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Closure under infinitely divisible distribution roots and the Embrechts–Goldie conjecture

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Abstract. We show that the distribution class $\mathcal{L}(\gamma) \setminus \mathcal{OS}$ is not closed under infinitely divisible distribution roots for $\gamma > 0$, that is, we provide some infinitely divisible distributions belonging to the class, whereas the corresponding Lévy distributions do not. In fact, one part of these Lévy distributions belonging to the class $\mathcal{OL}\setminus\mathcal{L}(\gamma)$ have different properties, and the other parts even do not belong to the class $\mathcal{OL}$. Therefore, combining with the existing related results, we give a completely negative conclusion for the subject and Embrechts–Goldie conjecture. Then we discuss some interesting issues related to the results of this paper.

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1 Introduction and main results

In this paper, all limit relations refer to $x \to \infty$, all distributions are supported on $[0, \infty)$, and the tailed distribution of a distribution V is denoted by $\overline{V} = 1 - V$, which respectively constitute the distribution function $V(x) = V(-\infty, x]$ and tail distribution function $\overline{V}(x) = V[x, \infty)$ for all $x \in (-\infty, \infty)$, unless otherwise stated.

Let H be an infinitely divisible distribution with the Laplace transform

$$\int_0^\infty \exp\{-\lambda y\}\, H(\mathrm{d}y) = \exp\left\{-a\lambda - \int_0^\infty \left(1 - \mathrm{e}^{\lambda y}\right) \nu(\mathrm{d}y)\right\}, \quad \operatorname{Re}\lambda \geqslant 0,$$

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where a is a nonnegative constant, and ν is a Borel measure on $(0,\infty)$, called a Lévy measure, that satisfies $\mu=\nu(1,\infty)<\infty$ and $\int_0^\infty \min\{1,y^2\}\nu(\mathrm{d}y)<\infty$ and generates the Lévy distribution F such that

$$F(x)=\frac{\nu(1,x]}{\mu}\mathbf{1}_{(1,\infty)}(x),\quad x\in(-\infty,\infty).$$

Then the distribution H admits the representation $H=H_1*H_2$, where $*$ denotes the convolution binary operation, $\overline{H_1}(x)=O(\mathrm{e}^{-\beta x})$ for all $\beta>0$, and

$$H_2(x)=\mathrm{e}^{-\mu}\sum_{k=0}^{\infty}\frac{F^{*k}(x)\mu^k}{k!},\quad x\in(-\infty,\infty). \tag{1.1}$$

The distribution H_2 is called the compound Poisson distribution generated by the Lévy distribution F and Poisson distribution with parameter μ. See, for example, Feller [11, pp. 450, 571], Embrechts et al. [10], or Sato [16, Chap. 4].

Suppose that $\mathcal{X}$ is a certain distribution class. If for an infinitely divisible distribution H and its Lévy distribution F, $H\in\mathcal{X}$ implies $F\in\mathcal{X}$, then the class is said to be closed under infinitely divisible distribution roots; otherwise, we say that the class is not closed under the roots.

We now introduce some distribution classes related to our theme. First, we introduce some notations.

Let g_1 and g_2 be eventually positive functions. We denote $g_{i,j}=\limsup g_i(x)/g_j(x)$ for $1\leqslant i\neq j\leqslant 2$. Next, denote $g_i(x)=O(g_j(x))$ if $g_{i,j}<\infty$, $g_i(x)\asymp g_j(x)$ if $g_{i,j}<\infty$ and $g_{j,i}<\infty$, $g_i(x)\lesssim g_j(x)$ if $g_{i,j}\leqslant 1$, $g_1(x)\sim g_2(x)$ if $g_{1,2}=g_{2,1}=1$, and $g_i(x)=o(g_j(x))$ if $g_{i,j}=0$.

We say that a distribution V belongs to the distribution class $\mathcal{L}(\gamma)$ for some $\gamma\geqslant 0$ if $\overline{V}(x)$ is positive and

$$\overline{V}(x-t)\sim \mathrm{e}^{\gamma t}\overline{V}(x)\quad \text{for each } t>0.$$

Further, if a distribution V belongs to the class $\mathcal{L}(\gamma)$ for some $\gamma\geqslant 0$, $m(V)=\int_0^\infty \mathrm{e}^{\gamma y}V(\mathrm{d}y)<\infty$, and

$$\overline{V^{*2}}(x)\sim 2m(V)\overline{V}(x),$$

then we say that the distribution V belongs to the distribution class $\mathcal{S}(\gamma)$. In particular, the classes $\mathcal{L}=\mathcal{L}(0)$ and $\mathcal{S}=\mathcal{S}(0)$ are called the long-tailed and subexponential distribution classes, respectively.

