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# The existence of solutions to higher-order differential equations with nonhomogeneous conditions

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**Abstract.** We prove the existence and uniqueness of solutions to the differential equations of higher order $x^{(l)}(s) + g(s, x(s)) = 0$, $s \in [c, d]$, satisfying three-point boundary conditions that contain a nonhomogeneous term $x(c) = 0$, $x'(c) = 0$, $x''(c) = 0, \ldots, x^{(l-2)}(c) = 0$, $x^{(l-2)}(d) - \beta x^{(l-2)}(\eta) = \gamma$, where $l \geqslant 3$, $0 \leqslant c < \eta < d$, the constants $\beta$, $\gamma$ are real numbers, and $g : [c, d] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. By using finer bounds on the integral of kernel, the Banach and Rus fixed point theorems on metric spaces are utilized to prove the existence and uniqueness of a solution to the problem.

*MSC:* 34B10, 34B15

*Keywords:* three-point boundary value problem, kernel, existence, uniqueness, fixed point theorems

## 1 Introduction

To study qualitative and quantitative properties of real-world problems, mathematical models are formulated. The majority of these models involve the rate of change of the dependent variable in relation to the independent variable, which results in differential equations under certain conditions. In general, the theory of differential equations can be found in the study of deflection of curved beams, flow of a viscous fluid, theory of plate deflection, and many more [1, 7, 8]. Because of its importance in theory and applications, there has been a lot of interest in investigating the existence of solutions to boundary value problems over the past few decades.

In 1988, Gupta [10] proved the existence of solutions for the deformation of a flexible beam with fixed ends, described by the equation

$$x^{(4)}(s) - \pi^4 x(s) + g\big(s, x(s)\big) = e(s), \quad 0 < s < 1,$$
$$x(0) = x(1) = x''(0) = x''(1) = 0.$$

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In 2003, Ma [14] studied the existence of multiple positive solutions for the deformation of a flexible beam with fixed one end and the other one free, described by the equation

$$x^{(4)}(s) = \lambda f\big(s, x(s), x'(s)\big), \quad 0 < s < 1,$$
$$x(0) = x'(0) = x''(1) = x'''(1) = 0.$$

In 2019, Li and Gao [13] demonstrated the solvability of deformation of a flexible beam with fixed ends in the equilibrium state, described by

$$x^{(4)}(s) = f\big(s, x(s), x''(s)\big), \quad s \in [0,1],$$
$$x(0) = x(1) = x''(0) = x''(1) = 0.$$

After that, the researchers studied the existence and uniqueness of solutions to the problems of third order; see [3, 5, 17, 20, 21], and for fourth order, see [4, 9, 19, 22].

In view of these interesting studies, we consider the higher-order differential equation

$$x^{(l)}(s) + g\big(s, x(s)\big) = 0, \quad s \in [c,d], \tag{1.1}$$

satisfying the nonhomogeneous conditions

$$\begin{aligned} &x(c) = 0, \quad x'(c) = 0, \quad x''(c) = 0, \quad \dots, \quad x^{(l-2)}(c) = 0, \\ &x^{(l-2)}(d) - \beta x^{(l-2)}(\eta) = \gamma, \end{aligned} \tag{1.2}$$

where $l \geqslant 3$, $0 \leqslant c < \eta < d$, the constants $\beta$, $\gamma$ are real numbers, and $g : [c,d] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The results are proved using fixed point theorems based on metrics. This problem generalizes many problems in the existing literature. Sun [23] and Almuthaybiri et al. [2] studied the third-order problems by taking $l = 3$ in (1.1) and (1.2). Sun et al. [24], Lakoud et al. [12], and Madhubabu et al. [15] addressed the fourth-order problems ($l = 4$).

The problem must be well posed to study the real-world problems under specific conditions. If the problem has a unique solution with given conditions, then several techniques can be used to verify the "well-posedness" of problem [26].

We now suppose that the following conditions are satisfied:

(A1) the constants $c$ and $d$ satisfy $(d-c)/(\eta-c) \neq \beta$, and
(A2) $|g(s,x) - g(s,z)| \leqslant \lambda|x-z|$ for all $(s,x), (s,z) \in [c,d] \times \mathbb{R}$, where $\lambda$ is a Lipschitz constant.

The paper is organized as follows. In Section 2, we obtain the solution of (1.1)–(1.2) by transforming it as a related integral equation that contains a kernel and then establish the estimates of kernels under integration. In Section 3, we establish the results on the existence of a unique solution to problem (1.1)–(1.2) using Banach and Rus fixed point theorems and also provide some examples.

## 2 Preliminary findings

We obtain the solution of (1.1) and (1.2) by writing the related equivalent integral equation involving a kernel. After that, we determine finer estimations of kernels under integration.

