${\sim_f}A \wedge_t {\sim_f}B \in \alpha$, and by t_t2 ${\sim_f}(A \wedge_t B) \in \alpha$. By definition of α^*, $A \wedge_t B \in \alpha^*$.

(2) ${\sim_f}A \in \alpha^* \Leftrightarrow {\sim_f}{\sim_f}A \in \alpha$ (by definition) $\Leftrightarrow A \in \alpha$ (by a_t8, a_t9).

(3) ($\Rightarrow$) Assume α is not prime. Then there are A and B such that $A \vee_t B \in \alpha$, $A \notin \alpha$, and $B \notin \alpha$. Then, by (2) above, ${\sim_f}(A \vee_t B) \in \alpha^*$, and ${\sim_f}A \notin \alpha^*$, and ${\sim_f}B \notin \alpha^*$. By $t4$, ${\sim_f}A \vee_t {\sim_f}B \in \alpha^*$, and hence α^* is not prime. ($\Leftarrow$) Assume α^* is not prime. Then there are A and B such that $A \vee_t B \in \alpha^*$, and $A \notin \alpha^*$, and $B \notin \alpha^*$. By definition of α^*, we have: ${\sim_f}(A \vee_t B) \in \alpha$, and ${\sim_f}A \notin \alpha$, and ${\sim_f}B \notin \alpha$. Arguing as above we conclude that α is not prime. □

The definition of α^* and Lemma 4.11 immediately call to mind the famous "Routley star operator" used for defining a negation operator in the "Australian semantics" for relevance logic. In fact, the Routley star $*$ represents an algebraic operation known as *involution* (see, e.g., [1, 11, 34]). In view of this, ${\sim_f}$ can be naturally interpreted as an object language *involution connective* with respect to $\vdash_t$, whereas ${\sim_t}$ stands for a negation relative to $\vdash_t$. It is quite remarkable that *SIXTEEN*$_3$ allows us to deal with ${\sim_f}$ and ${\sim_t}$ simultaneously, thereby delivering interesting new evidence for a deep interrelation between the "Australian" and the "American" semantics (cf. [18, pp. 45–47]).

For any prime theory α we define the canonical 16-valuation v_α^{16} as follows:

$\mathbf{N} \in v_\alpha^{16}(p)$	iff	${\sim_t}p \in \alpha$;	$\mathbf{F} \in v_\alpha^{16}(p)$	iff	${\sim_t}p \in \alpha^*$;
$\mathbf{T} \in v_\alpha^{16}(p)$	iff	$p \in \alpha$;	$\mathbf{B} \in v_\alpha^{16}(p)$	iff	$p \in \alpha^*$.[19]

LEMMA 4.12. *Let v_α^{16} be defined as above. Then for any formula $A \in \mathcal{L}_{t+\sim_f}$:*

$$\mathbf{N} \in v_\alpha^{16}(A) \quad \textit{iff} \quad \sim_t A \in \alpha; \qquad \mathbf{F} \in v_\alpha^{16}(A) \quad \textit{iff} \quad \sim_t A \in \alpha^*;$$
$$\mathbf{T} \in v_\alpha^{16}(A) \quad \textit{iff} \quad A \in \alpha; \qquad \mathbf{B} \in v_\alpha^{16}(A) \quad \textit{iff} \quad A \in \alpha^*.$$

Proof. We again, as in the proof of Lemma 4.6 consider only the case with negation. Let $A = \sim_f B$, and the lemma holds for B. Then we have:

$\mathbf{N} \in v_\alpha^{16}(\sim_f B) \Leftrightarrow \mathbf{F} \in v_\alpha^{16}(B)$ (Proposition 3.7) $\Leftrightarrow \sim_t B \in \alpha^*$ (inductive assumption) $\Leftrightarrow \sim_f \sim_t B \in \alpha$ (definition of α^*) $\Leftrightarrow \sim_t \sim_f B \in \alpha$ ($a_t 10$).

$\mathbf{F} \in v_\alpha^{16}(\sim_f (B)) \Leftrightarrow \mathbf{N} \in v_\alpha^{16}(B)$ (Proposition 3.7) $\Leftrightarrow \sim_t B \in \alpha$ (inductive assumption) $\Leftrightarrow \sim_f \sim_t B \in \alpha^*$ (Lemma 4.11(2)) $\Leftrightarrow \sim_t \sim_f B \in \alpha^*$($a_t 10$).