The classes $\mathcal{L}(\gamma)$ and $\mathcal{S}(\gamma)$ were respectively introduced by Chistyakov [3] for $\gamma=0$ and by Chover et al. [4, 5] for $\gamma>0$. Note that in the definition of the class $\mathcal{L}(\gamma)$, if $\gamma>0$ and the distribution V is lattice, then x and t are restricted to multiples of the lattice span; see Bertoin and Doney [1, Rem. 1]. If $\gamma=0$, then because V is supported on $[0,\infty)$ in this paper, the requirement $V\in\mathcal{L}$ is not needed in the definition of the class $\mathcal{S}$; see Chistyakov [3, Lemma 2].

Moreover, the class $\cup_{\gamma\geqslant 0}\mathcal{L}(\gamma)$ is properly contained in the following distribution class introduced by Shimura and Watanabe [18]. We say that a distribution V belongs to the class $\mathcal{OL}$ if $\overline{V}(x)$ is positive and

$$C^*(V,t)=\limsup\frac{\overline{V}(x-t)}{\overline{V}(x)}<\infty\quad \text{for each } t>0.$$

Correspondingly, the class $\cup_{\gamma\geqslant 0}\mathcal{S}(\gamma)$ is properly contained in the following distribution class introduced by Klüppelberg [13]. We say that a distribution V belongs to the class $\mathcal{OS}$ if

$$C^*(V)=\limsup\frac{\overline{V^{*2}}(x)}{\overline{V}(x)}<\infty.$$

Since V supported on $[0,\infty)$, $\overline{V}(x)>0$ for all $x\in(-\infty,\infty)$.

The class $\mathcal{S}(\gamma)$ is closed under infinitely divisible distribution roots; see Embrechts et al. [10] for $\gamma = 0$ and Sgibnev [17], Pakes [14, 15], and Watanabe [19] for $\gamma > 0$. In addition, Watanabe and Yamamuro [21] showed that the class $\mathcal{OS}$ is closed under infinitely divisible distribution roots on the condition that the Lévy distribution F is infinitely divisible. Recently, Cui et al. [7] proved that the class $\mathcal{L}(\gamma) \cap \mathcal{OS}$ for some $\gamma \geqslant 0$ is closed under infinitely divisible distribution roots with some appropriately restrictive conditions.

However, generally, the class $\mathcal{OS}$ is not closed under the infinitely divisible distribution roots; see Shimura and Watanabe [18, Prop. 1.1(iv)]. Moreover, Xu et al. [22, Thm. 2.2] for $\gamma = 0$ and Xu et al. [24, Thm. 1.1] for $\gamma > 0$ showed that the class $\mathcal{L}(\gamma) \cap \mathcal{OS}$ is also not closed under the corresponding roots. Recently, Wang, Cui, and Xu derived a similar result for the class $\mathcal{L} \setminus \mathcal{OS}$ in a book being written.

Therefore the closure problem corresponding to the class $\mathcal{L}(\gamma) \setminus \mathcal{OS}$ for some $\gamma > 0$ attracts our attention. To this end, we need to find some suitable Lévy distribution F to give a negative answer to this problem. These distributions have some different properties or satisfy some different conditions, for example,

$$\liminf \frac{\overline{F}(x-t)}{\overline{F}(x)} \geqslant \mathrm{e}^{\gamma t} \quad \text{for each } t > 0, \tag{1.2}$$

or

$$\overline{F}(x) = o\big(\overline{F^{*2}}(x)\big). \tag{1.3}$$

Then according to these properties or conditions, we can give rich and diverse distributions shown in the following three types:

(i) $F \in \mathcal{OL} \setminus (\mathcal{L}(\gamma) \cup \mathcal{OS})$, and both conditions (1.2) and (1.3) are satisfied;
(ii) $F \in \mathcal{OL} \setminus (\mathcal{L}(\gamma) \cup \mathcal{OS})$, and condition (1.3) is satisfied, but condition (1.2) is not;
(iii) $F \notin \mathcal{OL}$, and both conditions (1.2) and (1.3) are satisfied.

