DOI 10.1007/s10986-024-09635-1
*Lithuanian Mathematical Journal, Vol. 64, No. 2, April, 2024, pp. 138–162*

# Ground state solutions for quasilinear Schrödinger equations with critical Berestycki–Lions nonlinearities*

**Jian-Xin Han, Ming-Chao Chen, and Yan-Fang Xue**[1]

School of Mathematics and Statistics, Xin-Yang Normal University, Xinyang 464000, China
(e-mail: hjxin2008@163.com; 1432064866@qq.com; xueyanfang2015@163.com)

Received February 26, 2023; revised February 5, 2024

**Abstract.** We consider the quasilinear Schrödinger equation involving a general nonlinearity at critical growth. By using Jeanjean's monotonicity trick and the Pohozaev identity we get the existence results that generalize an earlier work [H. Liu and L. Zhao, Existence results for quasilinear Schrödinger equations with a general nonlinearity, *Commun. Pure Appl. Anal.*, 19(6):3429–3444, 2020] about the subcritical case to the critical case.

*MSC:* 35A15, 35J62, 35D30

*Keywords:* quasilinear Schrödinger equation, ground state solutions, critical exponent, Berestycki–Lions nonlinearity, Pohozaev identity

## 1 Introduction and main results

We consider the following quasilinear Schrödinger equation:

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\Delta\psi + V(x)\psi - \Delta\rho\bigl(|\psi|^2\bigr)\rho'\bigl(|\psi|^2\bigr)\psi - h(\psi), \tag{1.1}$$

where $V:\mathbb{R}^N\to\mathbb{R}$ is a given potential, and $\rho$ and $h$ are real functions. This modified version of the nonlinear Schrödinger equation has been derived to model several physical phenomena, such as superfluid films in plasma physics, condensed matter theory, etc. (see [15, 17, 23] for an explanation). We restrict ourselves to $\rho(s) = s$ and the stationary wave solutions, that is, the solutions of the form $\psi(t,x) = \exp(-\mathrm{i}Et)u(x)$, $E\in\mathbb{R}$, so that from equation (1.1) we get an equation of elliptic type of the formal structure

$$-\Delta u + V(x)u - \Delta\bigl(u^2\bigr)u = h(u), \tag{1.2}$$

where $x\in\mathbb{R}^N$, $N\geqslant 3$, $V$ is a potential well, and the nonlinearity $h$ is a general term with critical growth.

* The research is supported by National Natural Science Foundation of China (No. 11901499) and Nanhu Scholar Program for Young Scholars of XYNU (No. 201912).

[1] Corresponding author.

There are many results for problem (1.2) depending on different assumptions on the potential $V$ and nonlinearity $h$; for example, see [28] for the coercive potential, [31] for the radially symmetric potential, [6,25,26] for the periodic or asymptotically periodic potential, and [7] for the steep potential well. The nonlinearity $h$ satisfies some growth conditions, such as the Ambrosetti–Rabinowitz condition, a nonquadraticity condition, or a monotonicity condition. Up to our knowledge, there are few results on problem (1.2) under the Berestycki–Lions conditions (see [1, 2]), which are almost optimal for the existence of solutions. Here we mention [4, 14, 29, 30]. Colin and Jeanjean [4] proved the existence of a radially symmetric solution for problem (1.2) with $V(x) \equiv 0$ and the nonlinearity $h(u)$ satisfying the Berestycki–Lions conditions. This result [4, Thm. 1.2] can be regarded as the Berestycki–Lions theorem of the subcritical case for the quasilinear Schrödinger equation. Then we generalize the result to critical growth in [29]. Up to now, perhaps this is the only work on quasilinear Schrödinger equations with critical Berestycki–Lions growth that specifically addresses the case $V(x) \equiv 0$. The method is to analyze the behavior of solutions for subcritical problems and to take the limit as the exponent approaches the critical exponent, which was also used in [3, 19]. The remaining results on the Berestycki–Lions conditions [8, 10, 14, 30] pertain to the subcritical cases; specifically, [8] focuses on another type of quasilinear equation, whereas [10] deals with the Choquard equation. Moreover, He et al. [9] considered the strongly indefinite problem for the Hamiltonian elliptic systems with superquadratic or asymptotically quadratic condition and obtained ground state solutions by using the Nehari–Pankov-type constraint.

The main purpose of this paper is to extend the result in [14] to the case of critical exponents. We borrow an idea from [32] to obtain the ground state solution for problem (1.2). There are several difficulties in our paper. The main difficulty is caused by the second-order derivatives $\Delta(u^2)u$. To address this difficulty, various methods have been developed, such as a change of variables [4, 17, 19, 22, 25, 28], a constrained minimization argument [15, 18, 23, 24], and the perturbation method [16, 20, 21]. Here we use the change of variables in [4]. Besides, the nonlinear term $h$ in our paper satisfies very weak conditions, so it is hard to obtain the boundedness of (PS) sequence. We employ Jeanjean's monotonicity trick [11] to address it. At last, because of possible lack of compactness due to the criticality of the growth and unboundedness of the domain, to obtain the existence of the solutions, we will turn to the concentration compactness lemma due to Lions [12, 13].

We introduce some assumptions on the potential $V$ and nonlinearity $h$.

- (V) $0 < V_0 \leqslant V(x) \leqslant V_\infty := \lim_{|x|\to+\infty} V(x) < +\infty$, and there is $\sigma \in [1, 2)$ such that $\nabla V(x) \cdot x \in L^{2^*/(2^*-\sigma)}(\mathbb{R}^N)$.
- (h1) $h \in C(\mathbb{R}, \mathbb{R})$, $h(t) = 0$ for all $t \leqslant 0$, and $\lim_{t\to 0^+} h(t)/t = 0$.
- (h2) $\lim_{t\to\infty} h(t)/t^{2\cdot 2^*-1} = 1$.
- (h3) There exists $\beta > 0$ and $q \in (2, 2^*)$ if $N \geqslant 10$ or $q \in ((3N + 2)/(2(N - 2)),\ 2^*)$ if $3 \leqslant N < 10$ such that $h(t) \geqslant t^{2\cdot 2^*-1} + \beta t^{2q-1}$ for all $t \geqslant 0$.

It is well known that the critical exponent growth makes the problem very tough and more assumptions are of course needed. Now we state our main results. The first result of the present paper concerns the case of constant potential $V(x) \equiv V$, which plays an important role for studying the second result.

**Theorem 1.** *Suppose that* (h1)–(h3) *are satisfied and* $V(x) \equiv V > 0$. *Then problem* (1.2) *possesses a positive ground state solution.*

The second result in this paper with nonconstant $V(x)$ is as follows.

**Theorem 2.** *Suppose that conditions* (V) *and* (h1)–(h3) *hold. Then problem* (1.2) *possesses a positive ground state solution.*

*Remark 1.*

(i) These two theorems can be regarded as a form of the subcritical Berestycki–Lions theorem in [14] in the critical case. Due to the lack of compact embedding of $H^1(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$, for critical nonlinearity $h$, the existence of ground states of problem (1.2) becomes rather complicated. We borrow an idea from [32] to overcome this difficulty.

(ii) Our results also can be seen as an extension of semilinear porоblem in [32] to the quasilinear one.

NOTATION. In this paper, we use the following notations:

- $H^1(\mathbb{R}^N)$ is the usual Hilbert space endowed with the norm

$$\|u\|_H^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right) \mathrm{d}x.$$

- $L^s(\mathbb{R}^N)$ is the usual Banach space endowed with the norm

$$\|u\|_s^s = \int_{\mathbb{R}^N} |u|^s \,\mathrm{d}x, \quad s \in [1, +\infty).$$

- $E = \{u \in H^1(\mathbb{R}^N)\colon \int_{\mathbb{R}^N} V(x)u^2 \,\mathrm{d}x < \infty\}$ is endowed with the norm

$$\|u\|^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x.$$

- $B_r(y) := \{x \in \mathbb{R}^N\colon |x - y| < r\}$.
- $o_n(1)$ is a quantity tending to 0 as $n \to \infty$.
- $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$.
- $|\Omega|$ denotes the Lebesgue measure of a set $\Omega$.
- $C, C_1, C_2, \dots$ denote various positive (possibly, different) constants.

## 2 Preliminary results

Denoting $H(s) := \int_0^s h(t)\,\mathrm{d}t$, we observe that the natural variational functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x)u^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} u^2|\nabla u|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} H(u)\,\mathrm{d}x,$$

corresponding to equation (1.2), may be not well defined in the space $H^1(\mathbb{R}^N)$ when $N > 1$. We have to find a suitable functional space. Following [4], we define the function $f$ as follows:

$$f'(t) = \frac{1}{(1 + 2f^2(t))^{1/2}}, \quad t \in [0, +\infty),$$
$$f(t) = -f(-t), \quad t \in (-\infty, 0].$$

Then we obtain the functional

$$I(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(v)\,\mathrm{d}x - \int_{\mathbb{R}^N} H\big(f(v)\big)\,\mathrm{d}x,$$

where $I(v) = J(u) = J(f(v))$ is well defined on $E$, $I \in C^1(E, \mathbb{R})$ under the hypotheses (V) and (h1)–(h3). Moreover, we observe that if $v$ is a critical point of the functional $I$, then the function $u = f(v)$ is a solution of problem (1.2) (see [4]).

We now summarize the properties of $f$, which have been proved in [4, 5] and [25].

**Lemma 1.** *The function $f$ satisfies the following properties*:

(i) $f$ *is uniquely defined*, $C^\infty$, *and invertible*;
(ii) $|f'(t)| \leqslant 1$ *for all* $t \in \mathbb{R}$;
(iii) $|f(t)| \leqslant |t|$ *for all* $t \in \mathbb{R}$;
(iv) $f(t)/t \to 1$ *as* $t \to 0$;
(v) $f(t)/\sqrt{t} \to 2^{1/4}$ *as* $t \to \infty$;
(vi) $f(t)/2 \leqslant tf'(t) \leqslant f(t)$ *for all* $t > 0$;
(vii) $|f(t)| \leqslant 2^{1/4}|t|^{1/2}$ *for all* $t \in \mathbb{R}$;
(viii) *there exist constants* $M, R > 0$ *such that* $f^{22^*}(t) - 2^{2^*/2}t^{2^*} \geqslant -Mt^{2^*-1/2}$ *for all* $t \geqslant R$;
(ix) *there exists a positive constant* $C$ *such that* $|f(t)| \geqslant C|t|$ *for* $|t| \leqslant 1$ *and* $|f(t)| \geqslant C|t|^{1/2}$ *for* $|t| \geqslant 1$.

From items (iii) and (vii) of Lemma 1 we directly can get the following lemma.

**Lemma 2.** *For* $\alpha \in [1, 2)$, $|f(t)| \leqslant 2^{1/4}|t|^{\alpha/2}$ *for all* $t \in \mathbb{R}$.

**Lemma 3.** (*See* [33].) *Let* $k : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *be a continuous function such that for some constant* $C > 0$,

$$\lim_{s\to 0} \frac{k(x,s)}{s} \leqslant C, \qquad \lim_{s\to+\infty} \frac{k(x,s)}{s^{2^*-1}} = 0.$$

*Let* $\{v_n\} \subset E$ *be a bounded sequence such that* $v_n \rightharpoonup v$ *in* $E$. *Then*

$$\lim_{n\to+\infty}\left[\int\limits_{\mathbb{R}^N} K(x,v_n) - \int\limits_{\mathbb{R}^N} K(x,v) - \int\limits_{\mathbb{R}^N} K(x,\, v_n - v)\right] = 0,$$

*where* $K(x,v) = \int_0^v k(x,s)\,\mathrm{d}s$.

**Lemma 4.** (*See* [27].) *If* $v_n \rightharpoonup v$ *in* $D^{1,2}(\mathbb{R}^N)$ *and* $v \in L^\infty_{loc}(\mathbb{R}^N)$, *then*

$$|v_n|^{2^*-2}v_n - |v_n - v|^{2^*-2}(v_n - v) \to |v|^{2^*-2}v \quad \text{in } \bigl(D^{1,2}(\mathbb{R}^N)\bigr)^{-1}.$$

To get a bounded (PS) sequence for the functional $I$, we make use of the monotone method introduced by Jeanjean [11].