Let $\Psi \in C([c,d], \mathbb{R})$. Then we obtain a unique solution to the problem

$$x^{(l)}(s) + \Psi(s) = 0, \quad s \in [c,d], \tag{2.1}$$

with conditions specified in (1.2).

**Lemma 1.** (*See* [16,25].) *If condition* (A1) *holds, then the unique solution to equations* (2.1) *and* (1.2) *is given by*

$$x(s) = \frac{\gamma(s-c)^{l-1}}{\Delta} + \int_c^d \mathcal{R}(s,\theta)\Psi(\theta)\,\mathrm{d}\theta, \tag{2.2}$$

*where*

$$\mathcal{R}(s,\theta) = \mathcal{M}(s,\theta) + \frac{\beta(s-c)^{l-1}}{\Delta}\mathcal{N}(\eta,\theta), \tag{2.3}$$

$$\mathcal{M}(s,\theta) = \begin{cases} \frac{(s-c)^{l-1}(d-\theta)}{(l-1)!(d-c)} - \frac{(s-\theta)^{l-1}}{(l-1)!}, & c \leqslant \theta \leqslant s \leqslant d, \\ \frac{(s-c)^{l-1}(d-\theta)}{(l-1)!(d-c)}, & c \leqslant s \leqslant \theta \leqslant d, \end{cases}$$

$$\mathcal{N}(\eta,\theta) = \begin{cases} \frac{(\theta-c)(d-\eta)}{(d-c)}, & c \leqslant \theta \leqslant \eta \leqslant d, \\ \frac{(\eta-c)(d-\theta)}{(d-c)}, & c \leqslant \eta \leqslant \theta \leqslant d, \end{cases}$$

*and*

$$\Delta = (l-1)!\big[(d-c) - \beta(\eta-c)\big] \neq 0.$$

*Proof.* The integral equation corresponding to (2.1) is

$$x(s) = P_0 + P_1 s + P_2 s^2 + P_3 s^3 + \dots + P_{l-2}s^{l-2} + P_{l-1}s^{l-1} - \frac{1}{(l-1)!}\int_c^s (s-\theta)^{l-1}\Psi(\theta)\,\mathrm{d}\theta, \tag{2.4}$$

where $P_0, P_1, P_2, \dots, P_l$ are constants. Using conditions (1.2), we obtain the following equations:

$$\begin{aligned} &P_0 + P_1 c + P_2 c^2 + \dots + P_{l-1}c^{l-1} = 0, \\ &P_1 + 2P_2 c + 3P_3 c^2 + \dots + (l-2)P_{l-2}c^{l-3} + (l-1)P_{l-1}c^{l-2} = 0, \\ &2P_2 + 6P_3 c + \dots + (l-2)(l-3)P_{l-2}c^{l-4} + (l-1)(l-2)P_{l-1}c^{l-3} = 0, \\ &\quad \dots \\ &(l-2)!(1-\beta)P_{l-2} + (l-1)!(d-\beta\eta)P_{l-1} = \Gamma, \end{aligned}$$

where

$$\Gamma = \gamma + \int_c^d (d-\theta)\Psi(\theta)\,\mathrm{d}\theta - \beta\int_c^\eta (\eta-\theta)\Psi(\theta)\,\mathrm{d}\theta.$$

Solving the above, we get

$$P_{l-1} = \frac{\Gamma}{(l-1)![d-c-\beta(\eta-c)]}, \quad P_{l-2} = -\frac{(l-1)}{1!}cP_{l-1}, \quad P_{l-3} = -\frac{(l-1)(l-2)}{2!}c^2 P_{l-1}, \quad \dots,$$
$$P_1 = (-1)^{l-2}(l-2)(l-1)c^{l-2}P_{l-1}, \quad P_0 = (-1)^{l-1}c^{l-1}P_{l-1}.$$