$\mathbf{T} \in v_\alpha^{16}(\sim_f (B)) \Leftrightarrow \mathbf{B} \in v_\alpha^{16}(B)$ (Proposition 3.7) $\Leftrightarrow B \in \alpha^*$ (inductive assumption) $\Leftrightarrow \sim_f (B) \in \alpha$ (definition of α^*).

$\mathbf{B} \in v_\alpha^{16}(\sim_f (B)) \Leftrightarrow \mathbf{T} \in v_\alpha^{16}(B)$ (Proposition 3.7) $\Leftrightarrow B \in \alpha$ (inductive assumption) $\Leftrightarrow \sim_f (B) \in \alpha^*$ (Lemma 4.11(2)). □

THEOREM 4.13. *For any $A, B \in \mathcal{L}_{t+\sim_f}$: If $A \models_t^{16} B$, then $A \vdash_t B$.*

Proof. Let $A \models_t^{16} B$. Assume $A \nvdash_t B$. Then, by Lemma 4.5 there exists a prime theory α such that $A \in \alpha$ and $B \notin \alpha$. Taking the canonical valuation v_α^{16}, we have $\mathbf{T} \in v_\alpha^{16} A$ and $\mathbf{T} \notin v_\alpha^{16} B$. A contradiction (by Lemma 4.9). □

It is interesting to observe that by setting $\sim A := \sim_f \sim_t A$, we obtain another unary connective, which appears to be another kind of negation. It is not difficult to show that the following statements are theorems of $\mathbf{FDE}_t^{t+\sim_f}$:

$t_t 6$. $\sim(A \wedge_t B) \vdash_t \sim A \vee_t \sim B$;
$t_t 7$. $\sim A \wedge_t \sim B \vdash_t \sim(A \vee_t B)$;
$t_t 8$. $\sim A \vee_t \sim B \vdash_t \sim(A \wedge_t B)$;
$t_t 9$. $\sim(A \vee_t B) \vdash_t \sim A \wedge_t \sim B$;
$t_t 10$. $A \vdash_t \sim\sim A$;
$t_t 11$. $\sim\sim A \vdash_t A$.

Also contraposition holds:

$r_t 6$. $A \vdash_t B / \sim B \vdash_t \sim A$.

DEFINITION 4.14. For any A and B: $v^{16}(\sim A) = -_{tf} v^{16}(A)$.

4.2.2. *Conjunctions and Disjunctions*

Finally we add $\wedge_f$, $\vee_f$ to $\mathcal{L}_{t+\sim_f}$ and regain the whole language $\mathcal{L}_{tf}$. The main task is now to characterize $\wedge_f$, $\vee_f$ through the "truth-consequence" $\vdash_t$, but it is not quite clear how to approach this task successfully.

Since *SIXTEEN*$_3$ is interlaced and distributive, the monotonicity rules $A \vdash_t B/C \wedge_f A \vdash_t C \wedge_f B$ and $A \vdash_t B/C \vee_f A \vdash_t C \vee_f B$ as well as all the distributive laws $(A \wedge_f (B \vee_t C)) \vdash_t (A \wedge_f B) \vee_t (A \wedge_f C)$, etc.) must hold and may be postulated. Also the De Morgan laws $(\sim_f A \wedge_f \sim_f B \vdash_t \sim_f (A \vee_f B)$, etc.) are valid inferences. On the other hand the introduction/elimination postulates mixing falsity-connectives and truth-consequence (such as $A \wedge_f B \vdash_t A$, $A \vdash_t A \vee_f B$, etc.), fail to hold (otherwise $\wedge_f$ and $\wedge_t$ would be indistinguishable).

What is the sense of the presence of two different conjunctions (and disjunctions) in one system? Such a situation is not unusual. We may recall the presence of extensional and intensional conjunctions and disjunctions in various substructural logics (among them relevance logic). We believe that a careful investigation of the interrelations between $\wedge_t$, $\vee_t$, $\wedge_f$ and $\vee_f$ may throw additional light on this situation.[20]

We leave the following as an open problem:

PROBLEM 4.15. Axiomatize the whole $\mathbf{FDE}_t^{tf}$.

