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The first moment of quadratic Dirichlet L-functions at central values

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Abstract. We obtain an asymptotic formula for the smoothly weighted first moment of quadratic Dirichlet L-functions at central values, with explicit main terms and an error term that is "square-root" of the main term.

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1 Introduction

An important and well-studied problem in analytic number theory is estimating moments of families of L-functions. Considering the family of Dirichlet L-functions $L(s,\chi^{(d)})$ with a quadratic Dirichlet character $\chi^{(d)}$ associated with the fundamental discriminant d defined by the Kronecker symbol $\chi^{(d)}(n) = (\frac{d}{n})$, a problem is understanding the asymptotic behavior of

$$\sum_{0<d\leqslant X} L\bigl(s,\chi^{(d)}\bigr)^k$$

as $X \to \infty$. In this note, we mainly focus on the first moment of the family of quadratic Dirichlet L-functions at central values. We know that the Kronecker symbol restricted to a fundamental discriminant actually is a primitive character. For this family, Jutila [8] initially obtained an asymptotic formula summing over fundamental discriminants d for the first moment:

$$\sum_{0<d\leqslant X} L\left(\frac{1}{2},\chi^{(d)}\right) = \frac{P(1)}{4\zeta(2)} X\left(\log\frac{X}{\pi} + \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\right) + 4\gamma - 1 + 4\frac{P'}{P}(1)\right) + O\bigl(X^{3/4+\varepsilon}\bigr)$$

with

$$P(s) = \prod_p \left(1 - \frac{1}{p^s(p+1)}\right).$$

An error term of the same size was later given by Vinogradov and Takhtadzhyan [11] as a corollary. Precisely, they studied the asymptotic behavior of the sum $\sum_{|d|\leqslant X} L(\sigma,\chi^{(d)})$ for $\chi^{(d)}(-1)=\pm 1$ and $0\leqslant\sigma=\Re(s)\leqslant 1$. Using the method of double Dirichlet series and the theory of Eisenstein series, Goldfeld and Hoffstein [5] improved this result with a better error term:

$$\sum_{\substack{1<|d'|<X\\ d' \text{ squarefree}}} L\bigl(\sigma,\chi^{(d')}\bigr)=c_{1/2}X\log X+cX+O\bigl(X^{19/32+\varepsilon}\bigr)$$

for $\sigma=1/2$, where

$$c_{1/2}=\frac{3}{16}\prod_{p\geqslant 3}\bigl(1-2p^{-2}+p^{-3}\bigr).$$

They also dealt with the case of $\sigma>1/2$. We remark here that the definition of $\chi^{(d')}$ given in [5] is slightly different from the Kronecker symbol, which leads to the difference of the above two leading coefficients. It is implicit in [5] that we may obtain an error term of size $O(X^{1/2+\varepsilon})$ for the smoothed first moment. In [5], it is also pointed out that it seems unlikely to improve the error term without some substantial improvement in the zero-free region for the Riemann zeta function, simply due to the state of knowledge on the distribution of square-free integers. Restricting d to be odd and square-free, $\chi^{(8d)}$ are real primitive characters with conductor $8d$. Young [12] achieved the following result based on recursive arguments:

$$\sum_{\substack{(d,2)=1\\ d \text{ squarefree}}} L\left(\frac{1}{2},\chi^{(8d)}\right)\varPhi\left(\frac{d}{X}\right)=XP(\log X)+O\bigl(X^{1/2+\varepsilon}\bigr)$$

for some linear polynomial P depending on $\varPhi$, where the error term agrees with the implicit result in [5]. Goldfeld and Hoffstein [5] also conjectured that the optimal error term should be $O(X^{1/4+\varepsilon})$, and this has been observed in a numerical study by Alderson and Rubinstein [1].

In this note, we denote by $\chi^{(m)}=(\frac{m}{\cdot})$ the Kronecker symbol to distinguish it from the Jacobi symbol $\chi_n=(\frac{\cdot}{n})$. So let for any nonzero integer $m\equiv 0,1 \pmod 4$, called the discriminant, $\chi^{(m)}=(\frac{m}{\cdot})$ be the Kronecker symbol. Note that we can factor every such m uniquely into $m=dl^2$ with $l\geqslant 1$, where d is a fundamental discriminant, that is, d is either square-free and $d\equiv 1 \pmod 4$, or $d=4n$ with square-free $n\equiv 2,3 \pmod 4$. Then $\chi^{(d)}$ is a real primitive character of conductor $|d|$. We know that $\chi^{(d)}$ is even (resp., odd) if $d>0$ (resp., $d<0$). For any L-function, we write $L^{(c)}$ (resp., $L_{(c)}$) for the function given by the Euler product defining L but omitting those primes dividing (resp., not dividing) c. We reserve the letter p for a prime throughout this note and write L_p for $L_{(p)}$ for simplicity. Note that the Kronecker and Jacobi symbols are connected by the quadratic reciprocity law

$$\left(\frac{m}{n}\right)=\left(\frac{n}{m}\right)(-1)^{(m-1)/2\cdot(n-1)/2}.$$

So we have

$$\chi_n=\begin{cases}\chi^{(n)} & \text{if } n\equiv 1 \pmod 4,\\ \chi^{(-n)} & \text{if } n\equiv 3 \pmod 4,\end{cases}$$

and

$$L^{(2)}(s,\chi_n)=\begin{cases}L(s,\chi^{(4n)}) & \text{if } n\equiv 1 \pmod 4,\\ L(s,\chi^{(-4n)}) & \text{if } n\equiv 3 \pmod 4.\end{cases}$$

It is well known (see [9, Thm. 9.13]) that every primitive quadratic Dirichlet character is of the form $\chi^{(d)}$ for some fundamental discriminant d. For such d, the function $L(s,\chi^{(d)})$ has an analytic continuation to the entirety of $\mathbb{C}$, and so does $L^{(2)}(s,\chi_n)$.

We indicate that the characters of L-functions in the results mentioned above are in fact primitive. Gao and Zhao [4] took the nonprimitive characters into consideration. Let χ_n denote the quadratic character $(\frac{\cdot}{n})$ for an odd positive integer n defined by the Jacobi symbol. They evaluated the first moment of the family of $L^{(2)}(s,\chi_n)$ averaged over all odd positive integer n under the generalized Riemann hypothesis (GRH) as follows:

$$\sum_{(n,2)=1} L^{(2)}\left(\frac{1}{2}+\alpha,\, \chi_n\right) w\left(\frac{n}{X}\right) = c_1(\alpha)\widehat{w}(1)X + c_2(\alpha)\widehat{w}(1-\alpha)X^{1-\alpha} + O\big((1+|\alpha|)^{2+\varepsilon}X^{1/4+\varepsilon}\big), \tag{1.1}$$

where $0<\Re(\alpha)<1/2$, and $c_1(\alpha)$, $c_2(\alpha)$ are two constants depending on α. Let $\alpha\to 0$. For the smoothed first moment of L-functions at central values, they also deduced that

$$\sum_{(n,2)=1} L^{(2)}\left(\frac{1}{2},\chi_n\right) w\left(\frac{n}{X}\right) = XQ(\log X) + O\big(X^{1/4+\varepsilon}\big) \tag{1.2}$$

with a linear polynomial Q whose expression is omitted. Note that the error terms above are consistent with the conjectural size given in [5] and the characters in the summation are allowed to be nonprimitive. Additionally, they indicated that (1.1) and (1.2) also unconditionally hold with the error term $O(X^{1/2+\varepsilon})$.

