Journal of Philosophical Logic (2005) 34: 155–192
DOI: 10.1007/s10992-004-3758-3

HANNES LEITGEB

WHAT TRUTH DEPENDS ON

ABSTRACT. What kinds of sentences with truth predicate may be inserted plausibly and consistently into the T-scheme? We state an answer in terms of dependence: those sentences which depend directly or indirectly on non-semantic states of affairs (only). In order to make this precise we introduce a theory of dependence according to which a sentence φ is said to depend on a set Φ of sentences iff the truth value of φ supervenes on the presence or absence of the sentences of Φ in/from the extension of the truth predicate. Both φ and the members of Φ are allowed to contain the truth predicate. On that basis we are able define notions such as ungroundedness or self-referentiality within a classical semantics, and we can show that there is an adequate definition of truth for the class of sentences which depend on non-semantic states of affairs.

KEY WORDS: dependence, self-referentiality, supervenience, truth, ungroundedness

1. INTRODUCTION

We owe to Tarski [29, 30] both a positive and a negative result on truth: the positive one is that the extension of the truth predicate *Tr* can be defined in a formally correct and materially adequate manner if the vocabulary of the language for which it is to be defined does not contain *Tr*. The negative one is that if the vocabulary of a language does contain *Tr* and if the language is also "sufficiently expressive", then the extension of *Tr* cannot be defined so.

E.g., truth is definable inductively for the first-order language $\mathcal{L}$ of arithmetic in a way, such that all T-biconditionals for $\mathcal{L}$-sentences are derivable in a sufficiently strong metatheory, i.e.: since $Tr(\ulcorner\overline{2}+\overline{2}=\overline{4}\urcorner) \leftrightarrow \overline{2}+\overline{2}=\overline{4}$ can be derived from the definition, the latter is adequate with respect to $\overline{2}+\overline{2}=\overline{4}$ and accordingly for all further sentences of $\mathcal{L}$. On the other hand, the extension of *Tr* cannot be defined correctly and adequately for the language $\mathcal{L}_{Tr}$ that results from adding *Tr* to the vocabulary of $\mathcal{L}$. This is due to the fact that $\mathcal{L}_{Tr}$ is sufficiently expressive in order to contain self-referential or ungrounded sentences like the Liar sentences λ for which $\lambda \leftrightarrow \neg Tr(\ulcorner\lambda\urcorner)$ is derivable from arithmetic.

However, neither of these two results tells us anything about whether it is possible to define truth adequately for a set of sentences which lies somewhere "in between" $\mathcal{L}$ and $\mathcal{L}_{Tr}$: knowing that there is a correct and adequate definition of truth for $\mathcal{L}$, and knowing that there is no such definition for $\mathcal{L}_{Tr}$, we might still be looking for a formally correct definition of truth that is applicable to a class Φ of sentences with $\mathcal{L} \subsetneqq \Phi \subsetneqq \mathcal{L}_{Tr}$, such that the definition is materially adequate *with respect to the members of* Φ. As Horwich [20], p. 40, says, "permissible instantiations of the equivalence schema are restricted in some way so as to avoid paradoxical results." What could such a class Φ of "permissible instantiations" look like? What kinds of sentences *with* truth predicate may be inserted plausibly and consistently into the T-scheme?

One idea would be to search for *maximal* consistent sets Φ having the properties just outlined: as McGee [23] has shown, this actually leads nowhere. The great variety of maximally consistent sets of T-scheme instances is almost unconstrained as far as truth-theoretic interests are concerned. So instead of rushing a "top-down" approach we should perhaps rather see whether there is a "bottom-up" strategy of finding such a set Φ. We suggest turning to a class of sentences the truth or falsity of which may be said to be determined by, or to *depend on*, the truth or falsity of the sentences of $\mathcal{L}$.

For the moment let us focus just on atomic sentences. The standard semantics for first-order languages obeys a simple semantic rule as far as atomic sentences are concerned: the truth or falsity of, say, $P(t)$ depends on whether the entity denoted by t is a member of the extension of P or not; if it is, then $P(t)$ is true, if not, $P(t)$ is false. Therefore the truth or falsity of the sentence $Tr(\ulcorner \overline{2} + \overline{2} = \overline{4} \urcorner)$ of $\mathcal{L}_{Tr}$ depends on whether $\overline{2} + \overline{2} = \overline{4}$ is a member of the extension of Tr or not. Arguing accordingly, the truth or falsity of $Tr(\ulcorner Tr(\ulcorner \overline{2} + \overline{2} = \overline{4} \urcorner) \urcorner)$ depends on whether $Tr(\ulcorner \overline{2} + \overline{2} = \overline{4} \urcorner)$ is contained in the extension of Tr or not and generally the truth or falsity of every sentence of the form $Tr(\ulcorner \ldots \ulcorner Tr(\ulcorner \overline{2} + \overline{2} = \overline{4} \urcorner) \urcorner \ldots \urcorner)$ depends on whether its predecessor in this sequence of sentences is a member of the extension of Tr or it is not.

Returning again to the "bottom" of this progression, we see that whether $\overline{2} + \overline{2} = \overline{4}$ is a member of the extension of Tr or not is simply determined by our adequate definition of truth for $\mathcal{L}$ and thus ultimately depends on whether $2 + 2 = 4$ or not, i.e., on a non-semantic state of affairs. Since the truth or falsity of $Tr(\ulcorner \overline{2} + \overline{2} = \overline{4} \urcorner)$ depends on whether $\overline{2} + \overline{2} = \overline{4}$ is a member of the extension of Tr or not, and since the latter has already been settled adequately, we should actually revise the extension of Tr, which has been defined up to now solely for $\mathcal{L}$-sentences, in the way that we add $Tr(\ulcorner \overline{2} + \overline{2} = \overline{4} \urcorner)$ in case $\overline{2} + \overline{2} = \overline{4}$ is true, while otherwise leaving the original extension unchanged. In this case, of course, $Tr(\ulcorner \overline{2} + \overline{2} = \overline{4} \urcorner)$ ought to be added to the extension of Tr, because $\overline{2} + \overline{2} = \overline{4}$ is indeed true. Note that the original extension of *Tr* is thereby left unmodified as far as the sentences of $\mathcal{L}$ are concerned. Accordingly, since the truth or falsity of $Tr(\ulcorner Tr(\ulcorner \overline{2} + \overline{2} = \overline{4}\urcorner)\urcorner)$ depends on whether $Tr(\ulcorner \overline{2} + \overline{2} = \overline{4}\urcorner)$ is contained in the extension of *Tr* or not, which is determined by our last revision of the extension of *Tr*, we should also add $Tr(\ulcorner Tr(\ulcorner \overline{2} + \overline{2} = \overline{4}\urcorner)\urcorner)$ if $Tr(\ulcorner \overline{2} + \overline{2} = \overline{4}\urcorner)$ is a member of the extension of *Tr*, while leaving the extension unchanged otherwise; and so forth. In this way it is settled for every sentence $Tr(\ulcorner \ldots \ulcorner Tr(\ulcorner \overline{2} + \overline{2} = \overline{4}\urcorner)\urcorner \ldots \urcorner)$ whether the original extension of *Tr*, which is given by our adequate definition of truth for $\mathcal{L}$, is to be extended by adding $Tr(\ulcorner \ldots \ulcorner Tr(\ulcorner \overline{2} + \overline{2} = \overline{4}\urcorner)\urcorner \ldots \urcorner)$ or not. Every sentence of such a form depends directly or indirectly on non-semantic states of affairs. In this particular case, because of $2 + 2 = 4$, every such sentence would – and indeed should – be added to the extension of *Tr*.

What is the general scheme behind this line of reasoning? In order to outline a set Φ as referred to above, we consider a dependence relation by which the truth or falsity of sentences which contain *Tr* may be said to depend on whether some other sentences are members of the extension of *Tr* or not. Φ is determined as the set of sentences which, in this sense, depend directly or indirectly on the sentences of $\mathcal{L}$. Truth can be defined for the members of Φ by an iterative process that starts with the original extension of *Tr* for $\mathcal{L}$, i.e., with a set that is is determined by non-semantic states of affairs, and which proceeds furthermore according to the dependence relation.

The aim of this paper is to develop this informal account into a formal semantic theory. In Section 2 we first turn to the precise definition of our semantic dependence relation, which is not as obvious as the example above may have suggested: since the truth of complex sentences, in particular of quantified sentences, depends in general on the presence or absence of *more than just one sentence* in/from the extension of the truth predicate, we rather have to define a binary dependence relation holding between sentences and *sets* of sentences. E.g., for a predicate P in $\mathcal{L}$, sentences of the forms $\forall x(P(x) \rightarrow Tr(x))$ or $\exists x(P(x) \wedge Tr(x))$ will be seen to depend on the extension of P. Sentences such as $\forall x(Tr(x) \vee \neg Tr(x))$ depend on the empty set, since their truth or falsity (in this case: truth) does not depend at all on what sentences are contained in the extension of *Tr*. On the other hand, $\forall x(Tr(x) \rightarrow \neg Tr(\underset{\cdot}{\neg} x))$ depends on the complete language $\mathcal{L}_{Tr}$. After having defined our dependence relation properly, we study some of its formal properties and we consider various examples. Section 3 is devoted to the definition of our set Φ from above, which we will then call 'Φ_{lf}' since it is defined as the least fixed point of a certain set operator. Φ_{lf} is of course not a *language*, i.e., it is not closed under the usual syntactic rules which define the set of sentences of a first-order language – otherwise it would be identical to $\mathcal{L}_{Tr}$ – but it still has some nice closure properties. It may be described as the set of sentences of $\mathcal{L}_{Tr}$ *the truth or falsity of which depends on non-semantic states of affairs* (*only*); this excludes all ungrounded sentences like λ or versions of Yablo's [33] paradoxical sentences. Indeed we can even define notions such as self-referentiality or ungroundedness in terms of our dependency relation. Not surprisingly, any Liar sentence λ which is not just equivalent but even identical to $\neg Tr(\ulcorner\lambda\urcorner)$ turns out to be self-referential since it depends essentially on the singleton set that contains λ as its only member. In Section 4 we extend Tarski's definition of truth for $\mathcal{L}$ in the way that all T-biconditionals for sentences in Φ_{lf} are derivable from the definition plus the metatheory and thus we actually define the set Γ_{lf} of *true sentences that depend on non-semantic states of affairs* (*only*). Γ_{lf} is again characterized as being the least fixed point of a set operator and can be approximated from below by a "process" which resembles the one sketched in our atomic sentence example above. Section 5 concludes the paper by distinguishing our notion of dependence from related ones which have been introduced by Yablo [32], Gaifman [10] and other authors, by comparing our set Φ_{lf} to Herzberger's [18] and Kripke's [21] sets of grounded sentences, and by contrasting Γ_{lf} with the extensions of *Tr* as determined by other theories of truth such as Kripke's [21]; some more involved formal results concerning our own theory are stated at the end of the section. The proofs of most of the subsequent results are standard and can be left to the reader. We state only those proofs which are not so straight-forward and which are informative also for other reasons.

A few preliminaries and remarks on terminology: let $\mathcal{L}$ be the usual first-order language of arithmetic. $\mathcal{L}_{Tr}$ is $\mathcal{L}$ plus the additional unary truth predicate *Tr*. We often identify $\mathcal{L}$ and $\mathcal{L}_{Tr}$ with their corresponding sets of sentences. Adding *Tr* to the *arithmetic* language is handy for two reasons: we can use substitutional quantification due to the fact that every natural number is denoted by its numeral, which has the additional advantage that we may stick to a unary truth predicate where we would have to employ a binary satisfaction predicate otherwise. Secondly, the arithmetic language contains – modulo Gödelization – a theory of syntax for $\mathcal{L}_{Tr}$, which we take to be our object language. E.g., if φ is a sentence of $\mathcal{L}_{Tr}$, we have a singular term $\ulcorner\varphi\urcorner$ in $\mathcal{L}$ (and thus also in $\mathcal{L}_{Tr}$) that denotes the Gödel code of φ. Furthermore, there are means of denoting various syntactic functions and we use the usual dot notation in order to abbreviate some of these expressions: e.g., $\dot{\neg}$ is a function sign denoting the mapping which sends the code of φ to the code of $\neg\varphi$. Apart from such conveniences we could have used any other "ground language" as well. Nothing depends essentially on the details of the formal language which, after being extended by *Tr*, functions as the object language of our theory. Even a more detailed description of the arithmetic language is unnecessary, as long as all the examples which we refer to below can be expressed.

$\mathcal{L}_{Tr}$ is our object language and we take $\mathcal{L}_{Tr}$ extended by the language of set theory and by fragments of English to be our metalanguage. When we *use* the truth predicate in the metalanguage, we write 'true'; when we *mention* it as being a predicate of both object- and metalanguage, we write '*Tr*' or "true".

For $\varphi \in \mathcal{L}$, let $Val(\varphi)$ be the truth value of φ in the standard model $\mathfrak{M} = \langle \mathbb{N}, \mathfrak{I} \rangle$ of first-order arithmetic. We stick to classical logic throughout the paper, except for the discussion of Kripke's partial models in Section 5. For $\varphi \in \mathcal{L}_{Tr}$, let $Val_{\Phi}(\varphi)$ be the truth value of φ in the expansion of $\mathfrak{M}$ to $\mathfrak{M}_{Tr} = \langle \mathbb{N}, \mathfrak{I}_{Tr} \rangle$, where $\mathfrak{I}_{Tr}(Tr) = \Phi$, i.e.: given the standard interpretation of the arithmetic vocabulary and given that the extension of *Tr* is a subset Φ of the set of natural numbers, $Val_{\Phi}(\varphi)$ is the truth value of φ as defined by the usual semantic rules. Note that from now on we will simply say that $\Phi \subseteq \mathcal{L}_{Tr}$ when we should actually say that Φ is a set of *codes* of sentences of $\mathcal{L}_{Tr}$, i.e.: we identify syntactic entities with their codes.

We use 'c' as a metavariable for constant terms of $\mathcal{L}_{Tr}$, 'φ', 'ψ', 'ρ', 'χ', 'υ' as metavariables for sentences of $\mathcal{L}_{Tr}$ (including the sentences of $\mathcal{L}$), '$\varphi[x]$', '$\psi[x]$', '$\rho[x]$', etc. as metavariables for those formulas over the vocabulary of $\mathcal{L}_{Tr}$ which contain x freely, and 'Φ', 'Ψ', 'Γ', 'Π', 'Σ' as metavariables for subsets of $\mathcal{L}_{Tr}$ (in each case perhaps with additional indices). 'α', 'β', 'γ', 'δ' are metavariables for ordinals.

2. A Semantic Dependence Relation

"The" notion of dependence is one of the most frequently employed (and perhaps also abused) notions in philosophy. Different kinds of dependency have been considered in fields such as metaphysics, philosophy of mind, and philosophy of science, but few general, systematic theories of dependence seem to have been developed that are also applicable in a semantic context. The notion of dependence we are interested in is a semantic one, it is extensional, and it is also functional in so far as it can be expressed in terms of Grelling's [14] functional framework of his "Logical Theory of Dependence". However, it is not truth-functional in the sense that, e.g., what a disjunction depends on could be determined by applying a particular function associated with $\vee$ to the pair of sets that the corresponding disjuncts depend on. We are going to discuss other theories of semantic dependence in Section 5 after having introduced our own theory in sufficient detail.