5. THE LOGIC OF THE FALSITY ORDER

It is quite natural to suppose that the logic determined by the falsity order in a given language should be perfectly *dual* to the logic of the truth order in this language. Namely, we should be able to obtain syntactic and semantic presentations of these logics from one another simply by exchanging the 't' and 'f' subscripts in the postulates and definitions of the corresponding systems.

DEFINITION 5.1. $A \models_f^{16} B$ iff $\forall v^{16}(v^{16}(B) \leq_f v^{16}(A))$.

We again can consider the language $\mathcal{L}_f$ with propositional connectives $\wedge_f$, $\vee_f$ and $\sim_f$. A 16-valuation is extended to compound formulas of $\mathcal{L}_f$ by Definition 3.8(4)–(6) and Propositions 3.4(3), (4) and 3.7(2). We obtain the system $\mathbf{FDE}_f^f = (\mathcal{L}_f, \vdash_f)$ just by replacing $\wedge_t$, $\vee_t$, $\sim_t$ and $\vdash_t$ in the axioms and rules of $\mathbf{FDE}_t^t$ by $\wedge_f$, $\vee_f$, $\sim_f$ and $\vdash_f$, respectively.

The logic $(\mathcal{L}_{tf}, \models_f^{16})$ and the system $\mathbf{FDE}_f^{tf}$ are defined in analogy to the definitions of $(\mathcal{L}_{tf}, \models_t^{16})$ and $\mathbf{FDE}_t^{tf}$. On the way to axiomatizing $\mathbf{FDE}_f^{tf}$, we consider the language $\mathcal{L}_{f+\sim_t} := \{\wedge_f, \vee_f, \sim_f, \sim_t\}$. We introduce the system $\mathbf{FDE}_f^{f+\sim_t} = (\mathcal{L}_{f+\sim_t}, \vdash_f)$, where $\vdash_f$ is the binary relation on $\mathcal{L}_{f+\sim_t}$ satisfying the dualized versions of the axioms and rules of $(\mathcal{L}_{t+\sim_f}, \vdash_t)$.

As to the connective $\sim$ defined at the end of Subsection 4.2.1, it can be shown that in $\mathbf{FDE}_f^{f+\sim_t}$ the statements $t_f6 - t_f11$ hold, as well as the rule r_f6, which are obtained from $t_t6 - t_t11$ and r_t6 by replacing uniformly the subscript 't' by 'f'. Thus, whereas $\sim_t$ is a negation connective relative to $\leq_t$ but an involution connective relative to $\leq_f$, and $\sim_f$ is a negation for $\leq_f$ but an involution for $\leq_t$, $\sim$ is a negation connective with respect to *both* logical orderings. That is, $\sim$ can be naturally interpreted as a *generalized logical negation*.

Again, we have an open problem.

PROBLEM 5.2. Axiomatize the whole $\mathbf{FDE}_f^{tf}$.

6. THE BI-CONSEQUENCE LOGIC AND LOGICAL BILATTICES

Usually, a complete logical system comprises one syntactic deducibility relation $\vdash$ and one semantic entailment relation $\models$ such that for all formulas A and B of the formal language under consideration, $A \vdash B$ iff $A \models B$. The characterization of logic as the theory of valid inferences, however, does not preclude that there may be more than just one kind of valid inferences. The trilattice *SIXTEEN*$_3$ comes with *two* natural definitions of (non-equivalent) entailment relations reflecting increase of truth and decrease of falsity. Therefore it appears to be quite natural to conceive of the unified logic of *SIXTEEN*$_3$ as a *bi-consequence system* comprising two kinds of entailment relations.[21] This consideration leads us to the following definition:

DEFINITION 6.1. The bi-consequence logic $(\mathcal{L}_{tf}, \models_t^{16}, \models_f^{16})$ is the set of all true statements $A \models_x^{16} B$, where $A, B \in \mathcal{L}_{tf}$, and $x = t$ or $x = f$, (cf. Definitions 4.1 and 5.1, respectively).

The bi-consequence system $\mathbf{FDE}_{tf}^{tf} = (\mathcal{L}_{tf}, \vdash_t, \vdash_f)$ is to be defined syntactically just as $\mathbf{FDE}_t^{tf} \cup \mathbf{FDE}_f^{tf}$.