Substituting these values into (2.4), we have

$$
\begin{aligned}
x(s) &= \frac{\Gamma}{\Delta}\left[s^{l-1} + \frac{(l-1)}{1!}s^{l-2}c + \cdots + (-1)^{l-2}(l-2)(l-1)sc^{l-2} + (-1)^{l-1}c^{l-1}\right] \\
&\quad - \frac{1}{(l-1)!}\int_c^s (s-\theta)^{l-1}\Psi(\theta)\,\mathrm{d}\theta \\
&= \frac{(s-c)^{l-1}}{\Delta}\left[\gamma + \int_c^d (d-\theta)\Psi(\theta)\,\mathrm{d}\theta - \beta\int_c^\eta (\eta-\theta)\Psi(\theta)\,\mathrm{d}\theta\right] - \frac{1}{(l-1)!}\int_c^s (s-\theta)^{l-1}\Psi(\theta)\,\mathrm{d}\theta \\
&= \frac{\gamma(s-c)^{l-1}}{\Delta} + \frac{(s-c)^{l-1}[(d-c)-\beta(\eta-c)+\beta(\eta-c)]}{(l-1)![(d-c)-\beta(\eta-c)](d-c)}\int_c^d (d-\theta)\Psi(\theta)\,\mathrm{d}\theta \\
&\quad - \frac{\beta(s-c)^{l-1}}{\Delta}\int_c^\eta (\eta-\theta)\Psi(\theta)\,\mathrm{d}\theta - \frac{1}{(l-1)!}\int_c^s (s-\theta)^{l-1}\Psi(\theta)\,\mathrm{d}\theta \\
&= \frac{\gamma(s-c)^{l-1}}{\Delta} + \frac{(s-c)^{l-1}}{(l-1)!(d-c)}\int_c^d (d-\theta)\Psi(\theta)\,\mathrm{d}\theta + \frac{\beta(s-c)^{l-1}(\eta-c)}{(d-c)\Delta}\int_c^d (d-\theta)\Psi(\theta)\,\mathrm{d}\theta \\
&\quad - \frac{\beta(s-c)^{l-1}}{\Delta}\int_c^\eta (\eta-\theta)\Psi(\theta)\,\mathrm{d}\theta - \frac{1}{(l-1)!}\int_c^s (s-\theta)^{l-1}\Psi(\theta)\,\mathrm{d}\theta \\
&= \frac{\gamma(s-c)^{l-1}}{\Delta} + \int_c^s \left[\frac{(s-c)^{l-1}(d-\theta)}{(l-1)!(d-c)} - \frac{(s-\theta)^{l-1}}{(l-1)!}\right]\Psi(\theta)\,\mathrm{d}\theta + \int_s^d \left[\frac{(s-c)^{l-1}(d-\theta)}{(l-1)!(d-c)}\right]\Psi(\theta)\,\mathrm{d}\theta \\
&\quad + \frac{\beta(s-c)^{l-1}}{\Delta}\left\{\int_c^\eta \left[\frac{(\theta-c)(d-\eta)}{(d-c)}\right]\Psi(\theta)\,\mathrm{d}\theta + \int_\eta^d \left[\frac{(\eta-c)(d-\theta)}{(d-c)}\right]\Psi(\theta)\,\mathrm{d}\theta\right\} \\
&= \frac{\gamma(s-c)^{l-1}}{\Delta} + \int_c^d \mathcal{M}(s,\theta)\Psi(\theta)\,\mathrm{d}\theta + \frac{\beta(s-c)^{l-1}}{\Delta}\int_c^d \mathcal{N}(\eta,\theta)\Psi(\theta)\,\mathrm{d}\theta \\
&= \frac{\gamma(s-c)^{l-1}}{\Delta} + \int_c^d \mathcal{R}(s,\theta)\Psi(\theta)\,\mathrm{d}\theta.
\end{aligned}
$$

To prove the uniqueness of the solution, let $u$ be another solution of (2.1) and (1.2). Take $z(s) = x(s) - u(s)$. Then

$$z^{(l)}(s) = 0, \quad s \in [c,d], \tag{2.5}$$

$$z(c) = 0, \quad z'(c) = 0, \quad z''(c) = 0, \quad \ldots, \quad z^{(l-2)}(c) = 0, \qquad z^{(l-2)}(d) - \beta z^{(l-2)}(\eta) = 0. \tag{2.6}$$

The solution to (2.5) is

$$z(s) = Q_0 + Q_1 s + Q_2 s^2 + Q_3 s^3 + \cdots + Q_l s^l,$$

where $Q_0, Q_1, Q_2, \dots, Q_l$ are constants. Using conditions (2.6), it can be expressed as $PQ = O$, where

$$P = \begin{bmatrix} 1 & c & c^2 & \cdots & c^{l-1} \\ 0 & 1 & 2c & \cdots & (l-1)c^{l-2} \\ 0 & 0 & 1 & \cdots & (l-1)(l-2)c^{l-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & (d-\beta\eta)(l-1)! \end{bmatrix},$$

$$Q = [Q_0 \quad Q_1 \quad Q_2 \quad \cdots \quad Q_{l-1}]^{\mathrm{T}}, \quad \text{and} \quad O = [0 \quad 0 \quad 0 \quad \cdots \quad 0]^{\mathrm{T}}.$$

Since the determinant of $P$, $1 \cdot 2! \cdot 3! \cdots (l-3)! \cdot (l-2)!\,\Delta$, is nonzero, the system $PQ = O$ has a trivial solution only, and hence $z(s) \equiv 0$ for all $s \in [c, d]$. The proof is complete. □

*Remark 1.* Differentiating Eq. (2.2) with respect to $s$ up to $l$ times yields an equivalent boundary value problem (2.1) and (1.2).