Our notion of dependence may be circumscribed informally in the following ways: if φ is a sentence of $\mathcal{L}_{Tr}$ and Φ is a subset of $\mathcal{L}_{Tr}$, φ depends on Φ if and only if the truth value of φ depends on the presence or absence of the sentences that are contained in Φ in/from the extension of the truth predicate; or: no difference in the truth value of φ without a corresponding difference in the extension of the truth predicate as far as the members of Φ are concerned; or: for the truth value of φ it only makes a difference which members of Φ are contained in the extension of the truth predicate and which are not. These formulations show that the notion of dependence which we aim at is a kind of *supervenience*: the truth value of φ supervenes on which members of Φ are to be found in the extension of *Tr*. The supervenience of truth over non-semantic facts is hinted at by Davidson [9], pp. 214f., when he compares mental predicates with the truth predicate for a language $\mathcal{L}$ that contains some physical vocabulary plus vocabulary for a certain amount of mathematics and for its own syntax; both mental predicates and the truth predicate are said to supervene on $\mathcal{L}$ while not being reducible to $\mathcal{L}$ by means of law or definition. We are going to leave open this much-disputed relationship between supervenience and reduction and concentrate just on dependence.

When we say 'depends on', we do not refer to *partial* dependence, i.e.: if φ depends on Φ, then φ is *totally* determined by the status of Φ with respect to Φ's members being contained in the extension of *Tr*; φ does not really depend on anything "outside" of Φ then. Accordingly, if we say that a sentence depends on non-semantic states of affairs, we actually mean: it depends on non-semantic states of affairs *only*. 'φ depends on Φ' is also not intended to imply that Φ is *least* among the sets that φ depends on. Thus, adding "redundant" sentences to a set Φ which φ depends on does not have the effect that φ no longer depends on the resulting set. Later we are indeed going to introduce a notion of *essential* dependence in order to express an account of "least dependency". However, some sentences will be seen not to depend on *any* set essentially. Finally, we can restrict ourselves to a binary notion of *immediate* dependence (without furthermore qualifying 'dependence' in such a way): a binary notion of *indirect* dependence is just referred to implicitly by several of the constructions below, but it is not introduced by definition. On the other hand, by 'φ depends on non-semantic states of affairs' we actually mean 'φ depends *directly or indirectly* on non-semantic states of affairs'. Once the formal definitions are stated, each of these distinctions will get much clearer.[1]

Another word on terminology: when we speak of dependence on *states of affairs*, this should always be understood quasi-syntactically in the sense of Carnap, i.e., as a convenient manner of talking about *sentences* and its semantic properties. Sentences without truth predicate are said to describe *non-semantic* states of affairs[2]; sentences with truth predicate are said to describe *semantic* states of affairs.

Guided by these informal explanations we define for arbitrary $\varphi \in \mathcal{L}_{Tr}$ and $\Phi \subseteq \mathcal{L}_{Tr}$:

DEFINITION 1 (Dependence). φ depends on Φ :iff for all $\Psi_1, \Psi_2 \subseteq \mathcal{L}_{Tr}$: if $Val_{\Psi_1}(\varphi) \neq Val_{\Psi_2}(\varphi)$ then $\Psi_1 \cap \Phi \neq \Psi_2 \cap \Phi$.

Thus, in case of dependence, there is no difference with respect to the truth value of φ without a corresponding difference with respect to the presence or absence of the members of Φ in/from the extension of *Tr*.

There are the following obvious equivalent reformulations of this definition:

LEMMA 2 (Alternative formulations of dependence). *For all $\varphi \in \mathcal{L}_{Tr}$, $\Phi \subseteq \mathcal{L}_{Tr}$, the following sentences are equivalent:*

(1) *φ depends on Φ.*
(2) *For all $\Psi \subseteq \mathcal{L}_{Tr}$: $Val_{\Psi}(\varphi) = Val_{\Psi \cap \Phi}(\varphi)$.*
(3) *For all $\Psi_1, \Psi_2 \subseteq \mathcal{L}_{Tr}$: $Val_{\Psi_1}(\varphi) = Val_{\Psi_2}(\varphi)$ iff $Val_{\Psi_1 \cap \Phi}(\varphi) = Val_{\Psi_2 \cap \Phi}(\varphi)$.*

Lemma 2 shows that if φ depends on Φ, one only has to take a look at which members of Φ are contained in the extension of *Tr* in order to determine the truth value of φ.

From now on we will freely use the clauses stated in Definition 1 and Lemma 2 in order to express that φ depends on Φ; in particular item (2) of Lemma 2 will be used frequently. We state some examples of dependency after the introduction of the related notion of essential dependence below.

If we fix the first relatum of our dependence relation, we find that the relation has the following nice properties with respect to the alteration of the second relatum:

LEMMA 3 (Properties of dependence with respect to fixed first relatum). *For all $\varphi \in \mathcal{L}_{Tr}$, for all $\Phi, \Psi \subseteq \mathcal{L}_{Tr}$:*

(1) *If φ depends on Φ and $\Phi \subseteq \Psi$, then φ depends on Ψ.*
(2) *If φ depends on Φ and φ depends on Ψ, then φ depends on $\Phi \cap \Psi$.*
(3) *φ depends on $\mathcal{L}_{Tr}$.*

Let '$\wp$' denote the power set operation. We recall that $X \subseteq \wp(\mathcal{L}_{Tr})$ is called a 'filter' iff for all $\Phi, \Psi \subseteq \mathcal{L}_{Tr}$: (i) if $\Phi \in X$, $\Phi \subseteq \Psi$, then $\Psi \in X$, (ii) if $\Phi, \Psi \in X$, then $\Phi \cap \Psi \in X$. X is an ultrafilter iff X is a filter and (iii) for all $\Phi \subseteq \mathcal{L}_{Tr}$: $\mathcal{L}_{Tr} \backslash \Phi \in X$ iff $\Phi \notin X$.

In the following, we call $D(\varphi) := \{\Phi \subseteq \mathcal{L}_{Tr} \mid \varphi \text{ depends on } \Phi\}$ the *dependence filter of* φ (for $\varphi \in \mathcal{L}_{Tr}$). Lemma 3 entails that $D(\varphi)$ is a filter for all $\varphi \in \mathcal{L}_{Tr}$.

If $X \subseteq \wp(\mathcal{L}_{Tr})$, we call Φ *least in* X (or: X *is generated by* Φ) iff $\Phi \in X$ and for all $\Psi \in X$ it holds that $\Phi \subseteq \Psi$; we call Φ *minimal in* X iff $\Phi \in X$ and there is no $\Psi \in X$, such that $\Psi \subsetneqq \Phi$. Note that if $X \subseteq \wp(\mathcal{L}_{Tr})$ is a filter, then (precisely) one of the following holds: (i) there is a Φ which is least in X and $X = \{\Psi \subseteq \mathcal{L}_{Tr} \mid \Phi \subseteq \Psi\}$ – in this case X is called a *principal* filter; (ii) there is no Φ which is minimal in X.

In our context, least members of $D(\varphi)$ correspond to sets of sentences that φ depends on *essentially*, i.e., without taking into account any redundant sentences. So we define for $\varphi \in \mathcal{L}_{Tr}$, $\Phi \subseteq \mathcal{L}_{Tr}$:

DEFINITION 4 (Essential dependence). φ depends on Φ essentially :iff Φ is least in $D(\varphi)$.

Whether φ depends on some set essentially thus becomes an algebraic property of the filter $D(\varphi)$. If the filter is principal, then φ depends essentially on the generator of the filter. In the non-principal case φ does not depend essentially on *any* set: for every Φ that φ depends on there is some $\Phi' \subsetneqq \Phi$, such that φ also depends on Φ'; the members of $\Phi \setminus \Phi'$ are "redundant" and accordingly φ ought not be said to depend *essentially* on Φ (Examples 14 and 15 below are instances).

EXAMPLE LIST 1 puts some "flesh on the bones":

1. (Sentences without truth predicate)
   Let $\varphi \in \mathcal{L}$: then $D(\varphi) = \wp(\mathcal{L}_{Tr})$, i.e., φ depends on $\varnothing$ essentially. In particular, $\top$ (the verum) and $\bot$ (the falsum) depend on $\varnothing$ essentially.
2. (Atomic sentences with truth predicate)
   Let $\varphi = Tr(c)$, where $\mathfrak{I}(c) = \psi \in \mathcal{L}_{Tr}$: then $D(\varphi)$ is the (ultra-)filter which is generated by $\{\psi\}$, i.e., φ depends on $\{\psi\}$ essentially.
   Proof. For arbitrary $\Phi \subseteq \mathcal{L}_{Tr}$, it holds that: $Val_{\Phi}(Tr(c)) = 1$ iff $\psi \in \Phi$ iff $\psi \in \Phi \cap \{\psi\}$ iff $Val_{\Phi \cap \{\psi\}}(Tr(c)) = 1$. Thus for all $\Phi \subseteq \mathcal{L}_{Tr}$, $Val_{\Phi}(\varphi) = Val_{\Phi \cap \{\psi\}}(\varphi)$, and by item (2) of Lemma 2 φ depends on $\{\psi\}$. On the other hand, φ does not depend on $\varnothing$, because $Val_{\{\psi\}}(Tr(c)) = 1 \neq 0 = Val_{\{\psi\} \cap \varnothing}(Tr(c))$. Therefore $\{\psi\}$ is least in $D(\varphi)$. □
   Since $D(\varphi)$ is generated by a singleton, it is even an ultrafilter.

3. (A simple class of general sentences that involve the truth predicate)
   Let $\varphi = \forall x(P(x) \rightarrow Tr(x))$, where P is a predicate in the vocabulary of $\mathcal{L}$: then $D(\varphi)$ is the filter which is generated by $\mathfrak{I}(P)$, i.e., φ depends on $\mathfrak{I}(P)$ essentially.
   Proof. For arbitrary $\Phi \subseteq \mathcal{L}_{Tr}$ it is the case that: $Val_{\Phi}(\varphi) = 1$ iff $\mathfrak{I}(P) \subseteq \Phi$ iff $\mathfrak{I}(P) \subseteq \Phi \cap \mathfrak{I}(P)$ iff $Val_{\Phi\cap\mathfrak{I}(P)}(\varphi) = 1$. Thus for all $\Phi \subseteq \mathcal{L}_{Tr}$, $Val_{\Phi}(\varphi) = Val_{\Phi\cap\mathfrak{I}(P)}(\varphi)$, i.e., φ depends on $\mathfrak{I}(P)$. Furthermore, if Ψ is a proper subset of $\mathfrak{I}(P)$, φ does not depend on Ψ, since $Val_{\mathfrak{I}(P)}(\varphi) = 1 \neq 0 = Val_{\mathfrak{I}(P)\cap\Psi}(\varphi)$. So $\mathfrak{I}(P)$ is least in $D(\varphi)$. □
4. (Another simple class of general sentences involving the truth predicate)
   Let $\varphi = \exists x(P(x) \wedge Tr(x))$, where P is again a predicate in the vocabulary of $\mathcal{L}$: then $D(\varphi)$ is the filter which is generated by $\mathfrak{I}(P)$, i.e., φ depends on $\mathfrak{I}(P)$ essentially.
5. (A sentence expressing cases)
   Consider $\varphi = (Tr(\ulcorner\psi\urcorner) \rightarrow Tr(\ulcorner\rho\urcorner)) \wedge (\neg Tr(\ulcorner\psi\urcorner) \rightarrow Tr(\ulcorner\chi_1\urcorner) \wedge \cdots \wedge Tr(\ulcorner\chi_n\urcorner))$ for some $\psi, \rho, \chi_1, \ldots, \chi_n \in \mathcal{L}_{Tr}$: $D(\varphi)$ is the filter generated by $\{\psi, \rho, \chi_1, \ldots, \chi_n\}$ and thus φ depends on $\{\psi, \rho, \chi_1, \ldots, \chi_n\}$ essentially.
6. (Sentences which depend on $\mathcal{L}_{Tr}$ essentially)
   Consider $\forall x Tr(x)$, $\exists x Tr(x)$: their dependence filters are generated by $\mathcal{L}_{Tr}$, i.e., are identical to $\{\mathcal{L}_{Tr}\}$ and thus both sentences depend on $\mathcal{L}_{Tr}$ essentially.
7. (Further sentences which depend on $\mathcal{L}_{Tr}$ essentially)
   The dependence filters of $\forall x(Tr(x) \rightarrow \neg Tr(\dot{\neg} x))$ and $\exists x(Tr(x) \wedge Tr(\dot{\neg} x))$ are generated by $\mathcal{L}_{Tr}$, i.e., both sentences depend on $\mathcal{L}_{Tr}$ essentially.
   Proof. We show this only for the second sentence: assume that $\exists x(Tr(x) \wedge Tr(\dot{\neg} x))$ depends on some $\Phi \subseteq \mathcal{L}_{Tr}$, s.t. there is a sentence $\varphi \in \mathcal{L}_{Tr}$ with $\varphi \notin \Phi$. In this case $Val_{\{\varphi,\neg\varphi\}}(\exists x(Tr(x) \wedge Tr(\dot{\neg} x))) = 1$, but $Val_{\{\varphi,\neg\varphi\}\cap\Phi}(\exists x(Tr(x) \wedge Tr(\dot{\neg} x))) = 0$; this contradicts the assumed dependence on Φ. On the other hand, $\exists x(Tr(x) \wedge Tr(\dot{\neg} x))$ of course depends on $\mathcal{L}_{Tr}$ and therefore it depends essentially on $\mathcal{L}_{Tr}$. □
8. (Liar)
   Let $\mathfrak{I}(c_\lambda) = \lambda := \neg Tr(c_\lambda)$: then $D(\lambda)$ is the (ultra-)filter which is generated by $\{\lambda\}$, i.e., λ depends on $\{\lambda\}$ essentially.
9. (Truthteller)
   Let $\mathfrak{I}(c_\tau) = \tau := Tr(c_\tau)$: $D(\tau)$ is the (ultra-)filter which is generated by $\{\tau\}$, i.e., τ depends on $\{\tau\}$ essentially.
10. (Combinations of the Liar with sentences without truth predicate)

Let $\psi_1, \psi_2 \in \mathcal{L}$, s.t. $Val(\psi_1) = 1$, $Val(\psi_2) = 0$; let λ as in (8): then $D(\psi_1 \wedge \lambda) = D(\psi_2 \vee \lambda) = D(\lambda)$, i.e., both $\psi_1 \wedge \lambda$ and $\psi_2 \vee \lambda$ depend on $\{\lambda\}$ essentially, whereas $D(\psi_1 \vee \lambda) = D(\psi_2 \wedge \lambda) = \wp(\mathcal{L}_{Tr})$, i.e., both $\psi_1 \vee \lambda$ and $\psi_2 \wedge \lambda$ depend on $\varnothing$ essentially.