The peaceful co-existence of two entailment and two deducibility relations in one and the same logic is useful, because – as we have seen – it may well make a difference whether we move along the truth order or the non-falsity order.

Uniqueness of inference is attainable, if it is required that valuations are preserved by both orderings.

DEFINITION 6.2. $A \models^{16} B$ iff ($A \models_t^{16} B$ and $A \models_f^{16} B$).

PROBLEM 6.3. Investigate the set of all statements $A \models^{16} B$. Does it determine an autonomous system $\mathbf{FDE}^{tf} = (\mathcal{L}_{tf}, \vdash)$ (where $A \vdash B$ is defined as $A \vdash_t B$ and $A \vdash_f B$)? Is $\mathbf{FDE}^{tf}$ axiomatizable?

Definition 6.2 calls to mind the notion of a logical bilattice introduced by Arieli and Avron, [4–6].

DEFINITION 6.4. Let $\mathcal{B} = (B, \leq_1, \leq_2)$ be a bilattice, where $\sqcap_1$ and $\sqcup_1$ ($\sqcap_2$ and $\sqcup_2$) are the meet and join operations with respect to $\leq_1$ ($\leq_2$). A bifilter on $\mathcal{B}$ is a nonempty proper subset $\mathcal{F} \subset B$, such that

(1) $x \sqcap_1 y \in \mathcal{F}$ iff $x \in \mathcal{F}$ and $y \in \mathcal{F}$;
(2) $x \sqcap_2 y \in \mathcal{F}$ iff $x \in \mathcal{F}$ and $y \in \mathcal{F}$.

A bifilter $\mathcal{F}$ is said to be prime, if it satisfies

(1) $x \sqcup_2 y \in \mathcal{F}$ iff $x \in \mathcal{F}$ or $y \in \mathcal{F}$;
(2) $x \sqcup_2 y \in \mathcal{F}$ iff $x \in \mathcal{F}$ or $y \in \mathcal{F}$.

A pair $(\mathcal{B}, \mathcal{F})$ is called a logical bilattice, if $\mathcal{B}$ is a bilattice and $\mathcal{F}$ is a prime bifilter on $\mathcal{B}$.

Arieli and Avron [5, 6] show that $FOUR_2$ constitutes a logical bilattice with $\{\mathbf{T}, \mathbf{B}\}$ as the unique prime bifilter. They consider the language $\mathcal{L}^* = \{\wedge_t, \vee_t, \sim_t, \wedge_f, \vee_f\}$ (notation adjusted) and define entailment relations as follows:

DEFINITION 6.5. Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice, and let $\mathcal{L}^*(\mathcal{B})$ be $\mathcal{L}^*$ extended by a propositional constant for each element from $\mathcal{B}$. A valuation function v maps atoms to elements of $\mathcal{B}$ and is extended to compound formulas in the natural way (cf. the definition of 16-valuations). Let A, $B \in \mathcal{L}^*(\mathcal{B})$. Then $A \models_{BL(\mathcal{B},\mathcal{F})} B$ iff for every valuation v, $v(A) \in \mathcal{F}$ implies $v(B) \in \mathcal{F}$. Moreover, $A \models_{BL} B$ iff for every bilattice $(\mathcal{B}, \mathcal{F})$, we have $A \models_{BL(\mathcal{B},\mathcal{F})} B$.[22]

It turns out that entailment with respect to $FOUR_2$ and entailment with respect to the class of all logical bilattices coincide.

THEOREM 6.6 (Arieli and Avron [5]). *$A \models_{BL} B$ iff $A \models_{BL(FOUR_2,\{\mathbf{T},\mathbf{B}\})} B$, for all A, $B \in \mathcal{L}^*$.*

But why should we consider $(\mathcal{L}^*, \models_{BL})$ the logic of *logical* bilattices? The set of designated elements $\{\mathbf{T}, \mathbf{B}\}$ is the smallest bifilter originating from the logical (truth-and-falsity) order and the *information* order from $FOUR_2$. If entailment is understood as preservation of designated truth values, it seems quite natural to require that the definition of entailment in the logic of *logical* bilattices takes into account only *logical* orders. The set **16** offers an opportunity to meet this requirement. Namely, let us disregard for a while the information ordering of $SIXTEEN_3$ (i.e., simply "erase" the i-axis in Figure 4). Then $SIXTEEN_3$ turns into a *bilattice* $SIXTEEN_2 := (\mathbf{16}, \leq_t, \leq_f)$ with *two logical orders* – a truth order and a falsity order – which are independent of each other. We believe that it is much more appropriate to consider $SIXTEEN_2$ the proper structure that should define the basic logic of logical bilattices in the language $\mathcal{L}_{tf}$ instead of $\mathcal{L}^*$. The language $\mathcal{L}^*$ simply leaves out the negation operation supplied by the second ordering relation.