**Lemma 5.** (*See* [11].) *Let* $(E, \|\cdot\|)$ *be a Banach space, and let* $L \subset \mathbb{R}^+$ *be an interval. Consider a family of* $C^1$ *functionals on* $E$, $I_\lambda(v) = A(v) - \lambda B(v)$ *for all* $\lambda \in L$ *with* $B$ *nonnegative and suppose that either* $A(v) \to +\infty$ *or* $B(v) \to +\infty$ *as* $\|v\| \to \infty$. *For* $\lambda \in L$, *we set*

$$\Gamma_\lambda = \bigl\{\gamma \in C\bigl([0,1], E\bigr)\colon \gamma(0) = 0 \neq \gamma(1),\ I_\lambda\bigl(\gamma(1)\bigr) < 0\bigr\}.$$

*If for every* $\lambda \in L$, *the set* $\Gamma_\lambda$ *is nonempty and*

$$c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} I_\lambda\bigl(\gamma(t)\bigr) > \max\bigl\{I_\lambda\bigl(\gamma(0)\bigr), I_\lambda\bigl(\gamma(1)\bigr)\bigr\},$$

*then for almost every* $\lambda \in L$, *there is a sequence* $\{v_n\} \subset E$ *such that*

(i) $\{v_n\}$ *is bounded*;
(ii) $I_\lambda(v_n) \to c_\lambda$;
(iii) $I'_\lambda(v_n) \to 0$ *in the dual* $E^{-1}$ *of* $E$.

*Furthermore, the map* $\lambda \to c_\lambda$ *is left-continuous.*

For almost all $\lambda \in [1/2, 1]$, we search for bounded Palais–Smale sequences of the following perturbed functional:

$$I_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(v)\,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} H\big(f(v)\big)\,\mathrm{d}x.$$

In our case, we consider

$$A(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(v)\,\mathrm{d}x, \qquad B(v) := \int_{\mathbb{R}^N} H\big(f(v)\big)\,\mathrm{d}x.$$

Obviously, $B(v)$ is nonnegative. The next lemma ensures that $A$ is coercive and the functional $I_\lambda$ has a mountain pass geometry. Namely, the set $\Gamma_\lambda$ is nonempty, and $c_\lambda > 0$.

**Lemma 6.** *Suppose that* (V) *and* (h1)–(h3) *are satisfied. Then*:

(i) $\Gamma_\lambda = \{\gamma \in C([0,1], E)\colon\ \gamma(0) = 0 \neq \gamma(1),\ I_\lambda(\gamma(1)) < 0\} \neq \emptyset$ *for all* $\lambda \in [1/2, 1]$;
(ii) $c_\lambda = \inf_{\gamma\in\Gamma_\lambda} \max_{t\in[0,1]} I_\lambda(\gamma(t)) > 0$ *for all* $\lambda \in [1/2, 1]$;
(iii) $A(v) \to +\infty$ *as* $\|v\| \to \infty$.

*Proof.* (i) Set $\theta > 0$ sufficiently large and $w^\theta = w(\cdot/\theta), w \neq 0$. Define $\gamma : [0,1] \to E$ in the following way:

$$\gamma(t) = \begin{cases} w^t = w(\frac{\cdot}{t}) & \text{if } 0 < t \leqslant 1, \\ 0 & \text{if } t = 0. \end{cases}$$

Obviously, $\gamma$ is a continuous path from 0 to $w$. We consider the function

$$I_\lambda(w) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla w|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x) f^2(w)\,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} H\big(f(w)\big)\,\mathrm{d}x.$$

Let $\theta > 0$ large enough such that $|\theta w^\theta| > 1$. Then from Lemma 1(iii),(vii) and (h3) we have

$$\begin{aligned} I_\lambda\left(\theta w\left(\frac{x}{\theta}\right)\right) &\leqslant I_{1/2}\left(\theta w\left(\frac{x}{\theta}\right)\right) \leqslant \frac{\theta^N}{2}\int_{\mathbb{R}^N} |\nabla w|^2\,\mathrm{d}x + \frac{\theta^{N+2}}{2}\int_{\mathbb{R}^N} V(\theta x) w^2\,\mathrm{d}x \\ &\quad - \frac{1}{4\cdot 2^*}\int_{\mathbb{R}^N} f^{2\cdot 2^*}\left(\theta w\left(\frac{x}{\theta}\right)\right)\mathrm{d}x - \frac{\beta}{4q}\int_{\mathbb{R}^N} f^{2q}\left(\theta w\left(\frac{x}{\theta}\right)\right)\mathrm{d}x \\ &\leqslant \frac{\theta^N}{2}\int_{\mathbb{R}^N} |\nabla w|^2\,\mathrm{d}x + \frac{\theta^{N+2}}{2}\int_{\mathbb{R}^N} V(\theta x) w^2\,\mathrm{d}x \\ &\quad - C_1\theta^{N+2^*}\int_{\mathbb{R}^N} w^{2^*}\,\mathrm{d}x - C_2\theta^{N+q}\int_{\mathbb{R}^N} w^q\,\mathrm{d}x. \end{aligned}$$

Choose $w_0 = \theta w^\theta$ with $\theta > 0$ sufficiently large. Then $I_\lambda(w_0) \leqslant I_{1/2}(w_0) < 0$ for all $\lambda \in [1/2, 1]$. Defining $\gamma_0(\theta) = w_0 = \theta w^\theta$, we get that $\gamma_0 \in \Gamma_\lambda$.

(ii) Thanks to Lemma 1(ix), we can deduce that there is $C_3 > 0$ such that

$$f^2(t) \geqslant C_3\big(t^2 - |t|^{2^*}\big). \tag{2.1}$$

From (h1) and (h2) It follows that for any $\delta > 0$, there exists $C_\delta > 0$ such that

$$h(t) \leqslant \delta|t| + C_\delta|t|^{2\cdot 2^*-1}, \quad t \in \mathbb{R}, \tag{2.2}$$

and

$$H(t) \leqslant \frac{\delta}{2}|t|^2 + \frac{C_\delta}{2\cdot 2^*}|t|^{2\cdot 2^*}, \quad t \in \mathbb{R}. \tag{2.3}$$

By (2.1), (2.3), Lemma 1(iii),(vii), and the Sobolev inequality we have

$$\begin{aligned}
I_\lambda(v) \geqslant I_1(v) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} V(x)f^2(v)\,\mathrm{d}x - \int_{\mathbb{R}^N} H\big(f(v)\big)\,\mathrm{d}x \\
&\geqslant \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x + \frac{V_0C_3}{2}\int_{\mathbb{R}^N} \big(v^2 - |v|^{2^*}\big)\,\mathrm{d}x \\
&\quad - \frac{\delta}{2}\int_{\mathbb{R}^N} f^2(v)\,\mathrm{d}x - \frac{C_\delta}{2\cdot 2^*}\int_{\mathbb{R}^N} f^{2\cdot 2^*}(v)\,\mathrm{d}x \\
&\geqslant \min\left\{\frac{1}{2}, \frac{V_0C_3-\delta}{2}\right\}\|v\|^2 - \left(\frac{V_0C_3}{2} + \frac{C_\delta}{2\cdot 2^*}2^{2^*/2}\right)S^{-2^*/2}\|\nabla v\|_2^{2^*} \\
&\geqslant C_4\|v\|^2 - C_5\|v\|^{2^*},
\end{aligned}$$

where $\delta > 0$ is small enough, and $S$ is the best constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$. Since $2^* > 2$, $I_\lambda$ has a strict local minimum at $0$, and hence $c_\lambda = \inf_{\gamma\in\Gamma_\lambda}\max_{t\in[0,1]} I_\lambda(\gamma(t)) > 0$.

(iii) From Lemma 1(ix) we get that

$$\begin{aligned}
\|v\|^2 &= \int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x + \int_{\{x:\,|v(x)|\leqslant 1\}} V(x)v^2\,\mathrm{d}x + \int_{\{x:\,|v(x)|\geqslant 1\}} V(x)v^2\,\mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x + \frac{1}{C}\int_{\{x:\,|v(x)|\leqslant 1\}} V(x)f^2(v)\,\mathrm{d}x + V_\infty\int_{\{x:\,|v(x)|>1\}} |v|^{2^*}\,\mathrm{d}x \\
&\leqslant \int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x + \frac{1}{C}\int_{\mathbb{R}^N} V(x)f^2(v)\,\mathrm{d}x + C_6\left(\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x\right)^{2^*/2} \\
&\leqslant C_7\big(A(v) + A(v)^{2^*/2}\big),
\end{aligned}$$

which implies that $A(v) \to +\infty$ as $\|v\| \to \infty$. $\square$

Given $\epsilon > 0$, we consider the function $w_\epsilon : \mathbb{R}^N \to \mathbb{R}$ defined by

$$w_\epsilon(x) = C(N)\frac{\epsilon^{(N-2)/2}}{(\epsilon^2 + |x|^2)^{(N-2)/2}}, \quad C(N) = \big[N(N-2)\big]^{(N-2)/4}.$$

We observe that $\{w_\epsilon\}$ is a family of functions on which the infimum that defines the best constant $S$ for the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ is attained. Let $\phi \in C_0^\infty(\mathbb{R}^N, [0,1])$ be a cut-off function satisfying $\phi \equiv 1$ in $B_1(0)$ and $\phi \equiv 0$ in $\mathbb{R}^N \setminus B_2(0)$. Define

$$u_\epsilon = \phi w_\epsilon, \qquad v_\epsilon = \frac{u_\epsilon}{\|u_\epsilon\|_{2^*}}.$$

By a similar computation to that in [3, 25], as $\epsilon \to 0$, we have that

$$\int_{\mathbb{R}^N} |\nabla v_\epsilon|^2 \,\mathrm{d}x = S + O\big(\epsilon^{N-2}\big) \quad \text{if } N \geqslant 3, \tag{2.4}$$

$$\|v_\epsilon\|_{2^*-1/2}^{2^*-1/2} = O\big(\epsilon^{(N-2)/4}\big), \tag{2.5}$$

and

$$|v_\epsilon|_2^2 = \begin{cases} O(\epsilon) & \text{if } N = 3, \\ O(\epsilon^2 |\ln \epsilon|) & \text{if } N = 4, \\ O(\epsilon^2) & \text{if } N \geqslant 5. \end{cases} \tag{2.6}$$

There are positive constants $k_1, k_2$, and $\epsilon_0$ such that

$$k_1 < \int_{\mathbb{R}^N} |u_\epsilon|^{2^*} \,\mathrm{d}x < k_2 \quad \forall 0 < \epsilon < \epsilon_0. \tag{2.7}$$

**Lemma 7.** *Suppose that* (V) *and* (h1)–(h3) *are satisfied. Then the minimax level $c_\lambda$ satisfies*

$$c_\lambda < \frac{1}{2N} \lambda^{-(N-2)/2} S^{N/2}.$$

*Proof.* It suffices to show that there exists $v_0 \in E \setminus \{0\}$ such that

$$\max_{t \geqslant 0} I_\lambda(t v_0) < \frac{1}{2N} \lambda^{-(N-2)/2} S^{N/2}.$$

From the proof of Lemma 6 we have $\lim_{t\to\infty} I_\lambda(tv_\epsilon) = -\infty$ and $I_\lambda(tv_\epsilon) > 0$ for $t > 0$ small enough. Then there exists $t_\epsilon > 0$ such that $I_\lambda(t_\epsilon v_\epsilon) = \max_{t>0} I_\lambda(tv_\epsilon)$. We claim that there are constants $T_1$ and $T_2$ such that $0 < T_1 < t_\epsilon < T_2$. First, we prove that $t_\epsilon$ is bounded from below by a positive constant. Otherwise, if $t_\epsilon \to 0$ as $\epsilon \to 0$, then $t_\epsilon v_\epsilon \to 0$. Therefore $0 < c \leqslant \max_{t\geqslant 0} I_\lambda(tv_\epsilon) \to 0$, which is a contradiction. On the other hand, if $t_\epsilon \to +\infty$ as $\epsilon \to 0$, then similarly to the proof of Lemma 6, we can get $0 < c \leqslant I_\lambda(t_\epsilon v_\epsilon) \to -\infty$, which is a contradiction. Hence there is $T_2 > 0$ such that $t_\epsilon \leqslant T_2$ for $\epsilon$ small enough.