Following Gao and Zhao [4], we will unconditionally evaluate the first moment of a family of quadratic Dirichlet L-functions $L^{(2)}(s,\chi_n)$ averaged over all odd positive n and finally give an explicit expression of Q. Our main result is as follows.

Theorem 1. *Let $w(t)$ be a nonnegative Schwartz function with Mellin transform $\widehat{w}(s)$. Then for any $\varepsilon>0$ and $0<\Re(\alpha)<1/2$, we have*

$$\begin{aligned}\sum_{(n,2)=1} L^{(2)}\left(\frac{1}{2}+\alpha,\, \chi_n\right) w\left(\frac{n}{X}\right) &= X\widehat{w}(1)\frac{\zeta(1+2\alpha)}{\zeta(2+2\alpha)}\cdot\frac{1-2^{-1-2\alpha}}{2(1-2^{-2-2\alpha})}\\ &\quad + X^{1-\alpha}\widehat{w}(1-\alpha)\frac{\pi^{\alpha}\Gamma(\frac{1}{2}-\alpha)\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}\cdot\frac{\zeta(1-2\alpha)}{\zeta(2)}\cdot\frac{2^{2\alpha}}{6}\\ &\quad + O\big((1+|\alpha|)^{2+\varepsilon}X^{1/2+\varepsilon}\big).\end{aligned} \tag{1.3}$$

It is natural that the constants in the main term coincide with those in (1.1). Notice that the error term in (1.3) is uniform for α, and therefore we can take the limit as $\alpha\to 0^+$ to deduce the error term in the following asymptotic formula for the smoothed first moment of quadratic Dirichlet L-functions at central values.

Corollary 1. *With the notation as above, for any $\varepsilon>0$, we have*

$$\sum_{(n,2)=1} L^{(2)}\left(\frac{1}{2},\chi_n\right) w\left(\frac{n}{X}\right) = XQ(\log X) + O\big(X^{1/2+\varepsilon}\big).$$

where Q is a linear polynomial given by (4.4) *with explicit coefficients depending only on the absolute constants $\widehat{w}(1)$ and $\widehat{w}'(1)$.*

Note that our error term is consistent with the implicit result obtained in [5] and [12]. We obtain an explicit expression of Q.

2 Preliminaries

2.1 Gauss sums

We write $\psi_j = \chi^{(4j)}$ for $j = \pm 1, \pm 2$, where we recall that $\chi^{(d)} = (\frac{d}{\cdot})$ is the Kronecker symbol for integers $d \equiv 0, 1 \pmod 4$. Note that each ψ_j is a primitive character modulo $4j$. Let ψ_0 stand for the trivial character, that is, $\psi_0(n) = 1$ for all $n \in \mathbb{N}$.

Given any Dirichlet character χ modulo n and any integer q, the Gauss sum $\tau(\chi, q)$ is defined as

$$\tau(\chi, q) = \sum_{j \ (\mathrm{mod}\ n)} \chi(j) e\left(\frac{jq}{n}\right), \quad \text{where } e(z) = \exp(2\pi \mathrm{i} z).$$

If χ is primitive, then $\tau(\chi, q) = \overline{\chi}(q)\tau(\chi)$, where $\tau(\chi) = \tau(\chi, 1)$. For the evaluation of $\tau(\chi, q)$, we cite the following result from [3, Lemma 2.2].

Lemma 1.

(i) *If* $l \equiv 1 \pmod 4$, *then*

$$\tau\big(\chi^{(4l)}, q\big) = \begin{cases} 0 & \text{if } (q, 2) = 1, \\ -2\tau(\chi_l, q) & \text{if } q \equiv 2 \pmod 4, \\ 2\tau(\chi_l, q) & \text{if } q \equiv 0 \pmod 4. \end{cases}$$

(ii) *If* $l \equiv 3 \pmod 4$, *then*

$$\tau\big(\chi^{(4l)}, q\big) = \begin{cases} 0 & \text{if } 2 \mid q, \\ -2\mathrm{i}\tau(\chi_l, q) & \text{if } q \equiv 1 \pmod 4, \\ 2\mathrm{i}\tau(\chi_l, q) & \text{if } q \equiv 3 \pmod 4. \end{cases}$$

Recall that $\chi_n = (\frac{\cdot}{n})$ for any odd positive integer n. We define the associated Gauss sum $G(\chi_n, q)$ by

$$G(\chi_n, q) = \left(\frac{1 - \mathrm{i}}{2} + \left(\frac{-1}{n}\right)\frac{1 + \mathrm{i}}{2}\right)\tau(\chi_n, q) = \begin{cases} \tau(\chi_n, q) & \text{if } n \equiv 1 \pmod 4, \\ -\mathrm{i}\tau(\chi_n, q) & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

The advantage of $G(\chi_n, q)$ over $\tau(\chi_n, q)$ is that $G(\chi_n, q)$ is now a multiplicative function of n. Furthermore, we have the following result [10, Lemma 2.3].

Lemma 2. *If* $(m, n) = 1$, *then* $G(\chi_{mn}, q) = G(\chi_m, q)G(\chi_n, q)$. *Let* p^a *be the largest power of* p *dividing* q (*put* $a = \infty$ *if* $m = 0$), *and let* φ *be the Euler totient function. Then for* $k \geqslant 0$, *we have*

$$G(\chi_{p^k}, q) = \begin{cases} \varphi(p^k) & \text{if } k \leqslant a \text{ is even}, \\ 0 & \text{if } k \leqslant a \text{ is odd}, \\ -p^a & \text{if } k = a + 1 \text{ is even}, \\ (\frac{qp^{-a}}{p})p^a\sqrt{p} & \text{if } k = a + 1 \text{ is odd}, \\ 0 & \text{if } k \geqslant a + 2. \end{cases}$$

2.2 Functional equations for Dirichlet L-functions

Let $n \geqslant 1$, and let χ be a primitive character modulo n. If $n = 1$, then we have $\chi = 1$ and $L(s,\chi) = \zeta(s)$. Also, we put $\kappa = (1 - \chi(-1))/2$, so $\kappa = 0$ if χ is even (i.e., $\chi(-1) = 1$) and $\kappa = 1$ if χ is odd (i.e., $\chi(-1) = -1$). Then we have the following functional equation [7, Thm. 4.15].

Lemma 3. *With the above notations, $L(s,\chi)$ extends to a meromorphic function on $\mathbb{C}$, which is entire if $n \neq 1$. The completed L-function*

$$\Lambda(s,\chi) = \left(\frac{n}{\pi}\right)^{s/2} \Gamma\left(\frac{s+\kappa}{2}\right) L(s,\chi)$$

satisfies the functional equation

$$\Lambda(s,\chi) = \frac{\tau(\chi)}{\mathrm{i}^{\kappa}\sqrt{n}} \Lambda(1-s,\overline{\chi}).$$

Recall that the Gauss sum $\tau(\chi)$ is of modulus $\sqrt{n}$ for primitive χ. In particular,

$$\tau(\chi) = \mathrm{i}^{\kappa}\sqrt{n}$$

for odd and square-free n. So we have $\Lambda(s,\chi) = \Lambda(1-s,\chi)$ for any real primitive character.

