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ORIGINAL RESEARCH

Properties of propositional attitude operators

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Abstract
A simple model accounting for semantic properties of propositional attitude operators in negative contexts with no reference to possible worlds is proposed. Verbs occurring in such operators denote relations between individuals and specific sets of sentences (of a given natural language) and their negation is defined as the complement within a specific set of cognitively determined sentences. This approach avoids in particular the problem of intensionality of propositional attitude operators and allows to use many tools from the generalised quantifier theory. In that way the negation giving rise to factive presuppositions and to the neg-raising is defined in a natural way.

Keywords Propositional attitude verbs · Factivity · Middle point operators neg-transportability

1 Introduction

Many aspects of the semantics of verbs of propositional attitude and logical properties of operators formed from them remain mysterious and far from being well understood, in spite of the huge body of research. In this paper I want to discuss something which, as far as I can tell, has not been explicitly or extensively discussed in connection with verbs of propositional attitude: these are, roughly speaking, behaviour of propositional attitude verbs in negative contexts and semantic properties to which give rise various negations applied to verbs of propositional attitude or to their sentential arguments. Thus I will discuss in particular properties that lead to the well-known relation of presupposition of factive verbs and the lesser known, but related relation of intensional equivalence of propositional attitude verbs. I will also use the same tools to study the entailment associated with the so-called *neg-raising* or *neg-transportability* of certain

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verbs, which are also verbs of propositional attitude. Since in all these cases verbal negation is essentially involved, it will be shown that it is possible to use one notion of negation in order to define precisely the factive presupposition, neg-transportability and intensional equivalence. In addition it will be shown that the same notion of negation can be used to define in a natural way various classical logical properties such as consistency, completeness and duality. The importance of specific knowledge that individuals referred to by grammatical subjects of these verbs should have will be pointed out.

Two warnings are needed at the outset. First, though I will discuss some semantic properties of sentential operators formed from verbs of propositional attitude, I will try to avoid, as much as possible, the discussion of the lexical semantics of verbs forming such operators. This important part of their semantic description has many sides and is probably not separable from various pragmatic aspects. In addition the class of verbs which can be said to express propositional, or more generally, mental, attitudes is very large and it includes additionally quite naturally many adjectival predicates. Current literature on the clausal complementation and sentence embedding often takes into account these additional problems and provides fine descriptions of lexical properties of these items (cf. a.o. Anand and Hacquard 2014; Ciardelli et al. 2019; Egre 2008; Horn 1989; Mayr 2019; Uegaki 2019). I am interested here in classifying propositional attitude verb meanings according to their logical properties and consequently examples that will be given should be considered as only illustrating these properties.

The second warning concerns the type of objects or the category of expressions between which the relations to be discussed hold. Traditionally the relation of presupposition, or more generally of entailment, is considered as a relation between sentences. Since we are interested in logical properties of propositional operators and the relations between them it is preferable to talk about sentential operators that presuppose or entail and not about sentences in such a relation. Consequently I will use the generalised notion of entailment so that it applies to sentential or propositional operators. This is possible given the rule of type raising and in this case the entailment will correspond to the partial order of the Boolean algebra in which propositional operators are interpreted. Thus, given that sentential operators are of category S/S we will say in particular that expressions of category S/S presuppose expressions of the same category. For instance we will say that factives are operators F (of category S/S) such that F (cross-categorially) entails T and $not - F$ entails T, where T is the truth operators meaning *It is true that*. As we will see a precise statement of such definitions is possible also for other relations that we will discuss.

The propositional (or *sentential*) operators I will analyse are of the form A V *that/whether* where A is a proper name and V a verb of propositional attitudes. Thus the subject NP of the propositional operators refers to an individual. I will ignore the fact that such verbs can take quantified noun phrases as grammatical subjects. One of the reasons for this simplification is the fact that the negation of propositional operators with quantified subject NPs is in general more complex and is not related to the negation *sensu strictu* of the verb forming such an operator.

In spite of the fact that the propositional operators contain subject noun phrases (referring to an individual human being) I will treat such operators as syntactically and semantically in-decomposable wholes. The class of all sentential operators will be denoted by SO. They are of category S/S: they form sentences when applied to a sentence. This means that a sentential operator contains a complementizer. A proper sub-set of SO is the set PA, the set of propositional attitude operators. They differ from other SOs by the fact that verbs forming them are verbs of propositional attitude. A semantic property distinguishing PA operators from other SO operators will be given.

The following notation will be used from now on. Sentential operators will be noted O, O_1, O_2. Sentential operators formed from verbs whose NP subject denotes the human being a will be noted O_a or Q_a. Various "classical" operators will be noted in the standard way.

2 Factives, intensional equivalence and neg-raising

In this section I recall, mainly for the illustrative purposes, some semantic properties of propositional operators and entailments between sentences formed from such operators.

There are at least three cases of entailments under (negated or non-negated) intensional operators which are interesting from the semantic point of view. The first is the case of factive predicates presupposing their sentential complements, the second, that I will call *intensional equivalence*, is for instance, the entailment between two similar sentences with the same bare propositional operator in both sentences are differing only by the complementizer (*that* vs *whether*). Finally there is a case of "neg-transportation" entailments to which some intensional operators, called *neg-raising* operators, give rise. These operators have the property that when a negation applies to them (on the surface) it acts as a negation of the lower clause, embedded in the neg-raising predicate.

I start with presuppositions. Given the classical notion of presupposition and the transitivity of entailment if S presupposes T and T entails V then S presupposes V (where S, T and V are sentences). A presupposition which is not entailed by any other presupposition (of the same sentence) will be called *a maximal presupposition* and a presupposition which is entailed by another presupposition will be called *a secondary presupposition*. The class of constructions which illustrate well the relation of presupposition arising in the context of PA operators is represented by so-called *factives*. Since the publication of Kiparsky and Kiparsky (1970) (see also Fillmore 1965) the term *factive* refers in linguistic semantics to verbs and predicates like *to know, to forget, to regret, be sad, be strange*, etc. when they take sentential complements and the complementizer *that*. These predicates have an interesting semantic property: when completed by the grammatical subject referring to a human (which is usually the most natural case for them) they form sentential operators which presuppose the truth of the sentential complement that they take as argument. More specifically (positive forms of) sentences formed from factive predicates and their (natural) negations both entail the complement sentence. For instance (1a) and (1b) both seem to entail, on the most natural reading, that sentence P is true:

(1a) Leo knows/remembers that P
(1b) Leo does not know/not remember that P

Consequently we will consider that *Leo knows that* and *Leo remembers that* are factive PA operators, that is PA operators which presuppose their sentential argument.