The set $\{\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\}$ is, by Proposition 2.23 in [5], the smallest prime bifilter in $SIXTEEN_2$.[23]

DEFINITION 6.7. Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice, let $\mathcal{L}_{tf}(\mathcal{B})$ be $\mathcal{L}_{tf}$ extended by a propositional constant for each element from $\mathcal{B}$, and let $A, B \in \mathcal{L}_{tf}(\mathcal{B})$. Then $A \models_{bl(\mathcal{B},\mathcal{F})} B$ iff for every valuation v^{16}, $v^{16}(A) \in \mathcal{F}$ implies $v^{16}(B) \in \mathcal{F}$. Let $des := \{\boldsymbol{T}, \mathbf{NT}, \mathbf{TB}, \mathbf{NTB}\}$. Then $A \models_{bl} B$ iff $A \models_{(SIXTEEN_2, des)} B$.

It seems that the system $(\mathcal{L}_{tf}, \models_{bl})$ is a natural candidate for the title "basic logic of logical bilattices" in the sense of Arieli and Avron.

OBSERVATION 6.8. *It is not the case that for any* $A, B \in \mathcal{L}_{tf}$*:* $A \models^{16} B$ *if and only if* $A \models_{bl} B$.

Proof. Clearly, $A \wedge_t B \models_{bl} B$ but $A \wedge_t B \not\models^{16} B$, because $A \wedge_t B \not\models^{16}_f B$. □

Arieli's and Avron's strategy for defining an entailment relation consists in specifying a set of designated truth values and considering preservation of them. This is a general strategy extending the classical conception of valid inference. In the presence of two logical orderings, one for truth and one for (non-)falsity, however, a bi-consequence logic not only emerges naturally, but might be taken to be a welcome generalization of the standard conception of logic.

7. CONCLUDING REMARKS

In this paper we have argued in favor of introducing generalized truth values and generalizing Belnap's useful four-valued logic. It turned out that the logic generated separately by the algebraic operations under the truth order and the falsity order in the trilattice $SIXTEEN_3$ in fact coincide with the logic of first degree entailment $\mathbf{E}_{fde}$. However, we have also seen that directing the attention from $FOUR_2$ to the trilattice $SIXTEEN_3$ in a very natural way leads to richer propositional logics. In particular, we semantically defined the systems $\mathbf{FDE}^{tf}_f$ and $\mathbf{FDE}^{tf}_t$ and characterized their fragments $\mathbf{FDE}^{t+\sim_f}_t$ and $\mathbf{FDE}^{f+\sim_t}_f$. In addition, we have introduced a new bi-consequence system called $\mathbf{FDE}^{tf}_{tf}$. We expect this logic to be an interesting and useful instrument in information processing, especially when it becomes important to separately keep track of positive and negative information. As we have argued above, this may be the case when we deal with a net of hierarchically interconnected computers.

Definitions 4.1 and 5.1 can naturally be extended to define entailment as a relation between arbitrary sets of formulas (cf. [28, p. 417]):

DEFINITION 7.1. Let Γ, Δ be arbitrary sets of formulas of $\mathcal{L}_{tf}$. Then

$\Gamma \models^{16}_t \Delta$ iff
$\exists A_1, \ldots, A_m \in \Gamma, \exists B_1, \ldots, B_n \in \Delta\ (A_1 \wedge_t \cdots \wedge_t A_m \models^{16}_t B_1 \vee_t \cdots \vee_t B_n)$;

$\Gamma \models^{16}_f \Delta$ iff
$\exists A_1, \ldots, A_m \in \Gamma, \exists B_1, \ldots, B_n \in \Delta (A_1 \wedge_f \cdots \wedge_f A_m \models^{16}_f B_1 \vee_f \cdots \vee_f B_n)$.