Now, by (h3) and Lemma 1(iii),(viii) we observe that

$$\begin{aligned} I_\lambda(t_\epsilon v_\epsilon) &= \frac{1}{2} \int_{\mathbb{R}^N} \big|\nabla(t_\epsilon v_\epsilon)\big|^2 \,\mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^N} V(x) f^2(t_\epsilon v_\epsilon) \,\mathrm{d}x - \lambda \int_{\mathbb{R}^N} H\big(f(t_\epsilon v_\epsilon)\big) \,\mathrm{d}x \\ &\leqslant \frac{t_\epsilon^2}{2} \int_{\mathbb{R}^N} |\nabla v_\epsilon|^2 \,\mathrm{d}x + \frac{V_\infty}{2} \int_{\mathbb{R}^N} t_\epsilon^2 v_\epsilon^2 \,\mathrm{d}x \\ &\quad - \frac{\lambda}{2 \cdot 2^*} \int_{\mathbb{R}^N} f^{2 \cdot 2^*}(t_\epsilon v_\epsilon) \,\mathrm{d}x - \frac{\lambda\beta}{2q} \int_{\mathbb{R}^N} f^{2q}(t_\epsilon v_\epsilon) \,\mathrm{d}x \\ &\leqslant \frac{t_\epsilon^2}{2} \int_{\mathbb{R}^N} |\nabla v_\epsilon|^2 \,\mathrm{d}x + \frac{V_\infty}{2} t_\epsilon^2 \|v_\epsilon\|_2^2 - \frac{\lambda}{2^*} 2^{2^*/2-1} t_\epsilon^{2^*} \\ &\quad + \frac{\lambda M}{2 \cdot 2^*} t_\epsilon^{2^*-1} \int_{\mathbb{R}^N} v_\epsilon^{2^*-1/2} \,\mathrm{d}x - \frac{\lambda\beta}{2q} \int_{\mathbb{R}^N} f^{2q}(t_\epsilon v_\epsilon) \,\mathrm{d}x. \end{aligned}$$

Denote

$$l(t) := \frac{t^2}{2}\int_{\mathbb{R}^N} |\nabla v_\epsilon|^2\,\mathrm{d}x - \frac{t^{2^*}}{2^*}\lambda 2^{2^*/2-1}.$$

It is quite standard to get that $l(t)$ attains its maximum

$$l(t_0) = \frac{1}{2N}\lambda^{-(N-2)/2}\left[\int_{\mathbb{R}^N} |\nabla v_\epsilon|^2\,\mathrm{d}x\right]^{N/2} \tag{2.8}$$

at

$$t_0 = \left(\frac{1}{\lambda 2^{(2^*-2)/2}}\right)^{1/(2^*-2)}\left[\int_{\mathbb{R}^N} |\nabla v_\epsilon|^2\,\mathrm{d}x\right]^{1/(2^*-2)}.$$

It follows from (2.4) and (2.8) that

$$I_\lambda(t_\epsilon v_\epsilon) \leqslant \frac{1}{2N}\lambda^{-(N-2)/2}\big(S + O\big(\epsilon^{N-2}\big)\big)^{N/2} + C_1\|v_\epsilon\|_2^2 + C_2\|v_\epsilon\|_{2^*-1/2}^{2^*-1/2} - C_3\int_{\mathbb{R}^N} f^{2q}(t_\epsilon v_\epsilon)\,\mathrm{d}x.$$

Using the inequality

$$(a+b)^r \leqslant a^r + r(a+b)^{r-1}b, \quad a \geqslant 0,\ b \geqslant 0,\ r \geqslant 1,$$

we have

$$I_\lambda(t_\epsilon v_\epsilon) \leqslant \frac{1}{2N}\lambda^{-(N-2)/2}S^{N/2} + C_1\|v_\epsilon\|_2^2 + C_2\|v_\epsilon\|_{2^*-1/2}^{2^*-1/2} - C_3\int_{\mathbb{R}^N} f^{2q}(t_\epsilon v_\epsilon)\,\mathrm{d}x + O\big(\epsilon^{N-2}\big).$$

Let

$$\gamma(\epsilon) = \max\begin{cases}\{\epsilon,\ \epsilon^{(N-2)/4}\} & \text{if } N = 3,\\ \{\epsilon^2 \ln|\epsilon|,\ \epsilon^{(N-2)/4}\} & \text{if } N = 4,\\ \{\epsilon^2,\ \epsilon^{(N-2)/4}\} & \text{if } N \geqslant 5,\end{cases}$$

that is,

$$\gamma(\epsilon) = \begin{cases}\epsilon^{(N-2)/4} & \text{if } 3 \leqslant N < 10,\\ \epsilon^2 & \text{if } N \geqslant 10.\end{cases} \tag{2.9}$$

In view of (2.5), (2.6), and (2.9), we get that

$$I_\lambda(t_\epsilon v_\epsilon) \leqslant \frac{1}{2N}\lambda^{-(N-2)/2}S^{N/2} + C_4\gamma(\epsilon) - C_3\int_{\mathbb{R}^N} f^{2q}(t_\epsilon v_\epsilon)\,\mathrm{d}x.$$

Consider the function $\eta_\epsilon : [0,\infty) \to \mathbb{R}$ defined by

$$\eta_\epsilon(r) = \frac{\epsilon^{(N-2)/2}}{(\epsilon^2 + r^2)^{(N-2)/2}}.$$

Since $\phi \equiv 1$ in $B_1(0)$, in view of (2.7), there is a constant $C_5 > 0$ such that $v_\epsilon(x) \geqslant C_5\eta_\epsilon(|x|)$ for $|x| < 1$. Notice that $\eta_\epsilon$ is decreasing and $f$ is increasing, so that there is a positive constant $\alpha$ such that for $|x| < \epsilon$,

$$f\big(t_\epsilon v_\epsilon(x)\big) \geqslant f\big(T_1 C_5 \eta_\epsilon\big(|x|\big)\big) \geqslant f\big(T_1 C_5 \eta_\epsilon(\epsilon)\big) \geqslant f\big(\alpha\epsilon^{(2-N)/2}\big).$$

Then we can choose $\epsilon_1 > 0$ such that

$$\alpha\epsilon^{(2-N)/2} \geqslant 1, \qquad f\big(t_\epsilon v_\epsilon(x)\big) \geqslant f\big(\alpha\epsilon^{(2-N)/2}\big) \tag{2.10}$$

for $|x| < \epsilon$, $0 < \epsilon < \epsilon_1$. It follows from (2.10) and Lemma 1(ix) that

$$f^{2q}\big(t_\epsilon v_\epsilon(x)\big) \geqslant f^{2q}\big(\alpha\epsilon^{(2-N)/2}\big) \geqslant C\epsilon^{(2-N)q/2}$$

for $|x| < \epsilon$, $0 < \epsilon < \epsilon_1$. Hence

$$\begin{aligned}\int_{\mathbb{R}^N} f^{2q}(t_\epsilon v_\epsilon)\,\mathrm{d}x &= \int_{B_\epsilon(0)} f^{2q}(t_\epsilon v_\epsilon)\,\mathrm{d}x + \int_{\mathbb{R}^N\setminus B_\epsilon(0)} f^{2q}(t_\epsilon v_\epsilon)\,\mathrm{d}x \\ &\geqslant C\epsilon^{(2-N)q/2}\big|B_\epsilon(0)\big| = C\epsilon^{(2-N)q/2}\epsilon^N\omega_N.\end{aligned}$$

Then

$$\begin{aligned}I_\lambda(t_\epsilon v_\epsilon) &\leqslant \frac{1}{2N}\lambda^{-(N-2)/2}S^{N/2} + C_4\gamma(\epsilon) - C_3\int_{\mathbb{R}^N} f^{2q}(t_\epsilon v_\epsilon)\,\mathrm{d}x \\ &\leqslant \frac{1}{2N}\lambda^{-(N-2)/2}S^{N/2} + C_4\gamma(\epsilon) - C_3 C\epsilon^{(2-N)q/2+N}\omega_N.\end{aligned}$$

Since $(2-N)q/2+N < (N-2)/4$ when $3 \leqslant N < 10$, $(3N+2)/(2(N-2)) < q < 2^*$ and $(2-N)q/2+N < 2$ when $N \geqslant 10$, and thus $2 < q < 2^*$, we get our result. $\square$

*Remark 2.* In this paper, we assume that for almost every $\lambda \in [1/2, 1]$, if $\{v_n\} \subset E$ is a bounded $(\mathrm{PS})_{c_\lambda}$ sequence, then $v_n \geqslant 0$ in $E$. In fact, we have $(I'_\lambda(v_n), v_n^-) = o_n(1)$, and thus $\|v_n^-\| = o_n(1)$, from which we can derive that $\|v_n^+\| < \infty$, $I_\lambda(v_n^+) \to c_\lambda$, and $I'_\lambda(v_n^+) \to 0$.

**Lemma 8.** *Suppose that* (V) *and* (h1)–(h3) *are satisfied. Then there is a positive constant $\nu > 0$, independent of $\lambda \in [1/2, 1]$, such that if $v$ is a nontrivial critical point of the functional $I_\lambda$, then $\|v\| \geqslant \nu > 0$.*

*Proof.* By the equality $\langle I'_\lambda(v), v\rangle = 0$ we have

$$\int_{\mathbb{R}^N} |\nabla v|^2\,\mathrm{d}x + \int_{\mathbb{R}^N} V(x)f(v)f'(v)v\,\mathrm{d}x - \lambda\int_{\mathbb{R}^N} h\big(f(v)\big)f'(v)v\,\mathrm{d}x = 0.$$

Then by using condition (V), (2.2), and Lemma 1(vi),(vii) we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x + \frac{V_0}{2}\int_{\mathbb{R}^N} f^2(v)\,\mathrm{d}x \\
&\quad\leqslant \int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} V(x) f(v) f'(v) v \,\mathrm{d}x = \lambda \int_{\mathbb{R}^N} h\bigl(f(v)\bigr) f'(v) v \,\mathrm{d}x \\
&\quad\leqslant \lambda\delta \int_{\mathbb{R}^N} f^2(v)\,\mathrm{d}x + \lambda C_\delta \int_{\mathbb{R}^N} f^{2\cdot 2^*}(v)\,\mathrm{d}x \leqslant \lambda\delta \int_{\mathbb{R}^N} f^2(v)\,\mathrm{d}x + \lambda C_\delta 2^{2^*/2} \int_{\mathbb{R}^N} |v|^{2^*}\,\mathrm{d}x.
\end{aligned}$$

Combining this inequality and the Sobolev inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x + \left(\frac{V_0}{2} - \lambda\delta\right)\int_{\mathbb{R}^N} f^2(v)\,\mathrm{d}x \\
&\quad\leqslant \lambda C_\delta 2^{2^*/2} \int_{\mathbb{R}^N} |v|^{2^*}\,\mathrm{d}x \leqslant \lambda C \biggl(\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} f^2(v)\,\mathrm{d}x\biggr)^{2^*/2}.
\end{aligned}$$

Since $V_0/2 - \lambda\delta > 0$ with $\delta > 0$ small enough, from the last inequality and Lemma 1(iii) we obtain that there is a positive constant $\nu > 0$ such that

$$\|v\| \geqslant \biggl(\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x + \int_{\mathbb{R}^N} f^2(v)\,\mathrm{d}x\biggr)^{1/2} \geqslant \nu > 0. \qquad \square$$

We recall the Pohozaev identity with respect to problem (1.2). Since the proof is standard, we omit it.

**Lemma 9.** *For a.e.* $\lambda \in [1/2, 1]$, *if* $v$ *is a nontrivial critical point of* $I_\lambda$, *then* $v$ *satisfies*

$$\begin{aligned}
&\frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N} \nabla V(x)\cdot x f^2(v)\,\mathrm{d}x + \frac{N}{2}\int_{\mathbb{R}^N} V(x) f^2(v)\,\mathrm{d}x \\
&\quad= \lambda N \int_{\mathbb{R}^N} H\bigl(f(v)\bigr)\,\mathrm{d}x.
\end{aligned}$$

## 3 Proof of Theorem 1

In this section, we present the first result by some lemmas.