11. (Logical truths involving the Liar or the Truthteller)
    Let λ as in (8) and τ as in (9):
    then $D(\lambda \vee \neg\lambda) = D(\tau \vee \neg\tau) = D(\lambda \wedge \neg\lambda) = D(\tau \wedge \neg\tau) = \wp(\mathcal{L}_{Tr})$, i.e., $\lambda \vee \neg\lambda$, $\tau \vee \neg\tau$, $\lambda \wedge \neg\lambda$, $\tau \wedge \neg\tau$ depend on $\varnothing$ essentially.
12. (Liar cycle)
    Let $\mathfrak{I}(c_1) = Tr(c_2)$, $\mathfrak{I}(c_2) = Tr(c_3), \ldots, \mathfrak{I}(c_{n-1}) = Tr(c_n)$, $\mathfrak{I}(c_n) = \neg Tr(c_1)$:
    then $D(Tr(c_2))$ is generated by $\{Tr(c_3)\}$, $D(Tr(c_3))$ by $\{Tr(c_4)\}, \ldots,$ $D(Tr(c_{n-1}))$ by $\{Tr(c_n)\}$, $D(Tr(c_n))$ by $\{\neg Tr(c_1)\}$, and $D(\neg Tr(c_1))$ by $\{Tr(c_2)\}$ and therefore also the corresponding essential dependence claims hold.
13. (Yablo's paradox)
    Let $\upsilon_n = \neg\forall x(P_n(x) \rightarrow Tr(x))$ for all $n \in \mathbb{N}$, where P_n is a predicate in the vocabulary of $\mathcal{L}$, and $\mathfrak{I}(P_n) = \{\upsilon_{n+1}, \upsilon_{n+2}, \ldots\}$:
    then for all $n \in \mathbb{N}$, $D(\upsilon_n)$ is the filter which is generated by $\mathfrak{I}(P_n)$, i.e., υ_n depends on $\mathfrak{I}(P_n)$ essentially.
14. (A sentence without essential dependence)
    Let $\varphi = \forall x(S(x) \rightarrow \exists y(R(x, y) \wedge Tr(y)))$, where $\mathfrak{I}(S) = \{P(c_m) \mid m \in \mathbb{N}\}$, $\mathfrak{I}(R) = \{\langle P(c_m), P(c_n)\rangle \mid m, n \in \mathbb{N}, m < n\}$ (for S, R, P being members of the vocabulary of $\mathcal{L}$):
    then $D(\varphi)$ is not generated by any set, i.e., there is no $\Phi \subseteq \mathcal{L}_{Tr}$, s.t. φ depends on Φ essentially.
    Proof. First of all, φ depends on every set $\Psi_k = \{P(c_n) \mid k \leqslant n \in \mathbb{N}\}$, since: $Val_\Psi(\varphi) = 1$ iff for all $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$, s.t. $\langle P(c_m), P(c_n)\rangle \in \mathfrak{I}(R)$ and $P(c_n) \in \Psi$; the latter is equivalent to the fact that for all $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$, s.t. $\langle P(c_m), P(c_n)\rangle \in \mathfrak{I}(R)$ and $P(c_n) \in \Psi_k \cap \Psi$ (where k is an arbitrary natural number), because:
    ("$\rightarrow$") we show the contrapositive: assume that there is an $m \in \mathbb{N}$, s.t. for all $n > m$ $P(c_n) \notin \Psi_k \cap \Psi$. But then for all $n > \max(m, k)$ we have $P(c_n) \notin \Psi$, thus for $\max(m, k)$ there is no $n \in \mathbb{N}$, s.t. $\langle P(c_{\max(m,k)}), P(c_n)\rangle \in \mathfrak{I}(R)$ and $P(c_n) \in \Psi$, and therefore not for all $m \in \mathbb{N}$ there is an $n \in \mathbb{N}$, s.t. $\langle P(c_m), P(c_n)\rangle \in \mathfrak{I}(R)$ and $P(c_n) \in \Psi$.
    ("$\leftarrow$") Obvious.
    So we have that $Val_\Psi(\varphi) = 1$ iff $Val_{\Psi_k \cap \Psi}(\varphi) = 1$, i.e., $Val_\Psi(\varphi) = Val_{\Psi_k \cap \Psi}(\varphi)$, and therefore φ depends on Ψ_k. Since this is the case for every $k \in \mathbb{N}$, φ cannot depend on any set Ψ_k essentially, because $\Psi_1 \supsetneqq \Psi_2 \supsetneqq \Psi_3 \supsetneqq \cdots$.

If there were a Φ least in $D(\varphi)$, Φ would have to be a subset of each Ψ_k; the only such subset is the empty set, but φ does not depend on $\varnothing$, since $Val_{\varnothing}(\varphi) = 0 \neq 1 = Val_{\mathcal{L}_{Tr}}(\varphi)$. Thus φ does not depend on any set essentially. □

15. (Another sentence without essential dependence)
Let $\varphi = \forall x(S(x) \rightarrow \exists y(R'(x, y) \wedge Tr(y)))$, where
$\mathfrak{I}(S) = \{P(c_m) \mid m \in \mathbb{N}\}$, $\mathfrak{I}(R')$ is the well-founded strict total order $P(c_0), \varphi, P(c_1), \varphi, P(c_2), \varphi, \ldots$ (for S, R', P being members of the vocabulary of $\mathcal{L}$):
then $D(\varphi)$ is again not generated by any set, i.e., there is no $\Phi \subseteq \mathcal{L}_{Tr}$, s.t. φ depends on Φ essentially.
(The proof is analogous to the one for (14). The difference between (14) and (15) in terms of "self-referentiality" will show up in Section 3.)

Let us now fix the second relatum of the dependency relation. We find:

LEMMA 5 (Properties of dependence with respect to fixed second relatum). *For all $\chi \in \mathcal{L}$, φ, $\psi \in \mathcal{L}_{Tr}$, $\Phi \subseteq \mathcal{L}_{Tr}$ and all open formulas $\rho[x]$ over the vocabulary of $\mathcal{L}_{Tr}$:*

(1) *χ depends on Φ; in particular, $\top$ depends on Φ and $\bot$ depends on Φ.*
(2) *φ depends on Φ iff $\neg\varphi$ depends on Φ.*
(3) *If φ, ψ depend on Φ, then also $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi$ depend on Φ (and the same holds for all truth-functional compositions of φ, ψ).*
(4) *If for all constant terms t it holds that $\rho[t]$ depends on Φ,*
then also $\forall x \rho[x]$, $\exists x \rho[x]$ depend on Φ.
(5) *If $\varphi \vDash \psi$, $\psi \vDash \varphi$, then: φ depends on Φ iff ψ depends on Φ.*
(6) (*Strengthening of* 5)
If for all $\Psi \subseteq \mathcal{L}_{Tr}$ it holds that $Val_{\Psi}(\varphi) = Val_{\Psi}(\psi)$, then:
φ depends on Φ iff ψ depends on Φ.

By (6) of Lemma 5, if t is a constant term, such that $\mathfrak{I}(t) = \varphi$, then $Tr(t)$ and $Tr(\ulcorner\varphi\urcorner)$ are indistinguishable with respect to what they depend on. Thus we can restrict ourselves to stating theorems primarily for formulas of the form of $Tr(\ulcorner\varphi\urcorner)$, which refer to sentences by means of $\ulcorner$ and $\urcorner$.

Recall that a Lindenbaum–Tarski algebra is a Boolean algebra the members of which are congruence classes of sentences or formulas of a propositional or first-order language, where the congruence relation is defined by logical equivalence and where the Boolean operations are defined via their linguistic counterparts (i.e., Boolean complements are defined as the congruence classes of negations, etc.). We call a Lindenbaum–Tarski algebra which is defined on the basis of a first-order language *complete with respect to first-order-definable infima and suprema*, whenever the following is the case: if for all constant terms t it holds that the congruence class of $\rho[t]$ is a member of the algebra, also the congruence classes of $\forall x\rho[x]$ and $\exists x\rho[x]$ are members of the algebra.

For $\Phi \subseteq \mathcal{L}_{Tr}$, let us call $D^{-1}(\Phi) := \{\varphi \in \mathcal{L}_{Tr} \mid \varphi \text{ depends on } \Phi\}$ the *dependence algebra of* Φ. Obviously, dependence filters and dependence algebras are related in the following way: $\Phi \in D(\varphi)$ iff $\varphi \in D^{-1}(\Phi)$. But note that we use '$^{-1}$' for mainly mnemotechnical reasons, since D^{-1} is actually not the inverse mapping or relation of D; while D maps sentences to sets of sentences, D^{-1} maps sets of sentences to sets of sentences again.

It follows immediately from Lemma 5 and from our assumption of substitutional quantification that:

LEMMA 6. *For all* $\Phi \subseteq \mathcal{L}_{Tr}$: *up to taking congruence classes* (*by means of logical equivalence*), $D^{-1}(\Phi)$ *is a Lindenbaum–Tarski algebra which is complete with respect to first-order-definable infima and suprema.*

The subsequent lemma recalls that our definition of dependence satisfies our informal description of dependency stated in the introduction; furthermore it shows that the dependence algebra of Φ is closed under the application of *Tr* to the names of the members of Φ:

LEMMA 7 (Properties of dependence with respect to the truth predicate). *For all* $\varphi \in \mathcal{L}_{Tr}$, $\Phi \subseteq \mathcal{L}_{Tr}$: $\varphi \in \Phi$ *iff* $Tr(\ulcorner\varphi\urcorner)$ *depends on* Φ.

Equivalently: $\varphi \in \Phi$ *iff* $Tr(\ulcorner\varphi\urcorner) \in D^{-1}(\Phi)$.

3. THE SET OF SENTENCES THAT DEPEND ON NON-SEMANTIC STATES OF AFFAIRS

From studying the properties of the sets $D^{-1}(\Phi)$ we now turn to studying D^{-1} itself, i.e., the operator $D^{-1} : \wp(\mathcal{L}_{Tr}) \to \wp(\mathcal{L}_{Tr})$, $\Phi \mapsto D^{-1}(\Phi)$.

The most important fact about D^{-1} is that it is monotonic: for all Φ, $\Psi \subseteq \mathcal{L}_{Tr}$: if $\Phi \subseteq \Psi$, then $D^{-1}(\Phi) \subseteq D^{-1}(\Psi)$ (by (1) of Lemma 3). Actually, by Lemma 3, D^{-1} is even a homomorphism with respect to meet, i.e., $D^{-1}(\Phi \cap \Psi) = D^{-1}(\Phi) \cap D^{-1}(\Psi)$ for all Φ, $\Psi \subseteq \mathcal{L}_{Tr}$.

The monotonicity property of D^{-1} has some well-known consequences: (i) there is a least fixed point Φ_{lf} of D^{-1}; (ii) therefore: for all $\varphi \in \mathcal{L}_{Tr}$: $\varphi \in \Phi_{lf}$ iff φ depends on Φ_{lf}; (iii) Φ_{lf} can be reached from below as follows: let $\Phi_0 := \varnothing$, $\Phi_{\alpha+1} := D^{-1}(\Phi_\alpha)$, $\Phi_\beta := \bigcup_{\alpha<\beta} \Phi_\alpha$ (for β a limit); then $(\Phi_\alpha)_{\alpha \in Ord}$ is increasing and there is a least ordinal α^*, s.t. $\Phi_{\alpha^*} = \Phi_{lf}$. (Cf. the Knaster–Tarski Fixed Point Theorem in Tarski [31] and the general theory of inductive definitions in Moschovakis [24].) By 'increasing' we do not mean 'increasing *strictly*'. D^{-1} has of course other fixed points as well: its largest fixed point is $\mathcal{L}_{Tr} = D^{-1}(\mathcal{L}_{Tr})$; another fixed point is the one which is generated by iterating D^{-1} while starting with $\{\tau\}$, where $\tau = Tr(c_\tau)$ with $\mathfrak{I}(c_\tau) = \tau$ is a truthteller (note that $D^{-1}(\{\tau\}) \supsetneqq \{\tau\}$).

The progression (Φ_α) has the following obvious property concerning Tr:

LEMMA 8 (Properties of $(\Phi_\alpha)_{\alpha \in Ord}$ with respect to the truth predicate). *For all ordinals α, for all $\varphi \in \mathcal{L}_{Tr}$: $\varphi \in \Phi_\alpha$ iff $Tr(\ulcorner\varphi\urcorner) \in \Phi_{\alpha+1}$.*

By means of $(\Phi_\alpha)_{\alpha \in Ord}$ ordinal ranks may be assigned to the members of Φ_{lf}, such that the first occurrence of a sentence in the sequence (Φ_α) defines its rank of occurrence: for $\varphi \in \Phi_{lf}$, φ *has dependence rank* α :iff $\varphi \in \Phi_\alpha$, but for all $\beta < \alpha$: $\varphi \notin \Phi_\beta$. Every $\varphi \in \Phi_{lf}$ is a member of some Φ_α and there is also a least ordinal α such that $\varphi \in \Phi_\alpha$, since the class of ordinals is well-ordered; this least ordinal is precisely the dependence rank of φ and thus every member of Φ_{lf} has a unique dependence rank. Obviously, limit ordinals do not occur among these dependence ranks. Hence if β is a limit ordinal, then $\Phi_\beta = \bigcup_{\substack{\gamma<\beta,\\ \gamma=\delta+1}} \Phi_\gamma$. We will also make use of:

LEMMA 9 ($(\Phi_\alpha)_{\alpha \in Ord}$ and negation). *For all $\varphi \in \Phi_{lf}$, for all $\alpha \in Ord$: $\varphi \in \Phi_\alpha$ iff $\neg\varphi \in \Phi_\alpha$.*

Thus, for every $\varphi \in \Phi_{lf}$: the dependence ranks of φ and of $\neg\varphi$ coincide.

The formal machinery we have introduced can be put to work in order to define notions such as *dependence on non-semantic states of affairs*, *self-referentiality*, and *ungroundedness*. The definition of the latter two notions and in particular how to distinguish the one from the other is of special interest since questions such as 'is Yablo's paradox an instance of self-referentiality?' or 'what is a self-referential sentence?' have recently been debated hotly (see Priest [25], Sorenson [28], Beall [2], Bueno and Colyvan [4], Leitgeb [22]).

For $\varphi \in \mathcal{L}_{Tr}$ we define:

DEFINITION 10 (Dependence on non-semantic states of affairs).

(1) φ depends *directly* on non-semantic states of affairs :iff
φ depends on $\mathcal{L}$ (iff $\varphi \in D^{-1}(\mathcal{L})$).

(2) φ depends (directly or indirectly) on non-semantic states of affairs :iff $\varphi \in \Phi_{lf}$.

Due to Φ_{lf}'s being a fixed point under D^{-1}, dependence on non-semantic states of affairs can be restated as follows:

LEMMA 11 (Dependence on non-semantic states of affairs: variations). *For all $\varphi \in \mathcal{L}_{Tr}$: φ depends on non-semantic states of affairs iff either of the following holds*:

- $\varphi \in \Phi_{lf}$.
- $\varphi \in D^{-1}(\Phi_{lf})$.
- φ *depends on* Φ_{lf}.

While the set Φ_{lf} of sentences which depend on non-semantic states of affairs is not a language, it is actually not so "far" from being so, as Lemma 11 together with our Lemmata 6 and 7 on dependence algebras shows.

For $\varphi \in \mathcal{L}_{Tr}$ we can also define:

DEFINITION 12 (Self-referentiality, ungroundedness).

(1) φ is self-referential :iff
for all $\Phi \subseteq \mathcal{L}_{Tr}$: if φ depends on Φ, then $\varphi \in \Phi$.
(2) φ is ungrounded :iff
φ does not depend on non-semantic states of affairs.

Thus a sentence is self-referential iff it is a member of every set it depends on. Instead of calling a sentence 'depending on non-semantic states of affairs' we might just as well call it 'grounded', which would be short for 'grounded in the non-semantic part of the language' or 'grounded in non-semantic states of affairs'.

As we are going to state below, if there is a $\Phi \subseteq \mathcal{L}_{Tr}$, s.t. φ depends on Φ essentially and $\varphi \in \Phi$, then φ is self-referential according to Definition 12. But note that if we had *defined* self-referentiality by recurring to essential dependence, i.e., if we had defined φ to be self-referential iff there is a $\Phi \subseteq \mathcal{L}_{Tr}$, s.t. φ depends on Φ essentially and $\varphi \in \Phi$,[3] then all φ which do not depend on any set essentially would have turned out to be non-self-referential; e.g., Example 8 in Example List 2 below would not be self-referential, although it is so according to our Definition 12. That might have been interpreted as a technical defect rather than as a material feature of self-referentiality.[4] On the other hand, *every* φ would have turned out to be self-referential if we had defined φ to be self-referential iff there is a $\Phi \subseteq \mathcal{L}_{Tr}$, s.t. φ depends on Φ and $\varphi \in \Phi$ (since every φ depends on $\mathcal{L}_{Tr}$).

One could also introduce a notion of circularity, such that the members of a Liar cycle – which are *not* self-referential according to Definition 12 – turned out to be circular, but we omit such further qualifications.