An obvious line of further research is to construct Gentzen-style sequent calculi for the logics presented above. Another interesting task consists in introducing object language implication connectives $\rightarrow_t$ and $\rightarrow_f$ that correspond to the relations $\models^{16}_t$ and $\models^{16}_f$, and to formulate logics involving these additional connectives.

As we have emphasized repeatedly, the truth order in $FOUR_2$ is, in fact, a truth-and-falsity order, whereas the truth and falsity orderings in $SIXTEEN_3$ are independent of each other. Therefore, the starting point for generalizing the construction of the present paper should be $\mathcal{P}(\mathbf{4})$ instead of $\mathcal{P}(\mathbf{2})$. Let $\mathcal{P}^1(X) := \mathcal{P}(X)$ and $\mathcal{P}^n(X) := \mathcal{P}(\mathcal{P}^{n-1}(X))$ for $n > 1$. The information ordering on any set of generalized truth values is just the subset relation. In order to employ Definition 3.3 as a definition of the truth and the falsity orderings on $\mathcal{P}^n(\mathbf{4})$ for $n > 1$, we may use the following stipulations:

$$x^t_n := \{y_0 \in x \mid (\exists y_1 \in y_0)\,(\exists y_2 \in y_1) \ldots (\exists y_{n-1} \in y_{n-2})\ T \in y_{n-1}\}$$
$$x^{-t}_n := \{y_0 \in x \mid (\exists y_1 \in y_0)\,(\exists y_2 \in y_1) \ldots (\exists y_{n-1} \in y_{n-2})\ T \notin y_{n-1}\}$$
$$x^f_n := \{y_0 \in x \mid (\exists y_1 \in y_0)\,(\exists y_2 \in y_1) \ldots (\exists y_{n-1} \in y_{n-2})\ F \in y_{n-1}\}$$
$$x^{-f}_n := \{y_0 \in x \mid (\exists y_1 \in y_0)\,(\exists y_2 \in y_1) \ldots (\exists y_{n-1} \in y_{n-2})\ F \notin y_{n-1}\}$$

We can then define an infinite chain of trilattices by setting:

$$\mathcal{M}_3^n := (\mathcal{P}^n(\mathbf{4}), \sqcap_i, \sqcup_i, \sqcap_t, \sqcup_t, \sqcap_f, \sqcup_f)$$

and $\mathcal{M}_3^1 := SIXTEEN_3$. A plethora of further questions and problems now emerges quite naturally. For instance:

PROBLEM 7.2. Let $n > 1$. Determine the logic $(\mathcal{L}_t, \models_t^{4^{(2^n)}})$ of the truth order and the logic $(\mathcal{L}_f, \models_f^{4^{(2^n)}})$ of the falsity order in $\mathcal{M}_3^n$.

PROBLEM 7.3. Let $n > 1$. Determine the logic $(\mathcal{L}_{tf}, \models_t^{4^{(2^n)}})$ of the truth order and the logic $(\mathcal{L}_{tf}, \models_f^{4^{(2^n)}})$ of the falsity order in $\mathcal{M}_3^n$.

DEFINITION 7.4. $A \models^{4^{(2^n)}} B$ iff $(A \models_t^{4^{(2^n)}} B$ and $A \models_f^{4^{(2^n)}} B)$.

PROBLEM 7.5. Present a proof-theoretical characterization of $\models^{4^{(2^n)}}$ for $n > 1$.

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NOTES

[1] Dunn has developed this approach already in his doctoral dissertation [14] and then presented it in a number of conference talks and publications, most notably in [16] (see also [15]). The reader may consult [18] and [19] for a comprehensive account and systematization of Dunn's (and other) work in this area (cf. [41]). In the literature, the semantic strategy in question is sometimes called the "American Plan" as opposed to the so-called "Australian Plan". Both labels were brought into usage by R. Meyer [37] to contrast the *four-valued* approach of the "Americans" Dunn and Belnap to the *star semantics* of the "Australians" Routley and himself.

[2] Consider the set $\mathbf{2} = \{F, T\}$ to be the usual set of classical truth values. A standard classical valuation v^2 (a 2-valuation) is then a *function* from the set of sentences into **2** ascribing thus to every sentence *one and only one* element from **2**, i.e., either truth or falsity.