It is important to keep in mind that factives represent a new set of presupposing constructions in comparison with those traditionally studied in the context of definite descriptions for instance. These new constructions show that, as in the case of existential presuppositions, the notion of negation needed in the definition of presupposition must be different from classical negation. This is because, as has often been pointed out, the sentence *Leo does not know that P* can be considered as true in the case when P is false (cf. *Leo does not know that P because not-P*).

Kiparsky and Kiparsky (1970) distinguish, in part by syntactic means, two types of factive predicates: non-emotive and emotive ones. The first class includes verbs like *know, remember, realise* and to emotive factives belong verbs like *regret, resent* or predicates like *be sad, be strange, be interesting*, etc., when they take impersonal subjects. It is not clear to what extent the syntax alone is sufficient to distinguish clearly emotive from non-emotive factive predicates. In fact as we will see in the moment it is not clear whether there are emotive factives which take as subject NPs denoting human beings.

Concerning cognitive, or non-emotive, factives one observes that indeed they presuppose their complement sentence. However, one can also notice that there are very few non-emotive factives which can be considered as giving rise to "direct" presuppositions; most of such factives have their sentential complement as secondary presupposition. To see this let us look some examples. The verbs *remember* and *forget* are usually considered as non-emotive factives. However they are logically related to *know*: (2a) roughly means (2b) and (3a) means (3b). Furthermore, (2a) entails (2c) and (3a) entails (3c)

(2a) Bo remembers that $2 + 2 = 4$.
(2b) Bo continues to know that $2 + 2 = 4$.
(2c) Bo knows that $2 + 2 = 4$.
(3a) Bo forgot that $2 + 2 = 4$.
(3b) Bo does not know anymore that $2 + 2 = 4$.
(3c) Bo knew that $2 + 2 = 4$.

The above examples show that the meaning of non-emotive factives is non-trivially related to knowledge. Many other classically discussed cognitive factives are related to the predicate *know (that)* in a similar way. For instance *learn* is *come to know*, *prove* is *make known is a specific way*, *disclose* is make publicly known, *discover* - come to know in a specific way. In addition, as is well known, factivity depends on tense, aspect and modality. The above examples illustrate the fact that often the factive presupposition is a secondary presupposition, since it is an entailment of a maximal presupposition due to the verb *to know that*. Thus for many factive verbs there exists a "paraphrase" in which the "new" presupposition inducers such as *make, become*, aspectual verbs (*stop, continue, begin*), achievement verbs or temporal focus particles (*still, yet, not anymore*) occur and these presupposition inducers relate factivity presupposition to knowledge. These facts should be accounted for in any semantic description of factives and more generally, of PA operators.

As is well known, the negation giving rise to presuppositions cannot be interpreted by the classical propositional negation. For instance (4a) can be true and (4b) false in the situation in which Leo knows that not-P is true:

(4a) It is not true that Leo knows that P.
(4b) Leo knows whether P.

It does not seem that there are empirical grounds to consider that there are so-called emotive factives taking "ordinary", not impersonal, NP subjects. For instance, contrary to what is claimed in Zuber (1975/1977), the verb *regret* does not entail the knowledge since (5) is not contradictory. For this reason I will follow the observation made in Klein (1975) in reaction to Zuber (1975/1977), and developed in Schlenker (2005), that verbs like *regret* do not entail knowledge but entail belief: (6a) entails (6b):

(5) Bo does not know that life is sad but regrets that life is sad.
(6a) Bo regrets that life is sad.
(6b) Bo believes that life is sad.

There might be an additional complication with *regret* or its passive form, especially when used with impersonal subjects. I will leave this problem aside and consider that the truth the sentential complement of the verb *regret* is not its secondary presupposition.

To illustrate the notion of intensional equivalence I use the fact that many factive verbs can take two complements, *that* and *whether*, and constructions with such verbs differing just by the complementizer used, are in specific semantic relations to each other. Moreover only constructions in which the complementizer *that* occurs give rise to presuppositions.

Take for instance *know that* and *know whether*. The semantic relation between these two operators is indicated in (7a). Then, given that we accept bivalence and excluded middle, *to know whether* can be defined by *to know that* in the way indicated in (7b):

(7a) A knows whether P iff if P is true then A knows that P and if P is false then A knows that not P.
(7b) A knows whether P iff A knows that P or A knows that not P

It follows from (7a) and (7b) and from the fact that *know that* is a factive verb, that *know that* is intensionally equivalent to *know whether*, that is that the negation of *know that* entails the negation of *know whether*. To see this consider the following example:

(8a) Leo knows that life is sad.
(8b) Leo knows whether life is sad.
(9a) Leo does not know that life is sad.
(9b) Leo does not know whether life is sad.

One observes that (8a) entails (8b) and (9a) entails (9b). Indeed, if (9a) is true then by supposition its complement is also true and thus, given (7b) the sentence (9b) cannot be false. This means that (8a) is intensionally equivalent to (8b). Similar examples can be constructed with *remember* and *forget* but not with *say* or *tell*. It is important to keep in mind that the entailment from (9a) to (9b) holds only if the negation in (9a) is taken as, roughly speaking, the presupposition preserving negation.

The last case of intensional sentence-embedding predicates giving rise to specific entailments concerns the so-called *neg-raising* operators. I will be only interested in their semantic properties. These predicates have the following property: when negated sentences with such predicates imply a corresponding sentence in which the negation takes scope in the embedded clause. Thus, intuitively, (10a) entails (10b):

(10a) Leo does not think that life is sad.
(10b) Leo thinks that life is not sad.

The phenomenon illustrated by the above example has also played an important role in linguistic discussions. Various observations and proposals concerning the exact status of the relation between (10a) and (10b) and its representation in grammar have been made (Horn 1989; Collins and Postal 2018). I will consider that in the above example we have an intensional operator, formed from the verb of propositional attitude *to think*. This is not the case in general: some neg-transportable operators are not formed from verbs of propositional attitude. For instance the operator *It is true (false) that* is neg-transportable but is not a propositional attitude operator.

Notice that the negation in (10a) cannot be easily interpreted as just the metalinguistic or classical propositional negation. Indeed it is easy to see that (11) can be true (for instance when Leo is not in a state of thinking) and both (10a) and (10b), false:

(11) It is not true that Leo thinks that life is sad.