In some cases, we have to deal with the L-functions with nonprimitive characters. So we quote the following functional equation [3, Prop. 2.3] valid for all Dirichlet characters χ modulo n, which plays a key role in the proof of Theorem 1.

Lemma 4. *With the above notations, let χ be a Dirichlet character modulo n. Then we have*

$$L(s,\chi) = \frac{\pi^{s-\frac{1}{2}}}{\mathrm{i}^{\kappa} n^{s}} \frac{\Gamma\frac{1-s+\kappa}{2}}{\Gamma\frac{s+\kappa}{2}} K(1-s,\chi), \quad \text{where } K(s,\chi) = \sum_{q=1}^{\infty} \frac{\tau(\chi,q)}{q^{s}}. \tag{2.1}$$

Note that if χ is a primitive character, then Lemmas 4 and 3 coincide with each other.

2.3 Bounding L-functions

For a fixed quadratic character ψ modulo n, let $\widehat{\psi}$ be the primitive character inducing ψ. Then we have $\widehat{\psi} = \chi^{(d)}$ for some fundamental discriminant $d \mid n$ (see [9, Thm. 9.13]). We now gather some estimates of $L(s,\psi)$.

Write $n = n_1 n_2$ uniquely so that $(n_1, d) = 1$ and $p \mid n_2 \Rightarrow p \mid d$. The above notations imply that for any integer q,

$$L^{(q)}(s,\psi) = L(s,\widehat{\psi}) \prod_{p \mid q n_1} \left(1 - \frac{\widehat{\psi}(p)}{p^{s}}\right). \tag{2.2}$$

Observe that

$$\left|1 - \frac{\widehat{\psi}(p)}{p^{s}}\right| \leqslant 2 p^{\max(0,-\Re(s))}.$$

We then deduce that

$$\prod_{p \mid q n_1} \left(1 - \frac{\widehat{\psi}(p)}{p^{s}}\right) \ll 2^{\omega(q_1 n)} (q n_1)^{\max(0,-\Re(s))} \ll (q n_1)^{\max(0,-\Re(s))+\varepsilon}, \tag{2.3}$$

where $\omega(n)$ denotes the number of distinct prime factors of n. When d is a fundamental discriminant, we recall that the convexity bound for $L(s, \chi^{(d)})$ (see [4, Eq. (2.9)]) satisfies

$$L(s, \chi^{(d)}) \ll \begin{cases} 1 & \text{if } \Re(s) > 1, \\ (|d|(1+|s|))^{(1-\Re(s))/2+\varepsilon} & \text{if } 0 \leqslant \Re(s) < 1, \\ (|d|(1+|s|))^{1/2-\Re(s)+\varepsilon} & \text{if } \Re(s) < 0. \end{cases} \tag{2.4}$$

From (2.2), (2.3), and (2.4) we deduce that for any complex number s,

$$L^{(q)}(s, \psi) \ll (qn_1)^{\max(0, -\Re(s))+\varepsilon} \big(n\big(1+|s|\big)\big)^{\max\{1/2-\Re(s),\, (1-\Re(s))/2,\, 0\}+\varepsilon}. \tag{2.5}$$

We also need the following large sieve result for quadratic Dirichlet L-functions, which is a consequence of [6, Thm. 2].

Lemma 5. *Let $S(X)$ denote the set of real primitive characters χ with conductor not exceeding X. Then for any complex number s with $\Re(s) \geqslant 1/2$ and any $\varepsilon > 0$, we have*

$$\sum_{\chi \in S(X)} \big|L(s, \chi)\big| \ll X^{1+\varepsilon} |s|^{1/4+\varepsilon}. \tag{2.6}$$

Proof. From [6, Thm. 2] we get

$$\sum_{\chi \in S(X)} \big|L(s, \chi)\big|^4 \ll \big(X|s|\big)^{1+\varepsilon}.$$

The lemma now follows from the above and Hölder's inequality. □

2.4 Some results on multivariable complex functions

We gather here some results from multivariable complex analysis. We begin with the notation of a tube domain.

Definition 1. An open set $T \subset \mathbb{C}^n$ is a tube if there is an open set $U \subset \mathbb{R}^n$ such that

$$T = \big\{z \in \mathbb{C}^n\colon \Re(z) \in U\big\}.$$

For a set $U \subset \mathbb{R}^n$, we define $T(U) = U + \mathrm{i}\mathbb{R}^n \subset \mathbb{C}^n$ and then quote the following theorem from [2].

Lemma 6 [Bochner's tube theorem]. *Let $U \subset \mathbb{R}^n$ be a connected open set, and let $f(z)$ be a holomorphic function on $T(U)$. Then $f(z)$ has a holomorphic continuation to the convex hull of $T(U)$.*

We denote by $\widehat{T}$ the convex hull of an open set $T \subset \mathbb{C}^n$. The next result states the modulus of holomorphic continuations of multivariable complex functions [3, Prop. C.5].

Lemma 7. *Assume that $T \subset \mathbb{C}^n$ is a tube domain. Let $g, h : T \to \mathbb{C}$ be holomorphic functions, and let $\tilde{g}$, $\tilde{h}$ be their holomorphic continuations to $\widehat{T}$. If $|g(z)| \leqslant |h(z)|$ for all $z \in T$ and $h(z)$ is nonzero in T, then also $|\tilde{g}(z)| \leqslant |\tilde{h}(z)|$ for all $z \in \widehat{T}$.*

3 Proof of Theorem 1

For sufficiently large $\Re(s)$ and $\Re(w)$, we define the Dirichlet series

$$A(s,w)=\sum_{(n,2)=1}\frac{L^{(2)}(w,\chi_n)}{n^s}=\sum_{(nm,2)=1}\frac{\chi_n(m)}{m^w n^s}=\sum_{(m,2)=1}\frac{L(s,\chi^{(4m)})}{m^w}. \tag{3.1}$$

We next investigate the analytic properties of $A(s,w)$ following the arguments in [4].

3.1 First region of absolute convergence of $A(s,w)$

According to the first equality in (3.1), writing $n=n_0h^2$ and arguing similarly to (2.3), we derive

$$\begin{aligned}A(s,w)&=\sum_{(n,2)=1}\frac{L^{(2)}(w,\chi_n)}{n^s}=\sum_{(h,2)=1}\frac{1}{h^{2s}}\sum_{(n_0,2)=1}{}^{*}\frac{L^{(2)}(w,\chi_{n_0})\prod_{p|h}(1-\chi_{n_0}(p)p^{-w})}{n_0^s}\\&\ll\sum_{(h,2)=1}\frac{h^{\max(0,-\Re(w))+\varepsilon}}{h^{2s}}\sum_{(n,2)=1}{}^{*}\left|\frac{L^{(2)}(w,\chi_n)}{n^s}\right|,\end{aligned} \tag{3.2}$$

where $\sum^*$ henceforth denotes the sum over square-free integers. Note that for square-free odd integer n, χ_n is a primitive character modulo n. So we have

$$\left|L^{(2)}(w,\chi_n)\right|=\left|\left(1-\chi_n(2)2^{-w}\right)L(w,\chi_n)\right|\ll\left|L(w,\chi_n)\right|.$$