[3] In [40, p. 762] this kind of valuation has been called *multivaluation*.

[4] Bilattices have been studied by many authors (most notably by M. Fitting and A. Avron) in various contexts – see, e.g., [5–8, 21–27, 30, 39] and references therein.

[5] The information order is sometimes referred to as a "knowledge order" (denoted by $\leq_k$), which is not quite accurate from a philosophical point of view taking into account the classical definition of knowledge as justified *true* belief.

[6] J. M. Dunn informed us that this question was first raised by Manfred von Thun, when Dunn gave a lecture on "An Intuitive Semantics for First-Degree Entailments" at LaTrobe University in October 1975. As R. Meyer nicely put it: "... if we take seriously both true and false and neither true nor false separately, what is to prevent our taking them seriously conjunctively? As in 'It is both true and false and neither true nor false that snow is white' " [37, p. 19]. As we will argue below, Meyer's own answer to this question – "This way, in the end, lies madness" (*ibd.*) – appears a bit overhasty.

[7] Graham Priest, however, has presented in [38] an argument for taking seriously the set of truth values $\mathcal{P}(\{\{F\}, \{T\}, \{T, F\}\})$. See also [33].

[8] The idea to consider generalized truths values as subsets of a set containing more than two elements (**T** and **F**) has been also expressed by A. Karpenko in [35], p. 46.

[9] A "Belnap computer" is just a computer that uses Belnap's four-valued logic. Note also that it is not crucial for our example to have exactly *four* Belnap computers, there can well be more of them, or less (even one would be enough; the main point is that it should be connected to some "higher" computer).

[10] Note that our notion of a multilattice is extremely general. We do not impose any additional conditions that interconnect individual lattices of a given multilattice. According to our definition, a bilattice ($n = 2$) is just a set with two lattice-forming partial orders on it. Strangely enough there is no uniform and generally accepted definition of a bilattice in the literature. When Ginsberg first introduced this notion in [31], he defined it as a quintuple $(B, \wedge, \vee, \cdot, +)$ such that $(B, \wedge, \vee)$ and $(B, \cdot, +)$ are both lattices and each operation respects the lattice relations in the alternate lattice. (Fitting calls this latter condition the *interlacing condition*, i.e., a bilattice à la [31] is an interlaced bilattice in Fitting's sense.) But already in [32] we find a quite different definition according to which an operation of *negation* becomes a necessary element of *any* bilattice. This definition has been adopted in [5, 6, 23] and by some other authors. However, we believe that it is too narrow, because "there are interesting bilattice-like structures that do not have a notion of negation" [23, p. 241]. Fitting in [22] introduces the notion of a *pre-bilattice* just as a non-empty set with two partial orderings each giving this set the structure of a lattice. Then in [27] he defines a bilattice as a pre-bilattice with some "useful connections between orderings". We find this definition a bit vague. When exactly is a connection "useful"? Could we speak of "interesting" connections instead? And if we just omit the "usefulness requirement", then the definition is not very informative, for it is not difficult to introduce at any time *some* (maybe even trivial) kind of connection between given orderings. In fact, we do not think that the notion of a pre-bilattice is needed at all. What is really crucial here, is the number of different partial orderings defined on *one and the same* set. All other properties and conditions (including conditions that connect the ordering relations) can be specified later, thereby giving rise to different types of multilattices (and bilattices).

[11] There are also some other works in the literature which deal with the notion of a trilattice. Lakshmanan and Sadri introduced this notion in [36, p. 257], being directly motivated by enriching Ginsberg's bilattices with a *precision ordering*. However, they aim at constructing a probabilistic calculus as a suitable framework for probabilistic de-

ductive databases, thus dealing with an algebra defined on a set of *interval pairs* rather than on a set of generalized truth values. Another line of research concerning trilattices comes from *formal concept analysis*, a research area established and investigated by R. Wille, B. Ganter and their collaborators (see [12, 29, 42], and [44]). This tradition seems to be developing totally independently of investigations in the field of bilattices and multi-valued logic.

[12] Note that the information ordering so defined is apparently independent of any other partial order that could ever be introduced on a multilattice.