However, (10a) can be said to entail (11) and thus the negation in (10a) is stronger than the classical propositional negation since this negation applies to a normally intensional operator. We can thus consider that in (10a) we have an instance of attitude or intensionality preserving negation.

Clearly, entailments concerning presuppositions, intensional equivalence and neg-transportation, essentially involve negation. Moreover the cases we are considering involve intensional functional expressions (sentential operators). This means that in defining negation we have to take into account precisely that it has to apply to intensional sentential operators (functional expressions) and not to sentences. So first we have to make clear the notion of intensionality of such sentential operators and then define the appropriate negation of them. This is done in the next section.

3 Intensionality and propositional attitude operators

As is well known, PA operators give rise to strong intensionality which in particular makes it impossible to apply to their analysis tools from formal semantics compatible with the principle of compositionality (Pelletier 1994). In addition, Partee (1982) indicates that the full semantic description of verbs of propositional attitude, given their intensionality, makes it impossible to have a finitely representable semantics for natural languages. Since intensionality of propositional attitude operators will play an essential role in the analysis I propose, I start with examples illustrating the phenomenon in question. Examples (12a) and (12b) typically illustrate the intensionality of SOs formed from *know, believe* and *regret*:

(12a) Bo believes/knows/regrets that Brutus participated in the murder of Caesar.
(12b) Bo believes/knows/regrets that the oldest son of Sevilla participated in the murder of Caesar.

It is considered as historical truth that one of the participants of Caesar's killing was Brutus and that Brutus was the first son of Sevilla. However, these (very) specific (historical) facts are very likely not known by everybody and in particular Bo may be not aware of them: in other words Bo may not know that (13a) and (13b) are both true:

(13a) Brutus participated in the murder of Caesar.
(13b) The oldest son of Sevilla participated in the murder of Caesar.

Observe that a similar reasoning holds for rogative predicates, that is predicate taking only the complementizer *whether*: (14a) and (14b) may differ in their truth value:

(14a) Bo wonders whether Brutus participated in the murder of Caesar.
(14b) Bo wonders whether the oldest son of Sevilla participated in the murder of Caesar.

We will thus consider that the intensionality of PA predicates that will be studied here, is due to specific ignorance of the subject of such operators. Moreover I will consider that the above reasoning applies not only to PA verbs such as *know* or *believe* but also to many other PA verbs.

Intensionality of some PA operators necessitates some clarifications concerning the reading of sentences occurring in the scope of such predicates. We know that in such situations embedded sentences can have a "non literal" or non *de dicto* reading. For instance (15) in addition to the trivial reading given in (16a) has also a non-trivial reading in which the complement sentence is not interpreted as a contradiction. This reading is roughly indicated in (16b):

(15) Bo believes that Leo is older than he (Leo) is.
(16a) Bo believes that the sentence *Leo is older than he (Leo) is* is true.
(16b) Leo is of age A and Bo believes that Leo's age is higher than A.

In my proposal the above fact will be explicitly taken into account since, being interested in logical relations between PA predicates we will consider only *de dicto* readings, such as in (16a) of sentences embedded under PA operators.

It will be assumed that the intensionality of PA operators analysed is due to the specific ignorance of agents denoted by proper nouns forming such operators. The idea is as follows: a given situation in the world or a state of affairs can be expressed by different linguistically well-formed strings. In other words the world can be presented in different though equivalent ways and such different presentations of the world need not to be known to the agent involved in the PA operators.

More should be said about intensionality and its source. In particular, given that the agent has at his/her disposal an infinite number of expressions (their length is not limited) I assume implicitly that the number of possible ways of describing an event or a fact is also infinite and for this reason the agent cannot grasp all such possibilities. Hence the intensionality of PA predicates. I do not claim that ignorance is the unique source of intensionality PA operators but only that is is a sufficient one.

In logical approaches to the semantics of PA operators, which are basically predicates expressing knowledge and belief, various idealising assumptions concerning the logical competence of the agent, the subject NP of the PA operators, are made. Since I consider that the semantics of PA verbs belongs to the semantics of natural language, generalisations concerning logical competence of agents cannot be too strong and should be generally acceptable. In particular we cannot assume that agents are omniscient and that their attitudes such as knowledge, belief, regrets and so on, are closed under logical consequence. In fact the opposite will be assumed here: the fact that PA operators are intensional and that their intensionality is unavoidable should be related to the non-omniscience of human agents forming PA operators. Moreover, as we will see it is useful and empirically justified to consider that PA predicates have a negation which preserves the intensionality of the operator to which it applies and thus preserves the ignorance of the human agent forming the operator. As we will see such a negation is necessary to define neg-raising (neg-transportability) and factivity.

I will define the attitudinal character of *PA* operators and of their negation with the help of an abstract principle concerning the awareness by the agent of the attitude that she/he has. This principle echoes various "iteration principles" that are used in epistemic and modal logics. For instance the so-called KK principle says that for any sentence S if one knows that S (is true) then one knows that one knows that S. I will assume it and, moreover, I will assume that a similar principle that I call $ASAP$ (*attitudinal self-awareness principle*) holds with respect to any PA operator (cf. Zuber 2021):

ASAP: If any agent has a specific propositional attitude then he/she knows that he/she has this attitude.

The following examples illustrate more precisely what I mean by the ASAP. It seems to me that it is quite natural to suppose that (17a) entails (17b), when taken in *de dicto* readings:

(17a) Bo believes/knows/regrets/understands that life is sad.
(17b) Bo knows that she (Bo) believes/knows/regrets/understands that life is sad.

Similar entailments do not hold for verbs which intuitively are not verbs of propositional attitude: (18a) does not entail (18b) and (19a) does not entail (19b):

(18a) Bo left early.
(18b) Bo knows that she left early.
(19a) Bo proved successfully that S.
(19b) Bo knows that she proved successfully that S.

It follows from the above examples and from ASAP that we consider only propositional attitudes expressed by words or expressible by words and not implicit attitudes revealed only in behaviour of an agent. In addition, since we will also consider negative propositional attitudes, that is those expressed by (properly) negated propositional operators, I do not see how such "negative" attitudes could be revealed in (non linguistic) behaviour.

The condition expressed by ASAP belongs to logical, and more generally linguistic, competence of the agent having a specific propositional attitude. It is the characteristic property of PA operators. It can be considered as a generalisation of the KK principle frequently used in epistemic logic, even if occasionally considered controversial.