Ungroundedness follows from the existence of infinitely descending chains of essential dependence (but not necessarily vice versa as may be seen from constructing an ungrounded sentence that does not depend on any set essentially; cf. again 8 in Example List 2 below). Circles of essential dependence (such as the Liar cycles) are special instances of such infinitely descending chains:

LEMMA 13 (Ungroundedness and essential dependence). *For all $\varphi \in \mathcal{L}_{Tr}$: if there is an infinite sequence $(\psi_n)_{n\in\mathbb{N}}$ of sentences of $\mathcal{L}_{Tr}$, s.t.*

- $\psi_1 = \varphi$,
- *for every $n \in \mathbb{N}$ there is a set Ψ_{n+1}, s.t. ψ_n depends on Ψ_{n+1} essentially and $\psi_{n+1} \in \Psi_{n+1}$,*

then φ is ungrounded.

Proof. Assume there is such a sequence $(\psi_n)_{n\in\mathbb{N}}$ for $\varphi \in \mathcal{L}_{Tr}$ and suppose $\varphi \in \Phi_{lf}$: then there is a successor ordinal α_1+1, s.t. $\psi_1 = \varphi \in \Phi_{\alpha_1+1}$, and therefore ψ_1 depends on Φ_{α_1}. Since, by assumption, ψ_1 depends on a set Ψ_2 essentially, it follows that $\Psi_2 \subseteq \Phi_{\alpha_1}$. $\psi_2 \in \Psi_2$ is thus a member of Φ_{α_1}, and there is a successor ordinal $\alpha_2 + 1$, s.t. $\psi_2 \in \Phi_{\alpha_2+1}$ with $\alpha_2 + 1 < \alpha_1 + 1$, and ψ_2 depends on Φ_{α_2}. By iteration we find that there would have to be an infinitely descending sequence $(\alpha_n+1)_{n\in\mathbb{N}}$ of ordinals contradicting the fact that every set of ordinals is well-ordered. □

The next lemma collects some important properties of dependence, or lack of dependence, on non-semantic states of affairs:

LEMMA 14 (Properties of dependence or of lack of dependence on non-semantic states of affairs). *For all φ, $\psi \in \mathcal{L}_{Tr}$:*

(1) *If φ depends on Φ essentially, then:*
φ depends on non-semantic states of affairs iff $\Phi \subseteq \Phi_{lf}$.
(2) *Sentences without truth predicate depend on non-semantic states of affairs.*
(3) *Logical truths depend on non-semantic states of affairs.*
(4) *φ depends on non-semantic states of affairs iff $Tr(\ulcorner\varphi\urcorner)$ depends on non-semantic states of affairs.*
(5) *If there is a $\Phi \subseteq \mathcal{L}_{Tr}$, s.t. φ depends on Φ essentially and $\varphi \in \Phi$, then φ is self-referential.*
(6) *If φ is self-referential, then φ is ungrounded, i.e., φ does not depend on non-semantic states of affairs.*

(7) *The set of ungrounded sentences is closed under logical equivalence.*
(8) (*Strengthening of* 7)
If for all $\Psi \subseteq \mathcal{L}_{Tr}$ *it holds that* $Val_\Psi(\varphi) = Val_\Psi(\psi)$, *then:* φ *is ungrounded iff* ψ *is ungrounded.*
(9) (*Herzberger's result*)
There is no sentence $\varphi \in \mathcal{L}_{Tr}$, *s.t.* φ *depends on* Φ_{lf} *essentially* (*i.e., on the set of sentences that depend on non-semantic states of affairs*).

Proof. (6) follows by standard transfinite induction. We only show (9): assume that $\varphi \in \mathcal{L}_{Tr}$ depends on Φ_{lf} essentially: since φ consequently depends on Φ_{lf} by Definition 4, Lemma 11 entails that $\varphi \in \Phi_{lf}$. Therefore φ is self-referential by Definition 12 and thus ungrounded according to (6). So $\varphi \notin \Phi_{lf}$ after all and we have a contradiction. (Compare the analogous reasoning in Herzberger [18], pp. 151f.) □

We conclude with EXAMPLE LIST 2 (while presupposing what we have shown in Example List 1):

1. Let P be a predicate in the vocabulary of $\mathcal{L}$, where additionally $\Im(P) \subseteq \Phi_{lf}$ (in particular if $\Im(P) \subseteq \mathcal{L}$):
$\forall x(P(x) \to Tr(x))$ and $\exists x(P(x) \wedge Tr(x))$ depend on non-semantic states of affairs.
2. $\forall x Tr(x)$, $\exists x Tr(x)$, $\forall x(Tr(x) \to \neg Tr(\dot{\neg} x))$, $\exists x(Tr(x) \wedge Tr(\dot{\neg} x))$ are self-referential and thus ungrounded.
3. The Liar and the Truthteller are self-referential.
4. The Liar, the Truthteller, and all members of a Liar cycle are ungrounded.
5. Let $\psi_1, \psi_2 \in \mathcal{L}$, s.t. $Val(\psi_1) = 1$, $Val(\psi_2) = 0$; let λ as in 8:
then $\psi_1 \wedge \lambda$ and $\psi_2 \vee \lambda$ are ungrounded but not self-referential, whereas $\psi_1 \vee \lambda$ and $\psi_2 \wedge \lambda$ depend on non-semantic states of affairs.
6. The members of the sequence constituting Yablo's paradox are ungrounded though not self-referential.
7. The sentence $\forall x(S(x) \to \exists y(R(x, y) \wedge Tr(y)))$ from 14 in Example List 1 depends on non-semantic states of affairs:
Proof. In 14 of Example List 1 we have seen that $\forall x(S(x) \to \exists y(R(x, y) \wedge Tr(y)))$ depends on every set $\Psi_k = \{P(c_n) \mid k \leqslant n \in \mathbb{N}\}$. Since $\mathcal{L} \subseteq \Phi_{lf}$ and since every Ψ_k is a subset of $\mathcal{L}$, every Ψ_k is seen to be a subset of Φ_{lf}, therefore $\forall x(S(x) \to \exists y(R(x, y) \wedge Tr(y)))$ depends on Φ_{lf} and thus depends on non-semantic states of affairs. □
8. The sentence $\forall x(S(x) \to \exists y(R'(x, y) \wedge Tr(y)))$ from 15 in Example List 1 is self-referential:

Proof. We show the contrapositive: let $\varphi = \forall x(S(x) \to \exists y(R'(x, y) \wedge Tr(y)))$ and assume that $\varphi \notin \Psi$: then $Val_{\{\varphi\}}(\varphi) = 1 \neq 0 = Val_{\{\varphi\}\cap\Psi}(\varphi)$, which entails that φ does not depend on Ψ. □

Let $\lambda' = \forall x(x = \ulcorner\lambda\urcorner \to \neg Tr(x))$: while λ is self-referential, λ' is not, since λ' depends on $\{\lambda\}$ essentially by 3 of Example List 1 just as λ itself does, but $\lambda' \notin \{\lambda\}$. So we see that there are $\varphi, \psi \in \mathcal{L}_{Tr}$, such that for all $\Psi \subseteq \mathcal{L}_{Tr}$ it holds that $Val_\Psi(\varphi) = Val_\Psi(\psi)$ and where φ is self-referential while ψ is not (compare the discussion in Leitgeb [22] of such problems which affect the pre-theoretic notion of self-referentiality). If that were not approved, Definition 12 might be adapted as follows: φ is self-referential :iff *there is a* $\psi \in \mathcal{L}_{Tr}$, such that for all $\Psi \subseteq \mathcal{L}_{Tr}$ it holds that $Val_\Psi(\varphi) = Val_\Psi(\psi)$ and for all $\Phi \subseteq \mathcal{L}_{Tr}$ if ψ depends on Φ, then $\psi \in \Phi$. Since this notion of self-referentiality is still based on dependency, the trivialization results of [22] are avoided. But of course $\psi_1 \wedge \lambda$ and $\psi_2 \vee \lambda$ from above would now turn out to be self-referential, which is perhaps undesirable. On the other hand, assume we defined φ to be self-referential :iff *for all* $\psi \in \mathcal{L}_{Tr}$ we have that if for all $\Psi \subseteq \mathcal{L}_{Tr}$ it holds that $Val_\Psi(\varphi) = Val_\Psi(\psi)$, then for all $\Phi \subseteq \mathcal{L}_{Tr}$ if ψ depends on Φ, then $\psi \in \Phi$; then even λ would not count as self-referential because of the existence of λ'.

Furthermore, note that every sentence $Tr(c') \vee \neg Tr(c')$ with $\mathfrak{I}(c') = Tr(c') \vee \neg Tr(c')$ depends on non-semantic states of affairs only, according to (3) of Lemma 14. This is a consequence of the fact that our notion of dependency is invariant under logical and even arithmetic equivalence (recall (5) and (6) of Lemma 5).

Dependence ranks are of course not defined for ungrounded sentences and therefore also not for self-referential sentences.

4. Truth for Sentences That Depend on Non-Semantic States of Affairs

Now we turn to the precise definition of the progression of extensions for *Tr* which we referred to in the introductory section. For $(\Phi_\alpha)_{\alpha\in Ord}$ as defined in Section 3, let: $\Gamma_0 := \varnothing$, $\Gamma_{\alpha+1} := \{\varphi \in \Phi_{\alpha+1} \mid Val_{\Gamma_\alpha}(\varphi) = 1\}$, $\Gamma_\beta := \bigcup_{\alpha<\beta} \Gamma_\alpha$ (for β a limit). The sequence $(\Gamma_\alpha)_{\alpha\in Ord}$ is thus defined by a "jump" operation roughly in line with Kripke [21], but on the basis of a classical valuation mapping *Val* rather than a three-valued one. In contrast to Gupta's and Belnap's [16] revision sequences of classical structures we focus just on the members of (Φ_α) when we build up the sequence and we take unions at limit stages rather than set-theoretic limit inferiors.

The definition of (Γ_α) implies:

LEMMA 15 (Properties of $(\Gamma_\alpha)_{\alpha\in Ord}$ with respect to $(\Phi_\alpha)_{\alpha\in Ord}$). *For all $\alpha \in Ord$: $\Phi_\alpha = \Gamma_\alpha \cup \neg\Gamma_\alpha$ (where for $\Phi \subseteq \mathcal{L}_{Tr}$ we let $\neg\Phi := \{\varphi \in \mathcal{L}_{Tr} \mid \neg\varphi \in \Phi\}$).*

The progression (Γ_α) has properties similar to the ones described in our informal example in the introduction: for all ordinals α, for all $\varphi \in \mathcal{L}_{Tr}$: $\varphi \in \Gamma_\alpha$ iff $Tr(\ulcorner\varphi\urcorner) \in \Gamma_{\alpha+1}$.

Just as in the case of the dependency hierarchy we can introduce ranks with respect to the truth-theoretic hierarchy (Γ_α), but now only the members of $\bigcup_{\alpha\in Ord}\Gamma_\alpha$ have a truth rank: for $\varphi \in \bigcup_{\alpha\in Ord}\Gamma_\alpha$, φ has truth rank α :iff $\varphi \in \Gamma_\alpha$, but for all $\beta < \alpha$: $\varphi \notin \Gamma_\beta$ (obviously truth ranks are successor ordinals again).

The following theorem is essential for the justification of our later definition of truth:

THEOREM 16 (Convergence of $(\Gamma_\alpha)_{\alpha\in Ord}$). *For all $\alpha \in Ord$, for all $\beta \in Ord$ with $\beta < \alpha$: $\Gamma_\alpha \cap \Phi_\beta = \Gamma_\beta$.*

The proof of Theorem 16 is by standard transfinite induction. Theorem 16 implies directly that $(\Gamma_\alpha)_{\alpha\in Ord}$ is increasing. Consequently, $(\Gamma_\alpha)_{\alpha\in Ord}$ must converge to a limit: there is an $\alpha^+ \in Ord$, s.t. α^+ is the least $\alpha \in Ord$ with: for all $\beta \in Ord$ with $\beta > \alpha$, $\Gamma_\beta = \Gamma_\alpha$. So let us define $\Gamma_{lf} := \Gamma_{\alpha^+}$. (Wherever we have said '$\bigcup_{\alpha\in Ord}\Gamma_\alpha$' above, we thus might have said 'Γ_{lf}' as well.)

Moreover, we have for all $\varphi \in \Phi_\beta$, for all $\alpha > \beta$: $Val_{\Gamma_\alpha}(\varphi) = Val_{\Gamma_\beta}(\varphi)$. Thus, the truth value of every $\varphi \in \Phi_{lf}$ as given by $(Val_{\Gamma_\alpha}(\varphi))_{\alpha\in Ord}$ stabilizes after its dependence rank. From this we can conclude that for all $\varphi \in \Gamma_{lf}$ the truth rank of φ and the dependence rank of φ coincide. Therefore we can collapse our definitions of dependence ranks and truth ranks into the following definition: for $\varphi \in \Phi_{lf}$, φ has rank α :iff φ has dependence rank α (iff, for $\varphi \in \Gamma_{lf}$, φ has truth rank α).

The fixed point ranks of $(\Phi_\alpha)_{\alpha\in Ord}$ and $(\Gamma_\alpha)_{\alpha\in Ord}$ may now be seen to coincide as well: $\alpha^* = \alpha^+$. So we have actually have $\Gamma_{lf} = \Gamma_{\alpha^*}$ and $\Phi_{lf} = \Phi_{\alpha^*}$ (indeed: $\Phi_{lf} = \Gamma_{lf} \cup \neg\Gamma_{lf}$).

We find that Γ_{lf} has precisely the properties that we expect the extension of a truth predicate for Φ_{lf} to have:

THEOREM 17 (T-biconditionals and dependence on non-semantic states of affairs).

(1) *For all $\varphi \in \Phi_{lf}$: $\varphi \in \Gamma_{lf}$ iff $Val_{\Gamma_{lf}}(\varphi) = 1$.*
(2) *For all $\varphi \in \Phi_{lf}$: $Val_{\Gamma_{lf}}(Tr(\ulcorner\varphi\urcorner) \leftrightarrow \varphi) = 1$.*

We can finally define for $\varphi \in \mathcal{L}_{Tr}$:

DEFINITION 18. φ is true (in-$\mathcal{L}_{Tr}$) :iff $\varphi \in \Gamma_{lf}$.

By (1) of Theorem 17, Definition 18 is materially adequate with respect to the members of Φ_{lf}, i.e., with respect to the sentences that depend on non-semantic states of affairs. 'φ is true' is short for '$\varphi \in \Gamma_{lf}$', the semantic rules that define $Val_{\Gamma_{lf}}$ transform '$Val_{\Gamma_{lf}}(\varphi) = 1$' into a metalinguistic translation of φ in which Tr is replaced by '$\in \Gamma_{lf}$', i.e., by 'is true'. E.g., it follows: since $\overline{2} + \overline{2} = \overline{4} \in \Phi_{lf}$, $\overline{2} + \overline{2} = \overline{4}$ is true iff $\overline{2} + \overline{2} = \overline{4} \in \Gamma_{lf}$ iff $Val_{\Gamma_{lf}}(\overline{2} + \overline{2} = \overline{4}) = 1$ iff $2 + 2 = 4$. Moreover, because of $Tr(\ulcorner \overline{2} + \overline{2} = \overline{4} \urcorner) \in \Phi_{lf}$, $Tr(\ulcorner \overline{2} + \overline{2} = \overline{4} \urcorner)$ is true iff $Tr(\ulcorner \overline{2} + \overline{2} = \overline{4} \urcorner) \in \Gamma_{lf}$ iff $Val_{\Gamma_{lf}}(Tr(\ulcorner \overline{2} + \overline{2} = \overline{4} \urcorner)) = 1$ iff $\overline{2} + \overline{2} = \overline{4} \in \Gamma_{lf}$ iff $\overline{2} + \overline{2} = \overline{4}$ is true. (We have applied Definition 18, Theorem 17, and the definition of $Val_{\Gamma_{lf}}$.) But of course there are also much more complex sentences of $\mathcal{L}_{Tr}$ which may be shown to be members of $\Gamma_{lf} \subseteq \Phi_{lf}$: e.g., $Tr(\ulcorner \forall x(Sent_{\mathcal{L}}(x) \land Tr(x) \rightarrow \exists y(Tr(y) \land y = \dot{T}r(nam(x)))) \urcorner)$, which says that the very sentence is true that says about every true arithmetic sentence that there is a further true sentence which expresses the truth of the former, is true according to Definition 18 ($Sent_{\mathcal{L}}$ is an arithmetically definable predicate the extension of which is identical to $\mathcal{L}$; *nam* is an arithmetically definable function sign which denotes the mapping that assigns to n the code of the numeral $\overline{n}$ of n). On the other hand, the Liar sentence λ is entailed to be not true according to Definition 18 and the same holds for its negation; but that is alright since Definition 18 is claimed to be adequate just with respect to sentences which depend on non-semantic states of affairs.