It follows from (3.2) and the above estimate that

$$A(s,w)\ll\sum_{(h,2)=1}\frac{h^{\max(0,-\Re(w))+\varepsilon}}{h^{2s}}\sum_{(n,2)=1}{}^{*}\frac{|L(w,\chi_n)|}{|n^s|}. \tag{3.3}$$

Now (2.6) and partial summation implies that, except for a possible simple pole at $w=1$, both sums of the right-hand side expression in (3.3) are convergent for $\Re(s)>1$ and $\Re(w)\geqslant 1/2$. If $\Re(w)<1/2$, then it follows from the functional equation, (2.6), and partial summation that $A(s,w)$ is convergent for $\Re(2s)>1$, $\Re(2s+w)>1$, and $\Re(s+w)>3/2$. It follows that $A(s,w)$ converges absolutely in the region

$$S_0=\left\{(s,w)\colon\ \Re(s)>1,\ \Re(2s+w)>1,\ \Re(s+w)>\frac{3}{2}\right\},$$

which can be simplified to

$$S_0=\left\{(s,w)\colon\ \Re(s)>1,\ \Re(s+w)>\frac{3}{2}\right\},$$

since $\Re(2s+w)>1$ is contained in the other two conditions.

On the other hand, writing $m=m_0m_1^2$ with odd and square-free m_0, we can recast the last expression of (3.1) as

$$A(s,w)=\sum_{(m,2)=1}\frac{L(s,\chi^{(4m)})}{m^w}=\sum_{(m_1,2)=1}\frac{1}{m_1^{2w}}\sum_{(m_0,2)=1}{}^{*}\frac{L(s,\chi^{(4m_0)})\prod_{p|m_1}(1-\chi^{(4m_0)}(p)p^{-s})}{m_0^w}. \tag{3.4}$$

Using the same method as before, we see that $A(s,w)$ is also absolutely convergent in the region

$$S_1=\left\{(s,w)\colon\ \Re(s)\geqslant\frac{1}{2},\ \Re(w)>1\right\}\cup\left\{(s,w)\colon\ \Re(s)<\frac{1}{2},\ \Re(w+s)>\frac{3}{2},\ \Re(2w+s)>1\right\}$$

except for a simple pole at $s=1$ arising from the summands with $m=\bullet$, which means that m is a perfect square.

Notice that the convex hull of S_0 and S_1 is

$$S_2=\left\{(s,w)\colon\ \Re(s+w)>\frac{3}{2}\right\}.$$

Hence Lemma 6 implies that $(s-1)(w-1)A(s,w)$ converges absolutely in the region S_2.

3.2 Residue of $A(s,w)$ at $s=1$

We see that $A(s,w)$ has a pole at $s=1$ arising from the terms with $m=\bullet$ from (3.4). To compute the corresponding residue, we define the sum

$$A_1(s,w):=\sum_{\substack{(m,2)=1\\ m=\bullet}}\frac{L(s,\chi^{(4m)})}{m^w}=\sum_{\substack{(m,2)=1\\ m=\bullet}}\frac{\zeta(s)\prod_{p|2m}(1-p^{-s})}{m^w}.$$

For any $s\in\mathbb{C}$, let $a_s(n)$ be the multiplicative function such that $a_s(p^k)=1-1/p^s$ for any prime p and $k\geqslant 1$. Then we have

$$A_1(s,w)=\zeta(s)\bigl(1-2^{-s}\bigr)\sum_{\substack{(m,2)=1\\ m=\bullet}}\frac{\prod_{p|m}(1-p^{-s})}{m^w}=\zeta^{(2)}(s)\sum_{\substack{(m,2)=1\\ m=\bullet}}\frac{a_s(m)}{m^w}.$$

Writing the last sum above as an Euler product, we get

$$\begin{aligned}A_1(s,w)&=\zeta^{(2)}(s)\prod_{p>2}\sum_{\mu\geqslant 0}\frac{a_s(p^{2\mu})}{p^{2\mu w}}=\zeta^{(2)}(s)\prod_{p>2}\left(1+\left(1-\frac{1}{p^s}\right)\frac{p^{-2w}}{1-p^{-2w}}\right)\\&=\zeta^{(2)}(s)\prod_{p>2}\frac{1-p^{-s-2w}}{1-p^{-2w}}=\zeta^{(2)}(s)\zeta^{(2)}(2w)\prod_{p>2}\left(1-\frac{1}{p^{s+2w}}\right)\\&=:\zeta(s)\zeta(2w)P(s,w),\end{aligned}\tag{3.5}$$

where

$$P(s,w)=\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{2^{2w}}\right)\prod_{p>2}\left(1-\frac{1}{p^{s+2w}}\right).\tag{3.6}$$

It follows from (3.5) and (3.6) that except for a simple pole at $s=1$, the functions $P(s,w)$ and $A_1(s,w)$ are holomorphic in the region

$$S_3=\bigl\{(s,w)\colon\ \Re(s+2w)>1\bigr\}\tag{3.7}$$

and

$$\operatorname*{Res}_{s=1}A\left(s,\ \frac{1}{2}+\alpha\right)=\operatorname*{Res}_{s=1}A_1\left(s,\ \frac{1}{2}+\alpha\right)=\zeta(1+2\alpha)P\left(1,\ \frac{1}{2}+\alpha\right).\tag{3.8}$$

3.3 Second region of absolute convergence of $A(s, w)$

We infer from (3.4) that

$$\begin{aligned} A(s,w) &= \sum_{\substack{(m,2)=1\\ m=\bullet}} \frac{L(s,\chi^{(4m)})}{m^w} + \sum_{\substack{(m,2)=1\\ m\neq\bullet}} \frac{L(s,\chi^{(4m)})}{m^w} \\ &= \sum_{\substack{(m,2)=1\\ m=\bullet}} \frac{\zeta(s)\prod_{p|2m}(1-p^{-s})}{m^w} + \sum_{\substack{(m,2)=1\\ m\neq\bullet}} \frac{L(s,\chi^{(4m)})}{m^w} \\ &=: A_1(s,w) + A_2(s,w). \end{aligned} \tag{3.9}$$

Let us focus on $A_2(s,w)$. Observe that $\chi^{(4m)}$ is a Dirichlet character modulo $4m$ for any odd $m \geqslant 1$ such that $\chi^{(4m)}(-1) = 1$ but it may not be primitive. Now we can apply the functional equation (2.1) to $L(s,\chi^{(4m)})$ in the case $m \neq \bullet$ and arrive at

$$A_2(s,w) = \frac{\pi^{s-\frac{1}{2}}}{4^s} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} C(1-s,\ s+w), \tag{3.10}$$

where $C(s,w)$ is given by the double Dirichlet series

$$C(s,w) = \sum_{\substack{q,m\\ (m,2)=1\\ m\neq\bullet}} \frac{\tau(\chi^{(4m)},q)}{q^s m^w} = \sum_{\substack{q,m\\ (m,2)=1}} \frac{\tau(\chi^{(4m)},q)}{q^s m^w} - \sum_{\substack{q,m\\ (m,2)=1\\ m=\bullet}} \frac{\tau(\chi^{(4m)},q)}{q^s m^w}.$$