The set of true sentences will be noted T and any subset of T will be called a *veridical operator*. The set of all veridical operators will be denoted by VER. The set of sentences that the agent a knows to be true will be noted K_a and the set of sentences of which a knows whether they are true or not will be noted KW_a. If O is a sentential operator then $S \in O$ means that sentence formed from O and having S as sentential (propositional) argument is true. For instance $S \in T$ means *It is true that S* and $S \in K_a$ means that *a knows that S is true*. Moreover $S \in O$ and $S \notin O$ will be considered as sentences. The distinction between variables in the object language and metalanguage will often be intentionally not respected. In particular the distinction between an *operator* and its *denotation* will often be ignored.

4 Properties of propositional attitude operators

The universe from which we construct the denotational algebra of sentential operators is composed of (declarative) sentences of a given, fixed, natural language. Strictly speaking, not all sentences will belong to the universe of our model. We will exclude sentences with free variables, with indexicals and sentences with externally bound pronouns. In addition, as indicated below, some sentences will be identified. For instance, as we will see, we will identity sentences with an even number of negations according to the rule specified below in D4. Thus denotations of sentential operators will be constructed from elements of the power set algebra of this "simplified" set of sentences of a given natural language. If Σ is such a "simplified" set of sentences then $\wp(\Sigma)$ can be considered as a Boolean algebra whose elements are sets of sentences and Boolean operations correspond to operations on sets. of sentences. The unit of this algebra is the set Σ itself and the zero element is the empty set. Entailment corresponds to set inclusion between sets of sentences.

The sentential operators will be interpreted as sets of sentences, sub-sets of the universe given by the "simplified" set of sentences. Such an interpretation is compatible with the category S/S that propositional operators have. Informally, one can indeed consider that the propositional operator *It is true that* denotes the set of true sentences and the sentential operator *It is false that* denotes the set of false sentence. The reason is that (linguistic) objects of category S/S can be lifted and become of category $S/(S/S)$. I will accept this possibility and consider that propositional operators denote sets of sentences, taken independently of their truth-value.

Recall that SO denotes a set of sentential operators (that is expressions of category S/S and thus the set of PA operators is a sub-set of SO. Given ASAP we can define a PA operator as follows (Zuber 2021):

D 1: A sentential operator O_a is a *propositional attitude operator*, in short $O_a \in PA$, iff $\forall_{S\in\Sigma}(S \in O_a) \rightarrow (S \in O_a) \in K_a \wedge (S \notin O_a) \rightarrow (S \notin O_a) \in K_a$

Thus a propositional operator based on the agent a is a set of sentences such that for any sentence S, a knows whether S belongs to it or not. This means that *PA* operators are sentential operators whose members are determined by a specific knowledge of the agent on which they are based. Less formally, the agent has to be aware that she/he has the attitude expressed by the main verb of the operator. Since the operators such as *a wrote that* and *a proved that* do not have the property indicated in definition D1, they are not *PA* operators.

Since we suppose that no agent is omniscient, no PA operator will contain all sentences logically equivalent to any of its members.

Using the notion of a *PA* operator we define now the set U_a which will be considered as the "empirical universe" of sentences associated with the agent a and from which our model for various operations on *PA* operators will be constructed. By definition:

D2: $U_a = \{S : S \in \Sigma \wedge \forall_{O_a \in PA} KW_a(S \in O_a)$

The operator U_a corresponds to the set of (positive and negative) sentences with respect to which the agent a has any propositional attitude. We will consider that it corresponds to the set of sentences that a understands. Since we suppose that there are also sentences that a may not understand, any operation on PA operators has to be relativised to the set U_a (for any agent a). In particular, as we will see, the negation of PA operators will be relativised to the set U_a.

We have seen that intensionality of *PA* operators based on the agent a is also related to the knowledge of a. In that sense intensionality of PA operators is different from the intensionality of modals or intensionality induced by some adverbials, for instance. this type of intensionality will be called *normal intensionality*:

D3: O_a is *normally intensional*, $O_a \in NI$, iff $O_a \neq \emptyset$ and $\forall_{S \in U_a}$ if $O_a(S)$ is true then $\exists_{S_0 \in U_a}(S \equiv S_0)$ and $O_a(S_0)$ is false.

A typical, and in some sense basic, NI operator is the operator K_a, for any existing agent a. The reason is that normal intensionality is a consequence of a "strong non-omniscience" of human agents: informally it implies that if an agent knows the truth of a given sentence then there is always another sentence with the same logical value as a given sentence and such that the agent is not aware of its truth value.

More interestingly, given that PA operators are defined with the help of the operator K, we have:

Proposition 1 *Any PA operator is normally intensional.*

Proof Suppose, *a contrario* that $O_a \notin NI$. Then, by definition D3, there exists a sentence $S \in U_a$ such that $O_a(S)$ is true and for all sentences $S_0 \in U_a$ with the same truth value as S we would have $O_a(S_0)$ is true. But this is impossible because agent a is not omniscient.

We distinguish negated and non-negated sentences. Negated sentences are sentences which are of the form *It is not true that* S, where S is a sentence. The negation of S will be noted nS and the set of negated sentences will be noted NS. There is obvious semantic dependence between S and nS: S is true iff nS is false. We will also need a function corresponding to the deletion of the negation of a negated sentence. Thus:

D4: Let $S \in NS$. Then $(n)S = S_0$ if $S = nS_0$ and nS otherwise.

Thus $(n)S$ is the sentence obtained from a negated sentence by deleting its negation: $(n)(nS) = S$.

In addition to negated sentences we will also occasionally use complex sentences with other Boolean connectives. Thus if S_1 and S_2 are sentences then S_1 *or* S_2 is a sentence. The semantics of such complex sentences is induced in a standard way by the semantics of the corresponding Boolean connector.

The rule indicated in D4 is a rule which allows us to identify, roughly speaking, two sentences which differ just by an even number of negations. A similar rule could be given to identify two sentences which contain a conjunction or a disjunction and which differ just by the order of arguments of the conjunction or of the disjunction. I will ignore such rules since they do not matter for the basic argument of this paper.

The negation nS of S in D4 corresponds to the syntactic negation. We also need to define semantic negations. Since propositional operators are sets of Boolean objects they have two negations, the Boolean complement and the post-complement which corresponds to the set of their negated elements. More precisely:

D5: Let O be a sentential operator. Then $\neg O$, the Boolean complement of O is defined as $\neg O = \{S : S \notin O\}$. The post-complement of O, noted $O\neg$, is defined as $O\neg = \{nS : S \in O \wedge S \notin NS\} \cup \{(n)S : nS \in O\}$.