**Lemma 10.** *Suppose that* $V(x) \equiv V_0$ *and conditions* (h1)–(h3) *hold. Let* $\{v_n\} \subset E$ *be a bounded* $(\mathrm{PS})_{c_\lambda}$ *sequence for the functional* $I_\lambda$ *with*

$$0 < c_\lambda < \frac{1}{2N}\lambda^{-(N-2)/2} S^{N/2}.$$

*Then for each* $\lambda \in [1/2, 1]$, *there are a positive integer* $l \in \mathbb{N}\cup\{0\}$, *a subsequence of* $\{v_n\}$, *still denoted by* $\{v_n\}$, *and sequences* $\{y_n^k\} \subset \mathbb{R}^N$ *and* $\{\omega^k\} \subset E$ *for* $1 \leqslant k \leqslant l$ *such that*

(i) $v_n \rightharpoonup v$ *in* $E$ *with* $I'_\lambda(v) = 0$;

(ii) $y_n^k \to +\infty$ *and* $|y_n^k - y_n^{k'}| \to +\infty$ *for* $k \neq k'$ *and* $n \to +\infty$;

(iii) $\omega^k \neq 0$ *and* $I'_\lambda(\omega^k) = 0$ *for all* $1 \leqslant k \leqslant l$;
(iv) $\|v_n - v - \sum_{k=1}^{l} \omega^k(x - y_n^k)\| \to 0$;
(v) $I_\lambda(v_n) \to I_\lambda(v) + \sum_{k=1}^{l} I_\lambda(\omega^k)$, *where we agree that, in the case* $l = 0$, *the above conclusion holds without* $\omega^k$ *and* $\{y_n^k\}$.

*Proof.* Since $\{v_n\} \subset E$ is a bounded (PS)$_{c_\lambda}$ sequence, up to a subsequence, $v_n \rightharpoonup v$ in $E$, $v_n \to v$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$ $(1 \leqslant s < 2^*)$, and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^N$. By using the Lebesgue dominated convergence theorem, through the standard discussion, we can get that $I'_\lambda(v) = 0$. Thus (i) holds.

*Step 1*. Setting $v_n^1 := v_n - v$, the Brezis–Lieb lemma leads to

$$\|v_n\|^2 = \left\|v_n^1\right\|^2 + \|v\|^2 + o_n(1), \qquad \|v_n\|_{2^*}^{2^*} = \left\|v_n^1\right\|_{2^*}^{2^*} + \|v\|_{2^*}^{2^*} + o_n(1). \tag{3.1}$$

Next, we claim that

$$I_\lambda(v_n) - I_\lambda(v) = I_\lambda\big(v_n^1\big) + o_n(1) \tag{3.2}$$

and

$$I'_\lambda\big(v_n^1\big) \to 0 \quad \text{in } E^{-1}. \tag{3.3}$$

Denote

$$k(s) = V_0\big(s - f(s)f'(s)\big) + \lambda h\big(f(s)\big)f'(s) - \lambda 2^{2^*/2-1}|s|^{2^*-1}$$

and

$$K(s) = \int_0^s k(\tau)\,\mathrm{d}\tau = \frac{V_0}{2}\big[s^2 - f^2(s)\big] + \lambda H\big(f(s)\big) - \frac{\lambda}{2^*}2^{2^*/2-1}|s|^{2^*}.$$

The functions $k$ and $K$ enjoy the following properties under assumptions (V) and (h1)–(h3),

$$\lim_{s\to 0}\frac{k(s)}{s} = 0, \qquad \lim_{s\to+\infty}\frac{k(s)}{s^{2^*-1}} = 0, \tag{3.4}$$

$$\lim_{s\to 0}\frac{K(s)}{s^2} = 0, \qquad \lim_{s\to+\infty}\frac{K(s)}{s^{2^*}} = 0. \tag{3.5}$$

We are going to prove (3.4). From (h1), (h2), Lemma 1(iv),(v), and the fact that $f'(t) = (1 + 2f^2(t))^{-1/2}$ we have

$$\begin{aligned}\frac{k(s)}{s} &= V_0\left[1 - \frac{f(s)}{s}\cdot f'(s)\right] + \lambda\frac{h(f(s))}{f(s)}\cdot\frac{f(s)}{s}\cdot f'(s)\\ &\quad - \lambda 2^{2^*/2-1}|s|^{2^*-2} \to 0 \quad \text{as } s \to 0\end{aligned}$$

and

$$\begin{aligned}\frac{k(s)}{s^{2^*-1}} &= V_0\left[\frac{1}{s^{2^*-2}} - \frac{f(s)}{\sqrt{s}}\cdot\frac{\sqrt{s}}{s^{2^*-1}}f'(s)\right] + \lambda\frac{h(f(s))}{f^{2\cdot 2^*-1}(s)}\cdot\left(\frac{f^2(s)}{s}\right)^{2^*-1} f(s)f'(s)\\ &\quad - \lambda 2^{2^*/2-1} \to 0 \quad \text{as } s \to +\infty.\end{aligned}$$

The proof of (3.5) is similar to that of (3.4), so we omit it.

It follows from Lemma 3, (3.4), and (3.5) that

$$\lim_{n\to+\infty}\left[\int_{\mathbb{R}^N} K(v_n) - \int_{\mathbb{R}^N} K(v) - \int_{\mathbb{R}^N} K(v_n - v)\right] = 0. \tag{3.6}$$

Combining (3.1) and (3.6) with $v_n - v = v_n^1 \rightharpoonup 0$ in $E$, we get that $I_\lambda(v_n) - I_\lambda(v) = I_\lambda(v_n^1) + o_n(1)$, which gives (3.2).

It follows from (3.4) and (3.5) that for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$\begin{aligned}
\left|k(s)\right| &\leqslant \epsilon\left(|s| + |s|^{2^*-1}\right) + C_\epsilon |s|^{\alpha-1} \quad \forall 2<\alpha<2^*,\\
\left|k(s)s\right| &\leqslant \epsilon\left(|s|^2 + |s|^{2^*}\right) + C_\epsilon |s|^{\alpha} \quad \forall 2<\alpha<2^*, &(3.7)\\
\left|K(s)\right| &\leqslant \epsilon\left(|s|^2 + |s|^{2^*}\right) + C_\epsilon |s|^{\alpha} \quad \forall 2<\alpha<2^*. &(3.8)
\end{aligned}$$

Thanks to Lemmas 3 and 4, for any $\phi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} \left[k(v_n) - k\left(v_n^1\right) - k(v)\right]\phi \,\mathrm{d}x = o_n(1)\|\phi\|$$

and

$$\int_{\mathbb{R}^N} \left[|v_n|^{2^*-2}v_n - \left|v_n^1\right|^{2^*-2}v_n^1 - |v|^{2^*-2}v\right]\phi \,\mathrm{d}x = o_n(1)\|\phi\|.$$

Through the standard discussion, we can get that

$$\left\langle I_\lambda'\left(v_n^1\right), \phi\right\rangle + o_n(1) = \left\langle I_\lambda'(v_n) - I_\lambda'(v), \phi\right\rangle + o_n(1) = o_n(1).$$

Hence $\{v_n^1\}$ is a (PS) sequence of $I_\lambda$, which gives (3.3).

Now there are two cases that may occur.

(I) Vanishing: $\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|v_n^1|^2\,\mathrm{d}x = 0$;

(II) Nonvanishing: $\limsup_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|v_n^1|^2\,\mathrm{d}x > 0$.

In case (I), it follows from the Lions lemma that $v_n^1 \to 0$ in $L^\alpha(\mathbb{R}^N)$ with $\alpha \in (2, 2^*)$. By (3.7), (3.8), and the Lebesgue dominated convergence theorem we get

$$\int_{\mathbb{R}^N} K\left(v_n^1\right)\mathrm{d}x = o_n(1), \qquad \int_{\mathbb{R}^N} k\left(v_n^1\right)v_n^1\,\mathrm{d}x = o_n(1). \tag{3.9}$$

Then from (3.1), (3.6), and (3.9) we get

$$\begin{aligned}
I_\lambda(v_n) - I_\lambda(v) &= \frac{1}{2}\left(\|v_n\|^2 - \|v\|^2\right) - \int_{\mathbb{R}^N}\left[K(v_n) - K(v)\right]\mathrm{d}x\\
&\quad - \frac{\lambda}{2^*}2^{2^*/2-1}\int_{\mathbb{R}^N}\left(|v_n|^{2^*} - |v|^{2^*}\right)\mathrm{d}x\\
&= \frac{1}{2}\|v_n^1\|^2 - \frac{\lambda}{2^*}2^{2^*/2-1}\int_{\mathbb{R}^N}\left|v_n^1\right|^{2^*}\mathrm{d}x + o_n(1) &(3.10)
\end{aligned}$$

and

$$\begin{aligned}\langle I'_\lambda(v_n), v_n\rangle - \langle I'_\lambda(v), v\rangle &= \|v_n\|^2 - \|v\|^2 - \int\limits_{\mathbb{R}^N} \big[k(v_n)v_n - k(v)v\big]\,\mathrm{d}x \\ &\quad - 2^{2^*/2-1}\lambda \int\limits_{\mathbb{R}^N} \big[|v_n|^{2^*} - |v|^{2^*}\big]\,\mathrm{d}x \\ &= \big\|v_n^1\big\|^2 - 2^{2^*/2-1}\lambda \int\limits_{\mathbb{R}^N} \big|v_n^1\big|^{2^*}\,\mathrm{d}x + o_n(1). \end{aligned} \tag{3.11}$$

We may assume that $\|v_n^1\|^2 \to b$. Then by (3.11) and the fact that $\langle I'_\lambda(v_n), v_n\rangle - \langle I'_\lambda(v), v\rangle \to 0$ we obtain

$$2^{2^*/2-1}\lambda \int\limits_{\mathbb{R}^N} \big|v_n^1\big|^{2^*}\,\mathrm{d}x \to b.$$

By the definition of $S$ we can get that

$$S\Bigg(\int\limits_{\mathbb{R}^N} \big|v_n^1\big|^{2^*}\,\mathrm{d}x\Bigg)^{2/2^*} \leqslant \int\limits_{\mathbb{R}^N} \big|\nabla v_n^1\big|^2\,\mathrm{d}x \leqslant \big\|v_n^1\big\|^2, \quad \text{that is,} \quad S\bigg(\frac{b}{2^{2^*/2-1}\lambda}\bigg)^{2/2^*} \leqslant b,$$

where either $b = 0$, or $b \geqslant \lambda^{-(N-2)/2}S^{N/2}/2$.

Assume that $b \geqslant \lambda^{-(N-2)/2}S^{N/2}/2$. Then from (3.10) we obtain that

$$c_\lambda \geqslant \bigg(\frac{1}{2} - \frac{1}{2^*}\bigg)b \geqslant \frac{1}{2N}\lambda^{-(N-2)/2}S^{N/2},$$

which contradicts to Lemma 7. Thus $b = 0$. Therefore Lemma 10 is true with $l = 0$.

In case (II), there exist a positive constant $\delta^1 > 0$ and a sequence $y_n^1 \in \mathbb{R}^N$ such that $\int_{B_1(y_n^1)} |v_n^1|^2\,\mathrm{d}x = \delta^1/2 > 0$.

Let $w_n^1(x) := v_n^1(x + y_n^1)$. Then $\{w_n^1\}$ is bounded in $E$, and there exists a function $w^1 \in E \setminus \{0\}$ such that $w_n^1 \rightharpoonup w^1$ in $E$. It follows from $v_n^1 \rightharpoonup 0$ in $E$ that $y_n^1$ is unbounded, so we can assume that $|y_n^1| \to +\infty$. Moreover, wee can verify that $I'_\lambda(w^1) = 0$.