Note that $C(s,w)$ is initially convergent for sufficiently large $\Re(s)$ and $\Re(w)$. To extend this region, we rewrite $C(s,w)$ as

$$C(s,w) = \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{(m,2)=1} \frac{\tau(\chi^{(4m)},q)}{m^w} - \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{\substack{(m,2)=1\\ m=\bullet}} \frac{\tau(\chi^{(4m)},q)}{m^w} =: C_1(s,w) - C_2(s,w). \tag{3.11}$$

Let ψ, ψ' be two Dirichlet characters with conductors dividing 8. We define

$$\begin{aligned} C_1(s,w;\psi,\psi') &:= \sum_{m,q\geqslant 1} \frac{G(\chi_m,q)\psi(m)\psi'(q)}{m^w q^s}, \\ C_2(s,w;\psi,\psi') &:= \sum_{m,q\geqslant 1} \frac{G(\chi_{m^2},q)\psi(m)\psi'(q)}{m^{2w} q^s}. \end{aligned} \tag{3.12}$$

Then following the arguments contained in [3, §6.4] and applying Lemma 1, we obtain that

$$\begin{aligned} C_1(s,w) &= -2^{-s}\big(C_1(s,w;\psi_2,\psi_1) + C_1(s,w;\psi_{-2},\psi_1)\big) \\ &\quad + 4^{-s}\big(C_1(s,w;\psi_1,\psi_0) + C_1(s,w;\psi_{-1},\psi_0)\big) \\ &\quad + C_1(s,w;\psi_1,\psi_{-1}) - C_1(s,w;\psi_{-1},\psi_{-1}), \\ C_2(s,w) &= -2^{1-s}C_2(s,w;\psi_1,\psi_1) + 2^{1-2s}C_2(s,w;\psi_1,\psi_0). \end{aligned} \tag{3.13}$$

Now we write every integer $q \geqslant 1$ uniquely as $q = q_1 q_2^2$ with square-free q_1 to derive that

$$C_i(s, w; \psi, \psi') = \sum_{q_1}^{*} \frac{\psi'(q_1)}{q_1^s} \cdot D_i(s, w; q_1, \psi, \psi'), \quad i = 1, 2, \tag{3.14}$$

where

$$\begin{aligned} D_1(s, w; q_1, \psi, \psi') &= \sum_{m, q_2 = 1}^{\infty} \frac{G(\chi_m, q_1 q_2^2)\psi(m)\psi'(q_2^2)}{m^w q_2^{2s}}, \\ D_2(s, w; q_1, \psi, \psi') &= \sum_{m, q_2 = 1}^{\infty} \frac{G(\chi_{m^2}, q_1 q_2^2)\psi(m)\psi'(q_2^2)}{m^{2w} q_2^{2s}}. \end{aligned} \tag{3.15}$$

The following result gives the required analytic properties of $D_i(s, w; q_1, \psi, \psi')$.

Lemma 8. *With the notation as above and for $\psi \neq \psi_0$, the functions $D_i(s, w; q_1, \psi, \psi')$, $i = 1, 2$, have meromorphic continuations to the region*

$$\left\{(s, w)\colon \Re(s) > \frac{1}{2},\ \Re(w) > 1\right\}. \tag{3.16}$$

Moreover, the only pole of $D_1(s, w; q_1, \psi, \psi')$ in this region occurs when $q_1 = 1$ and $\psi = \psi_1$ at $w = 3/2$, and this pole is simple. For $\Re(s) \geqslant 1/2 + \varepsilon$ and $\Re(w) \geqslant 1 + \varepsilon$, away from the possible poles, we have

$$\begin{aligned} &D_1(s, w; q_1, \psi, \psi') \ll \big(q_1\big(1 + |w|\big)\big)^{\max\{(3/2 - \Re(w))/2,\, 0\} + \varepsilon}, \\ &D_2(s, w; q_1, \psi, \psi') \ll q_1^{\varepsilon}. \end{aligned} \tag{3.17}$$

Proof. We first focus on $D_1(s, w; q_1, \psi, \psi')$. By Lemma 2 the summands in (3.15) are jointly multiplicative functions of m and q_2. Moreover, we may assume that m is odd since $\psi \neq \psi_0$ with conductor dividing 8. These observations enable us to write $D_1(s, w; q_1, \psi, \psi')$ as an Euler product such that

$$D_1(s, w; q_1, \psi, \psi') = \prod_p D_{1,p}(s, w; q_1, \psi, \psi'), \tag{3.18}$$

where

$$D_{1,p}(s, w; q_1, \psi, \psi') = \begin{cases} \sum_{k=0}^{\infty} \frac{\psi'(2^{2k})}{2^{2ks}} & \text{if } p = 2, \\ \sum_{l,k=0}^{\infty} \frac{\psi(p^l)\psi'(p^{2k})G(\chi_{p^l}, q_1 p^{2k})}{p^{lw + 2ks}} & \text{if } p > 2. \end{cases} \tag{3.19}$$

It is easy to see that for $p = 2$,

$$\sum_{k=0}^{\infty} \frac{\psi'(2^{2k})}{2^{2ks}} = \begin{cases} 1 & \text{if } \psi' \neq \psi_0, \\ (1 - 2^{-2s})^{-1} & \text{if } \psi' = \psi_0. \end{cases} \tag{3.20}$$

Now, for any fixed $p > 2$,

$$\sum_{l,k=0}^{\infty} \frac{\psi(p^l)\psi'(p^{2k})G(\chi_{p^l}, q_1 p^{2k})}{p^{lw + 2ks}} = \sum_{l=0}^{\infty} \frac{\psi(p^l)G(\chi_{p^l}, q_1)}{p^{lw}} + \sum_{l \geqslant 0, k \geqslant 1} \frac{\psi(p^l)\psi'(p^{2k})G(\chi_{p^l}, q_1 p^{2k})}{p^{lw + 2ks}}. \tag{3.21}$$

In fact, the above sums only have finitely many nonzero terms. Recalling that q_1 is square-free, we deduce from Lemma 2 that $G(\chi_{p^l}, q_1p^{2k})$ has the trivial bound

$$G\big(\chi_{p^l}, q_1p^{2k}\big) \ll p^l$$

and $G(\chi_{p^l}, q_1p^{2k}) = 0$ for $l \geqslant 2k+3$. The above estimates allow us to obtain that when $\Re(s) > 1/2$ and $\Re(w) > 1$,

$$\begin{aligned}\sum_{l\geqslant 0, k\geqslant 1} \frac{\psi(p^l)\psi'(p^{2k})G(\chi_{p^l}, q_1p^{2k})}{p^{lw+2ks}} &= \sum_{k\geqslant 1} \frac{\psi'(p^{2k})G(\chi_1, q_1p^{2k})}{p^{2ks}} + \sum_{l,k\geqslant 1} \frac{\psi(p^l)\psi'(p^{2k})G(\chi_{p^l}, q_1p^{2k})}{p^{lw+2ks}} \\ &\ll p^{-2\Re(s)} + \left| \sum_{k=1}^{\infty} \sum_{1\leqslant l\leqslant 2k+2} \frac{1}{p^{l(w-1)+2ks}} \right| \\ &\ll p^{-2\Re(s)} + \left| \sum_{k=1}^{\infty} \frac{2k+2}{p^{2ks}} \left(\frac{1}{p^{w-1}} + \frac{1}{p^{(2k+2)(w-1)}} \right) \right| \\ &\ll p^{-2\Re(s)} + p^{-2\Re(s)-\Re(w)+1} + p^{-2\Re(s)-4\Re(w)+4}. \end{aligned} \tag{3.22}$$