Informally, we get the post-negation of an element by replacing all sentences which belong to it by their negations. Thus the notion of post-complement corresponds to the negation of the sentential argument of sentential operators. It will be used to determine various properties of propositional operators. For instance the relationship between the operators K_a and KW_a can be expressed as $KW_a = K_a \cup K_a\neg$.

It is easy to verify that the post-complement of Booleanly complex propositional operators has the following properties:

Fact 1: Let O_1 and O_2 be sentential operators. Then:

(i) $(O_1 \cap O_2)\neg = O_1\neg \cap O_2\neg$
(ii) $(O_1 \cup O_2)\neg = O_1\neg \cup O_2\neg$
(iii) $(O\neg)\neg = O$
(iv) $O_1 \subseteq O_2$ iff $O_1\neg \subseteq O_2\neg$

The attitudinal character of PA operators is preserved by the post-complement:

Fact 2: $O_a \in PA$ iff $O_a\neg \in PA$.

The reason is that *a knows whether S* iff *a knows whether nS*.

The Boolean complement does not need to preserve the attitudinal character of PA operators. Neither does the Boolean complement preserve intensionality nor presupposition, as we have seen. So we need a specific negation of PA operators. Syntactically this negation should be a modifier of propositional operators: it should take a propositional operator as argument and give a propositional operator as result. Consequently it should be of category $(S/S)/(S/S)$. Semantically, we want such a negation, in addition to invert, roughly speaking, the truth values, to preserve the attitudinal force of the operator to which it applies. Using the ASAP principle and definition D2 we have the following definition of the negation $\sim$ of a PA operator O_a:

D6: $\sim O_a = \neg O_a \cap U_a$.

The negation defined in D6 will be called PA preserving negation. Indeed, given that $\sim O_a \subseteq U_a$, if $O_a \in PA$ then $\sim O_a \in PA$. Thus the negation defined in D6 preserves the attitudinal character of the operator to which it applies. For this reason we will consider that the ASAP principle applies to negated as well as to non-negated operators.

Observe that the negation given in D6 is stronger than "ordinary" negation corresponding to the Boolean complement. The reason is that we want it to preserve the attitudinal aspect of the operator which the ordinary negation does not. For instance it may be true that agent a does not believe that some S holds because a may be ignorant of some facts, as in the case of the intensionality illustrated above but we would not say that in this case a doubts that S. With propositional attitude preserving negation of *believe* we do want to say that *a does not believe that S* entails *a doubts that S*.

Since PA negation is attitude preserving negation Proposition 1 indicates that PA negation is also a negation that preserves normal intensionality of the operator to which it applies.

Let us see some properties of PA negation. For simplicity all *PA* operators will be based an a fixed agent a and consequently O_a, Q_a, O_1, O_2, etc. denote *PA* operators all based on the agent a.

First, PA preserving negation gives rise to what can be called *weak intensional contraposition*, as indicated in:

Fact 3: If $O_a \subseteq Q_a$ then $\sim Q_a \subseteq\sim O_a$.

Second, the PA preserving negation does not commute with post-negation. In the generalized quantifier theory the order of application of the Boolean negation and of the post-negation does not matter since for any quantifier Q we have $\neg(Q\neg) = (\neg Q)\neg$. In the case of PA operators this holds only within the universe U_a, as shown by Fact 4:

Fact 4: $U_a \cap \sim (O_a\neg) = U_a \cap (\sim O_a)\neg$.

More importantly, PA preserving negation can be used to define formally various semantic relations discussed above. First we define a presupposition:

D7: Let O_1 be a PA operator and O_2 be a propositional operator. Then O_1 presupposes O_2 iff $O_1 \subseteq O_2$ and $\sim O_1 \subseteq O_2$.

Definition D7 is of course a version of the classical definition of presupposition.

The following facts are obvious given the definition of a *PA* operator and of its (presupposition preserving) negation:

Fact 5: Any PA operator O_a presupposes $O_a \cup \sim O_a$.
Fact 6: If O_1 presupposes O_2 and $O_2 \subseteq O_3$ then O_1 presupposes O_3.

Thus any PA operator O_a has a *presupposed part*, the set $O_a \cup \sim O_a$. This set will be called *presupposed part* of O_a and denoted by $PP(O_a)$.

The important point is that $O_a \cup \sim O_a \neq T$ and thus the presupposition indicated in Fact 5 (or in Fact 6) is not trivial. Given Fact 4 we have to conclude that (20a) presupposes (20b) and from Fact 4 it follows that (21a) presupposes (21b):

(20a) Bo regrets that life is sad.

(20b) Bo regrets that life is sad or Bo does not regret that life is sad.
(21a) Bo regrets/believes/doubts/knows that it is raining.
(21b) Bo knows whether she regrets/believes/doubts/knows that it is raining.

Moreover, $PP(O_a)$ is the maximal presupposition of O_a, as indicated in Fact 7:

Fact 7: O_a presupposes Q_a iff $PP(O_a) \subseteq Q_a$.

Factive operators, or *factives*, are specific presupposing PA operators:

D8: A PA operator O_a is factive, $O_a \in FACT$, iff O_a presupposes T.

The following fact indicates that the attitude preserving negation also preserves factivity:

Fact 8: O_a is factive iff $\sim O_a$ is factive.

A more interesting property of presupposing operators is indicated in:

Proposition 2 *Any veridical PA operator is factive.*

I indicate a proof based on Zuber (1980). Let O_a be a veridical propositional attitude operator. We have to show that $\sim O_a \subseteq T$. Suppose *a contrario* that this is not the case, that is that there exists an S such that $S \in \sim O_a$ and $S \notin T$. Given Definition D8 and Proposition 1 $\sim O_a$ is normally intensional. This means that there exist a sentence S_0 which is false (because S is false) and such that $S_0 \notin \sim O_a$. But then, given the definition D6 of attitudinal negation we have $S_0 \in O_a$. Contradiction, since O_a is veridical.