**Lemma 2.** *For all $s, \theta \in [c, d]$, the kernel $\mathcal{M}(s, \theta)$ is nonnegative.*

*Proof.* The positivity of $\mathcal{M}(s, \theta)$ is proved using simple algebraic computations. □

**Lemma 3.** *The kernel $\mathcal{M}(s, \theta)$ satisfies the integral inequality*

$$\int_c^d \mathcal{M}(s, \theta)\,\mathrm{d}\theta \leqslant \frac{(l-1)^{l-1}(d-c)^l}{l!2^l} \quad \textit{for all } s \in [c, d]. \tag{2.7}$$

*Proof.* Consider

$$\begin{aligned} \int_c^d \mathcal{M}(s, \theta)\,\mathrm{d}\theta &= \int_c^s \left[ \frac{(s-c)^{l-1}(d-\theta)}{(l-1)!(d-c)} - \frac{(s-\theta)^{l-1}}{(l-1)!} \right] \mathrm{d}\theta + \int_s^d \frac{(s-c)^{l-1}(d-\theta)}{(l-1)!(d-c)}\,\mathrm{d}\theta \\ &= \left[ -\frac{(s-c)^{l-1}(d-\theta)^2}{(l-1)!2(d-c)} + \frac{(s-\theta)^l}{l(l-1)!} \right]_c^s + \left[ -\frac{(s-c)^{l-1}(d-\theta)^2}{(l-1)!2(d-c)} \right]_s^d \\ &= \frac{(s-c)^{l-1}(d-c)}{2(l-1)!} - \frac{(s-c)^l}{l!}. \end{aligned}$$

Let $\Phi(s) = (s-c)^{l-1}(d-c)/(2(l-1)!) - (s-c)^l/l!$. Then the maximum value of $\Phi(s)$ is attained at $s = (d-c)(l-1)/2 + c$ by application of results from the fundamental calculus and is given by

$$\max_{s \in [c,d]} \Phi(s) = \max_{s \in [c,d]} \left[ \frac{(s-c)^{l-1}(d-c)}{2(l-1)!} - \frac{(s-c)^l}{l!} \right] = \frac{(l-1)^{l-1}(d-c)^l}{l!2^l},$$

which gives inequality (2.7). □

**Lemma 4.** *The kernel $\mathcal{N}(\eta, \theta)$ satisfies the integral inequality*

$$\int_c^d \left|\mathcal{N}(\eta, \theta)\right| \mathrm{d}\theta \leqslant \frac{1}{2}(d-c)^2.$$

*Proof.* We have

$$\begin{aligned}
\int_c^d \left|\mathcal{N}(\eta,\theta)\right| \mathrm{d}\theta &= \int_c^\eta \left|\mathcal{N}(\eta,\theta)\right| \mathrm{d}\theta + \int_\eta^d \left|\mathcal{N}(\eta,\theta)\right| \mathrm{d}\theta \\
&= \int_c^\eta \left|\frac{(\theta-c)(d-\eta)}{(d-c)}\right| \mathrm{d}\theta + \int_\eta^d \left|\frac{(\eta-c)(d-\theta)}{(d-c)}\right| \mathrm{d}\theta \\
&= \int_c^\eta \frac{(\theta-c)(d-\eta)}{(d-c)} \,\mathrm{d}\theta + \int_\eta^d \frac{(\eta-c)(d-\theta)}{(d-c)} \,\mathrm{d}\theta \\
&= \left[\frac{(\theta-c)^2(d-\eta)}{2(d-c)}\right]_c^\eta + \left[-\frac{(\eta-c)(d-\theta)^2}{2(d-c)}\right]_\eta^d \\
&= \frac{1}{2}(\eta-c)(d-\eta) \leqslant \frac{1}{2}(d-c)^2. \qquad \square
\end{aligned}$$

**Lemma 5.** *The kernel $\mathcal{R}(s,\theta)$ satisfies the integral inequality*

$$\int_c^d \left|\mathcal{R}(s,\theta)\right| \mathrm{d}\theta \leqslant (d-c)^l \left[\frac{(l-1)^{l-1}}{l!2^l} + \frac{|\beta|(d-c)}{(l-1)!2|(d-c)-\beta(\eta-c)|}\right].$$