More precisely, it follows from Lemma 2 that for $p \nmid q_1$, the Gauss sum

$$G\big(\chi_{p^l}, q_1\big) = \begin{cases} 1 & \text{if } l = 0, \\ \chi^{(q_1)}(p)\sqrt{p} & \text{if } l = 1, \\ 0 & \text{if } l \geqslant 2;, \end{cases} \tag{3.23}$$

and for $p \mid q_1$, $p \neq 2$,

$$G\big(\chi_{p^l}, q_1\big) = \begin{cases} 1 & \text{if } l = 0, \\ 0 & \text{if } l = 1 \text{ or } l \geqslant 3, \\ -p & \text{if } l = 2. \end{cases} \tag{3.24}$$

Combining the above estimates together yields that for $p \nmid 2q_1$ and $\Re(w) > 1$,

$$\begin{aligned}\sum_{l=0}^{\infty} \frac{\psi(p^l)G(\chi_{p^l}, q_1)}{p^{lw}} &= 1 + \frac{\psi(p)\chi^{(q_1)}(p)}{p^{w-1/2}} = \left(1 - \frac{\psi(p)\chi^{(q_1)}(p)}{p^{w-1/2}}\right)^{-1}\left(1 - \frac{1}{p^{2w-1}}\right) \\ &= \frac{L_p(w-\frac{1}{2}, \psi\chi^{(q_1)})}{\zeta_p(2w-1)} \end{aligned} \tag{3.25}$$

since $(\psi(p)\chi^{(q_1)}(p))^2 = 1$. Then we derive from (3.19)–(3.25) that for $p \nmid 2q_1$, $\Re(s) > 1/2$, and $\Re(w) > 1$,

$$\begin{aligned} D_{1,p}(s, w; q_1, \psi, \psi') = {}& \frac{L_p(w-\frac{1}{2}, \chi^{(4q_1)}\psi)}{\zeta_p(2w-1)} \\ & \times \big(1 + O\big(p^{-2\Re(s)} + p^{-2\Re(s)-\Re(w)+1} + p^{-2\Re(s)-4\Re(w)+4}\big)\big). \end{aligned} \tag{3.26}$$

Now we deduce the first statement of the lemma from (3.18), (3.19), and the above. We can see that the only pole that is simple in the region given by (3.16) is at $w = 3/2$, and this occurs when $q_1 = 1$ and $\psi = \psi_1$.

According to (3.24), we further obtain that when $p \mid q_1, p \neq 2$,

$$\sum_{l=0}^{\infty} \frac{\psi(p^l)G(\chi_{p^l}, q_1)}{p^{lw}} = 1 - \frac{\psi(p^2)}{p^{2w-1}} = 1 + O\big(p^{-2\Re(w)+1}\big). \tag{3.27}$$

It follows from (3.19), (3.22), and (3.27) that for $p \mid q_1$, $p \neq 2$, $\Re(s) > 1/2$, and $\Re(w) > 1$,

$$D_{1,p}(s, w; q_1, \psi, \psi') = 1 + O\big(p^{-2\Re(w)+1} + p^{-2\Re(s)} + p^{-2\Re(s)-\Re(w)+1} + p^{-2\Re(s)-4\Re(w)+4}\big)$$

and the finite product

$$\prod_{p \mid 2q_1} D_{1,p}(s, w; q_1, \psi, \psi') \ll 2^{\omega(2q_1)} \ll q_1^{\varepsilon}. \tag{3.28}$$

We conclude from (3.18), (3.19), (3.26), and (3.28) that for $\Re(s) \geqslant 1/2 + \varepsilon$ and $\Re(w) \geqslant 1 + \varepsilon$,

$$\begin{aligned} D_1(s, w; q_1, \psi, \psi') &\ll q_1^{\varepsilon} \left| \frac{L^{(2q_1)}(w - \frac{1}{2}, \psi\chi^{(q_1)})}{\zeta^{(2q_1)}(2w-1)} \right| \ll q_1^{\varepsilon} \left| \frac{L(w - \frac{1}{2}, \psi\chi^{(q_1)})}{\zeta(2w-1)} \right| \\ &\ll \big(q_1\big(1 + |w|\big)\big)^{\max\{(3/2 - \Re(w))/2,\, 0\} + \varepsilon}, \end{aligned}$$

where the last bound follows from (2.5) and the absolute convergence of $\zeta^{-1}(2w-1)$. This leads to the estimate in (3.17).

Using the same method, we give a sketch of the proof for $D_2(s, w; q_1, \psi, \psi')$. For $\Re(s) \geqslant 1/2 + \varepsilon$ and $\Re(w) \geqslant 1 + \varepsilon$, we have

$$\begin{aligned} D_2(s, w; q_1, \psi, \psi') &= \prod_{p \nmid 2q_1} D_{2,p}(s, w; q_1, \psi, \psi') \prod_{p \mid 2q_1} D_{2,p}(s, w; q_1, \psi, \psi') \\ &\ll \prod_{p \mid 2q_1} D_{2,p}(s, w; q_1, \psi, \psi') \ll q_1^{\varepsilon}, \end{aligned}$$

since (3.23) and (3.24) give that

$$\sum_{l=0}^{\infty} \frac{\psi(p^l)G(\chi_{p^{2l}}, q_1)}{p^{2lw}} = \begin{cases} 1 & \text{if } p \nmid 2q_1, \\ 1 - \frac{\psi(p)}{p^{2w-1}} & \text{if } p \mid q_1,\ p \neq 2. \end{cases}$$

This completes the proof of the lemma. □

Now applying Lemma 8 with (3.13) and (3.14), we see that $(w - 3/2)C(s, w)$ is defined in the region

$$\left\{(s, w)\colon\ \Re(s) > \frac{1}{2},\ \Re(w) > 1,\ \Re\left(s + \frac{w}{2}\right) > \frac{7}{4}\right\}.$$

This, together with (3.7) and (3.9), now implies that $(s-1)(w-1)(s+w-3/2)A(s, w)$ can be extended to the region

$$S_4 = \left\{(s, w)\colon\ \Re(s + 2w) > 1,\ \Re(s) < \frac{1}{2},\ \Re(s + w) > 1,\ \Re(w - s) > \frac{3}{2}\right\},$$

in which the condition $\Re(s+2w)>1$ coming from S_3 is redundant. So

$$S_4=\left\{(s,w)\colon\ \Re(s)<\frac{1}{2},\ \Re(s+w)>1,\ \Re(w-s)>\frac{3}{2}\right\}.$$

We then easily observe that the convex hull of S_2 and S_4 equals

$$S_5=\{(s,w)\colon\ \Re(s+w)>1\}.$$

Now Lemma 6 gives that $(s-1)(w-1)(s+w-3/2)A(s,w)$ converges absolutely in the region S_5.