*Step 2*. Let $v_n^2(x) = v_n(x) - v(x) - w^1(x - y_n^1)$. Similarly to step 1, we can get

$$\begin{aligned} \big\|v_n^2\big\|^2 &= \|v_n\|^2 - \|v\|^2 - \big\|w^1\big\|^2 + o_n(1), \\ \big\|v_n^2\big\|_{2^*}^{2^*} &= \|v_n\|_{2^*}^{2^*} - \|v\|_{2^*}^{2^*} - \big\|w^1\big\|_{2^*}^{2^*} + o_n(1), \\ I_\lambda(v_n) - I_\lambda(v) - I_\lambda(w^1) &= I_\lambda\big(v_n^2\big) + o_n(1), \end{aligned} \tag{3.12}$$

and

$$I'_\lambda\big(v_n^2\big) \to 0 \quad \text{in } E^{-1}. \tag{3.13}$$

Define

$$\delta^2 = \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int\limits_{B_1(y)} \big|v_n^2\big|^2\,\mathrm{d}x.$$

If vanishing occurs, then $\|v_n^2\|^2 \to 0$, that is, $v_n(x) - v(x) - w^1(x - y_n^1) \to 0$ in $E$. From (3.12) we can get

$$I_\lambda(v_n) = I_\lambda(v) + I_\lambda\big(w^1\big) + o_n(1),$$

and Lemma 10 holds with $l = 1$. If nonvanishing occurs, then there are a function $w^2 \in E\backslash\{0\}$ and a sequence $\{y_n^2\} \subset \mathbb{R}^N$ such that $w_n^2(x) = v_n^2(x + y_n^2) \rightharpoonup w^2(x)$ in $E$. Thus by (3.13) we have $I'_\lambda(w^2) = 0$. Moreover, $v_n^2 \rightharpoonup 0$ in $E$, which tells us that $|y_n^2| \to +\infty$ and $|y_n^2 - y_n^1| \to +\infty$.

At last, by iteration we obtain the results of the lemma. □

**Lemma 11.** *Suppose that* $V(x) \equiv V_0$ *and* (h1)–(h3) *hold. Then for almost every* $\lambda \in [1/2, 1]$, $I_\lambda$ *has a positive critical point.*

*Proof.* From Lemmas 5, 6, and 7 we see that $I_\lambda$ possesses a bounded $(\mathrm{PS})_{c_\lambda}$ sequence $\{v_n\}$ and $0 < c_\lambda < \lambda^{-(N-2)/2}S^{N/2}/(2N)$. By Remark 2 we suppose that $v_n \geqslant 0$ in $E$. Then there exists a subsequence of $v_n$, still denoted by $v_n$, satisfying $v_n \rightharpoonup v$ in $E$. If $v \neq 0$, then the result is obvious. Otherwise, we assume that $v_n \rightharpoonup 0$ in $E$. Then we claim that there exists a positive constant $\sigma > 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |v_n|^2 \,\mathrm{d}x \geqslant \sigma > 0. \tag{3.14}$$

Otherwise, applying the Lions lemma, we obtain that $v_n \to 0$ in $L^\alpha(\mathbb{R}^N)$ for all $\alpha \in (2, 2^*)$. Similarly to (3.9), we have

$$\int_{\mathbb{R}^N} K(v_n) = o_n(1), \qquad \int_{\mathbb{R}^N} k(v_n)v_n = o_n(1).$$

Since $v_n$ is a $(\mathrm{PS})_{c_\lambda}$ sequence, we obtain

$$\begin{aligned} I_\lambda(v_n) - I_\lambda(v) &= \frac{1}{2}\big(\|v_n\|^2 - \|v\|^2\big) - \int_{\mathbb{R}^N} \big[K(v_n) - K(v)\big]\,\mathrm{d}x \\ &\quad - \frac{\lambda}{2^*}2^{2^*/2-1} \int_{\mathbb{R}^N} \big(|v_n|^{2^*} - |v|^{2^*}\big)\,\mathrm{d}x \\ &= \frac{1}{2}\|v_n\|^2 - \frac{\lambda}{2^*}2^{2^*/2-1} \int_{\mathbb{R}^N} |v_n|^{2^*}\,\mathrm{d}x + o_n(1) \end{aligned}$$

and

$$\begin{aligned} \big\langle I'_\lambda(v_n), v_n\big\rangle - \big\langle I'_\lambda(v), v\big\rangle &= \|v_n\|^2 - \|v\|^2 - \int_{\mathbb{R}^N} \big[k(v_n)v_n - k(v)v\big]\,\mathrm{d}x \\ &\quad - 2^{2^*/2-1}\lambda \int_{\mathbb{R}^N} \big[v_n^{2^*} - v^{2^*}\big]\,\mathrm{d}x \\ &= \|v_n\|^2 - 2^{2^*/2-1}\lambda \int_{\mathbb{R}^N} |v_n|^{2^*}\,\mathrm{d}x + o_n(1), \end{aligned}$$

that is,

$$c_\lambda + o_n(1) = \frac{1}{2}\|v_n\|^2 - \frac{\lambda}{2^*}2^{2^*/2-1}\int\limits_{\mathbb{R}^N} |v_n|^{2^*}\,\mathrm{d}x, \tag{3.15}$$

$$\|v_n\|^2 = 2^{2^*/2-1}\lambda\int\limits_{\mathbb{R}^N} |v_n|^{2^*}\,\mathrm{d}x + o_n(1). \tag{3.16}$$

We may assume that $\|v_n\|^2 \to \nu$. Then by (3.16) we obtain $2^{2^*/2-1}\lambda\int_{\mathbb{R}^N}|v_n|^{2^*}\mathrm{d}x \to \nu$.

If $\nu > 0$, then by the Sobolev embedding theorem we get that

$$\nu \geqslant \frac{1}{2}\lambda^{-(N-2)/2}S^{N/2}. \tag{3.17}$$

Joining (3.15), (3.16), and (3.17), we get that $c_\lambda \geqslant \lambda^{-(N-2)/2}S^{N/2}/(2N)$, which is a contradiction. Therefore we get (3.14), which implies that there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $|y_n| \to +\infty$ and

$$\int\limits_{B_1(y_n)} |v_n|^2\,\mathrm{d}x \geqslant \frac{\sigma}{2} > 0. \tag{3.18}$$

Let $w_n(x) = v_n(x+y_n)$. Then $\{w_n\}$ is also a $(\mathrm{PS})_{c_\lambda}$ sequence for $I_\lambda$. Furthermore, $w_n \rightharpoonup w \neq 0$ in $E$. The maximum principle implies $w > 0$, which ends the proof. □

*Proof of Theorem* 1. In view of the proof of Lemma 11, $I_\lambda$ has a bounded $(\mathrm{PS})_{c_\lambda}$ sequence $\{v_n\}$ for a.e. $\lambda \in [1/2, 1]$. Moreover, $v_n \geqslant 0$ and $0 < c_\lambda < \lambda^{-(N-2)/2}S^{N/2}/(2N)$. Lemma 10 implies that

$$c_\lambda = I_\lambda(v_n) + o_n(1) = I_\lambda(v) + \sum_{k=1}^{l} I_\lambda\big(\omega^k\big), \quad I'_\lambda(v) = 0, \quad I'_\lambda\big(\omega^k\big) = 0, \quad 1 \leqslant k \leqslant l.$$

By Lemma 9 we have $I_\lambda(v) > 0$ and $I_\lambda(\omega^k) \geqslant 0, 1 \leqslant k \leqslant l$. Hence $c_\lambda \geqslant I_\lambda(v) > 0$. Therefore there exist $\lambda_n \in [1/2, 1]$, $0 < c_{\lambda_n} < \lambda_n^{-(N-2)/2}S^{N/2}/(2N)$, and $v_{\lambda_n} \in E$ satisfying $\lambda_n \to 1$, $v_{\lambda_n} > 0$, $I'_{\lambda_n}(v_{\lambda_n}) = 0$, $0 < I_{\lambda_n}(v_{\lambda_n}) \leqslant c_{\lambda_n}$. By $I'_{\lambda_n}(v_{\lambda_n}) = 0$ and Lemma 9 we have

$$c_{\lambda_n} \geqslant I_{\lambda_n}(v_{\lambda_n}) = \frac{1}{N}\int\limits_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2\,\mathrm{d}x > 0.$$

Thus by the Sobolev embedding theorem we get the boundedness of $\|v_{\lambda_n}\|_{2^*}$. It follows from the Pohozaev identity, (2.3), and Lemma 1(vii) that

$$\begin{aligned}
&\frac{N-2}{2N}\int\limits_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2\,\mathrm{d}x + \frac{V_0}{2}\int\limits_{\mathbb{R}^N} f^2(v_{\lambda_n})\,\mathrm{d}x \\
&\quad = \lambda\int\limits_{\mathbb{R}^N} H\big(f(v_{\lambda_n})\big)\,\mathrm{d}x \leqslant \frac{\delta}{2}\int\limits_{\mathbb{R}^N} f^2(v_{\lambda_n})\,\mathrm{d}x + \frac{C_\delta}{2\cdot 2^*}\int\limits_{\mathbb{R}^N} f^{2\cdot 2^*}(v_{\lambda_n})\,\mathrm{d}x \\
&\quad \leqslant \frac{\delta}{2}\int\limits_{\mathbb{R}^N} f^2(v_{\lambda_n})\,\mathrm{d}x + \frac{C_\delta}{2^*}2^{2^*/2-1}\int\limits_{\mathbb{R}^N} |v_{\lambda_n}|^{2^*}\,\mathrm{d}x,
\end{aligned}$$

which implies that

$$\left(\frac{V_0}{2}-\frac{\delta}{2}\right)\int\limits_{\mathbb{R}^N} f^2(v_{\lambda_n})\,\mathrm{d}x \leqslant \frac{C_\delta}{2^*}2^{2^*/2-1}\int\limits_{\mathbb{R}^N}|v_{\lambda_n}|^{2^*}\,\mathrm{d}x-\frac{N-2}{2N}\int\limits_{\mathbb{R}^N}|\nabla v_{\lambda_n}|^2\,\mathrm{d}x.$$

Letting $\delta < V_0$, from the boundedness of $\|\nabla v_{\lambda_n}\|_2$ and $\|v_{\lambda_n}\|_{2^*}$ we get that $\int_{\mathbb{R}^N} f^2(v_{\lambda_n})\,\mathrm{d}x$ is bounded. Namely, there is $C_1 > 0$ such that $\int_{\mathbb{R}^N} f^2(v_{\lambda_n})\,\mathrm{d}x \leqslant C_1$, and then we get that $\int_{\mathbb{R}^N}|v_{\lambda_n}|^2\,\mathrm{d}x$ is bounded. In fact, according to (ix) of Lemma 1 and the Sobolev inequality we can get

$$\int\limits_{\{x:\,|v_{\lambda_n}(x)|\leqslant 1\}} v_{\lambda_n}^2\,\mathrm{d}x \leqslant \frac{1}{C^2}\int\limits_{\{x:\,|v_{\lambda_n}(x)|\leqslant 1\}} f^2(v_{\lambda_n})\,\mathrm{d}x \leqslant \frac{C_1}{C^2}$$

and

$$\begin{aligned}\int\limits_{\{x:\,|v_{\lambda_n}(x)|\geqslant 1\}} v_{\lambda_n}^2\,\mathrm{d}x &\leqslant \int\limits_{\{x:\,|v_{\lambda_n}(x)|\geqslant 1\}} v_{\lambda_n}^{2^*}\,\mathrm{d}x \leqslant C_2\Bigg(\int\limits_{\{x:\,|v_{\lambda_n}(x)|\geqslant 1\}}|\nabla v_{\lambda_n}|^2\,\mathrm{d}x\Bigg)^{2^*/2}\\ &\leqslant C_2\bigg(\int\limits_{\mathbb{R}^N}|\nabla v_{\lambda_n}|^2\,\mathrm{d}x\bigg)^{2^*/2}.\end{aligned}$$

So there exists a constant $C_3$ such that

$$\int\limits_{\mathbb{R}^N} v_{\lambda_n}^2\,\mathrm{d}x = \int\limits_{\{x:\,|v_{\lambda_n}(x)|\leqslant 1\}} v_{\lambda_n}^2\,\mathrm{d}x + \int\limits_{\{x:\,|v_{\lambda_n}(x)|\geqslant 1\}} v_{\lambda_n}^2\,\mathrm{d}x \leqslant C_3.$$

Then we get that $\{v_{\lambda_n}\}$ is bounded in $E$. Without loss of generality, we may assume that the limit of $I_{\lambda_n}(v_{\lambda_n})$ exists. By Lemma 5 we know that $\lambda \to c_\lambda$ is continuous from the left, and thus we have

$$0 \leqslant \lim_{n\to+\infty} I_{\lambda_n}(v_{\lambda_n}) \leqslant c_1 < \frac{1}{2N}S^{N/2}.$$