Theorem 1. *In types* (i), (ii), *and* (iii)*, there respectively exists a Lévy distribution F with $m(F) = \infty$ such that H, H_2, and F^{*k} for all $k \geqslant 2$ belong to the class $\mathcal{L}(\gamma) \setminus \mathcal{OS}$ for some $\gamma > 0$.*

Combining with the corresponding results mentioned above and Theorem 1 of this paper, we know that for each $\gamma \geqslant 0$, the classes $\mathcal{L}(\gamma) \cap \mathcal{OS}$ and $\mathcal{L}(\gamma) \setminus \mathcal{OS}$ are not closed under infinitely divisible distribution roots, where the corresponding Lévy distribution F belongs to the class $\mathcal{OL} \setminus (\mathcal{L}(\gamma) \cup \mathcal{OS})$ or $\mathcal{OL}^c$.

Remark 1. Here we make some explanations for conditions (1.2) and (1.3).

(i) Conditions (1.2) and (1.3) play an important role in the proof of the theorem; see Lemmas 1, 2, and 4. Condition (1.2) also is used on other occasions; for example, see Lemma 7 and Theorem 7 of Foss and Korshunov [12], Lemma 1, Theorem 2, Corollary 1, Theorem 6, and Remark 5(ii) of Xu et al. [24], and Theorem 1.1 of Cui et al. [7].

(ii) Theorem 1.1 of Xu et al. [24] shows that for each $\gamma > 0$, there is a Lévy distribution $F \in \mathcal{OL} \setminus (\mathcal{L}(\gamma) \cup \mathcal{OS})$ such that $m(F) < \infty$ and condition (1.2) is satisfied, whereas condition (1.3) does not hold, and H, H_2, and F^{*k} for all $k \geqslant 2$ belong to the class $(\mathcal{L}(\gamma) \cap \mathcal{OS}) \setminus \mathcal{S}(\gamma)$. Combining with the properties of distributions in type (ii), we know that conditions (1.2) and (1.3) cannot imply each other.

(iii) Clearly, condition (1.2) is satisfied when $\gamma = 0$ or $F \in \mathcal{L}(\gamma)$ for some $\gamma > 0$. Of course, many distributions outside $\mathcal{L}(\gamma)$ also meet the condition; see, for example, distributions in the proof for types (i) and (iii) below. However, some distributions that satisfy neither condition (1.2) nor condition (1.3) have been found in Example 1 of Foss and Korshunov [12], Proposition 3.2 and Remark 4.1 of Chen et al. [2], and Theorem 1.1 of Watanabe [20].

We organize the paper as follows. We prove Theorem 1 in Section 2. In Section 3, we deeply discuss some interesting questions for some distributions and conditions in the paper.

2 Proof of Theorem 1

2.1 Proof for type (i)

We first construct some light-tailed Lévy distributions that come from the following heavy-tailed distributions.

For any $s \geqslant 1$ and $\alpha \in [1/2, 1)$, we take $r = 1 + 1/\alpha$ and a large enough such that $a^{r-1} > 2^{s+2}$. Define the sequence of numbers

$$A = \{a_0 = a,\ a_n = a^{r^n},\ n \geqslant 1\}.$$

Then we take $b > s$ and define the heavy-tailed distribution with density

$$f_0(x) = Cbs^{-1} \sum_{n=0}^{\infty} x^{1/s-1} a_n^{-\alpha-1/s} \big((x a_n^{-1})^{1/s} - 1 \big)^{b-1} \mathbf{1}_{[a_n, 2^s a_n)}(x), \quad x \in (-\infty, \infty).$$

Integrating the density gives the distribution F_0 such that

$$\begin{aligned}\overline{F_0}(x) = {} & \mathbf{1}_{(-\infty, a_0)}(x) \\ & + \sum_{n=0}^{\infty} \Bigg(\Bigg(\sum_{i=n}^{\infty} C a_i^{-\alpha} - C a_n^{-\alpha} \bigg(\Big(\frac{x}{a_n} \Big)^{1/s} - 1 \bigg)^{b} \Bigg) \mathbf{1}_{[a_n, 2^s a_n)}(x) \\ & + \Bigg(\sum_{i=n+1}^{\infty} C a_i^{-\alpha} \Bigg) \mathbf{1}_{[2^s a_n, a_{n+1})}(x) \Bigg), \quad x \in (-\infty, \infty),\end{aligned} \tag{2.1}$$

where C is a constant such that $\overline{F}(a_0) = 1$. Clearly, when $x \in [a_n, 2^s a_n)$, by $b > s \geqslant 1$ and

$$f_0(x) = Cbs^{-1} a_n^{-\alpha-1/s} \big(x^{(1/s-1)/(b-1)+1/s} a_n^{-1/s} - x^{(1/s-1)/(b-1)} \big)^{b-1}$$

we know that f_0 is an increasing function. In addition, f_0 is locally long-tailed in the sense that if $x, x - t \in [a_n, 2^s a_n)$, then $f_0(x - t) \sim f_0(x)$ as $n \to \infty$.