So we have finally succeeded in finding what Horwich [20], p. 42, calls "principles governing our selection of excluded instances" of the T-scheme: we exclude all instantiations with sentences that do not depend on non-semantic states of affairs. This selection satisfies the first of Horwich's demands for such principles, i.e., it avoids "liar-type contradictions"; on the other hand, it is not so clear whether it also satisfies his other two postulates, i.e., that the set of excluded instances be as small as possible (compare Subsection 5.4) and that there be a constructive specification of the excluded instances that is as simple as possible (compare our discussion of the complexity of groundedness in Subsections 5.2 and 5.5).

Furthermore we have:

COROLLARY 19 (Φ_{lf} and truth). *For every $\varphi \in \Phi_{lf}$: φ is true or $\neg\varphi$ is true, but not both of them.*

Thus, Definition 18 settles for every sentence that depends on non-semantic states of affairs whether it is true or whether its negation is true. Γ_{lf} is the set of true sentences that depend on non-semantic states of affairs.

Theorem 17 can actually be strengthened in the following respect:

THEOREM 20 (Strengthening of Theorem 17). *For all ordinals* α, *for all* $\varphi \in \Phi_\alpha$: $Val_{\Gamma_\alpha}(Tr(\ulcorner \varphi \urcorner) \leftrightarrow \varphi) = 1$.

5. Discussion

Now we turn to a more thorough discussion of Sections 2, 3, and 4; in Subsection 5.5 we add some formal results.

5.1. *Dependence*

The upshot of Section 2 is that a notion of dependence is "hidden" within the usual semantic rules governing the semantics of a language such as $\mathcal{L}_{Tr}$. In Section 2, $\mathcal{L}_{Tr}$ is considered as being interpreted just with respect to its purely arithmetic part, i.e., with respect to $\mathcal{L}$, while the additional predicate *Tr* is regarded as uninterpreted and thus as a kind of variable. In this way, the semantic covariation between the sentences of $\mathcal{L}_{Tr}$ and the sets which they depend on can be studied. Since *Tr* is left uninterpreted and is a fortiori not yet regarded as expressing truth, our theory of dependence is not specifically a part of a theory of truth but its scope is more general; other applications of the theory are conceivable. The general idea is that those entities that are evaluated according to our semantic rules, i.e., *sentences*, are said to depend on (sets of) those entities that are contained in the intended universe of discourse. If the aims of this paper had been different ones, the latter entities might indeed have been chosen to be physical entities, or mental entities, or sets, or something else. The only aspect of Section 2 that is really specific to our later interpretation of *Tr* as being a truth predicate for a proper fragment of $\mathcal{L}_{Tr}$ is that the sentences of $\mathcal{L}_{Tr}$ are members of the intended domain of $\mathcal{L}_{Tr}$ *themselves* (modulo coding). In this way, a sentence φ may be said to depend not just on any set Φ of members of the domain but rather on a set of *sentences* and our central Definition 1 is restricted accordingly. For the same reason, the sentences which φ depends on may themselves be said to depend on further sentences and so forth. The output of the theory is therefore not just a one-level construction of sentences which depend on non-sentences, but a complex scaffold of dependency relationships between sentences and other sentences. As Section 3 has shown, if the domain of this dependence relation is restricted to the set of sentences that depend on non-semantic states of affairs, a well-founded hierarchy of dependency up to the ordinal level of α^* is determined. This is of course analogous to the situation in set theory, where the membership relation may either be regarded as "flat", such that only atoms or urelements are permissible as being members of sets, or where sets may themselves be members of sets as is now our standard understanding of set theory. In the latter case, if the domain of the epsilon relation is restricted to the class of well-founded sets, membership turns out to be arranged according to a similar ordinal system of levels (though, of course, of "unbounded height").[5] In that respect, the semantic dependence relation between sentences plays a similar role for our theory as the membership relation between sets does for set theory.

Let us now turn to those traits of our dependency relation that are specific for a *semantic* relation: its relata, i.e., sentences, are *about* something (while, e.g., sets are not "about" anything). Can we understand the notion of dependence introduced above as an elaboration of *what* the sentences of $\mathcal{L}_{Tr}$ are about, i.e.: may 'φ depends on Φ' be interpreted as 'φ is *about* the members of Φ'? Well, not precisely: first of all, if dependency is a form of "aboutness" at all, it is certainly aboutness in a particular respect only. E.g.: the sentence '$\ulcorner \overline{2}+\overline{2}=\overline{4} \urcorner$ is a sentence' is, or rather may be reworded as, a sentence of pure arithmetic and thus is a member of $\mathcal{L}$. '$\ulcorner \overline{2}+\overline{2}=\overline{4} \urcorner$ is a sentence' may furthermore be said to be about the sentence $\overline{2}+\overline{2}=\overline{4}$, in fact, about the manner in which $\overline{2}+\overline{2}=\overline{4}$ is classified syntactically. However, since it is a sentence of $\mathcal{L}$, it depends essentially on the empty set (cf. 1 of Example List 1). So what kind of aboutness, if any, might be expressed by means of 'φ depends on Φ'? If we reconsider the instances of dependence in Example List 1, we find that many of them match our intuitions concerning aboutness *with respect to truth-theoretic concerns*. Actually, this is not so suprising: only *Tr* is regarded as "variable" within our theory of dependence and consequently only dependence or aboutness concerning truth-theoretic properties can be expressed by 'φ depends on Φ'. If we restrict the theory of semantics for the moment just to the theory of truth, we might say that because '$\ulcorner \overline{2}+\overline{2}=\overline{4} \urcorner$ is a sentence' is not about $\overline{2}+\overline{2}=\overline{4}$ *in a semantic respect*, the former does not depend on the latter according to our theory. On the other hand, $Tr(c)$ is about *semantic properties* of the entity denoted by c and this is reflected by the truth of the corresponding dependency statement. The same holds in other cases such as $\forall x(P(x) \rightarrow Tr(x))$ and $\exists x(P(x) \wedge Tr(x))$ (with P in $\mathcal{L}$), which are about semantic properties of the members of the extension of P; accordingly, $\forall x Tr(x)$, $\exists x Tr(x)$, $\forall x(Tr(x) \rightarrow \neg Tr(\dot{\neg} x))$, $\exists x(Tr(x) \wedge Tr(\dot{\neg} x))$ are indeed about all sentences of $\mathcal{L}_{Tr}$ with respect to semantic concerns; the Liar λ and the Truthteller τ are about their own semantic properties; etc. (cf. Example List 1). So we see that aboutness concerning semantic properties is at least one aspect of our notion of dependency. However, dependence is not to be *identified* with this aspect, since there are other instances of dependence which undermine such an identification: e.g., $\lambda \vee \neg\lambda$, i.e., $\neg Tr(\ulcorner\lambda\urcorner) \vee \neg\neg Tr(\ulcorner\lambda\urcorner)$, may be said to be about the semantic properties of λ. However, $\lambda \vee \neg\lambda$ depends on the empty set again by 11 of Example List 1 (see also Lemma 5). Obviously, our pre-theoretic intuitions concerning aboutness are more fine-grained than any *semantic* notion of dependence such as ours can ever be. While our relation of dependency is closed under logical equivalence and even arithmetic equivalence (see (5) and (6) of Lemma 5), such that logically equivalent sentences depend on the same sets of sentences, this is not the case according to the intuitions described above: $\tau \vee \neg\tau$ may said to be about the semantic properties of τ and not of λ, despite the fact that $\lambda \vee \neg\lambda$ and $\tau \vee \neg\tau$ are logically equivalent; $\overline{2} + \overline{2} = \overline{4} \vee Tr(c)$ is about the entity denoted by c, or so it seems, while the arithmetically equivalent $\overline{2} + \overline{2} = \overline{4}$ is not, but both depend on the empty set. Contrary to our notion of dependency, the informal notion of aboutness is not just a matter of semantics, but also of syntax: what a sentence is about does not only depend on the proposition that the sentence expresses but also on *how* this proposition is expressed.[6] The relation of dependence as given by Definition 1 may thus – at best – be regarded as a formal *substitute* for the pre-theoretic relation of aboutness concerning semantic properties. Such a substitution, which certainly exceeds the boundaries of an explicatory definition, seems to be admissible in view of the difficulties of grasping the proper notion of aboutness since the latter is perhaps even inherently unclear (compare the related discussion concerning the notion of self-referentiality at the end of Section 3; for a general discussion on the problems of defining ‘about’ see Putnam [26], Goodman [12, 13]).[7]

Another possible understanding of dependence that should be distinguished from the one which is determined by Definition 1 is *dependence by compositionality*: e.g., a sentence $\varphi \wedge \psi$ may be said to depend on φ and ψ since its truth value is determined by the truth values of φ and ψ; in order to compute the truth value of $\varphi \wedge \psi$, we first have to compute the truth values of its conjuncts. Moreover, the truth value of $\varphi \wedge \psi$ supervenes on the truth values of φ and ψ in the sense that there is no difference with respect to the truth value of $\varphi \wedge \psi$ without a corresponding difference with respect to the truth value of either φ or ψ. But according to Definition 1, $Tr(\ulcorner\rho\urcorner) \wedge Tr(\ulcorner Tr(\ulcorner\rho\urcorner)\urcorner)$ depends on $\{\rho, Tr(\ulcorner\rho\urcorner)\}$, not on $\{Tr(\ulcorner\rho\urcorner), Tr(\ulcorner Tr(\ulcorner\rho\urcorner)\urcorner)\}$, i.e.: by applying D^{-1} to $Tr(\ulcorner\rho\urcorner) \wedge$

$Tr(\ulcorner Tr(\ulcorner \rho \urcorner)\urcorner)$ all outer occurrences of *Tr* are eliminated. So we see that $Tr(\ulcorner \rho \urcorner) \wedge Tr(\ulcorner Tr(\ulcorner \rho \urcorner)\urcorner)$ rather depends on what $Tr(\ulcorner \rho \urcorner)$ and $Tr(\ulcorner Tr(\ulcorner \rho \urcorner)\urcorner)$ depend on, not on $Tr(\ulcorner \rho \urcorner)$ and $Tr(\ulcorner Tr(\ulcorner \rho \urcorner)\urcorner)$ themselves. In that respect our theory of dependence is similar to Yablo's [32]: according to the latter, the so-called "fact" $\langle Tr(\ulcorner \rho \urcorner) \wedge Tr(\ulcorner Tr(\ulcorner \rho \urcorner)\urcorner), t\rangle$ depends on each member of some set S of facts, such that S contains $\langle \rho, t\rangle$ and $\langle Tr(\ulcorner \rho \urcorner), t\rangle$. Yablo's main intention is to thereby point out and analyze the dependency component of Kripke's [21] notion of grounding: "the dependence aspect is the one behind the attempt to picture grounding in terms of the *understanding* of 'true'. What do we do when we have to evaluate a sentence – say "The sentence 'Snow isn't white' is true or the sentence 'The sentence 'Snow is white' is true' is not true" – involving complicated attributions of truth? Evidently, we try to figure out what its truth-value *depends* on, and then what *that* depends on, and so on and so forth ..." (Yablo [32], p. 118). In order to clarify this notion of dependency, Yablo defines a family of binary dependence relations Δ between facts of the form $\langle \varphi, v\rangle$ with $v = t$ or $v = f$, such that the set of facts that $\langle \varphi, v\rangle$ depends on is a member of $\langle \varphi, v\rangle$'s so-called sufficiency set $\mathbb{S}(\langle \varphi, v\rangle)$ (which we assume to be non-empty for the sake of simplicity). $\mathbb{S}(\langle \varphi, v\rangle)$ is the set of sets S of facts that are sufficient for $\langle \varphi, v\rangle$ in the sense that φ has truth value v in the partial model in which S determines the extension of *Tr*. A dependence tree for a fact $\langle \varphi, v\rangle$ and for a dependence relation Δ is a particular graphical realization of Δ in which $\langle \varphi, v\rangle$ occurs at the top; φ is shown to be grounded in the Kripkean sense iff there is a dependence tree for $\langle \varphi, t\rangle$ or $\langle \varphi, f\rangle$, such that every branch that starts at the top of the tree terminates after finitely many steps.

Despite the similarity concerning what $Tr(\ulcorner \rho \urcorner) \wedge Tr(\ulcorner Tr(\ulcorner \rho \urcorner)\urcorner)$ depends on, our account of dependency differs from Yablo's in several respects. First of all, our dependency relation is a relation between sentences and sets of sentences, not between what Yablo calls "facts". Secondly, our notion of dependency is defined in terms of supervenience, not in terms of sufficiency of being true or false. Furthermore, Yablo's dependency is a form of partial dependence: e.g., given that $N(\ulcorner \varphi \urcorner)$ is a true sentence of the ground language, there is a dependence relation Δ, such that $\langle \exists x(N(x) \wedge \neg Tr(x)), t\rangle$ depends on $\langle \varphi, f\rangle$ *and nothing else*. On the other hand, according to our theory, $\exists x(N(x) \wedge \neg Tr(x))$ depends on the *total* extension of N and φ is just one member of this extension. Thus, the notion of total dependency is split by Yablo into a class of partial dependence relations. Moreover, the set of sets of facts that a given fact $\langle \varphi, v\rangle$ depends on is determined compositionally, i.e., by the syntax of φ and by the sufficiency sets associated with the subformulas of φ. E.g.,

$\mathbb{S}(\langle\varphi \vee \psi, t\rangle) = \mathbb{S}(\langle\varphi, t\rangle) \cup \mathbb{S}(\langle\psi, t\rangle)$ and $\mathbb{S}(\langle\varphi \wedge \psi, f\rangle) = \mathbb{S}(\langle\varphi, f\rangle) \cap \mathbb{S}(\langle\psi, f\rangle)$.[8] In contrast, the only operator in our theory that is vaguely analogous to Yablo's $\mathbb{S}$, i.e., D, is not compositional at all: $D(\lambda \vee \neg\lambda) = \wp(\mathcal{L}_{Tr})$ differs from $D(\lambda \vee \lambda) = \{\Psi \mid \lambda \in \Psi\}$, although $\lambda \vee \lambda$ results from replacing $\neg\lambda$ in $\lambda \vee \neg\lambda$ by λ, where $\neg\lambda$ and λ have the same D-value (since $D(\lambda) = D(\neg\lambda) = \{\Psi \mid \lambda \in \Psi\}$). Note that $\mathbb{S}(\lambda) \neq \mathbb{S}(\neg\lambda)$ in Yablo's theory according to which $\langle\lambda \vee \neg\lambda, t\rangle$ depends on $\langle\lambda, t\rangle$ or on $\langle\lambda, f\rangle$; the latter demonstrates again the compositional component of this notion of dependence. Finally, Yablo's dependence is also partial in the sense that it is based on a three-valued background semantics governed by the Strong Kleene scheme. This is mainly in order to adapt the dependency framework to Kripke's theory of truth, but some of Yablo's results really depend on the monotonicity of Kripke's jump operator for partial evaluations. However, other parts of Yablo's theory – in particular his Sections 4 and 6 – are also applicable in the case of a classical semantics and it would be interesting to know whether our notion of dependency can be somehow related to Yablo's in such a new setting. In any case, due to its compositionality, Yablo's notion of dependence is perhaps closer to the informal aboutness relation sketched above than our relation of dependence is, although also Yablo's dependency obeys a principle of closure under logical equivalence: if $\varphi, \psi \in \mathcal{L}_{Tr}$ are logically equivalent according to whatever semantic background scheme is used, $\langle\varphi, v\rangle$ and $\langle\psi, v\rangle$ have the same image under $\mathbb{S}$ (for arbitrary v).