Observing that

$$I(v_{\lambda_n}) = I_{\lambda_n}(v_{\lambda_n}) + (\lambda_n - 1)\int\limits_{\mathbb{R}^N} H\big(f(v_{\lambda_n})\big)\,\mathrm{d}x,$$

where $I = I_1$, together with the boundedness of $\|v_{\lambda_n}\|$, we have

$$0 \leqslant \lim_{n\to+\infty} I(v_{\lambda_n}) \leqslant c_1 < \frac{1}{2N}S^{N/2}, \qquad \lim_{n\to+\infty} I'(v_{\lambda_n}) = 0.$$

By Lemma 8, $\|v_{\lambda_n}\| \geqslant \nu > 0$, where $\nu > 0$ is independent of $\lambda_n$. Note that $\|v_{\lambda_n}\|$ is bounded. Then following the lines as in the proof of Lemma 11, we obtain that problem (1.2) has a positive solution $v_0$. Moreover, by Lemma 10

$$I(v_0) \leqslant \lim_{n\to+\infty} I(v_{\lambda_n}) \leqslant c_1 < \frac{1}{2N}S^{N/2}.$$

Finally, we try to find the ground state solution. Let

$$m = \inf\big\{I(v)\colon\ v \in E,\ v \neq 0,\ I'(v) = 0\big\}.$$

Since $I'(v_0) = 0$, we have $m \leqslant I(v_0) < S^{N/2}/(2N)$. Lemma 9 implies that $m \geqslant 0$. Hence $0 \leqslant m \leqslant I(v_0) < S^{N/2}/(2N)$. By the definition of $m$ there exists $\{w_n\} \subset E$ such that $w_n \neq 0, I(w_n) \to m$, and $I'(w_n) = 0$. Lemma 8 implies that $\|w_n\| \geqslant \nu > 0$. Note that $\|w_n\| \leqslant C$. Following the same lines as in the proof of Lemma 11 and Remark 2, we have that there is $\{u_n\} \subset E$ such that $u_n \geqslant 0$, $u_n \rightharpoonup u_0 > 0$ in $E$, $I(u_n) \to m$, and $I'(u_n) = 0$. By Lemma 10, $I'(u_0) = 0$ and $I(u_0) \leqslant m$. The fact $I'(u_0) = 0$ implies that $I(u_0) \geqslant m$. Therefore $u_0 \neq 0$ satisfies $I(u_0) = m$ and $I'(u_0) = 0$. The proof is complete. □

## 4 Proof of Theorem 2

In this section, we assume that $V(x)$ is not identical to a constant. We discuss the family of functionals

$$I_\lambda(v) = \frac{1}{2}\int\limits_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x + \frac{1}{2}\int\limits_{\mathbb{R}^N} V(x) f^2(v)\,\mathrm{d}x - \lambda \int\limits_{\mathbb{R}^N} H\big(f(v)\big)\,\mathrm{d}x,$$

and

$$I_\lambda^\infty(v) = \frac{1}{2}\int\limits_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x + \frac{1}{2}\int\limits_{\mathbb{R}^N} V_\infty f^2(v)\,\mathrm{d}x - \lambda \int\limits_{\mathbb{R}^N} H\big(f(v)\big)\,\mathrm{d}x,$$

where $\lambda \in [1/2, 1]$.

**Lemma 12.** *For $\lambda \in [1/2, 1]$, let $v \in E$ be a nontrivial critical point for $I_\lambda^\infty$. Then there is a path $\gamma \in C([0,1], E)$ such that $\gamma(0) = 0$, $I_\lambda^\infty(\gamma(1)) < 0$, $0 \notin (\gamma(0,1])$, $v \in \gamma([0,1])$, and* $\max_{t\in[0,1]} I_\lambda^\infty(\gamma(t)) = I_\lambda^\infty(v)$.

*Proof.* Define

$$v^t(x) = \begin{cases} v(\frac{x}{t}) & \text{if } t > 0, \\ 0 & \text{if } t = 0, \end{cases}$$

where $v \in E$ is a nontrivial critical point for $I_\lambda^\infty$.

It follows from the Pohozaev identity that

$$\frac{N-2}{2}\int\limits_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x + \frac{N}{2}\int\limits_{\mathbb{R}^N} V_\infty f^2(v)\,\mathrm{d}x = \lambda N \int\limits_{\mathbb{R}^N} H\big(f(v)\big)\,\mathrm{d}x$$

and

$$\begin{aligned} I_\lambda^\infty\big(v^t\big) &= \frac{1}{2}\int\limits_{\mathbb{R}^N} \big(\big|\nabla v^t\big|^2 + V_\infty f^2\big(v^t\big)\big)\,\mathrm{d}x - \lambda \int\limits_{\mathbb{R}^N} H\big(f\big(v^t\big)\big)\,\mathrm{d}x \\ &= \frac{t^{N-2}}{2}\int\limits_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x + \frac{t^N}{2}\int\limits_{\mathbb{R}^N} V_\infty f^2(v)\,\mathrm{d}x - \lambda t^N \int\limits_{\mathbb{R}^N} H\big(f(v)\big)\,\mathrm{d}x \\ &= \left(\frac{t^{N-2}}{2} - \frac{N-2}{2N} t^N\right) \int\limits_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x. \end{aligned}$$

Hence

$$\max_{t>0} I_\lambda^\infty\big(v^t\big) = I_\lambda^\infty(v), \qquad I_\lambda^\infty\big(v^t\big) \to -\infty \quad \text{as } t \to +\infty,$$

and

$$\|v^t\|^2 \leqslant \|\nabla v^t\|_2^2 + V_\infty \|v^t\|_2^2 = t^{N-2}\|\nabla v\|_2^2 + t^N V_\infty \|v\|_2^2 \to 0 \quad \text{as } t \to 0.$$

Choose $t_0 > 1$ such that $I_\lambda^\infty(v^{t_0}) < 0$ and set $\gamma(t) = v^{t_0 t}$ for $t \in [0,1]$ and $\gamma(0) = 0$. This is the desired path. □

*Remark 3.* From Theorem 1 we know that $I_\lambda^\infty(v)$ has a ground state solution for $\lambda \in [1/2, 1]$.

**Lemma 13.** *Assume that condition* (V) *and* (h1)–(h3) *are satisfied. Then for almost every* $\lambda \in [1/2,1]$, $I_\lambda(v)$ *has a positive critical point.*

*Proof.* For a.e. $\lambda \in [1/2,1]$, it follows from Lemma 7 and Remark 2 that there exists $\{v_n\} \in E$, $v_n \geqslant 0$, such that $v_n \rightharpoonup v$ in $E$, $I_\lambda(v_n) \to c_\lambda$, $I'_\lambda(v_n) \to 0$, and

$$0 < c_\lambda < \frac{1}{2N}\lambda^{-(N-2)/2} S^{N/2}.$$

We claim that $v \neq 0$. Suppose by contradiction that $v = 0$. Similarly to the proof of Lemma 11, there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that $|y_n| \to +\infty$ and $w_n(x) = v_n(x + y_n) \rightharpoonup w(x) \neq 0$ in $E$. Since $v_n \rightharpoonup 0$ in $E$, we have

$$I_\lambda^\infty(v_n) \to c_\lambda, \qquad \big(I_\lambda^\infty\big)'(v_n) \to 0.$$

Thus

$$I_\lambda^\infty(w_n) \to c_\lambda, \qquad \big(I_\lambda^\infty\big)'(w_n) \to 0.$$

Since $w_n \rightharpoonup w \neq 0$ in $E$, we have $(I_\lambda^\infty)'(w) = 0$. From Lemma 10, $c_\lambda \geqslant I_\lambda^\infty(w)$. Remark 3 implies that $I_\lambda^\infty$ has a ground state solution $u$. Thus $c_\lambda \geqslant I_\lambda^\infty(u)$. It follows from Lemma 12 that

$$c_\lambda \geqslant I_\lambda^\infty(u) = \max_{t\in[0,1]} I_\lambda^\infty\big(\gamma(t)\big),$$

where $\gamma \in C([0,1], E)$ is such that $\gamma(0) = 0$, $I_\lambda^\infty(\gamma(1)) < 0$, $0 \notin \gamma((0,1])$, $u \in \gamma([0,1])$. Thanks to condition (V), we have $I_\lambda(\gamma(t)) < I_\lambda^\infty(\gamma(t))$ for all $t \in (0,1]$. Thus the definition of $c_\lambda$ implies that

$$c_\lambda \leqslant \max_{t\in[0,1]} I_\lambda\big(\gamma(t)\big) < \max_{t\in[0,1]} I_\lambda^\infty\big(\gamma(t)\big) \leqslant c_\lambda,$$

which is a contradiction. Then we obtain $v \neq 0$. A standard argument shows that $v > 0$. Thus Lemma 13 holds. □

**Lemma 14.** *Suppose that conditions* (h1)–(h3) *and* (V) *hold. Let* $\{v_n\} \subset E$ *be a bounded* $(\mathrm{PS})_{c_\lambda}$ *sequence for the functional* $I_\lambda$ *with*

$$v_n \geqslant 0, \qquad 0 < c_\lambda < \frac{1}{2N}\lambda^{-(N-2)/2} S^{N/2}.$$

*Then there is a subsequence of* $\{v_n\}$, *still denoted by* $\{v_n\}$, *such that*

(i) $v_n \rightharpoonup v$ *in* $E$ *with* $I'_\lambda(v) = 0$;
(ii) $c_\lambda \geqslant I_\lambda(v)$.

*Proof.* Since $\{v_n\} \subset E$ is bounded, up to a subsequence, we may assume that $v_n \rightharpoonup v$ in $E$. Then we can get that $I'_\lambda(v) = 0$. Thus (i) holds.

Set $v_n^1 = v_n - v$. The Brezis–Lieb lemma leads to

$$\|v_n\|^2 = \|v_n^1\|^2 + \|v\|^2 + o_n(1), \qquad \|v_n\|_{2^*}^{2^*} = \|v_n^1\|_{2^*}^{2^*} + \|v\|_{2^*}^{2^*} + o_n(1). \tag{4.1}$$

Next, we claim that

$$I_\lambda(v_n) - I_\lambda(v) = I_\lambda^\infty\big(v_n^1\big) + o_n(1) \tag{4.2}$$

and

$$\big(I_\lambda^\infty\big)'\big(v_n^1\big) \to 0 \quad \text{in } E^{-1}. \tag{4.3}$$

Denote

$$k_\infty(s) = V_\infty\big(s - f(s)f'(s)\big) + \lambda h\big(f(s)\big)f'(s) - \lambda 2^{2^*/2-1}|s|^{2^*-1},$$
$$k(x,s) = V(x)\big(s - f(s)f'(s)\big) + \lambda h\big(f(s)\big)f'(s) - \lambda 2^{2^*/2-1}|s|^{2^*-1},$$
$$K_\infty(s) = \int_0^s k_\infty(\tau)\,\mathrm{d}\tau = \frac{V_\infty}{2}\big[s^2 - f^2(s)\big] + \lambda H\big(f(s)\big) - \frac{\lambda}{2^*}2^{2^*/2-1}|s|^{2^*},$$

and

$$K(x,s) = \int_0^s k(x,\tau)\,\mathrm{d}\tau = \frac{V(x)}{2}\big[s^2 - f^2(s)\big] + \lambda H\big(f(s)\big) - \frac{\lambda}{2^*}2^{2^*/2-1}|s|^{2^*}.$$

Then

$$I_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^N}\big(|\nabla v|^2 + V(x)v^2\big)\,\mathrm{d}x - \int_{\mathbb{R}^N} K(x,v)\,\mathrm{d}x - \frac{\lambda}{2^*}2^{2^*/2-1}\int_{\mathbb{R}^N}|v|^{2^*}\,\mathrm{d}x,$$

and

$$I_\lambda^\infty(v) = \frac{1}{2}\int_{\mathbb{R}^N} t\big(|\nabla v|^2 + V_\infty v^2 t\big)\,\mathrm{d}x - \int_{\mathbb{R}^N} K_\infty(v)\,\mathrm{d}x - \frac{\lambda}{2^*}2^{2^*/2-1}\int_{\mathbb{R}^N}|v|^{2^*}\,\mathrm{d}x.$$