From now on, we denote by $\mathcal{F}_1(0)$ the set of all distributions of the form (2.1). Further, for some $\gamma > 0$ and $F_0 \in \mathcal{F}_1(0)$, we define the distribution F related to γ and F_0 by

$$\overline{F}(x) = \mathbf{1}_{(-\infty,0)}(x) + \mathrm{e}^{-\gamma x} \overline{F_0}(x) \mathbf{1}_{[0,\infty)}(x), \quad x \in (-\infty, \infty). \tag{2.2}$$

Let

$$\mathcal{F}_1(\gamma) = \{F \text{ in (2.2): } F_0 \in \mathcal{F}_1(0)\}.$$

Then we take $F \in \mathcal{F}_1(\gamma)$ to prove Theorem 1 for type (i) in three steps.

For the pair of distributions F_0 and F_1 mentioned above, it is easy to show that $F_0 \in \mathcal{OL} \backslash \mathcal{L}$ with expectation $\mu(F_0) = \infty$. Thus $m(F) = \infty$ and $F \in \mathcal{OL} \setminus (\mathcal{L}(\gamma) \cup \mathcal{OS})$. Further, H_2, H, and F^{*k} for all $k \geqslant 1$ do not belong to the class $\mathcal{OS}$. Condition (1.2) holds by $F_0 \in \mathcal{F}_0$ and (2.2), whereas condition (1.3) results from $\mu(F_0) = \infty$ and the following lemma, which is Lemma 4(iii) of Xu et al. [24].

Lemma 1. *Let F_0 and F be a pair of distributions defined by* (2.2)*. If $\mu(F_0) = \infty$, then* (1.3) *holds for the distribution F.*

Secondly, we prove that $F^{*k} \in \mathcal{L}(\gamma)$ for all $k \geqslant 2$. To this end, we need another lemma, which is due to Corollary 1(ii) of Xu et al. [24].

Lemma 2. *For some $n \geqslant 2$, $F^{*k} \in \mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$ and all $k \geqslant n+1$ if $F^{*n} \in \mathcal{L}(\gamma)$ and either of the following two cases is true*:

(i) *condition* (1.2) *holds, and* $|\overline{F}(x-t) - \mathrm{e}^{\gamma t}\overline{F}(x)| = o(\overline{F^{*n}}(x))$ *for each* $t > 0$*;*
(ii) *condition* (1.3) *holds.*

According to condition (1.3) and Lemma 2(ii), we only need to show that F^{*2} belongs to the class $\mathcal{L}(\gamma)$. To this end, we also need the following result, which comes from Theorem 7 of Xu et al. [24]. As Remark 6 of Xu et al. [24] points out, neither F_0 and F are required to belong to the class $\mathcal{L}(\gamma)$ in the following lemma.

Lemma 3. *Let F_0 be an absolutely continuous distribution with density f_0. Assume that for all $t > 0$, there are constants $x_0 > 0$ and $C = C(F_0, t, x_0)$ such that*

$$\overline{F_0}(x-t) - \overline{F_0}(x) \leqslant C\big(f_0(x-t) + f_0(x)\big) \quad \textit{for all } x \geqslant x_0 \tag{2.3}$$

and

$$\int\limits_{[x/2,\,x]} \overline{F}_0(x-y)\,F_0(\mathrm{d}y) = o\bigg(\int\limits_{[x/2,\,x]} \overline{F}_0(x-y)\overline{F}_0(y)\,\mathrm{d}y\bigg). \tag{2.4}$$

If $\gamma > 0$ and F is of the form (2.2)*, then $F^{*2} \in \mathcal{L}(\gamma)$.*

Remark 2. Here we make some explanations for conditions (2.3) and (2.4).

(i) Conditions (2.3) and (2.4) cannot imply each other. See Proposition 1 for details. This fact is not revealed by Xu et al. [24].
(ii) In Theorem 7 of Xu et al. [24] the condition corresponding to (2.3) is

$$\overline{F_0}(x-t) - \overline{F_0}(x) = O\big(f_0(x-t) + f_0(x)\big). \tag{2.5}$$

However, this difference between conditions (2.3) and (2.5) has no effect on the conclusion of the lemma. In fact, condition (2.3) is more rigorous, because $f_0(x-t) + f_0(x) = 0$ in some cases; see the following proof.