Gupta employs a notion of semantic dependence which is restricted to quantifier-free sentences φ, ψ of $\mathcal{L}_{Tr}$ (see Gupta and Belnap [16], p. 111, and the slightly different account in Gupta [15], p. 197): φ immediately depends on ψ :iff either (i) $\varphi = \psi$, or (ii) ψ is a sentence that is denoted by some name c and $Tr(c)$ is a subformula of φ; (indirect) dependence is defined as the transitive closure of immediate dependence. As our examples of Example List 1 together with Lemma 5 show, this relation of immediate dependence often coincides with our notion of dependence in the sense that several instances of sentences which are members of the set that φ depends on essentially, are among the sentences that φ depends on immediately. However, $Tr(c)$ depends on $Tr(c)$ immediately also in the case where $Tr(c)$ is not a Truthteller and both $Tr(c) \vee \neg Tr(c)$ and $\neg(Tr(c) \vee \neg Tr(c))$ depend immediately on the sentence denoted by c (in case there is one), although the sentences are non-contingent; obviously, we are back to the discussion of aboutness and dependence above. Moreover, as we pointed out at the beginning of Section 2, we are not really in need of a formal notion of *indirect* dependence.

Gaifman [10] ([11] is the more recent version of the theory), Simmons [27], Bolander [3], and Cook [8] introduce further concepts of semantic dependence.[9] We are going to concentrate just on the first and the third[10]: they share with Yablo's account (i) a graph-theoretical approach according to which dependency graphs are defined where the nodes of these graphs are identified with certain linguistic items and where the edges of these graphs are intended to express some sort of dependency between these linguistic items, (ii) for every node that is identified with a complex linguistic item p there are edges which lead to subnodes that belong to linguistic subitems associated with p, (iii) a three-valued semantics is used in order to evaluate these linguistic items (the Strong Kleene scheme being the default choice). In each of these respects, our theory differs from Gaifman's and Bolander's (for the same reason why it differs from Yablo's theory). In the case of Gaifman, the linguistic items that figure as the nodes of dependency graphs[11] are not sentence types but sentence *tokens* which "point" to their sentence types. The truth predicate is not to be applied to names of sentence types but to the names of sentence tokens, accordingly. Gaifman's semantic rules assign the values T, F, *GAP* (for 'truth value *gap*') to an item in such a way that not only the values of its subitems are relevant but also its "position" in the dependency graph; for that reason Gaifman call his pointer semantics "essentially non-Tarskian" (Gaifman [10], p. 235). On the other hand, if p points to a disjunction $\varphi \vee \psi$, then there always two derived pointers which point to the disjuncts φ and ψ respectively, e.g., if p points to $Tr(r) \vee Tr(s)$, where r and s are now names for tokens, then p has two derived pointers $p1$ and $p2$ which point to $Tr(r)$ and $Tr(s)$ respectively. Hence the pointer network for these tokens includes an edge from p to $p1$ and another one from p to $p2$. We see that some kind of dependence compositionality is preserved in the sense that what a complex linguistic items depends on is still sensitive to what its subitems depend on; as we have already pointed out, this is not necessarily the case in our theory. As far as the assignment of truth values is concerned, Gaifman's choice of sentence tokens as the bearers of truth values together with his choice of semantic rules leads to a nice handling of the so-called "two-line puzzles": let p and q be distinct pointers which both point to $\neg Tr(p)$; then p is evaluated GAP while q is evaluated *true* (by Gaifman's "Jump Rule": see [10], p. 231). This is in contrast with our account in which both the Liar sentence λ and the sentence $Tr(c)$ with $\mathfrak{I}(c) = \lambda$ are not true according to Definition 18. We do not regret this consequence of our theory because our intention has only been to define truth in a classical framework for all those sentences that depend on non-semantic states of affairs; neither λ

nor $Tr(c)$ is among these sentences. Chapuis [7], p. 36f presents an example which is intended to show that Gaifman's semantic rules do not work properly in all cases: let p point to $Tr(r)$, q to $\neg Tr(r)$, and r to $(Tr(p) \wedge \neg Tr(q)) \vee (\neg Tr(p) \wedge Tr(q)) \vee (\neg Tr(p) \wedge \neg Tr(q))$ ("Gupta's Puzzle"). This gives rise to a graph that includes each of the three tokens in one closed loop. By Gaifman's "Closed Loop Rule" each of the tokens is assigned *GAP*. But intuitively, or so Chapuis argues, since $Tr(r)$ and $\neg Tr(r)$ contradict each other, only one of p and q can be true, therefore r is definitely true and so is also p. His conclusion is that the mere occurrence of a loop in the dependency graph should not preclude the assignment of classical truth values to the linguistic items that are involved. Let us compare this to our theory of semantic dependence: consider the sentences $Tr(r)$, $\neg Tr(r)$, and $(Tr(p) \wedge \neg Tr(q)) \vee (\neg Tr(p) \wedge Tr(q)) \vee (\neg Tr(p) \wedge \neg Tr(q))$ (call the last of the three sentences 'ρ' for short) with $\mathfrak{I}(r) = \rho$, $\mathfrak{I}(p) = Tr(r)$, and $\mathfrak{I}(q) = \neg Tr(r)$. It is easily seen that both $Tr(r)$ and $\neg Tr(r)$ depend on $\{\rho\}$, while ρ depends on $\{Tr(r), \neg Tr(r)\}$, i.e., in this case our theory yields precisely the same results as Gaifman's and thus Chapuis' point applies to both theories if it applies to any of them. If we had defined dependence in a way, such that 'φ depends on Φ iff for all *consistent* $\Psi \subseteq \mathcal{L}_{Tr}$: $Val_{\Psi}(\varphi) = Val_{\Psi \cap \Phi}(\varphi)$' would have been a corollary to that definition, then ρ would depend on the empty set and would in fact turn out to be true. But we did not want to build any truth-theoretic presumptions such as *the extension of Tr is consistent* into our notion of dependence; at the same time we are solely interested in a definition of truth for those sentences which depend on non-semantic states of affairs according to this very notion of dependence. (Compare Section 5.4 below for a related discussion.)

Bolander [3] refers to theories in theoretical computer science as his paragons.[12] Sentences figure as the nodes in his dependency graphs, edges are regarded as expressing dependence or reference. Every complex sentence "refers" to its subsentences, where in the case of quantified sentences arbitrary constant singular terms are used as a substitute for free variables: e.g., $\forall x(P(x) \rightarrow Tr(x))$ refers to all sentences of the form $P(\overline{n}) \rightarrow Tr(\overline{n})$ where $\overline{n}$ is an arbitrary numeral.[13] Sentences are evaluated on the basis of partial interpretations in accordance with the Strong Kleene scheme. Bolander introduces the following terminology (which we only sketch while omitting a detailed description of the underlying framework): a sentence is grounded iff it does not contain any open formula of the form $Tr(x)$ as a subformula; a sentence is regular iff there is no path in the dependency graph that leads from it to an ungrounded sentence; finally, a sentence is protected iff every expression of the form $Tr(x)$ that occurs in the sentence actually occurs as a part of an $\mathcal{L}_{Tr}$-subformula of the form

$\forall x(\varphi[x] \rightarrow \psi[x])$, where $\varphi[x]$ is an open formula over the vocabulary of $\mathcal{L}$ and where the extension of $\varphi[x]$ in the standard model of arithmetic is a set of regular sentences. Bolander proves that Robinson's arithmetic, i.e., an important subsystem of full first-order arithmetic, is consistent with an arbitrary set of T-biconditionals for *protected* sentences ([3], p. 135). There is an analogous result for our theory: if $\forall x(\varphi[x] \rightarrow \psi[x])$ is a sentence of $\mathcal{L}_{Tr}$, such that $\varphi[x]$ is an open formula over $\mathcal{L}$ and the extension of $\varphi[x]$ in the standard model of arithmetic is a set of sentences which depend on non-semantic states of affairs, then $\forall x(\varphi[x] \rightarrow \psi[x])$ depends itself on non-semantic states of affairs (Example 1 of Example List 2 is an instance); therefore the corresponding T-biconditional $Tr(\ulcorner \forall x(\varphi[x] \rightarrow \psi[x])\urcorner) \leftrightarrow \forall x(\varphi[x] \rightarrow \psi[x])$ follows from our Truth Definition 18 together with our metatheory. The same holds for all sentences of the form $\exists x(\varphi[x] \wedge \psi[x])$ for $\varphi[x]$ as stated before. It would be interesting to know more about this sort of correspondence between protectedness on the one hand and membership in Φ_{lf} on the other, in particular because protectedness is to a large extent a syntactic constraint on formulas while dependence on non-semantic states of affairs is a semantic one.

5.2. *Groundedness*

In Section 3 we have characterized the set Φ_{lf} of sentences that depend on non-semantic states of affairs as the limit of the sequence

$$\varnothing, D^{-1}(\varnothing), D^{-1}(D^{-1}(\varnothing)), \ldots$$

where $D^{-1}(\Phi)$ is the set of sentences that depend on Φ. We have furthermore suggested that Φ_{lf} might just as well be called the set of "grounded sentences". This is in line with Herzberger [18], who characterizes groundedness in terms of so-called "domains" of sentences: every (meaningful) sentence is assumed to be *about* a set of entities, its domain; e.g., the Liar sentence λ is about itself, thus its domain is the unit set whose sole member is λ itself. As Herzberger admits, "The general notion of a domain is more readily indicated than explicated, but the analysis to follow depends on no problematic cases, and ultimately proves independent of any particular explication of 'domain"' ([18], p. 148). On this basis, 'φ is groundless' can be defined as abbreviating 'φ is the first member of some infinite sequence of sentences, each of which belongs to the domain of its predecessor'.[14] Turning to our own theory again, we find that Herzberger's notion of a domain can indeed be explicated in terms of dependence, however, this explication is subject to some reservations: as we have seen in the last subsection, aboutness is just one aspect of dependency among others; furthermore, there is not always *the* distinguished set Φ that a sentence φ depends on.

In case of essential dependence, the domain of φ may be identified with the unique set that φ depends on essentially. However we already know that there are also sentences that do not depend on *any* set essentially (recall 14 of Example List 1). Given these constraints, Lemma 13 seems to be as close to Herzberger's definition of groundlessness as possible: if φ is the first member of some infinite sequence of sentences, each of which belongs to the very set that its predecessor depends on essentially (and given that such a set exists!), then φ is ungrounded. The other direction is not necessarily the case, contrary to Herzberger's intended definition.

For many authors, the significance of groundedness lies in what Herzberger calls the "Semantic Grounding Condition": "Any given sentence determines a statement only if it is grounded or is nonsemantic (in the sense of incorporating in purely referential position no semantic term)" ([18], p. 149). In our context, a corresponding claim would be: any given sentence of $\mathcal{L}_{Tr}$ determines a statement only if it is a member of Φ_{lf}. The latter condition includes, according to our definition, also all non-semantic sentences, i.e., the sentences of $\mathcal{L}$. The main argument in favour of the Semantic Grounding Condition is that it excludes sentences such as the Liar, the Truthteller, or the members of Yablo's sequence from the class of sentences that are admissible of being evaluated true or false. It might be argued that there are instances of self-referential, or, more generally, ungrounded sentences which we are actually inclined to assign a truth value to, e.g., general semantic principles such as $\forall x(Sent_{\mathcal{L}_{Tr}}(x) \rightarrow (Tr(\dot{\neg}x) \leftrightarrow \neg Tr(x)))$ or their negations. But even for such principles the question arises whether we do not actually think of their corresponding restrictions to the set of grounded sentences (or subsets thereof): the only plausible confirming instances of $\forall x(Sent_{\mathcal{L}_{Tr}}(x) \rightarrow (Tr(\dot{\neg}x) \leftrightarrow \neg Tr(x)))$ we can think of also seem to be confirming instances of $\forall x(Sent_{\mathcal{L}_{Tr}}(x) \wedge Grounded(x) \rightarrow (Tr(\dot{\neg}x) \leftrightarrow \neg Tr(x)))$ ($Sent_{\mathcal{L}_{Tr}}$ is an arithmetically definable predicate the extension of which is identical to $\mathcal{L}_{Tr}$). Note that the latter principle is derivable metatheoretically from our Definition 18 and our background theory (but note that *Grounded* cannot be expressed arithmetically; see the discussion below). The former one is not derivable and neither is its negation as may be seen from the fact that both are self-referential (compare 2 in Example List 2).

Is the Semantic Grounding Condition therefore a plausible semantic hypothesis? Let us consider this question more precisely: first of all, we understand 'φ determines a statement' just as 'φ is true or $\neg\varphi$ is true' (footnote 6 of Herzberger [18], p. 148, might be read as hinting at such an interpretation). Thus 'any given sentence of $\mathcal{L}_{Tr}$ determines a statement only if it is a member of Φ_{lf}' is equivalent to 'for all $\varphi \in \mathcal{L}_{Tr}$: φ is true or

$\neg\varphi$ is true only if φ is a member of Φ_{lf}'. This last version of the Semantic Grounding Condition is indeed derivable from Definition 18 and our theory of dependence and might even be strengthened into 'any given sentence of $\mathcal{L}_{Tr}$ determines a statement *if and only if* it is a member of Φ_{lf}'. Its status is that of a conceptual, though non-trivial theorem. However, can we also derive the *truth* of the Semantic Grounding Condition from our theory? This is not the case: 'φ is a member of Φ_{lf}' simply cannot be expressed synonymously in $\mathcal{L}_{Tr}$, since the latter lacks the necessary set-theoretic expressions. Of course, 'φ is a member of Φ_{lf}' can be replaced by the extensionally equivalent 'φ is true or $\neg\varphi$ is true' because of Corollary 19; the resulting principle 'for all $\varphi \in \mathcal{L}_{Tr}$: φ is true or $\neg\varphi$ is true only if φ is true or $\neg\varphi$ is true' can therefore be expressed as a sentence of $\mathcal{L}_{Tr}$, i.e., $\forall x(Sent_{\mathcal{L}_{Tr}}(x) \to (Tr(x) \vee Tr(\dot{\neg}x) \to Tr(x) \vee Tr(\dot{\neg}x)))$, and we can finally derive '$\forall x(Sent_{\mathcal{L}_{Tr}} \to (Tr(x) \vee Tr(\dot{\neg}x) \to Tr(x) \vee Tr(\dot{\neg}x)))$ is true' from Definition 18. But that is just because $\forall x(Sent_{\mathcal{L}_{Tr}} \to (Tr(x) \vee Tr(\dot{\neg}x) \to Tr(x) \vee Tr(\dot{\neg}x)))$ is a *logical* truth, which shows that it cannot be the adequate translation of the metalinguistic Semantic Grounding Condition into our object language $\mathcal{L}_{Tr}$.