Similarly to the proof of (3.4), the functions $k_\infty(s)$, $k(x,s)$, $K_\infty(s)$, and $K(x,s)$ enjoy the following properties under assumptions (V) and (h1)–(h3):

$$\lim_{s\to 0}\frac{k_\infty(s)}{s} = 0, \qquad \lim_{s\to+\infty}\frac{k_\infty(s)}{s^{2^*-1}} = 0, \qquad \lim_{s\to 0}\frac{k(x,s)}{s} = 0, \qquad \lim_{s\to+\infty}\frac{k(x,s)}{s^{2^*-1}} = 0, \tag{4.4}$$
$$\lim_{s\to 0}\frac{K_\infty(s)}{s^2} = 0, \qquad \lim_{s\to+\infty}\frac{K_\infty(s)}{s^{2^*}} = 0, \qquad \lim_{s\to 0}\frac{K(x,s)}{s^2} = 0, \qquad \lim_{s\to+\infty}\frac{K(x,s)}{s^{2^*}} = 0. \tag{4.5}$$

Since $v_n^1 = v_n - v$ and $I_\lambda(v_n) \to c_\lambda$, by Lemma 3 and (4.1) we have

$$c_\lambda - I_\lambda(v) = \frac{1}{2}\|v_n^1\|^2 - \int_{\mathbb{R}^N} K\big(x, v_n^1\big)\,\mathrm{d}x - \frac{\lambda}{2^*}2^{2^*/2-1}\int_{\mathbb{R}^N}\big|v_n^1\big|^{2^*}\,\mathrm{d}x + o_n(1). \tag{4.6}$$

It follows from (4.4) and Lemma 3 that

$$\lim_{n\to+\infty}\left[\int_{\mathbb{R}^N} K(x,v_n)\,\mathrm{d}x-\int_{\mathbb{R}^N} K(x,v)\,\mathrm{d}x-\int_{\mathbb{R}^N} K\big(x,v_n^1\big)\,\mathrm{d}x\right]=0. \tag{4.7}$$

Combining (4.7) with $v_n^1\rightharpoonup 0$ in $E$, we get that

$$I_\lambda(v_n)-I_\lambda(v)=I_\lambda\big(v_n^1\big)+o_n(1)=I_\lambda^\infty\big(v_n^1\big)+o_n(1), \tag{4.8}$$

which gives (4.2).

It follows from (4.4) and (4.5) that for any $\epsilon>0$, there exists a constant $C_\epsilon>0$ such that

$$\big|k_\infty(s)\big|,\big|k(x,s)\big|\leqslant\epsilon\big(|s|+|s|^{2^*-1}\big)+C_\epsilon|s|^{\alpha-1}\quad\forall 2<\alpha<2^*. \tag{4.9}$$

By an argument similar to that in the proof of Lemma 8.1 in [27], for any $\phi\in C_0^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N}\big[k(x,v_n)-k\big(x,v_n^1\big)-k(x,v)\big]\phi\,\mathrm{d}x=o_n(1)\|\phi\|. \tag{4.10}$$

Thanks to (4.6)–(4.10) and the fact $v_n^1\rightharpoonup 0$ in $E$, we get that

$$\big\langle I_\lambda'\big(v_n^1\big),\phi\big\rangle+o_n(1)=\big\langle I_\lambda'(v_n)-I_\lambda'(v),\,\phi\big\rangle+o_n(1)=o_n(1),$$

which implies that $I_\lambda'(v_n^1)=o_n(1)$. Hence

$$c_\lambda-I_\lambda(v)=I_\lambda^\infty\big(v_n^1\big)+o_n(1),\quad \big(I_\lambda^\infty\big)'\big(v_n^1\big)=o_n(1), \tag{4.11}$$

which gives (4.3).

We will consider two cases.

*Case 1*. Vanishing: $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|v_n^1|^2\,\mathrm{d}x=0$.

The Lions lemma implies that $v_n^1\to 0$ in $L^\alpha(\mathbb{R}^N)$ with $\alpha\in(2,2^*)$. It follows from (4.10) and (4.11) that

$$c_\lambda-I_\lambda(v)=\frac12\int_{\mathbb{R}^N}\big(\big|\nabla v_n^1\big|^2+V_\infty\big|v_n^1\big|^2\big)\,\mathrm{d}x-\frac{\lambda}{2^*}2^{2^*/2-1}\int_{\mathbb{R}^N}\big|v_n^1\big|^{2^*}+o_n(1) \tag{4.12}$$

and

$$\int_{\mathbb{R}^N}\big(\big|\nabla v_n^1\big|^2+V(x)\big|v_n^1\big|^2\big)\,\mathrm{d}x-\lambda 2^{2^*/2-1}\int_{\mathbb{R}^N}\big|v_n^1\big|^{2^*}=o_n(1). \tag{4.13}$$

Thus we can get from (4.12) and (4.13) that

$$c_\lambda-I_\lambda(v)=\frac12\int_{\mathbb{R}^N}\big(V_\infty-V(x)\big)\big|v_n^1\big|^2\,\mathrm{d}x+\left(\frac12-\frac1{2^*}\right)2^{2^*/2-1}\int_{\mathbb{R}^N}\big|v_n^1\big|^{2^*}\,\mathrm{d}x+o_n(1)\geqslant 0,$$

that is, $c_\lambda\geqslant I_\lambda(v)$.

*Case 2*. Nonvanishing: $\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B_1(y)}|v_n^1|^2\,\mathrm{d}x > 0$. There exist a constant $\beta_1 > 0$ and $y_n^1 \in \mathbb{R}^N$ such that

$$\int\limits_{B_1(y_n^1)} \left|v_n^1\right|^2 \mathrm{d}x \geqslant \frac{\beta_1}{2} > 0, \qquad |y_n^1| \to +\infty.$$

Then $v_n^1(x+y_n^1) \rightharpoonup v^1(x) \neq 0$ in $E$,

$$c_\lambda - I_\lambda(v) = I_\lambda^\infty\big(v_n^1\big(x+y_n^1\big)\big) + o_n(1), \quad \big(I_\lambda^\infty\big)'\big(v_n^1\big(x+y_n^1\big)\big) = o_n(1),$$

and thus $(I_\lambda^\infty)'(v^1) = 0$. If

$$c_\lambda - I_\lambda(v) < \frac{1}{2N}\lambda^{-(N-2)/2}S^{N/2},$$

then similarly to the proof of Lemma 10, we can get this lemma. Otherwise, let $v_n^2(x) = v_n^1(x+y_n^1) - v^1(x)$. Then

$$c_\lambda - I_\lambda(v) - I_\lambda^\infty\big(v^1\big) + o_n(1) = I_\lambda^\infty\big(v_n^2\big), \qquad \big(I_\lambda^\infty\big)'\big(v_n^2\big) = o_n(1),$$

and either

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int\limits_{B_1(y)} \left|v_n^2\right|^2 \mathrm{d}x = 0, \tag{4.14}$$

or there is a positive constant $\beta_2 > 0$ such that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int\limits_{B_1(y)} \left|v_n^2\right|^2 \mathrm{d}x \geqslant \frac{\beta_2}{2} > 0. \tag{4.15}$$

If (4.14) holds, then by case 1 and Lemma 10 we have

$$c_\lambda - I_\lambda(v) - I_\lambda^\infty\big(v^1\big) \geqslant 0, \qquad I_\lambda^\infty\big(v^1\big) \geqslant 0.$$

Thus $c_\lambda \geqslant I_\lambda(v)$. So we may assume that (4.15) holds, and continuing the above process, we get $v_n^i \in E$, $y_n^i \in \mathbb{R}^N$, $|y_n^i| \to +\infty,\ i \in \mathbb{N}$, such that

$$\big(I_\lambda^\infty\big)'\big(v^i\big) = 0, \quad v_n^i\big(x+y_n^i\big) \rightharpoonup v^i(x) \neq 0 \quad \text{in } E,$$

and

$$c_\lambda - I_\lambda(v) - \sum_{i=1}^{j} I_\lambda^\infty\big(v^i\big) + o_n(1) = I_\lambda^\infty\big(v_n^{j+1}\big), \qquad \big(I_\lambda^\infty\big)'\big(v_n^{j+1}\big) = o_n(1),$$

where $v_n^{j+1}(x) = v_n^j(x+y_n^j) - v^j(x)$. Since $(I_\lambda^\infty)'(v^i) = 0$, we get

$$I_\lambda^\infty\big(v^i\big) = \frac{1}{N}\int\limits_{\mathbb{R}^N} \left|\nabla v^i\right|^2 \mathrm{d}x. \tag{4.16}$$

We claim that there is $\tau > 0$, independent of $i$, such that

$$\int_{\mathbb{R}^N} \left|\nabla v^i\right|^2 \mathrm{d}x \geqslant \tau > 0. \tag{4.17}$$

In fact, by (h1)–(h3) and $(I_\lambda^\infty)'(v^i) = 0$, for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$\int_{\mathbb{R}^N} \left(\left|\nabla v^i\right|^2 + V_\infty\left|v^i\right|^2\right) \mathrm{d}x \leqslant \epsilon \int_{\mathbb{R}^N} \left|v^i\right|^2 \mathrm{d}x + C_\epsilon \int_{\mathbb{R}^N} \left|v^i\right|^{2^*} \mathrm{d}x.$$

Then we have

$$\int_{\mathbb{R}^N} \left(\left|\nabla v^i\right|^2 + (V_\infty - \epsilon)\left|v^i\right|^2\right) \mathrm{d}x \leqslant C_\epsilon \int_{\mathbb{R}^N} \left|v^i\right|^{2^*} \mathrm{d}x.$$

Choosing $\epsilon \in (0, V_\infty)$ and combining this inequality with the Sobolev inequality, we have

$$\int_{\mathbb{R}^N} \left|\nabla v^i\right|^2 \mathrm{d}x \leqslant \int_{\mathbb{R}^N} \left(\left|\nabla v^i\right|^2 + (V_\infty - \epsilon)\left|v^i\right|^2\right) \mathrm{d}x \leqslant C_\epsilon \int_{\mathbb{R}^N} \left|v^i\right|^{2^*} \mathrm{d}x \leqslant C\left(\int_{\mathbb{R}^N} \left|\nabla v^i\right|^2 \mathrm{d}x\right)^{2^*/2},$$

which shows that (4.17) holds. By (4.16) and (4.17), at some $j = l$, we get

$$c_\lambda - I_\lambda(v) - \sum_{i=1}^{j} I_\lambda^\infty(v^i) < \frac{1}{2N}\lambda^{-(N-2)/2}S^{N/2}.$$

Similarly to the proof of Lemma 10, we obtain the results of Lemma 14. □

*Proof of Theorem* 2. By Lemma 13, for a.e. $\lambda \in [1/2, 1]$, $I_\lambda$ has a bounded sequence $\{v_n\} \subset E$ such that

$$I_\lambda(v_n) \to c_\lambda, \quad I'_\lambda(v_n) \to 0, \quad 0 < c_\lambda < \frac{1}{2N}\lambda^{-(N-2)/2}S^{N/2}, \quad v_n \rightharpoonup v \neq 0 \quad \text{in } E.$$

Then from Lemma 14 we obtain $c_\lambda \geqslant I_\lambda(v)$ and $I'_\lambda(v) = 0$. Therefore there exist $\lambda_n \in [1/2, 1]$ and $v_{\lambda_n} \in E$, $v_{\lambda_n} \neq 0$, satisfying

$$\lambda_n \to 1, \qquad I'_{\lambda_n}(v_{\lambda_n}) = 0, \qquad 0 < c_{\lambda_n} < \frac{1}{2N}\lambda_n^{-(N-2)/2}S^{N/2}, \qquad 0 < I_{\lambda_n}(v_{\lambda_n}) \leqslant c_{\lambda_n} \leqslant c_{1/2}.$$

Then we can deduce that $\{v_{\lambda_n}\}$ is bounded in $E$. It follows from $0 < I_{\lambda_n}(v_{\lambda_n}) \leqslant c_{\lambda_n} \leqslant c_{1/2}$ and the Pohozaev identity that

$$\int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2 \mathrm{d}x \leqslant \frac{1}{2}\int_{\mathbb{R}^N} \nabla V(x) \cdot x f^2(v_{\lambda_n}) \mathrm{d}x + Nc_{1/2}.$$