We now continue to prove Theorem 1 for type (i).

Note that $f_0(x)$ is an increasing function on $x \in [a_n, 2^s a_n)$, so if $t > 0$ and $x \in [a_n,\, 2^s a_n)$ is large enough, then

$$\begin{aligned}\overline{F}_0(x-t) - \overline{F}_0(x) &= \int\limits_{(x-t,\,x]} f_0(y)\,\mathrm{d}y = \int\limits_{(\max\{x-t,\,a_n\},x]} f_0(y)\,\mathrm{d}y \\ &\leqslant t f_0(x) = O\big(f_0(x-t) + f_0(x)\big),\end{aligned}$$

where the second equality is due to $a_n - 2^s a_{n-1} \to \infty$ as $n \to \infty$. This ensures that if $t > 0$ is fixed and x is sufficiently large, then $x - t > 2^s a_{n-1}$.

If $x \in [2^s a_n, 2^s a_n + t)$, then because $f_0(x)$ is a locally long-tailed and increasing function on $[a_n, 2^s a_n)$ and $a_n \leqslant 2^s a_n - t \leqslant x - t < 2^s a_n$, we have

$$\begin{aligned}\overline{F}_0(x-t) - \overline{F}_0(x) &= \int\limits_{(x-t,\,\min\{x,\,2^s a_n\}]} f_0(y)\,\mathrm{d}y \leqslant t f_0\big(2^s a_n\big) \\ &\sim t f_0(x-t) = O\big(f_0(x-t) + f_0(x)\big).\end{aligned}$$

If $x \in [2^s a_n + t,\ a_{n+1})$, then clearly

$$\overline{F}_0(x-t) - \overline{F}_0(x) = 0 = f_0(x-t) + f_0(x).$$

The above three facts imply that (2.3) is true. Thus, to prove that $F^{*2} \in \mathcal{L}(\gamma)$, by Lemma 3 it suffices to prove that

$$W_0(x) = \int_{[x/2,\,x]} \overline{F}_0(x-y)\, F_0(\mathrm{d}y) = o\biggl(\int_{[x/2,\,x]} \overline{F}_0(x-y)\overline{F}_0(y)\,\mathrm{d}y\biggr) = o\bigl(T_0(x)\bigr), \tag{2.6}$$

where

$$T_0(x) = \int_{[x/2,\,x]} \overline{F_0}(x-y)\overline{F_0}(y)\,\mathrm{d}y.$$

To this end, for all integer $n \geqslant 1$, since $W_0(x) = 0$ for $x \in [2^{s+1}a_n,\ a_{n+1})$, we only need to deal with $W_0(x)$ for x in the following two cases:

(i) $x \in [a_n,\ 2^{s+1}a_n - a_n^{5/6})$ and

(ii) $x \in [2^{s+1}a_n - a_n^{5/6},\ 2^{s+1}a_n)$.

In case (i), by (2.1), since $2x \geqslant a_n^{5/6}$ and

$$\overline{F_0}\biggl(\frac{x + a_n^{5/6}/2}{2}\biggr) \geqslant \overline{F_0}\biggl(\frac{2^s a_n - a_n^{5/6}}{4}\biggr),$$

we have

$$\begin{aligned} W_0(x) &= \int_{[x/2,\,x]} \overline{F_0}(x-y) f_0(y)\,\mathrm{d}y \leqslant f_0\bigl(2^s a_n\bigr) \int_{[x/2,\,x]} \overline{F_0}(x-y)\,\mathrm{d}y \\ &\leqslant 2Cbs^{-1}a_n^{-\alpha-1} \int_{[0,\,x/2]} \overline{F_0}(y)\,\mathrm{d}y \leqslant 2Cbs^{-1}a_n^{-\alpha-1} \int_{[0,2^s a_n]} \overline{F_0}(y)\,\mathrm{d}y = O\bigl(a_n^{-2\alpha}\bigr), \end{aligned}$$