A property similar to Proposition 2 has been established in Zuber (1980) and Zuber (1982) using somewhat different tools. It can be interpreted as a strong constraint on the set of natural language PA operators: if a propositional operator is not factive but is veridical it cannot be a propositional attitude operator and thus is not normally intensional. For instance *It is true that* and *It is necessary that* are veridical but they cannot be considered as PA operators (they do not take a human subject NP) and the operators like *Bo behaved in such a way that*, *Bo was right (by saying, by writing) that* and *Bo (successfully) proved that* are veridical but are not an attitude expressing ones and thus are not factive. For instance from *Bo is right that S* it does not follow that *Bo knows whether she is right (or not) that S*. Consequently *Bo is right that* does not satisfy the definition of PA operators. Similarly with the operators formed from *truly believe* or *correctly predict*. Though these operators are veridical they are not normally intensional and thus they are not factive.

The following proposition indicates the relation between knowledge and factivity:

Proposition 3 *If $O_a \in PA$ and $O_a \subseteq K_a$ or $O_a \subseteq\sim K_a$ then O_a is factive.*

Proof Observe that if $O_a \subseteq K_a$ then O_a is veridical and thus, given Proposition 2, O_a is factive. Similarly, if $O_a \subseteq\sim K_a$ then O_a is veridical and thus factive.

Proposition 3 can be illustrated by the well-known examples of factive verbs: *know that, realise that*, *reveal that* and *remember that* entail *know that* and thus propositional operators formed from them are factive. Similarly *forget, not remember* and *not know* entail *not know* and thus are factive verbs. An interesting question is whether there are factives which are not related to knowledge in the way indicated in Proposition 3.

Factivity relates some propositional operators to truth. It is also clear that factive operators are veridical ones. A more general class of sentential operators related to the truth of its elements is a sub-set of operators that I will call, following Westerståhl (2012), who comments on some work of Keenan on type $\langle 1 \rangle$ quantifiers, *midpoint* operators (MP operators, for short). Midpoint quantifiers are quantifiers denoted by NPs like *exactly half (of) the, some but not all CN*, etc. In a sentence containing such NPs in the subject position it is possible to negate the VP without changing the truth-value of the sentence. For instance *Exactly half the students danced* and *Exactly half the students did not dance* are logically equivalent. In Zuber (2007) this property is called *constancy on complements*.

Midpoint SOs are defined as follows:

D9: A sentential operator O is a midpoint operator, $O \in MP$, iff $O = O\neg$.

It follows from the definition of the post-complement that MP operators can be defined as in Fact 9:

Fact 9: $O \in MP$ iff $\forall_S (S \in O) \equiv (nS \in O)$.

The definitional property of MP operators is similar to the definition of type $\langle 1 \rangle$ midpoint quantifiers. Consequently it should not be surprising that MP operators have other properties similar to those of midpoint quantifiers.

Observe first that, given the properties of the post-complement and definition D9, the set of MP operators is closed under Boolean operations and the post-complement:

Fact 10: If $O_1, O_2 \in MP$ then $\neg O_1 \in MP$, $O_1\neg \in MP$, $(O_1 \cup O_2) \in MP$ and $(O_1 \cap O_2) \in MP$.

Facts 9 and 10 allow us to show that MP operators are specific unions and intersections of sets of sentences as indicated in:

Proposition 4 *A sentential operator O is a MP operator iff there exists a sentential operator Q such that $O = Q \cup Q\neg$ (or $O = Q \cap Q\neg$).*

Proof The implication from right to left is obvious since it follows from the definition of MP that $Q \cup Q\neg$ and $Q \cap Q\neg$ are MP operators.

To prove the implication from left to right we use Fact 9. We decompose the set O into two sets, the set of non-negated sentences which are its elements and the set of the corresponding negated sentences. The first set of sentences constitutes a sentential operator, say Q and the second set constitutes the operator $Q\neg$. The second disjunct of the necessary condition of the proposition is a consequence of Fact 10.

Following Proposition 4 we will say that a MP operator O is decomposable into $Q \cup Q\neg$ or into $Q \cap Q\neg$ or that it is decomposable into a disjunctive or into a conjunctive form. A disjunctive or a conjunctive form of a MP operator is non-trivial iff it is a disjunction or a conjunction of two operators which are not MP operators.

The set MP has an empirically important subset of SOs that I will call *indirectly truth telling* operators:

D10: A sentential operator O is an *indirectly truth telling* operator, $O \in ITT$ for short, iff $O \in MP$ and O is decomposable into $Q \cup Q\neg$ and $Q \in VER$.

Thus ITT operators are MP operators that are composed of two parts such that one part contains only true sentences and the other part contains only false ones. In that sense operators T and $\neg T$ are trivially ITT operators.

In order to see some non-trivial ITT operators consider the following examples:

(22a) Bo told/informed Dan whether Lea left (or not).
(22b) Bo told/informed Dan that Lea left or Bo told Dan that Lea did not leave.
(23a) Bo knows/remembers whether Lea left (or not).
(23b) Bo knows/remembers that Lea left or Bo remembers that Lea did not leave.

In the above examples the possible disjunction indicated in parenthesis is not a disjunction between sentences but an exclusive disjunction between propositional operators. The above examples show that operators based on *tell whether, inform whether* and *remember whether* are all MP operators. However, there is a difference between *tell* and *inform whether* on the one hand, and *know* and *remember whether* on the other hand: only the second group belongs to the ITT class because only in this group factive, and thus veridical, operators are used.

Though sentences in (22a) do not contain ITT operators, in the sense of definition D13 (because *inform* and *tell* do not form veridical operators), they seem to express indirectly a truth related to their sentential complement: they seem to imply in some way that Bo told the truth (or gave true information) concerning Lea's leaving. Karttunen (1977) and Vendler (1980) have made, independently, such a claim concerning the operator *to tell whether*. Karttunen's and Vendler's observations gave rise to what may be called *Karttunen-Vendler* thesis which is that verbs of communications (such as *tell, indicate, inform, disclose*, etc.) when used with the complementizer *whether* give rise to "truth conveying" operators.

The Karttunen-Vendler thesis was challenged by Tsohatzidis (1993) who indicates that the subject of verbs of communication can be mistaken or can even lie when such verbs are used with the complementizer *whether*. An example Tsohatzidis gave later (p.c. 2019) is given in (24):

(24) John did tell us whether Mary is in Rome but I don't trust him: he is always mistaken about where Mary is.

The exact status of verbs of communication is out of scope of this paper since it touches to other important problems in the semantics of natural languages such as indirect discourse, relations between semantics and pragmatics and even even cross-linguistic differences (cf. Egré 2008; Spector and Egré 2015; Stokke 2013; Tsohatzidis 1997). For our purpose it is enough to observe that verbs of communication are different from other verbs forming PA operators.