By the Hölder inequality, the Sobolev inequality, assumption (V), and Lemma 2 we get

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2\,\mathrm{d}x &\leqslant \frac{1}{2}\big|\nabla V(x)\cdot x\big|_{2^*/(2^*-\sigma)}\biggl(\int_{\mathbb{R}^N} f^{2\cdot 2^*/\sigma}(v_{\lambda_n})\,\mathrm{d}x\biggr)^{\sigma/2^*} + Nc_{1/2}\\ &\leqslant C_1\biggl(\int_{\mathbb{R}^N} |v_{\lambda_n}|^{2^*}\,\mathrm{d}x\biggr)^{\sigma/2^*} + Nc_{1/2} \leqslant C_2\biggl(\int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2\,\mathrm{d}x\biggr)^{\sigma/2} + Nc_{1/2}.\end{aligned}$$

Since $\sigma \in [1,2)$, we can get that

$$\int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2\,\mathrm{d}x \leqslant C_3$$

for some $C_3 > 0$. Then using the same arguments as in the proof of Lemma 8, we obtain

$$\begin{aligned}A(v_{\lambda_n}) &\leqslant C_4\int_{\mathbb{R}^N} \big(|\nabla v_{\lambda_n}|^2 + f^2(v_{\lambda_n})\big)\,\mathrm{d}x \leqslant C_5\int_{\mathbb{R}^N} |v_{\lambda_n}|^{2^*}\,\mathrm{d}x\\ &\leqslant C_6\biggl(\int_{\mathbb{R}^N} |\nabla v_{\lambda_n}|^2\,\mathrm{d}x\biggr)^{2^*/2} \leqslant C_6 C_3^{2^*/2}.\end{aligned}$$

Combining this inequality with the coercivity of $A$, we get that $\{v_{\lambda_n}\}$ is bounded in $E$. Set

$$I(v_{\lambda_n}) = I_{\lambda_n}(v_{\lambda_n}) + (\lambda_n - 1)\int_{\mathbb{R}^N} H\big(f(v_{\lambda_n})\big)\,\mathrm{d}x,$$

where $I = I_1$. Then we have

$$0 \leqslant \lim_{n\to+\infty} I(v_{\lambda_n}) \leqslant c_1 < \frac{1}{2N}S^{N/2}, \qquad \lim_{n\to+\infty} I'(v_{\lambda_n}) = 0.$$

By Lemma 8, $\|v_{\lambda_n}\| \geqslant \nu > 0$, where $\nu > 0$ is independent of $\lambda_n$. Note that $\|v_{\lambda_n}\|$ is bounded. Then following the same lines as in the proof of Lemma 14, we can get that $v_{\lambda_n} \rightharpoonup v_0 \neq 0$ in $E$. Moreover, by Lemma 14 we get

$$I(v_0) \leqslant \lim_{n\to+\infty} I(v_{\lambda_n}) \leqslant c_1 < \frac{1}{2N}S^{N/2}, \quad I'(v_0) = 0.$$

At last, we try to find the ground state solution. Define

$$m = \inf\big\{I(v)\colon\ v \in E,\ v \neq 0,\ I'(v) = 0\big\}.$$

Since $I'(v_0) = 0$, $m \leqslant I(v_0) < S^{N/2}/(2N)$. By the definition of $m$ there exists $\{w_n\} \subset E$ such that $w_n \neq 0$, $I(w_n) \to m$, and $I'(w_n) = 0$. Lemma 8 implies that $\|w_n\| \geqslant \nu > 0$. Similarly to the proof of the boundedness of $\|v_{\lambda_n}\|$ above, we have $\|w_n\| \leqslant C$. Following the same lines as the proof of Lemma 13, we have $w_n \rightharpoonup w$ in $E$, $w \neq 0$. From Lemma 14 we get $I'(w) = 0$ and $I(w) \leqslant m$. Moreover, $I'(w) = 0$ implies that $I(w) \geqslant m$. Therefore $w \neq 0$ satisfies $I(w) = m$ and $I'(w) = 0$. The proof is complete. □

## References


1. H. Berestycki and P.-L. Lions, Nonlinear scalar field equations, I. Existence of a ground state, *Arch. Ration. Mech. Anal.*, **82**(4):313–345, 1983, https://doi.org/10.1007/BF00250555.

2. H. Berestycki and P.-L. Lions, Nonlinear scalar field equations, II. Existence of infinitely many solutions, *Arch. Ration. Mech. Anal.*, **82**(4):347–375, 1983, https://doi.org/10.1007/BF00250556.

3. H. Brezis and L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, *Commun. Pure Appl. Math.*, **36**(4):437–477, 1983, https://doi.org/10.1002/cpa.3160360405.

4. M. Colin and L. Jeanjean, Solutions for a quasilinear Schrödinger equation: A dual approach, *Nonlinear Anal., Theory Methods Appl.*, **56**(2):213–226, 2004, https://doi.org/10.1016/j.na.2003.09.008.

5. J.M. do Ó and U. Severo, Quasilinear Schrödinger equations involving concave and convex nonlinearities, *Commun. Pure Appl. Anal.*, **8**(2):621–644, 2009, https://doi.org/10.3934/cpaa.2009.8.621.

6. X.-D. Fang and A. Szulkin, Multiple solutions for a quasilinear Schrödinger equation, *J. Differ. Equations*, **254**(4): 2015–2032, 2013, https://doi.org/10.1016/j.jde.2012.11.017.

7. Y. Guo and Z. Tang, Ground state solutions for the quasilinear Schrödinger equation, *Nonlinear Anal., Theory Methods Appl.*, **75**(6):3235–3248, 2012, https://doi.org/10.1016/j.na.2011.12.024.

8. Y. He, J. Chen, and Q. Gao, A nonexistence result for a class of quasilinear Schrödinger equations with Berestycki–Lions conditions, *Filomat*, **37**(5):1497–1506, 2023, https://doi.org/10.2298/FIL2305497H.

9. Y. He, D. Qin, and D. Chen, Ground state solutions for Hamiltonian elliptic systems with super or asymptotically quadratic nonlinearity, *Bound. Value Probl.*, **158**(1):1–20, 2019, https://doi.org/10.1186/s13661-019-1270-5.

10. Y.B. He, J.L. Zhou, and X.Y. Lin, Existence of ground state solutions for quasilinear Schrödinger equations with general Choquard type nonlinearity, *Bound. Value Probl.*, **2019**:158, 2020, https://doi.org/ 10.1186/s13661-020-01420-6.

11. L. Jeanjean, On the existence of bounded Palais–Smale sequences and application to a Landesman–Lazer-type problem set on $\mathbb{R}^N$, *Proc. R. Soc. Edinb., Sect. A, Math.*, **129**(4):787–809, 1999, https://doi.org/10.1017/S0308210500013147.

12. P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. I, *Ann. Inst. H. Poincaré, Anal. Non Linéaire*, **1**(2):109–145, 1984, https://doi.org/10.1016/S0294-1449(16)(16)30428-0.

13. P.-L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. II, *Ann. Inst. H. Poincaré, Anal. Non Linéaire*, **1**(4):223–283, 1984, https://doi.org/10.1016/S0294-1449(16)30422-X.

14. H. Liu and L. Zhao, Existence results for quasilinear Schrödinger equations with a general nonlinearity, *Commun. Pure Appl. Anal.*, **19**(6):3429–3444, 2020, https://doi.org/10.3934/cpaa.2020059.

15. J. Liu and Z.-Q. Wang, Soliton solutions for quasilinear Schrödinger equations, I, *Proc. Am. Math. Soc.*, **131**(2):441–448, 2003, https://doi.org/10.1090/S0002-9939-02-06783-7.

16. J.-Q. Liu, X.-Q. Liu, and Z.-Q. Wang, Multiple sign-changing solutions for quasilinear elliptic equations via perturbation method, *Commun. Partial Differ. Equations*, **39**(12):2216–2239, 2014, https://doi.org/10.1080/03605302.2014.942738.

17. J.-Q. Liu, Y.-Q. Wang, and Z.-Q. Wang, Soliton solutions for quasilinear Schrödinger equations, II, *J. Differ. Equations*, **187**(2):473–493, 2003, https://doi.org/10.1016/S0022-0396(02)00064-5.

18. J.-Q. Liu, Y.-Q. Wang, and Z.-Q. Wang, Solutions for quasilinear Schrödinger equations via the Nehari method, *Commun. Partial Differ. Equations*, **29**(5-6):879–901, 2004, https://doi.org/10.1081/PDE-120037335.

19. X. Liu, J. Liu, and Z.-Q. Wang, Ground states for quasilinear Schrödinger equations with critical growth, *Calc. Var. Partial Differ. Equ.*, **46**(3–4):641–669, 2013, https://doi.org/10.1007/s00526-012-0497-0.

20. X.-Q. Liu, J.-Q. Liu, and Z.-Q. Wang, Quasilinear elliptic equations via perturbation method, *Proc. Am. Math. Soc.*, **141**(1):253–263, 2013, https://doi.org/10.1090/S0002-9939-2012-11293-6.

21. X.-Q. Liu, J.-Q. Liu, and Z.-Q. Wang, Quasilinear elliptic equations with critical growth via perturbation method, *J. Differ. Equations*, **254**(1):102–124, 2013, https://doi.org/10.1016/j.jde.2012.09.006.

22. A. Moameni, Existence of soliton solutions for a quasilinear Schrödinger equation involving critical exponent in $\mathbb{R}^N$, *J. Differ. Equations*, **229**(2):570–587, 2006, https://doi.org/10.1016/j.jde.2006.07.001.

23. M. Poppenberg, K. Schmitt, and Z.-Q. Wang, On the existence of soliton solutions to quasilinear Schrödinger equations, *Calc. Var. Partial Differ. Equ.*, **14**(3):329–344, 2002, https://doi.org/10.1007/s005260100105.

24. D. Ruiz and G. Siciliano, Existence of ground states for a modified nonlinear Schrödinger equation, *Nonlinearity*, **23**(5):1221–1233, 2010, https://doi.org/10.1088/0951-7715/23/5/011.

25. E.A.B. Silva and G.F. Vieira, Quasilinear asymptotically periodic Schrödinger equations with critical growth, *Calc. Var. Partial Differ. Equ.*, **39**(1–2):1–33, 2010, https://doi.org/10.1007/s00526-009-0299-1.

26. E.A.B. Silva and G.F. Vieira, Quasilinear asymptotically periodic Schrödinger equations with subcritical growth, *Nonlinear Anal., Theory Methods Appl.*, **72**(6):2935–2949, 2010, https://doi.org/10.1016/j.na.2009.11.037.

27. M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996, https://doi.org/10.1007/978-1-4612-4146-1.

28. X. Wu, Multiple solutions for quasilinear Schrödinger equations with a parameter, *J. Differ. Equations*, **256**(7):2619–2632, 2014, https://doi.org/10.1016/j.jde.2014.01.026.

29. Y.-F. Xue, J.-X. Han, and X.-C. Zhu, Existence results for quasilinear Schrödinger equations under a general critical growth term, *Ann. Pol. Math.*, **126**(2):177–196, 2021, https://doi.org/10.4064/ap200709-11-1.

30. Y.-F. Xue, Y. Lv, and C.-L.Tang, Existence and nonexistence results for quasilinear Schrödinger equations under a general nonlinear term, *Ann. Pol. Math.*, **120**(3):271–293, 2017, https://doi.org/10.4064/ap170502-2-12.

31. X. Zeng, Y. Zhang, and H.-S. Zhou, Positive solutions for a quasilinear Schrödinger equation involving Hardy potential and critical exponent, *Commun. Contemp. Math.*, **16**(6):1450034, 2014, https://doi.org/10.1142/S0219199714500345.

32. J. Zhang and W.M. Zou, The critical case for a Berestycki–Lions theorem, *Sci. China, Math.*, **57**(3):541–554, 2014, https://doi.org/10.1007/s11425-013-4687-9.

33. X. Zhu and D. Cao, The concentration-compactness principle in nonlinear elliptic equations, *Acta. Math. Sci.*, **9**(3): 307–328, 1989, https://doi.org/10.1016/S0252-9602(18)30356-4.