Moreover, there is no arithmetic formula $\rho[x]$, s.t. Φ_{lf} is the extension of $\rho[x]$, since otherwise $\forall x(\rho[x] \to Tr(x) \vee Tr(\dot{\neg}x))$ could be shown to be both grounded (by 3 of Example List 1 and (1) of Lemma 14) and not grounded (by (6) of Lemma 14). More generally, as (9) of Lemma 14 tells us, there is no sentence of $\mathcal{L}_{Tr}$ which depends on the set of grounded sentences essentially, as the Semantic Grounding Condition might be expected to do (compare Herzberger's paradoxes of grounding in [18] and the corresponding set-theoretic fact that the class of grounded sets may be proved to be proper). We could also make use of the considerations on complexity in Subsection 5.5 in order to derive the same result. So we see that the Semantic Grounding Condition is a plausible semantic thesis, that it is indeed derivable from our theory, but that the sentence which expresses its truth cannot be derived so. Since the Semantic Grounding Condition cannot be formulated adequately as a sentence of $\mathcal{L}_{Tr}$, our theory also does not imply that the Semantic Grounding Condition determines a statement. But that does not undermine its status as being a plausible thesis of semantics. We are not looking for a materially adequate theory of truth for a semantically closed language in the sense that all consequences of the theory would also be derivable from the theory as being true. This paper has much more modest intentions: to characterize a plausible class of sentences with truth predicate for which Tarski's adequacy criterion can indeed be satisfied by a correct definition of truth in a standard first-order language. We do not deny that there are sentences of $\mathcal{L}_{Tr}$ which are not contained in this class (like λ and $\neg\lambda$) and we also do not deny the (trivial) fact that there are sentences beyond $\mathcal{L}_{Tr}$ which are not contained in this class; the Semantic Grounding Condition is among the latter.

Let us now turn to the most well-known and perhaps also most elaborate semantic theory of grounding that is to be found in the literature, i.e., Kripke's [21]. Kripke replaces Tarski's hierarchy of languages, metalanguages, metametalanguages, ... by a flexible hierarchy of levels building up the set of grounded sentences of $\mathcal{L}_{Tr}$. The extension of *Tr* is the least fixed point (we disregard other ones) of a set operator based on a monotonic scheme of three-valued logic; a sentence is grounded if and only if it is either itself a member of the extension of *Tr* or its negation is. As we have noted in the last subsection, Yablo [32] has characterized these grounded sentences as those for which there is a grounded dependence tree, where 'grounded sentence' is understood in the sense of Kripke and 'grounded dependence tree' is understood in the sense of Yablo. The basic idea that underlies this characterization is contained in the extension of our so far incomplete quotation of Yablo's on p. 25: "What do we do when we have to evaluate a sentence ... involving complicated attributions of truth? Evidently, we try to figure out what its truth-value *depends* on, and then what *that* depends on, and so on and so forth in the hope of eventually making our way down to sentences not containing 'true' which can be evaluated by conventional means." (Yablo [32], p. 118.) While precisely the same idea is behind our own theory of truth, the resulting sets of grounded sentences differ: partly because the notions of dependence differ, partly because of the different semantic frameworks. Since the set of grounded sentences may be reconstructed uniquely from the set of *true* grounded sentences both in Kripke's and in our account, we postpone the comparison of the two sets of grounded sentences to the subsequent subsection, where we focus on grounded truth.

5.3. *Grounded Truth*

Γ_{lf} has been defined to be the limit of the sequence $(\Gamma_\alpha)_{\alpha \in Ord}$ defined in Section 4; a sentence of $\mathcal{L}_{Tr}$ is true according to Definition 18 if and only if it is a member of Γ_{lf}. As far as the sentences which depend on non-semantic states of affairs are concerned, choosing Γ_{lf} as the extension of 'true' has been proved to be adequate by Theorem 17. We simply do not care about the sentences "outside" of Φ_{lf}; we are satisfied with narrowing 'true-in-$\mathcal{L}_{Tr}$' to 'true-in-$\mathcal{L}_{Tr}$ and grounded' since that leaves us with a plausible set of true sentences that is still much broader than the extension of 'true-in-$\mathcal{L}$'. A different option would have to been to turn Definition 18 into a conditional definition of the form 'if $\varphi \in \Phi_{lf}$ then ...', but then deriving adequacy from Theorem 17 would have been problematic.

Theorem 17 shows that defining truth in terms of Γ_{lf} is materially adequate with respect to all sentences which depend on non-semantic states of affairs, or, as we might also say, that Definition 18 is dependence-adequate in the following sense:

A definition of truth for an object language $\mathcal{L}_o$ within a metalanguage $\mathcal{L}_m$ is dependence-adequate iff it implies all $\mathcal{L}_m$-biconditionals of the form

$$Tr(s) \leftrightarrow A$$

where 's' is replaced by a singular term t in $\mathcal{L}_m$, s.t. t denotes a sentence $\varphi \in \mathcal{L}_o$, *where* φ *depends on non-semantic states of affairs* and where 'A' is replaced by φ.

In our case, $\mathcal{L}_o = \mathcal{L}_{Tr}$; note that if an object language $\mathcal{L}_o$ does not contain semantic terms, e.g., if $\mathcal{L}_o = \mathcal{L}$, the criterion simply coincides with Tarski's original one.

Γ_{lf} is related set-theoretically to Kripke's sets of true grounded sentences according to the Strong Kleene and to the Supervaluation scheme as follows:

THEOREM 21 (Grounded truth compared).

(1) *Kripke's least fixed point E_∞ of true grounded sentences according to the Strong Kleene scheme is set-theoretically incomparable to Γ_{lf}. Thus, also Φ_{lf} is set-theoretically incomparable to the set $E_\infty \cup \neg E_\infty$ of sentences which are grounded according to the Strong Kleene scheme* (*recall that for $\Phi \subseteq \mathcal{L}_{Tr}$, $\neg\Phi := \{\varphi \in \mathcal{L}_{Tr} \mid \neg\varphi \in \Phi\}$*).

(2) *Kripke's least fixed point E'_∞ of true grounded sentences according to the Supervaluation scheme* (*in the version of Cantini* [6]) *is a proper superset of Γ_{lf}. Φ_{lf} is a proper subset of the set $E'_\infty \cup \neg E'_\infty$ of sentences which are grounded according to the Supervaluation scheme. The iteration of the supervaluation operator starting with the initial set Γ_{lf} converges to E'_∞; moreover: $\Gamma_{lf} = \Phi_{lf} \cap E'_\infty \subsetneq E'_\infty$.*

Proof. (1) is proved easily (but we presuppose acquaintance with E_∞): the logical truth $Tr(\ulcorner\lambda\urcorner) \vee \neg Tr(\ulcorner\lambda\urcorner)$ is certainly a member of Γ_{lf} but not of E_∞, and the same holds for the non-logical truth $Tr(\ulcorner\lambda \vee \neg\lambda\urcorner)$. On the other hand, $Tr(\ulcorner\overline{2} + \overline{2} = \overline{4}\urcorner) \vee \lambda$ is a member of E_∞ but not of Γ_{lf}. The latter may be seen by showing that $Tr(\ulcorner\overline{2} + \overline{2} = \overline{4}\urcorner) \vee \lambda \notin \Phi_{lf}$: (i) obviously $Tr(\ulcorner\overline{2} + \overline{2} = \overline{4}\urcorner) \vee \lambda$ depends on $\{\overline{2} + \overline{2} = \overline{4}, \lambda\}$, since for every $\Phi \subseteq \mathcal{L}_{Tr}$: $Val_\Phi(Tr(\ulcorner\overline{2} + \overline{2} = \overline{4}\urcorner) \vee \lambda) = Val_{\Phi\cap\{\overline{2}+\overline{2}=\overline{4},\lambda\}}(Tr(\ulcorner\overline{2} + \overline{2} = \overline{4}\urcorner) \vee \lambda)$ and we can apply Lemma 2. (ii) But $Tr(\ulcorner\overline{2} + \overline{2} = \overline{4}\urcorner) \vee \lambda$

neither depends on $\{\overline{2}+\overline{2}=\overline{4}\}$, since $Val_{\{\lambda\}}(Tr(\ulcorner\overline{2}+\overline{2}=\overline{4}\urcorner)\vee\lambda)\neq Val_{\{\lambda\}\cap\{\overline{2}+\overline{2}=\overline{4}\}}(Tr(\ulcorner\overline{2}+\overline{2}=\overline{4}\urcorner)\vee\lambda)$, nor on $\{\lambda\}$, since $Val_{\{\overline{2}+\overline{2}=\overline{4},\lambda\}}(Tr(\ulcorner\overline{2}+\overline{2}=\overline{4}\urcorner)\vee\lambda)\neq Val_{\{\overline{2}+\overline{2}=\overline{4},\lambda\}\cap\{\lambda\}}(Tr(\ulcorner\overline{2}+\overline{2}=\overline{4}\urcorner)\vee\lambda)$, nor on the empty set, since $Val_{\{\lambda\}}(Tr(\ulcorner\overline{2}+\overline{2}=\overline{4}\urcorner)\vee\lambda)\neq Val_{\{\lambda\}\cap\varnothing}(Tr(\ulcorner\overline{2}+\overline{2}=\overline{4}\urcorner)\vee\lambda)$. Thus $Tr(\ulcorner\overline{2}+\overline{2}=\overline{4}\urcorner)\vee\lambda$ depends on $\{\overline{2}+\overline{2}=\overline{4},\lambda\}$ essentially. Because of that and because $\lambda\notin\Phi_{lf}$ implies that $\{\overline{2}+\overline{2}=\overline{4},\lambda\}\nsubseteq\Phi_{lf}$, $Tr(\ulcorner\overline{2}+\overline{2}=\overline{4}\urcorner)\vee\lambda$ does not depend on Φ_{lf} and so we finally have that $Tr(\ulcorner\overline{2}+\overline{2}=\overline{4}\urcorner)\vee\lambda\notin\Phi_{lf}$ by Corollary 11. The claim concerning Φ_{lf} and $E_\infty\cup\neg E_\infty$ follows from taking into account the negations of these examples.

So we turn to (2): let E'_∞ be the least supervaluation fixed point as defined by Cantini [6], p. 250, i.e., E'_∞ is the least fixed point of the operator $FV:\wp(\mathcal{L}_{Tr})\to\wp(\mathcal{L}_{Tr})$, s.t.

$$FV(\Phi):=\{\varphi\in\mathcal{L}_{Tr}\mid \text{ for all consistent } \Psi\supseteq\Phi\text{: } Val_\Psi(\varphi)=1\}.$$

Since E'_∞ is a fixed point of FV,

$$E'_\infty=\{\varphi\in\mathcal{L}_{Tr}\mid \text{ for all consistent } \Psi\supseteq E'_\infty\text{: } Val_\Psi(\varphi)=1\}.$$

- First we show that $\Gamma_{lf}\subseteq FV(\Gamma_{lf})$:
  Let $\varphi\in\Gamma_{lf}$. Assume that $\varphi\notin FV(\Gamma_{lf})$: then there is a consistent extension Ψ of Γ_{lf}, s.t. $Val_\Psi(\varphi)=0$. Since $\varphi\in\Gamma_{lf}$ also $\varphi\in\Phi_{lf}$ and we conclude that $Val_\Psi(\varphi)=Val_{\Psi\cap\Phi_{lf}}(\varphi)$. But $\Psi\cap\Phi_{lf}=\Gamma_{lf}$: ("$\subseteq$") If $\psi\in\Psi\cap\Phi_{lf}$, then $\psi\in\Phi_{lf}$ and thus either ψ or $\neg\psi$ is a member of Γ_{lf} by Corollary 19. In the latter case, Ψ would have to contain both $\neg\psi$ and ψ, contradicting its consistency; thus $\psi\in\Gamma_{lf}$. ("$\supseteq$") If $\psi\in\Gamma_{lf}$, then $\psi\in\Psi\cap\Phi_{lf}$ by our assumption on Ψ and $\Gamma_{lf}\subseteq\Phi_{lf}$. So we see that $Val_\Psi(\varphi)=Val_{\Gamma_{lf}}(\varphi)$, i.e., since $Val_\Psi(\varphi)=0$ also $Val_{\Gamma_{lf}}(\varphi)=0$, which contradicts $\varphi\in\Gamma_{lf}$ by Theorem 17. Therefore $\Gamma_{lf}\subseteq FV(\Gamma_{lf})$. (Note that $\Gamma_\alpha\subseteq FV(\Gamma_\alpha)$ follows analogously for all ordinals α, just by replacing 'Γ_{lf}' by 'Γ_α' and 'Φ_{lf}' by 'Φ_α'.)
- But $FV(\Gamma_{lf})\nsubseteq\Gamma_{lf}$, since $Tr(\ulcorner\overline{2}+\overline{2}=\overline{4}\urcorner)\vee\lambda$ is a member of $FV(\Gamma_{lf})$ but not of Φ_{lf} and thus it is also not a member of Γ_{lf} (therefore Γ_{lf} is also not identical to E'_∞). Note that for the same purpose we could have considered, e.g., $\forall x(Sent_{\mathcal{L}_{Tr}}(x)\to(Tr(x)\to\neg Tr(\dot{\neg}x)))\in\mathcal{L}_{Tr}$.
- The latter two items imply that $\Gamma_{lf}\neq FV(\Gamma_{lf})$ and thus $\Gamma_{lf}\subsetneq FV(\Gamma_{lf})$.
- $D^{-1}(E'_\infty\cup\neg E'_\infty)\subseteq E'_\infty\cup\neg E'_\infty$:
  for assume that $\varphi\in D^{-1}(E'_\infty\cup\neg E'_\infty)$ though $\varphi\notin E'_\infty\cup\neg E'_\infty$. Thus there is a consistent $\Psi\supseteq E'_\infty$, s.t. $Val_\Psi(\varphi)=0$, and there is a consistent $\Pi\supseteq E'_\infty$, s.t. $Val_\Pi(\neg\varphi)=0$, i.e., $Val_\Pi(\varphi)=1$. Since φ depends on $E'_\infty\cup\neg E'_\infty$, $Val_\Psi(\varphi)=Val_{\Psi\cap(E'_\infty\cup\neg E'_\infty)}(\varphi)=0$ and

$Val_{\Pi}(\varphi) = Val_{\Pi\cap(E'_{\infty}\cup\neg E'_{\infty})}(\varphi) = 1$. But because Ψ and Π are consistent supersets of E'_{∞} and E'_{∞} is consistent, $\Psi \cap (E'_{\infty} \cup \neg E'_{\infty}) = E'_{\infty}$ and $\Pi \cap (E'_{\infty} \cup \neg E'_{\infty}) = E'_{\infty}$, so we have a contradiction.