*Proof.* We have

$$\begin{aligned}
\int_c^d \left|\mathcal{R}(s,\theta)\right| \mathrm{d}\theta &= \int_c^d \left|\mathcal{M}(s,\theta) + \frac{\beta(s-c)^{l-1}}{\Delta}\mathcal{N}(\eta,\theta)\right| \mathrm{d}\theta \\
&\leqslant \int_c^d \left|\mathcal{M}(s,\theta)\right| \mathrm{d}\theta + \left|\frac{\beta(s-c)^{l-1}}{\Delta}\right| \int_c^d \left|\mathcal{N}(\eta,\theta)\right| \mathrm{d}\theta \\
&\leqslant \frac{(l-1)^{l-1}(d-c)^l}{l!2^l} + \frac{|\beta|(d-c)^{l-1}}{|\Delta|} \cdot \frac{(d-c)^2}{2} \\
&= (d-c)^l \left[\frac{(l-1)^{l-1}}{l!2^l} + \frac{|\beta|(d-c)}{(l-1)!2|(d-c)-\beta(\eta-c)|}\right]. \qquad \square
\end{aligned}$$

The following Banach and Rus fixed point theorems are crucial in demonstrating our results.

**Theorem 1.** (*See* [6].) *Let $\rho$ be a metric on a nonempty set $H$, and suppose that the pair $(H,\rho)$ constitutes a complete metric space. If $G: H \to H$ satisfies the inequality*

$$\rho(Gx, Gz) \leqslant \beta\rho(x,z)$$

*for all $x, z \in H$, where $0 < \beta < 1$, then $G$ has a unique fixed point $x^* \in H$, that is, such that $Gx^* = x^*$.*

**Theorem 2.** (*See* [18].) *Let $\rho$ and $\delta$ be two metrics on a nonempty set $H$, and suppose the pair $(H,\rho)$ constitutes a complete metric space. Suppose $G: H \to H$ is a continuous function with respect to the metric $\rho$*

*on $H$ and satisfies the following inequalities for $x, z \in H$:*

$$\rho(Gx, Gz) \leqslant \Upsilon \delta(x, z), \quad \textit{where } \Upsilon > 0, \tag{2.8}$$

*and*

$$\delta(Gx, Gz) \leqslant \kappa \delta(x, z), \quad \textit{where } 0 < \kappa < 1. \tag{2.9}$$

*Then $G$ has a unique fixed point $x^* \in H$, that is, such that $Gx^*=x^*$.*

## 3 Main results depending on metrics

Let $H$ be the set of all continuous real-valued functions on $[c, d]$. Define the metric $\rho$ on $H$ by

$$\rho(x, z) = \max_{s \in [c,d]} \bigl|x(s) - z(s)\bigr|, \tag{3.1}$$

and let

$$\delta(x, z) = \Biggl(\int_c^d \bigl|x(s) - z(s)\bigr|^p \,\mathrm{d}s\Biggr)^{1/p}, \quad p > 1,$$

for $x, z \in H$. Then we can see that the ordered pair $(H, \rho)$ is a complete metric space, and the ordered pair $(H, \delta)$ is a metric space. It is clear that the useful relationship between the two metrics $\rho$ and $\delta$ is given by

$$\delta(x, z) \leqslant (d - c)^{1/p} \rho(x, z) \quad \text{for all } x, z \in H. \tag{3.2}$$

Define the operator $G : H \to H$ by

$$Gx(s) = \frac{\gamma(s - c)^{l-1}}{\Delta} + \int_c^d \mathcal{R}(s, \theta) g\bigl(\theta, x(\theta)\bigr) \,\mathrm{d}\theta \quad \text{for all } s \in [c, d],$$

where the kernel $\mathcal{R}(s, \theta)$ is given in (2.3).

Note that $x$ is a solution of (1.1)–(1.2) if and only if it satisfies

$$x(s) = \frac{\gamma(s - c)^{l-1}}{\Delta} + \int_c^d \mathcal{R}(s, \theta) g\bigl(\theta, x(\theta)\bigr) \,\mathrm{d}\theta \quad \text{for all } s \in [c, d]. \tag{3.3}$$

**Theorem 3.** *Suppose conditions* (A1) *and* (A2) *are satisfied. If $c$ and $d$ satisfy the inequality*

$$(d - c)^l \biggl[\frac{(l - 1)^{l-1}}{l!2^l} + \frac{|\beta|(d - c)}{(l - 1)!2|(d - c) - \beta(\eta - c)|}\biggr] < \frac{1}{\lambda}, \tag{3.4}$$

*then problem* (1.1)–(1.2) *admits a unique solution.*

*Proof.* For establishing the uniqueness of solution to the boundary value problem (1.1)–(1.2), the operator $G$ has a unique fixed point $x^* \in H$, that is, $Gx^* = x^*$. Such a fixed point will also lie in $C^{(l)}([c, d])$, as we can directly see by differentiating the integral equation (3.3).