[13] As Fitting put it: "[W]e might call a truth value t_1 *less-true-or-more-false* than t_2 if t_1 contains *false* but t_2 doesn't, or t_2 contains *true* but t_1 doesn't" [24, p. 94].

[14] Note that we could introduce sets x^{-t} and x^{-f} for $FOUR_2$ as well, for example as follows:

$$x^{-t} := \{z \in x \mid z \neq T\}; \qquad x^{-f} := \{z \in x \mid z \neq F\}.$$

It turns then out, however, that $x^{-t} = x^f$ and $x^{-f} = x^t$, which once again confirms our observation that truth and falsity in $FOUR_2$ are still interdependent.

[15] Similarly, **T** and **F** from **4** can be viewed as analogues (or *representatives*) of the classical values T and F.

[16] Thus, Belnap's informational interpretation of generalized truth values is not just an incidental *façon de parler*, but expresses the very essence of his construction. Therefore it is not by chance that this semantics has found so many fruitful applications in theoretical computer science and other areas related to information theory.

[17] Note that according to Propositions 3.4 and 3.7 any 16-valuation for an arbitrary formula A can be unambiguously modeled by a certain combination of the expressions $\mathbf{N} \in v^{16}(A)$, $\mathbf{F} \in v^{16}(A)$, $\mathbf{T} \in v^{16}(A)$, $\mathbf{B} \in v^{16}(A)$ and their negations. For example $v^{16}(A) = \mathbf{NT}$ is representable as $\mathbf{N} \in v^{16}(A)$, $\mathbf{F} \notin v^{16}(A)$, $\mathbf{T} \in v^{16}(A)$ and $\mathbf{B} \notin v^{16}(A)$, etc. This will greatly simplify the whole semantic exposition.

[18] This definition introduces a relation between (two) formulas. It is not difficult to extend it to a relation between arbitrary sets of formulas (see Definition 7.1 below).

[19] Incidentally, it turns out that it is possible to apply a similar construction in the completeness proof for $\mathbf{FDE}_t^t$, too. Namely, consider $\alpha^* := \{A \mid \sim_t A \notin \alpha\}$. Then, even for formulas of the "pure" language $\mathcal{L}_t$, we can simply define a canonical valuation v_α^{16} for any prime theory α as above (instead of dealing with arbitrary pairs of prime theories and the canonical valuation $v_{\mathcal{T}}^{16}$). However, in this case such a construction is not necessary, and one can content oneself just with pairs of theories which are totally independent of each other.

[20] It is interesting to note that $\wedge_t$ and $\wedge_f$ (as well as $\vee_t$ and $\vee_f$) behave virtually identically with respect to Belnap's **T** and **F** (the representatives of classical truth values). The only difference between the "parallel" conjunctions and disjunctions concerns their contrasting behaviour with respect to gaps and gluts (see Proposition 3.4). Thus, e.g., for any $x, y \in \mathbf{16}$, if $\mathbf{N} \in x$ and $\mathbf{B} \in y$, then $\mathbf{N} \in x \wedge_t y$ and $\mathbf{B} \notin x \wedge_t y$, but $\mathbf{B} \in x \wedge_f y$ and $\mathbf{N} \notin x \wedge_f y$.

[21] Cf. the formalism of "biconsequence relations" developed by Bochman in [13].

[22] Actually, Arieli and Avron define more general notions. Let Γ, Δ be finite sets of formulas from $\mathcal{L}^*(\mathcal{B})$. $\Gamma \models_{BL(\mathcal{B},\mathcal{F})} \Gamma$ iff for every valuation v such that $v(A) \in \mathcal{F}$ for each $A \in \Gamma$, there exists some $B \in \Delta$ with $v(B) \in \mathcal{F}$. $\Gamma \models_{BL} \Delta$ iff for every bilattice $(\mathcal{B}, \mathcal{F})$, $\Gamma \models_{BL(\mathcal{B},\mathcal{F})} \Delta$.

[23] Note that if we consider again the information order and return to $SIXTEEN_3$, we can analogously define the notion of a (prime) *trifilter* (and generally, for multilattices the notion of a *multifilter*). Then the set {***T***, ***B***, **NT**, **TB**, **NFT**, **NTB**, **FTB**, **A**} turns out to be the smallest prime trifilter in $SIXTEEN_3$.

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