We now have enough tools to define more precisely and study the class of neg-transportable operators:

D11: Let O_a be a PA operator. Then O_a is neg-transportable, $O_a \in NTR$ for short, iff $\sim O_a \subseteq O_a\neg \cap U_a$.

Observe first that factive operators cannot be neg-transportable:

Proposition 5 *No factive operator is neg-transportable.*

Proof Suppose $O_a \in FACT \cap NTR$. Then $\sim O_a \subseteq T$ since $O_a \in FACT$ and $\sim O_a \subseteq O_a\neg$, since $O_a \in NTR$. This is impossible, however, because $O_a\neg \cap T = \emptyset$.

Similarly, one can prove that:

Proposition 6 *No neg-transportable operator is a midpoint operator.*

Proof Suppose *a contrario* that some $O_a \in NTR \cap MP$. Then $\sim O_a \subseteq O_a\neg$. Since $O_a \in MP$ we have $O_a\neg = O_a$. Hence $\sim O_a \subseteq O_a$. Contradiction.

The class of NTR operators can be equivalently defined as in:

Proposition 7 *If $O_a \in PA$ then $O_a \in NTR$ iff $O_a \cap U_a$ presupposes $(O_a \cup O_a\neg) \cap U_a$.*

Proof (a) Suppose first that $O_a \in NTR$. Clearly $O_a \cap U_a \subseteq (O_a \cap U_a) \cup \sim O_a$. Then, since $\sim O_a \subseteq (O_a\neg \cap U_a)$ we get (1) $(O_a \cap U_a) \cup (O_a\neg \cap U_a)$. Since $\sim (O_a \cap U_a) = \sim O_a$, it is also true that $\sim (O_a \cap U_a) \subseteq \sim O_a \cup (O_a\neg \cap U_a)$. Hence, given that $O_a \in NTR$ we have (2) $\sim (O_a \cap U_a) \subseteq ((O_a \cup O_a\neg) \cap U_a)$. Thus given (1) and (2) the sufficient condition of Proposition 7 holds.

(b) Suppose now that $O_a \cap U_a$ presupposes $(O_a \cup O_a\neg) \cap U_a$. We have to show that $O_a \in NTR$. Given the supposition we have $\sim (O_a \cap U_a) \subseteq (O_a \cup O_a\neg) \cap U_a$. Given that $\sim (O_a \cap U_a) = \sim O_a$ and since $\sim O_a \cap (O_a \cap U_a) = \emptyset$ we have $\sim O_a \subseteq (O_a\neg \cap U_a)$. Thus $O_a \in NTR$.

The following fact is an obvious consequence of Proposition 7 and of Fact 1 (iii):

Fact 11: $O_a \in NTR$ iff $O_a\neg \in NTR$.

The presupposition indicated in Proposition 7 corresponds to the so-called *excluded middle* presupposition. This presupposition has been proposed by Bartsch (1973) who indicates that it entails neg-transportability. Observe that Proposition 7 states that excluded middle presupposition is equivalent to neg-transportability, and, furthermore that excluded middle presupposition, given Fact 5, is a particular case of the presupposition proper to PA operators in general.

The excluded middle presupposition is related to the truth expressed by the *tertium non datur* principle applied to propositional operators or to the ("classical") *completeness* of quantifiers, that is when the universe of discourse is not restricted. Since we use the universe restricted by the set U_a our definition of completeness has to take into account this fact. Thus:

D12: The propositional operator O_a is complete relative to the set U_a, or O_a is u-complete, or $O_a \in UCMPL$, iff for any sentence $S \in U_a$, either $S \in O_a$ or $nS \in O_a$.

The following proposition indicates the relationship between u-completeness and neg-transportability:

Proposition 8 *Any u-complete PA operator is neg-transportable.*

Proof Let $O_a \in UCMPL$ and, *a contrario* $O_a \notin NTR$. Then for some S we have $S \in \sim O_a$ and $S \notin O_a\neg$. Hence $nS \notin O_a$. But then, since O_a is u-complete, $S \in O_a$. Contradiction.

Since the operator formed from the verb *regret* is not neg-transportable, we have to conclude, given proposition 8, that the operator *a regrets that* is not u-complete.

In the case of neg-transportable operators the negation "moves" from the operator to its argument. The question one can ask is whether there are operators where the negation can "move" in the other direction: from the argument to the operator and, additionally, whether there are operators with two-ways transportable negation. Such an inverse "movement" of the negation is illustrated in (25):

(25) *Leo thinks that life is not sad* entails *Leo does not think that life is sad*

The excluded middle presupposition and neg-transportability are related to consistency or non-contradiction and to completeness. In the generalized quantifier theory a quantifier Q is consistent iff for no property A we have A and A' (the complement of A) belong to Q. Similarly, Q is complete iff any property A or its complement A' belong to Q. In order to extend the notion of consistency to PA operators we have also to take into account the fact that the intensional negation is stronger that the classical negation. To do this we relativise the notion of completeness and of consistency to the set U_a. Thus:

D13: Let $O_a \in PA$. Then O_a is consistent relative to the set U_a, or O_a is u-consistent, or $Q_a \in UCONST$ for short, iff $\forall_{S \in U_a} \neg(S \in O_a \wedge nS \in O_a)$.

A simple, or *absolute* consistency (and completeness) is defined in the same way without the restriction to the set U_a. Fact 12 indicates a class of (absolutely) consistent operators:

Fact 12: Any veridical operator is consistent.

It follows from Fact 13 that factives are consistent operators.

SOs formed from verbs of communication are neither complete nor consistent and, thus, neither u-consistent nor u-complete. Concerning MP operators we have:

Proposition 9 *No non-trivial MP operator entails a consistent operator.*

Proof This follows from the fact that no non-trivial MP operator is consistent.

The transportability of negation from the argument to the operator entails the consistency of this operator:

Proposition 10 *If $O_a\neg \subseteq\sim O$ then $O_a \in CONST$.*

Proof Suppose *a contrario* that $O_a \notin CONST$. Then for some S we have $S \in O_a$ and $nS \in O_a$. Hence $nS \in O_a\neg$ and thus $nS \in\sim O_a$. Contradiction.

In order to indicate further the importance of PA negation I analyse briefly *dual* operators. In the generalized quantifier theory one defines the relation of duality between two quantifiers as follows: quantifier Q_1 is the dual to the quantifier Q_2 iff $\neg Q_1\neg = Q_2$. A quantifier which is dual to itself is called *self-dual*. One shows easily that consistent and complete quantifiers are self-dual. In order to extend the notion of duality and self-duality to PA operators we have also to take into account the fact that the intensional negation is stronger that the classical negation:

D14: Let $O_a, Q_a \in PA$. Then O_a is the dual of Q_a iff $U_a \cap \sim O_a\neg = U_a \cap Q_a$.