For all $x, z \in H$ and $s \in [c,d]$, using (A2), we obtain

$$\begin{aligned}
\big|Gx(s) - Gz(s)\big| &= \left|\frac{\gamma(s-c)^{l-1}}{\Delta} + \int_c^d \mathcal{R}(s,\theta) g\big(\theta, x(\theta)\big)\,\mathrm{d}\theta - \frac{\gamma(s-c)^{l-1}}{\Delta} - \int_c^d \mathcal{R}(s,\theta) g\big(\theta, z(\theta)\big)\,\mathrm{d}\theta\right| \\
&\leqslant \int_c^d \big|\mathcal{R}(s,\theta)\big|\big|g\big(\theta,x(\theta)\big) - g\big(\theta,z(\theta)\big)\big|\,\mathrm{d}\theta \\
&\leqslant \lambda \int_c^d \big|\mathcal{R}(s,\theta)\big|\big|x(\theta) - z(\theta)\big|\,\mathrm{d}\theta \leqslant \lambda \int_c^d \big|\mathcal{R}(s,\theta)\big|\rho(x,z)\,\mathrm{d}\theta \\
&\leqslant \lambda(d-c)^l\left[\frac{(l-1)^{l-1}}{l!2^l} + \frac{|\beta|(d-c)}{(l-1)!2|(d-c) - \beta(\eta-c)|}\right]\rho(x,z).
\end{aligned}$$

From this it follows that

$$\rho(Gx, Gz) \leqslant \Upsilon\rho(x,z),$$

where

$$\Upsilon = \lambda(d-c)^l\left[\frac{(l-1)^{l-1}}{l!2^l} + \frac{|\beta|(d-c)}{(l-1)!2|(d-c) - \beta(\eta-c)|}\right].$$

By applying (3.4) we can see that $\Upsilon < 1$, and thus the operator $G$ satisfies the conditions of Theorem 1. This shows that the operator $G$ has a unique fixed point and is a solution of (1.1)–(1.2).

In accordance with Rus' theorem, we employ two metrics to establish the uniqueness of (1.1)–(1.2).

**Theorem 4.** *Suppose that conditions* (A1) *and* (A2) *are satisfied. If two constants* $p > 1$ *and* $q > 1$ *satisfy* $1/p + 1/q = 1$ *and*

$$\lambda\left(\int_c^d\left(\int_c^d \big|\mathcal{R}(s,\theta)\big|^q d\theta\right)^{p/q}\mathrm{d}s\right)^{1/p} < 1, \tag{3.5}$$

*then problem* (1.1)–(1.2) *admits a unique solution.*

*Proof.* For establishing the uniqueness of solution to the boundary value problem (1.1)–(1.2), the operator $G$ has a unique fixed point $x^* \in H$, that is, such that $Gx^* = x^*$. Such a fixed point will also lie in $C^{(l)}([c,d])$, as we can directly see by differentiating the integral equation (3.3). First, we demonstrate that (2.8) of Theorem 2 holds.

Using (A2) and Hölder's inequality [11], for all $x, z \in H$ and $s \in [c,d]$, we obtain

$$\begin{aligned}
\big|Gx(s) - Gz(s)\big| &\leqslant \int_c^d \big|\mathcal{R}(s,\theta)\big|\big|g\big(\theta,x(\theta)\big) - g\big(\theta,z(\theta)\big)\big|\,\mathrm{d}\theta \\
&\leqslant \int_c^d \big|\mathcal{R}(s,\theta)\big|\lambda\big|x(\theta) - z(\theta)\big|\,\mathrm{d}\theta
\end{aligned}$$

$$\leqslant \left( \int_c^d \left| \mathcal{R}(s,\theta) \right|^q \mathrm{d}\theta \right)^{1/q} \lambda \left( \int_c^d \left| x(\theta) - z(\theta) \right|^p \mathrm{d}\theta \right)^{1/p}$$

$$\leqslant \lambda \max_{s \in [c,d]} \left( \int_c^d \left| \mathcal{R}(s,\theta) \right|^q \mathrm{d}\theta \right)^{1/q} \delta(x,z).$$

Now we define

$$\Upsilon = \lambda \max_{s \in [c,d]} \left( \int_c^d \left| \mathcal{R}(s,\theta) \right|^q \mathrm{d}\theta \right)^{1/p}.$$

We conclude that

$$\rho(Gx, Gz) \leqslant \Upsilon \delta(x,z) \quad \text{for some } \Upsilon > 0 \text{ and for all } x, z \in H, \tag{3.6}$$

and so inequality (2.8) of Theorem 2 is satisfied. Now applying inequality (3.2) to (3.6), we get

$$\rho(Gx, Gz) \leqslant \Upsilon \delta(x,z) \leqslant \Upsilon (d-c)^{1/p} \rho(x,z) \quad \text{for all } x, z \in H.$$

Thus, for any given $\epsilon > 0$, we can take $\Omega = \epsilon/(\Upsilon(d-c)^{1/p})$ such that $\rho(x,z) < \Omega$, which implies that $\rho(Gx, Gz) < \epsilon$. Hence the operator $G$ is continuous on $H$ with respect to the metric $\rho$ given in (3.1).