3.4 Residue of $A(s,w)$ at $s=3/2-w$

With the notation as above, we deduce from (3.11), (3.13), (3.14), and Lemma 8 that $C(s,w)$ has a pole at $w=3/2$ and

$$\operatorname*{Res}_{w=3/2} C(s,w)=4^{-s}\operatorname*{Res}_{w=3/2} D_1(s,w;1,\psi_1,\psi_0)+\operatorname*{Res}_{w=3/2} D_1(s,w;1,\psi_1,\psi_{-1}). \tag{3.29}$$

It follows from Lemma 2 that for $p\neq 2$, $\psi=\psi_1$, and $\psi'=\psi_0$ or ψ_{-1},

$$\begin{aligned}\sum_{l\geqslant 0,\,k\geqslant 1}\frac{\psi(p^l)\psi'(p^{2k})G(\chi_{p^l},p^{2k})}{p^{3l/2+2ks}}&=\sum_{k\geqslant 1}\frac{G(\chi_1,p^{2k})}{p^{2ks}}+\sum_{l,k\geqslant 1}\frac{G(\chi_{p^l},p^{2k})}{p^{3l/2+2ks}}\\&=\sum_{k\geqslant 1}\frac{1}{p^{2ks}}+\sum_{k\geqslant 1}\frac{1}{p^{2ks}}\left(\sum_{l=1}^{k}\frac{\varphi(p^{2l})}{p^{3l}}+\frac{p^{2k}\sqrt{p}}{p^{3(2k+1)/2}}\right)\\&=\sum_{k\geqslant 1}\frac{1}{p^{2ks}}+\frac{1}{p}\sum_{k\geqslant 1}\frac{1}{p^{2ks}}=\left(1+\frac{1}{p}\right)\frac{p^{-2s}}{1-p^{-2s}}.\end{aligned} \tag{3.30}$$

Furthermore, we derive from (3.19), (3.21), (3.25), and (3.30) that for $p\neq 2$,

$$\begin{aligned}D_{1,p}(s,w;1,\psi_1,\psi_0)=D_{1,p}(s,w;1,\psi_1,\psi_{-1})&=1+\frac{\psi_1(p)\chi^{(1)}(p)}{p^{w-1/2}}+\left(1+\frac{1}{p}\right)\frac{p^{-2s}}{1-p^{-2s}}\\&=1+\frac{1}{p^{w-1/2}}+\left(1+\frac{1}{p}\right)\frac{p^{-2s}}{1-p^{-2s}}=\zeta_p\left(w-\frac{1}{2}\right)Q_p(s,w),\end{aligned} \tag{3.31}$$

where

$$\begin{aligned}Q_p(s,w)|_{w=3/2}&=\left(1-\frac{1}{p^{w-1/2}}\right)\left(1+\frac{1}{p^{w-1/2}}+\left(1+\frac{1}{p}\right)\frac{p^{-2s}}{1-p^{-2s}}\right)\\&=\left(1-\frac{1}{p^2}\right)\left(1-p^{-2s}\right)^{-1}.\end{aligned} \tag{3.32}$$

For $p=2$, $D_{1,2}(s,w;1,\psi_1,\psi')$ is given by (3.20). It follows from (3.18)–(3.20), (3.31), and (3.32) that

$$\begin{aligned}D_1(s,w;1,\psi_1,\psi_0)&=\zeta\left(w-\frac{1}{2}\right)Q(s,w),\\D_1(s,w;1,\psi_1,\psi_{-1})&=\left(1-2^{-2s}\right)\zeta\left(w-\frac{1}{2}\right)Q(s,w),\end{aligned} \tag{3.33}$$

where

$$Q(s,w)|_{w=3/2} = \frac{(1-2^{-(w-1/2)})\zeta(2s)}{(1-2^{-2})\zeta(2)}\bigg|_{w=3/2} = \frac{2\zeta(2s)}{3\zeta(2)}. \tag{3.34}$$

We deduce from (3.29), (3.33), and (3.34) that

$$\operatorname*{Res}_{w=3/2} C(s,w) = 4^{-s}Q\left(s,\frac{3}{2}\right) + \left(1-2^{-2s}\right)Q\left(s,\frac{3}{2}\right) = \frac{2\zeta(2s)}{3\zeta(2)}.$$

Now (3.9), the functional equation (3.10), and the above lead to

$$\operatorname*{Res}_{s=3/2-w} A(s,w) = \operatorname*{Res}_{s=3/2-w} A_2(s,w) = \frac{2\cdot\pi^{1-w}}{3\cdot 4^{3/2-w}}\frac{\Gamma(\frac{w-1/2}{2})}{\Gamma(\frac{3/2-w}{2})}\frac{\zeta(2w-1)}{\zeta(2)}.$$

Setting $w = 1/2+\alpha$ gives

$$\operatorname*{Res}_{s=1-\alpha} A\left(s,\frac{1}{2}+\alpha\right) = \frac{2^{2\alpha-1}\pi^{1/2-\alpha}\Gamma(\frac{\alpha}{2})}{3\Gamma(\frac{1-\alpha}{2})}\frac{\zeta(2\alpha)}{\zeta(2)}. \tag{3.35}$$

Note that the functional equation in Lemma 3 for $n=1$ implies that

$$\zeta(2\alpha) = \pi^{2\alpha-1/2}\frac{\Gamma(\frac{1}{2}-\alpha)}{\Gamma(\alpha)}\zeta(1-2\alpha).$$

This allows us to rewrite e expression (3.35) as

$$\operatorname*{Res}_{s=1-\alpha} A(s,\frac{1}{2}+\alpha) = \frac{\pi^{\alpha}\Gamma(\frac{1}{2}-\alpha)\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{1-\alpha}{2})\Gamma(\alpha)}\cdot\frac{\zeta(1-2\alpha)}{\zeta(2)}\cdot\frac{2^{2\alpha}}{6}. \tag{3.36}$$

3.5 Bounding $A(s,w)$ in vertical strips

We will estimate $|A(s,w)|$ in vertical strips following the arguments in [4]. For the previously defined regions S_j, we set

$$\widetilde{S}_j = S_{j,\delta}\cap\left\{(s,w)\colon \Re(s) > -\frac{5}{2},\ \Re(w) > \frac{1}{2}-\delta\right\},$$

where δ is a fixed number with $0<\delta<1/1000$ and $S_{j,\delta} = \{(s,w)+\delta(1,1),\ (s,w)\in S_j\}$. We further set

$$p(s,w) = (s-1)(w-1)\left(s+w-\frac{3}{2}\right), \qquad \tilde{p}(s,w) = 1+\left|p(s,w)\right|.$$

Observe that $p(s,w)A(s,w)$ is analytic in the regions under our consideration. We apply (2.6) and partial summation to bound the expression for $A(s,w)$ given in (3.3) if $\Re(w)\geqslant 1/2$ and apply the functional equation to convert the case $\Re(w)<1/2$ back to the case $\Re(w)>1/2$. This gives that in the region $\widetilde{S}_0$,

$$p(s,w)A(s,w) \ll \tilde{p}(s,w)\left(1+|w|\right)^{\max\{1/2-\Re(w),\,0\}+\varepsilon}.$$

Similarly, we bound the expression for $A(s,w)$ given in (3.4) to see that in the region $\widetilde{S}_1$,

$$p(s,w)A(s,w) \ll \tilde{p}(s,w)\big(1+|s|\big)^{\max\{1/2-\Re(s),\,0\}+\varepsilon}.$$