D15: O_a is self-dual, $O_a \in SDUAL$, iff $U_a \cap O_a = U_a \cap \sim O_a\neg$.

With the above definitions one can show many similarities between dual and self-dual PA operators and dual and self-dual quantifiers. Facts 13 and 14 indicate some properties of dual operators and Proposition 11 indicates the relation between complete, consistent and self-dual PA operators:

Fact 13: O_a is the dual of Q_a iff Q_a is the dual of O_a.
Fact 14: O_a is the dual of Q_a iff $\sim O_a = Q_a\neg$.

Proposition 11: $O_a \in SDUAL$ iff $O_a \in UCONST \cap UCMPL$.

Proof of Proposition 11 *If*-part: Suppose *a contrario* that $O_a \in SDUAL$ and $O_a \notin UCONST$. This means that for some S we have $S \in O_a$ and $nS \in O_a$. Hence $S \in O_a\neg$ and thus $S \in \sim O_a$. Contradiction.

Suppose now, *a contrario* that $O_a \notin UCMPL$. This means that for some $S \in U_a$ we have $S \notin O_a$ and $nS \notin O_a$. Hence $S \notin O_a\neg$ and thus $S \notin \sim O_a$. Contradiction.
Only-if-part: Suppose that $O_a \in UCONST \cap UCMPL$. We show first that $\sim O_a \subseteq O_a\neg$. Let $S \in \sim O_a$. This means that $S \notin O_a$ and thus, since O_a is complete, $nS \in O_a$. Hence $S \in O_a\neg$.

It remains to be shown that $O_a\neg \subseteq \sim O_a$ whenever $O_a \in UCONST \cap UCMPL$. Suppose $S \in O_a\neg$. Hence $nS \in O_a$ and thus, given that O_a is complete, $S \notin O_a$. This means, given that $S \in U_a$, that $S \in \sim O_a$.

Clearly factive operators do not have duals. The operators formed from *believe that* and *doubt that* are duals of each other.

We conclude this section by studying briefly the relation of intensional equivalence between two PA operators:

D16: A PA operator O_1 is intensionally equivalent to a PA operator O_2 iff $O_a \subseteq O_2$ and $\sim O_1 \subseteq \sim O_2$.

Observe that the binary relation that definition D16 establishes between PA operators is not symmetric.

We have seen that intensional equivalence holds between ITT operators and their components. More precisely, the following is true:

Proposition 11 *Let $O_a \in ITT$ and $O_a = Q_a \cup Q_a\neg$. Then Q_a is intensionally equivalent to O_a.*

Proof Clearly $Q_a \subseteq O_a$. Suppose now, *a contrario* that $S \in \sim Q_a$ and $S \notin \sim O_a$. Then $S \notin O_a$ and $S \in (\neg(Q_a \cup Q_a\neg) \cap PP(Q_a \cap Q_a\neg)$. But this is impossible because $S \in T$ and $Q \in FACT$.

Interestingly not only ITT operators give rise to the intensional equivalence. For instance it seems true that (26a) is intensionally equivalent to (26b):

(26a) Bo hopes that S.
(26b) Bo thinks that S.

One notices that (26a) has a presupposition, roughly *Bo wishes that S* which is not a presupposition of (26b). More generally if O_a is intensionally equivalent to Q_a then any presupposition of Q_a is a presupposition of O_a and the maximal presupposition of O_a is not a presupposition of Q_a. Thus means, informally, that the relation of intensional equivalence reduces to "ordinary" logical equivalence in all situations in which the main presupposition of the first argument of the relation is satisfied. More formally this can be expressed by the following fact, which can also be considered as justifying the name of *intensional equivalence*:

Fact 15: O_a is intensionally equivalent to Q_a iff $O_a = PP(O_a) \cap Q_a$.

In Zuber (2020) the relation of implicational equivalence between sentences is defined: S_1 is implicationally equivalent to S_2 iff S_1 and S_2 have the same truth values in all models in which their presuppositions are satisfied and consistent. This notion was introduced to account for the semantic relation between sentences which differ just by the fact that in one of them a two-way implicative is used, as for instance (27a) and (27b):

(27a) Bo managed to prove the theorem.
(27b) Bo proved the theorem.

One notices that the negation of (27a) entails the negation of (27b). Given our definition of intensional equivalence and Fact 15 we can consider that intensional equivalence is a particular case of implicational equivalence.

5 Conclusion

The leading approach in the semantics of constructions involving verbs of propositional attitude is what can be dubbed *propositionalism*: denotations of clausal complements in such constructions are defined with the help of propositions or sets of propositions. In particular declarative and interrogative complements denote sets of propositions. This means that propositional operators based on proper nouns denote relations between individuals and sets of propositions. This is true in the classical semantics of propositional attitude verbs and in more recent attempts to unify the semantics of declaratives and interrogatives (Ciardelli et al. 2019; Uegaki 2019).

My proposal in this paper is more like the sententialists proposal (cf. Higginbotham 2006) though I do not postulate a reference to specific sentences but to a set of specific sentences. This approach allow us in particular to avoid serious problems related to the intensionality of propositional attitude operators. More importantly, it allows us to fully use the natural machinery proper to Boolean algebras and, in particular, various tools from the generalized quantifier theory. This move appears to be very fruitful since many properties of *PA* operators could be formulated in a very simple way. In addition formal results thus obtained and various empirical observations give enough arguments to formulate and defend various universal semantic constraints concerning natural languages. We have seen, for instance, that very likely in natural languages any factive operator entails knowledge or excludes knowledge, any veridical operator formed from naturally intensional verb is factive, any (non-trivial) ITT operator is composed from a factive verb, etc. What is important is the fact that formulation and justification of these universal constraints could be expressed with technically simple vocabulary and without recourse to ontologically charged tools.

Propositional operators that I have considered are lexically simple. Many verbs of propositional attitude can be modified by various verbal modifiers. We have for instance: *try to remember, desire to know, want to forget, cease to regret, continue to believe*, etc. The semantic contribution of such verbal modifiers is by no means trivial and clearly necessitates additional study. A starting point might be the study the of semantics of implicative verbs as proposed in Zuber (2020).

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