Now we show that inequality (2.9) of Theorem 2 is satisfied. For all $x, z \in H$ and for $s \in [c,d]$, we obtain

$$\left( \int_c^d \left| Gx(s) - Gz(s) \right|^p \mathrm{d}s \right)^{1/p}$$

$$\leqslant \left( \int_c^d \left[ \left( \int_c^d \left| \mathcal{R}(s,\theta) \right|^q \mathrm{d}\theta \right)^{1/q} \lambda \left( \int_c^d \left| x(\theta) - z(\theta) \right|^p \mathrm{d}\theta \right)^{1/p} \right]^p \mathrm{d}s \right)^{1/p}$$

$$\leqslant \lambda \left( \int_c^d \left( \int_c^d \left| \mathcal{R}(s,\theta) \right|^q \mathrm{d}\theta \right)^{p/q} \mathrm{d}s \right)^{1/p} \delta(x,z),$$

and so we conclude

$$\delta(Gx, Gz) \leqslant \lambda \left( \int_c^d \left( \int_c^d \left| \mathcal{R}(s,\theta) \right|^q \mathrm{d}\theta \right)^{p/q} \mathrm{d}s \right)^{1/p} \delta(x,z) = \kappa \delta(x,z),$$

where

$$\kappa = \lambda \left( \int_c^d \left( \int_c^d \left| \mathcal{R}(s,\theta) \right|^q \mathrm{d}\theta \right)^{p/q} \mathrm{d}s \right)^{1/p}.$$

By applying (3.5) we can see that $\kappa < 1$, and, consequently, the operator $G$ satisfies all of the conditions of Theorem 2. This shows that the operator $G$ has only one fixed point, which is a solution of (1.1)–(1.2). □

Examples are provided to support our established results.

*Example 1.* Consider the third-order differential equation

$$x''' + \frac{1}{2s}x + s^2 e^{2s} = 0, \quad s \in [1,2], \tag{3.7}$$

with

$$x(1) = 0, \quad x'(1) = 0, \qquad x'(2) - \frac{1}{2}x'\left(\frac{3}{2}\right) = 0. \tag{3.8}$$

Clearly, $\Delta = 2![(d-c) - \beta(\eta - c)] = 3/2 \neq 0$. Then

$$\left|\frac{\partial g(s,x)}{\partial x}\right| = \left|\frac{1}{2s}\right| \leqslant 1$$

and

$$(d-c)^3\left[\frac{1}{12} + \frac{|\beta|(d-c)}{4|(d-c) - \beta(\eta - c)|}\right] = \frac{1}{4} = 0.25 < \frac{1}{\lambda}.$$

Hence by Theorem 3 problem (3.7)–(3.8) has a unique solution.

To determine the numerical solution of the above problem, from Eq. (3.7) we obtain

$$x'''(1) = -e^2 \quad \text{and} \quad x^{(4)}(1) = -4e^2.$$

Therefore the Taylor series solution is

$$x(s) \equiv \frac{x''(1)}{2}(s-1)^2 - \frac{e^2}{6}(s-1)^3 - \frac{e^2}{6}(s-1)^4.$$

Using the condition $x'(2) - x'(3/2)/2 = 0$, we get

$$x''(1) = \frac{17}{12}e^2.$$

Hence

$$x(s) \equiv \frac{17}{12}e^2(s-1)^2 - \frac{e^2}{6}(s-1)^3 - \frac{e^2}{6}(s-1)^4.$$

The graph of the solution $x$ is displayed in Fig. 1.

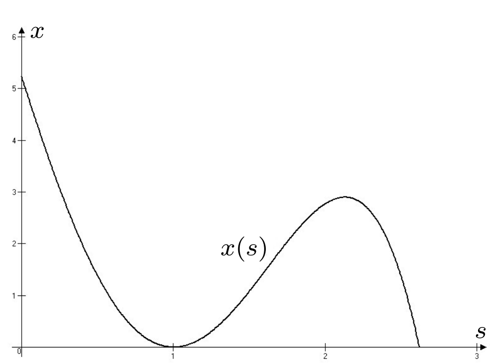


**Figure 1.**

*Example 2.* Consider the fourth-order differential equation

$$x^{(4)} + 6e^{3s} + s^2 + \cos x = 0, \quad s \in [0,1], \tag{3.9}$$

with

$$x(0) = 0, \quad x'(0) = 0, \quad x''(0) = 0, \qquad x''(1) - x''\left(\frac{1}{2}\right) = 0. \tag{3.10}$$