Using the above estimates, we apply Lemma 7 to deduce that in the convex hull $\widetilde{S}_2$ of $\widetilde{S}_0$ and $\widetilde{S}_1$,

$$p(s,w)A(s,w) \ll \tilde{p}(s,w)\big(1+|w|\big)^{\max\{1/2-\Re(w),\,0\}+\varepsilon}\big(1+|s|\big)^{3+\varepsilon}. \tag{3.37}$$

Moreover, using estimates (2.5) for $\zeta(s)$ and $\zeta(2w)$ (corresponding to the case with $\psi=\psi_0$ being the trivial character) to bound $A_1(s,w)$ given in (3.5), we see that in the region $\widetilde{S}_3$,

$$A_1(s,w) \ll \big(1+|w|\big)^{\max\{1/2-\Re(2w),\,(1-\Re(2w))/2,\,0\}+\varepsilon}\big(1+|s|\big)^{\max\{1/2-\Re(s),\,(1-\Re(s))/2,\,0\}+\varepsilon}. \tag{3.38}$$

Also, we deduce from (3.11)–(3.15) and Lemma 8 that

$$C(s,w) \ll \big(1+|w|\big)^{\max\{(3/2-\Re(w))/2,\,0\}+\varepsilon} \tag{3.39}$$

in the region

$$\left\{(s,w)\colon\ \Re(s) \geqslant \frac{1}{2}+\varepsilon,\ \Re(w) \geqslant 1+\varepsilon,\ \Re\left(s+\frac{w}{2}\right) \geqslant \frac{7}{4}+\varepsilon\right\}.$$

Now applying (3.9) and the functional equation (3.10) together with estimates (3.38) and (3.39), we obtain that in the region $\widetilde{S}_4$,

$$\begin{aligned} p(s,w)A(s,w) &\ll \tilde{p}(s,w)\big(1+|s+w|\big)^{\max\{(3/2-\Re(s+w))/2,\,0\}+\varepsilon}\big(1+|s|\big)^{3+\varepsilon} \\ &\ll \tilde{p}(s,w)\big(1+|w|\big)^{2+\varepsilon}\big(1+|s|\big)^{5+\varepsilon}. \end{aligned} \tag{3.40}$$

Finally, we conclude from (3.37), (3.40), and Lemma 7 that in the convex hull $\widetilde{S}_5$ of $\widetilde{S}_2$ and $\widetilde{S}_4$,

$$p(s,w)A(s,w) \ll \tilde{p}(s,w)\big(1+|w|\big)^{2+\varepsilon}\big(1+|s|\big)^{5+\varepsilon}.$$

Dividing by $p(s,w)$, we obtain the following bound valid in $\widetilde{S}_5$ and away from the poles of $A(s,w)$:

$$A(s,w) \ll \big(1+|w|\big)^{2+\varepsilon}\big(1+|s|\big)^{5+\varepsilon}. \tag{3.41}$$

3.6 Completing the proof

Using the Mellin inversion on $w(n/X)$, we see that for the function $A(s,w)$ defined in (3.1),

$$\sum_{(n,2)=1} L^{(2)}\left(\frac{1}{2}+\alpha,\,\chi_n\right) w\left(\frac{n}{X}\right) = \frac{1}{2\pi \mathrm{i}} \int\limits_{(2)} A\left(s,\,\frac{1}{2}+\alpha\right) X^s \widehat{w}(s)\,\mathrm{d}s, \tag{3.42}$$

where $\widehat{w}$ is the Mellin transform of w given by

$$\widehat{w}(s) = \int\limits_0^\infty w(t)t^s \frac{\mathrm{d}t}{t}.$$

Integration by parts renders that for any integer $E \geqslant 0$,

$$\widehat{w}(s) \ll \frac{1}{(1+|s|)^E}. \tag{3.43}$$

We shift the line of integration in (3.42) to $\Re(s) = 1/2 + \varepsilon$. The integral on the new line can be absorbed into the O-term in (1.3) upon using (3.41) and (3.43). We also encounter two simple poles at $s = 1$ and $s = 1 - \alpha$ with the corresponding residues given by (3.8) and (3.36), respectively. Direct computations now lead to the main terms given in (1.3). This completes the proof of Theorem 1.

4 Proof of Corollary 1

Following the proof of Theorem 1, we derive from (3.9) and (3.10) that for $w = 1/2$,

$$A\left(s, \frac{1}{2}\right) = \sum_{(m,2)=1} \frac{L(s, \chi^{(4m)})}{m^{1/2}} = \frac{\pi^{s-1/2}}{4^s} \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} C\left(1-s,\, s+\frac{1}{2}\right) \tag{4.1}$$

with

$$C\left(1-s,\, s+\frac{1}{2}\right) = \sum_{\substack{q,m\geqslant 1\\(m,2)=1}} \frac{\tau(\chi_{4m}, q)}{q^{1-s} m^{s+1/2}}.$$

It follows from (3.11)–(3.15) and Lemma 8 that for $\Re(s) \geqslant 1/2 + \varepsilon$, the pole of the integrand occurs when $q_1 = 1$ and $\psi = \psi_1$. We derive from (3.13) that at this time,

$$C\left(1-s,\, s+\frac{1}{2}\right) = \zeta(s) \cdot \frac{(1-2^{-s})\zeta(2-2s)}{(1-2^{-2})\zeta(2)}. \tag{4.2}$$

Inserting into (4.1) and using the Mellin formula, we obtain that

$$\sum_{(n,2)=1} L^{(2)}\left(\frac{1}{2}, \chi_n\right) w\left(\frac{n}{X}\right) = \frac{1}{2\pi \mathrm{i}} \int_{(2)} A\left(s, \frac{1}{2}\right) X^s \widehat{w}(s)\, \mathrm{d}s. \tag{4.3}$$

Similarly, shifting the line of integration to $\Re(s) = 1/2 + \varepsilon$, we will encounter a unique pole $s = 1$ of order 2 and deduce the same error term as in Theorem 1. It suffices to estimate the main term. According to the expansions of Laurent series

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1), \qquad \Gamma\left(\frac{1-s}{2}\right) = \frac{-2}{s-1} - \gamma + O(s-1),$$

we see that the residue at $s = 1$ gives the main term of (4.3). The residue is given by

$$\lim_{s\to 1} \frac{\mathrm{d}}{\mathrm{d}s} (s-1)^2 A\left(s, \frac{1}{2}\right) X^s \widehat{w}(s).$$

We derive from the above, together with (4.1), (4.2), and the Laurent series, that

$$\begin{aligned} Q(\log X) &= \lim_{s\to 1} \frac{\mathrm{d}}{\mathrm{d}s}(s-1)^2 A\left(s, \frac{1}{2}\right)\widehat{w}(s)X^{s-1} \\ &= \frac{\widehat{w}(1)}{6\zeta(2)}\log X + \left(\frac{\gamma}{4} + \frac{\log\frac{\pi}{2}}{6} + \frac{\sqrt{\pi}}{12\Gamma'(\frac{1}{2})} + \frac{2\zeta'(0)}{3}\right)\frac{\widehat{w}(1)}{\zeta(2)} + \frac{\widehat{w}'(1)}{6\zeta(2)} \end{aligned} \tag{4.4}$$

as $\zeta(0) = -1/2$ and $\Gamma(1/2) = \sqrt{\pi}$. This gives an explicit expression of Q and finishes the proof of Corollary 1.

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