- Therefore $\{\varphi \in D^{-1}(E'_{\infty} \cup \neg E'_{\infty}) \mid Val_{E'_{\infty}}(\varphi) = 1\} \subseteq E'_{\infty}$:
  $Val_{E'_{\infty}}(\varphi) = 1$ contradicts $\varphi \in \neg E'_{\infty}$, because otherwise $\neg\varphi \in E'_{\infty} = \{\psi \in \mathcal{L}_{Tr} \mid$ for all consistent $\Psi \supseteq E'_{\infty}$: $Val_{\Psi}(\psi) = 1\}$ and therefore $Val_{E'_{\infty}}(\neg\varphi) = 1$.
- Furthermore, $\Phi_{lf} \subseteq E'_{\infty} \cup \neg E'_{\infty}$:
  since $D^{-1}(E'_{\infty} \cup \neg E'_{\infty}) \subseteq E'_{\infty} \cup \neg E'_{\infty}$ by what we have seen before, and since D^{-1} is monotonic, it is a well-know fact that transfinite iteration of D^{-1} on $E'_{\infty} \cup \neg E'_{\infty}$ leads to a fixed point of D^{-1} which is a subset of $E'_{\infty} \cup \neg E'_{\infty}$; but Φ_{lf} is the least such fixed point.
- $\Phi_{lf} \neq E'_{\infty} \cup \neg E'_{\infty}$ and thus $\Phi_{lf} \subsetneq E'_{\infty} \cup \neg E'_{\infty}$:
  $\Phi_{lf} = E'_{\infty} \cup \neg E'_{\infty}$ is not the case because of $Tr(\ulcorner\overline{2}+\overline{2} = \overline{4}\urcorner) \vee \lambda \notin \Phi_{lf}$, whereas $Tr(\ulcorner\overline{2}+\overline{2} = \overline{4}\urcorner) \vee \lambda \in E'_{\infty} \subseteq E'_{\infty} \cup \neg E'_{\infty}$.
- This in turn entails that $\Gamma_{lf} \cap \neg E'_{\infty} = \varnothing$ and $\Gamma_{lf} \subsetneq E'_{\infty}$ because:
  at first, $\Gamma_{lf} \subseteq \Phi_{lf} \subseteq E'_{\infty} \cup \neg E'_{\infty}$ by what we have seen before. Now if $\Gamma_{lf} \cap \neg E'_{\infty} \neq \varnothing$, there is a least successor $\alpha+1$, s.t. $\Gamma_{\alpha+1} \cap \neg E'_{\infty} \neq \varnothing$ but $\Gamma_{\alpha} \cap \neg E'_{\infty} = \varnothing$ (note that $\Gamma_0 \cap \neg E'_{\infty} = \varnothing \cap \neg E'_{\infty} = \varnothing$). Let $\varphi \in \Gamma_{\alpha+1} \cap \neg E'_{\infty}$: thus on the one hand $Val_{\Gamma_{\alpha}}(\varphi) = 1$ by definition of $\Gamma_{\alpha+1}$ and on the other hand for all consistent supersets Ψ of E'_{∞}: $Val_{\Psi}(\varphi) = 0$; in particular, $Val_{E'_{\infty}}(\varphi) = 0$. But since $\Gamma_{\alpha} \cap \neg E'_{\infty} = \varnothing$ and $\Gamma_{\alpha} \subseteq \Gamma_{lf} \subseteq \Phi_{lf} \subseteq E'_{\infty} \cup \neg E'_{\infty}$ from before, we have $\Gamma_{\alpha} \subseteq E'_{\infty}$. Hence also $\Gamma_{\alpha} = E'_{\infty} \cap \Phi_{\alpha}$, for: left to right is clear now; if $\psi \in E'_{\infty} \cap \Phi_{\alpha}$ and $\psi \notin \Gamma_{\alpha}$ then by Corollary 19 $\neg\psi \in \Gamma_{\alpha} \subseteq E'_{\infty}$ and therefore both ψ and $\neg\psi$ are members of E'_{∞} which contradicts the consistency of E'_{∞}. So we can continue: since $\varphi \in \Gamma_{\alpha+1} \subseteq \Phi_{\alpha+1} = D^{-1}(\Phi_{\alpha})$, φ depends on Φ_{α}, and it follows that $0 = Val_{E'_{\infty}}(\varphi) = Val_{E'_{\infty}\cap\Phi_{\alpha}}(\varphi) = Val_{\Gamma_{\alpha}}(\varphi)$ contradicting $Val_{\Gamma_{\alpha}}(\varphi) = 1$ above.
  Thus we find that $\Gamma_{lf} \cap \neg E'_{\infty} = \varnothing$ and therefore we also have $\Gamma_{lf} \subseteq E'_{\infty}$. Because $\Phi_{lf} \neq E'_{\infty} \cup \neg E'_{\infty}$ we know that $\Gamma_{lf} \neq E'_{\infty}$, and therefore $\Gamma_{lf} \subsetneq E'_{\infty}$.
- Because of $\Gamma_{lf} \subsetneq E'_{\infty}$ and the monotonicity of FV, E'_{∞} is the fixed point that is generated by transfinite iteration of FV on the basis of Γ_{lf}.
- $\Gamma_{lf} = \Phi_{lf} \cap E'_{\infty}$:
  Left to right follows from what we have just proved. ("$\supseteq$") If $\varphi \in \Phi_{lf} \cap E'_{\infty}$, then because of $\varphi \in \Phi_{lf}$ and Corollary 19 either $\varphi \in \Gamma_{lf}$ or $\neg\varphi \in \Gamma_{lf}$. But in the latter case, $\neg\varphi \in \Gamma_{lf} \subseteq E'_{\infty}$ contradicting again the consistency of E'_{∞}. Thus $\varphi \in \Gamma_{lf}$ is the case. □

So we see that Γ_{lf} is a proper subset of E'_∞, Φ_{lf} is a proper subset of $E'_\infty \cup \neg E'_\infty$ and the true sentences that depend on non-semantic states of affairs are precisely those "super-true" sentences that depend on non-semantic states of affairs. Note that the members of Γ_{lf} are therefore also stably true over all (Belnap-)revision sequences as implied by Burgess [5], Proposition 10.2b. Cantini's [6] axiom system VF is correct with respect to the Supervaluation scheme: if we consider the more convenient version of VF stated in Halbach [17], p. 182, we see that several of the axioms of VF are indeed derivable in our theory, though not all of them. E.g., the truth-analogue of the modal axiom scheme K is not derivable: $Tr(\ulcorner\overline{2}+\overline{2}=\overline{4}\urcorner) \in \Gamma_{lf}$, $Tr(\ulcorner\overline{2}+\overline{2}=\overline{4}\urcorner) \rightarrow (Tr(\ulcorner\overline{2}+\overline{2}=\overline{4}\urcorner) \vee \lambda) \in \Gamma_{lf}$, but $Tr(\ulcorner\overline{2}+\overline{2}=\overline{4}\urcorner) \vee \lambda \notin \Gamma_{lf}$.

5.4. *A Weakness of the Theory?*

In the proof of Theorem 21 we have made use of the fact that the set Γ_{lf} of true sentences which depend on non-semantic states of affairs is rather restrictive in a certain respect, which is in turn a consequence of the fact that already Φ_{lf} is restrictive: let $\varphi \in \mathcal{L}$, such that $Val(\varphi) = 1$ (in the proof of Theorem 21 we have set $\varphi := \overline{2}+\overline{2}=\overline{4}$). Intuitively, $Tr(\ulcorner\varphi\urcorner) \vee \lambda$, where λ is the Liar again, might be thought of as being a member of Γ_{lf}, but in fact it is not, since $Tr(\ulcorner\varphi\urcorner) \vee \lambda \notin \Phi_{lf}$ (the proof is analogous to the one for $\varphi = \overline{2}+\overline{2}=\overline{4}$). Accordingly, also $Tr(\ulcorner Tr(\ulcorner\varphi\urcorner)\urcorner) \vee \lambda \notin \Phi_{lf}$, etc. On the other hand, $\varphi \vee \lambda$ is indeed a member of Γ_{lf} and thus of Φ_{lf}.

One may reply to this either in a positive or in a negative manner: viewed positively, the truth values of $Tr(\ulcorner\varphi\urcorner) \vee \lambda$, $Tr(\ulcorner Tr(\ulcorner\varphi\urcorner)\urcorner) \vee \lambda, \ldots$, i.e., of $Tr(\ulcorner\varphi\urcorner) \vee \neg Tr(\ulcorner\lambda\urcorner)$, $Tr(\ulcorner Tr(\ulcorner\varphi\urcorner)\urcorner) \vee \neg Tr(\ulcorner\lambda\urcorner), \ldots$ simply depend on whether the Liar is a member of the extension of Tr or not; thus they ought to be excluded from the set Φ_{lf} of sentences that depend on non-semantic states of affairs *only*. Moreover, given that φ is a true member of the ground language, and if looked at from a purely *semantic* viewpoint, the sentence $\varphi \vee \lambda$ is indistinguishable from φ; since the latter depends on non-semantic states of affairs, the same must be true of the former. Accordingly, sets such as $E'_\infty \cup \neg E'_\infty$ or E'_∞ above are overly inclusive in having sentences as their members which do not only depend on non-semantic states of affairs.

The negative reaction would be: the reason why $Tr(\ulcorner\varphi\urcorner) \vee \lambda$ is considered not to be grounded is that the truth value of one of its components depends on its own presence in the extension of Tr. The reason why we would nevertheless like to count $Tr(\ulcorner\varphi\urcorner) \vee \lambda$ as true is that $Tr(\ulcorner\varphi\urcorner)$ is indeed grounded and a member of Γ_{lf}, and therefore the truth value of $Tr(\ulcorner\varphi\urcorner) \vee \lambda$ is not "really" dependent any longer on the whether the Liar is a member of the extension of *Tr* or not. Put differently: $Tr(\ulcorner\varphi\urcorner) \vee \lambda$ is *not* dependent on the Liar *given* that φ is contained in the extension of *Tr*; one might thus say that $Tr(\ulcorner\varphi\urcorner) \vee \lambda$ is *conditionally* grounded under the latter assumption and that is also why it should not be excluded from the extension of *Tr*. Consequently, our sets Φ_{lf} and Γ_{lf} are overly restrictive in not allowing for members such as $Tr(\ulcorner\varphi\urcorner) \vee \lambda$.

The latter reasoning leads us to a possible refinement of our theory of semantic dependence by a notion of conditional dependence:

For $\varphi \in \mathcal{L}_{Tr}$, for all $\Sigma, \Phi \subseteq \mathcal{L}_{Tr}$:

DEFINITION 22 (Conditional dependence). φ depends on Φ given Σ :iff for all $\Psi_1, \Psi_2 \subseteq \mathcal{L}_{Tr}$ with $\Sigma \subseteq \Psi_1, \Psi_2$: if $Val_{\Psi_1}(\varphi) \neq Val_{\Psi_2}(\varphi)$ then $\Psi_1 \cap \Phi \neq \Psi_2 \cap \Phi$.

By Definition 22, dependence simpliciter corresponds precisely to conditional dependence given the empty set.

As far as we can see, a theory that is similar to the one of plain dependence can be developed also for conditional dependence. The analogues of the sequences (Φ_α) and (Γ_α) now have to be determined by simultaneous recursion, such that $\Phi_{\alpha+1}$ is defined as the set of sentences that depend on Φ_α *given* Γ_α. We postpone the elaboration of this theory to another paper.

5.5. *Some Further Technical Issues*

Finally, let us collect some results concerning formal complexity and ordinal number issues related to our theory.

First of all, D^{-1}, Φ_{lf}, Γ_{lf} are Π^1_1 and $\alpha^* \leqslant \omega_1^{CK}$: $\varphi \in D^{-1}(\Phi)$ iff for all Ψ: $(T(\varphi, \Psi) \wedge T(\varphi, \Psi \cap \Phi)) \vee (\neg T(\varphi, \Psi) \wedge \neg T(\varphi, \Psi \cap \Phi))$ where '$T(\varphi, \Psi)$' expresses that $Val_\Psi(\varphi) = 1$; but '$T(\varphi, \Psi)$' is Δ^1_1, as is well-known. Thus, by the usual complexity considerations, also $(T(\varphi, \Psi) \wedge T(\varphi, \Psi \cap \Phi)) \vee (\neg T(\varphi, \Psi) \wedge \neg T(\varphi, \Psi \cap \Phi))$ is Δ^1_1, and therefore it can be written down in the form that it starts with universal second-order quantifiers which are followed by an elementary arithmetic formula. In this way we see that '$\varphi \in D^{-1}(\Phi)$' is equivalent to a formula given by a sequence of universal second-order quantifiers followed by an elementary formula in the arithmetic language, i.e., it is Π^1_1. Since additionally D^{-1} is monotonic, we can apply Spector's theorem (compare Moschovakis [24], p. 25) and get: Φ_{lf}, which is identical to Moschovakis' $I_{D^{-1}}$, is Π^1_1 and hence inductive by Kleene's well-known theorem. Furthermore, by Spector's theorem, the closure ordinal of D^{-1} is smaller-than-equals the least non-recursive ordinal ω_1^{CK}. By (2) of Theorem 21, $\varphi \in \Gamma_{lf}$ iff $\varphi \in \Phi_{lf}$ and $\varphi \in E'_\infty$. Therefore, since Φ_{lf} is Π^1_1 and E'_∞ is Π^1_1 (see Burgess [5]), we conclude that also Γ_{lf} is Π^1_1.

As Philip D. Welch has demonstrated to us in personal communication, D^{-1} is even Δ^1_1, i.e., hyperarithmetic; Φ_{lf}, Γ_{lf} are not Δ^1_1, in fact, Φ_{lf} is complete Π^1_1; from these facts he derives that α^* is identical to ω_1^{CK} (that Φ_{lf} and Γ_{lf} are not arithmetic follows already from the reasoning on p. 32; the fact concerning ω_1^{CK} may also be seen from constructing a progression of sentences in $\mathcal{L}_{Tr}$, where each sentence depends on the ones earlier in the progression and where the whole progression has length ω_1^{CK}). Thus, our theory of truth resembles Kripke's with respect to complexity and fixed point ordinals.

ACKNOWLEDGMENT

We would like to thank V. Halbach, A. Hieke, L. Horsten, P. D. Welch, and an anonymous referee for various enlightening comments on the paper.

NOTES

[1] The phrase 'φ depends on Φ' is certainly a technical one and may be said to go slightly against natural language. In particular, it is actually an abbreviation of '*the truth value of* φ depends on Φ' which in turn could and perhaps should be replaced by 'Φ determines the truth value of φ' (as has been remarked by an anonymous referee). We hope the latter is excusable in view of 'x depends on y' and 'y determines x' being synonymous in many linguistic contexts.

[2] Therefore, also sentences which express properties of the arithmetically definable reference relation between singular terms and natural numbers count as *non-semantical*. Our reason for doing so is precisely the fact that reference is definable within pure arithmetic which we have chosen to be our "ground language".

[3] This might be called a notion of *essential self-referentiality*.

[4] We want to thank an anonymous referee for pointing this out clearly.

[5] Here we look at sets from the viewpoint of ZFC *without* the axiom of foundation (or regularity). If the latter is added, the set-theoretic universe is restricted to well-founded sets right from the start.

[6] When we say 'proposition', we think of sets of possible worlds in the line of Carnap, not structured propositions having individuals or attributes as their components.

[7] An anonymous referee has pointed out to us that the informal notion of aboutness does perhaps not only depend on syntax and semantics but also on pragmatics, i.e., context, awareness of speaker and so forth.

[8] $\vee$ is actually not a primitive sign of Yablo's [32] vocabulary, but it might be introduced in a standard manner.

[9] As far as the latter two authors are concerned, we thank an anonymous referee for calling our attention to them.

[10] The four of them share certain features; in particular, they all use versions of "dependency networks". Bolander briefly compares his account to Cook's in [3], p. 119.

[11] Gaifman [10] himself does not use the terms 'dependency graph' and 'dependence' but rather says that a pointer *calls* (*directly*) another pointer when he wants to express that the former depends on the latter. I follow Bolander [3] in presenting Gaifman's approach as one that employs a "dependency graph"-theoretical framework.

[12] Dependency graphs for linguistic items have, e.g., been used in logic programming; see Apt and Bol [1].

[13] Bolander [3] actually uses a predicate K the interpretation of which is left open; it might express knowledge or belief rather than truth. In the case of truth, Bolander's so-called "reflection principle" for K is the just the T-scheme $Tr(\ulcorner \ldots \urcorner) \leftrightarrow \ldots$; T-biconditionals are thus just the instances of his reflection principle for Tr (see [3], p. 67f).

[14] In Herzberger [18] the notion of groundlessness is defined differently, however, the inadequacy of the original definition is realized and corrected by Herzberger's erratum [19], where it is replaced by the definition we have stated.

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