Clearly, $\varDelta = 3![(d-c) - \beta(\eta - c)] = 3 \neq 0$. Then

$$\left|\frac{\partial g(s,x)}{\partial x}\right| = \sin x \leqslant 1.$$

For convenience, taking $p = 2$ and $q = 2$, by algebraic computations we obtain

$$\int_0^1 \left|\mathcal{R}(s,\theta)\right|^2 \mathrm{d}\theta = \frac{2}{15}s^5 + \frac{1}{3}s^4(1-s)^4,$$

$$\int_0^1 \left(\int_0^1 \left|\mathcal{R}(s,\theta)\right|^2 \mathrm{d}\theta\right) \mathrm{d}s = \frac{43}{1890} = 0.022751,$$

and so

$$\left(\int_0^1 \left(\int_0^1 \left|\mathcal{R}(s,\theta)\right|^2 \mathrm{d}\theta\right) \mathrm{d}s\right)^{1/2} = 0.150834346 < \frac{1}{\lambda}.$$

Hence by Theorem 4 problem (3.9)–(3.10) has a unique solution.

To determine the numerical solution of the above problem, from Eq. (3.9) we obtain

$$x^{(4)}(0) = -7.$$

Therefore the Taylor series solution is

$$x(s) \equiv \frac{x'''(0)}{6}s^3 - \frac{7}{24}s^4.$$

Using the condition $x''(1) - x''(1/2) = 0$, we get

$$x'''(0) = \frac{49}{18}.$$

Hence

$$x(s) \equiv \frac{49}{108}s^3 - \frac{7}{2}s^4.$$

The graph of the solution $x$ is displayed in Fig. 2.

*Remark 2.* The Rus theorem has two metrics that are not necessarily equal. The space in the Rus theorem is considered to be complete with respect to the first of these metrics but not necessarily complete with respect to the second one. In addition, the operator is assumed to be contractive concerning the second metric. Hence the Rus theorem applies to a wider class of problems even though the interval length is large when compared to the Banach theorem.

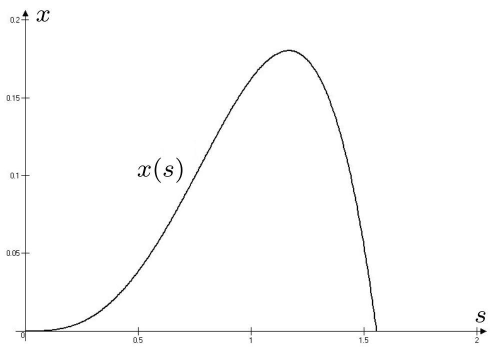


**Figure 2.**

Let us consider an example illustrating Remark 2.

*Example 3.* Consider the third-order differential equation

$$x''' + \frac{x}{3s^2} + \mathrm{e}^{3s}s^4 = 0, \quad s \in [1,3], \tag{3.11}$$

with

$$x(0) = 0, \quad x'(0) = 0, \qquad x'(3) - \frac{1}{4}x'(2) = 0. \tag{3.12}$$

Clearly, $\Delta = 2![(d-c) - \beta(\eta - c)] = 7/2 \neq 0$. Then

$$\left|\frac{\partial g(s,x)}{\partial x}\right| = \left|\frac{1}{3s^2}\right| \leqslant 1,$$

and

$$(d-c)^3\left[\frac{1}{12} + \frac{|\beta|(d-c)}{4|(d-c) - \beta(\eta - c)|}\right] = \frac{26}{21} = 1.238095 > \frac{1}{\lambda}.$$

Therefore Theorem 3 is not applicable to problem (3.11)–(3.12).

For Theorem 4, taking $p = 2$ and $q = 2$, by algebraic computations we obtain

$$\int_1^3 \left|\mathcal{R}(s,\theta)\right|^2 \mathrm{d}\theta = \frac{1}{105840}(s-1)^5\left(605s^2 - 4675s + 9362\right) + \frac{17}{168}\left(s^2 - 4s + 3\right)^2,$$

$$\int_1^3 \left(\int_1^3 \left|\mathcal{R}(s,\theta)\right|^2 \mathrm{d}\theta\,\mathrm{d}s\right) = \frac{1492}{6615} = 0.22555,$$

and so

$$\left(\int_1^3 \left(\int_1^3 \left|\mathcal{R}(s,\theta)\right|^2 \mathrm{d}\theta\right)\mathrm{d}s\right)^{1/2} = 0.474921 < \frac{1}{\lambda}.$$

Hence by Theorem 4 problem (3.11)–(3.12) has a unique solution.

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