and since $f_0(y)$ is an increasing function when $y \in [a_n, 2^s a_n)$,

$$\begin{aligned} T_0(x) &= \int_{[x/2,\,x]} \overline{F_0}(x-y)\overline{F_0}(y)\,\mathrm{d}y \geqslant \int_{[x/2,\,(x+2^{-1}a_n^{5/6})/2]} \overline{F_0}(x-y)\overline{F_0}(y)\,\mathrm{d}y \\ &\geqslant a_n^{5/6}\overline{F_0}\Bigl(\frac{x}{2}\Bigr)\frac{\overline{F_0}((x+2^{-1}a_n^{5/6})/2)}{4} \geqslant a_n^{5/6}\frac{\overline{F_0}^2(2^s a_n - 4^{-1}a_n^{5/6})}{4} \\ &= a_n^{5/6}\frac{(\int_{[2^s a_n - 4^{-1}a_n^{5/6},\,\infty]} f_0(y)\,\mathrm{d}y)^2}{4} \geqslant a_n^{5/6}\frac{(\int_{[2^s a_n - 4^{-1}a_n^{5/6},\,2^s a_n]} f_0(y)\,\mathrm{d}y)^2}{4} \\ &\geqslant a_n^{5/6}\frac{(4^{-1}a_n^{5/6} f_0(2^s a_n - 4^{-1}a_n^{5/6}))^2}{4} \asymp a_n^{-2\alpha+1/2}, \end{aligned}$$

which implies

$$\frac{W_0(x)}{T_0(x)} = \frac{\int_{[x/2,\,x]} \overline{F_0}(x-y) f_0(y)\,\mathrm{d}y}{T_0(x)} = O\bigl(a_n^{-1/2}\bigr) \to 0, \quad n \to \infty. \tag{2.7}$$

In case (ii), by (2.1), since $\overline{F_0}(2^s a_n) = \sum_{i=n+1}^{\infty} C a_i^{-\alpha}$, $x/2 \leqslant 2^s a_n$, and $a_{n+1} = a_n^r$ for all $n \geqslant 1$, we have

$$\begin{aligned}T_0(x) &= \int\limits_{[x/2,\, x]} \overline{F_0}(x-y)\overline{F_0}(y)\,\mathrm{d}y \geqslant \overline{F_0}\biggl(\frac{x}{2}\biggr) \int\limits_{[0,\, x/2]} \overline{F_0}(y)\,\mathrm{d}y \\ &\geqslant \overline{F_0}\bigl(2^s a_n\bigr) \int\limits_{[0,\, x/2]} \overline{F_0}(y)\,\mathrm{d}y \geqslant C a_{n+1}^{-\alpha} \int\limits_{[0,\, x/2]} \overline{F_0}(y)\,\mathrm{d}y \\ &= C a_n^{-\alpha-1} \int\limits_{[0,\, x/2]} \overline{F_0}(y)\,\mathrm{d}y.\end{aligned}$$

Thus, also by (2.1), since

$$x - 2^s a_n \geqslant 2^s a_n - a_n^{5/6} \quad \text{and} \quad 2^s a_n - a_n^{5/6} \leqslant \frac{x}{2} \leqslant 2^s a_n,$$

we have

$$\begin{aligned}\frac{W_0(x)}{T_0(x)} &\leqslant \frac{\int_{[x/2,\, 2^s a_n]} \overline{F_0}(x-y) f_0(y)\,\mathrm{d}y}{C a_n^{-\alpha-1} \int_{[0,\, x/2]} \overline{F_0}(y)\,\mathrm{d}y} \leqslant b s^{-1} \frac{\int_{[2^s a_n - a_n^{5/6},\, 2^s a_n]} \overline{F_0}(y)\,\mathrm{d}y}{\int_{[a_n/2,\, a_n]} \overline{F_0}(y)\,\mathrm{d}y} \\ &\leqslant b s^{-1} \frac{\overline{F_0}(2^s a_n - a_n^{5/6}) a_n^{5/6}}{\overline{F_0}(a_n) a_n/2} = O\bigl(a_n^{-1/3}\bigr) \to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{2.8}$$

Therefore by (2.7) and (2.8), (2.6) holds.

Finally, we prove that H_2 and H belong to the class $\in \mathcal{L}(\gamma)$. For H_2, by (1.1) it is easy to see that for any $0 < \varepsilon < 1$, there is a positive integer $M = M(F, \varepsilon)$ such that

$$\sum_{k=M}^{\infty} \frac{\mathrm{e}^{-\mu} \mu^{k+1} \overline{F^{*k}}(x)}{(k+1)!} \leqslant \varepsilon \overline{H_2}(x), \quad x \in [0, \infty). \tag{2.9}$$

Therefore, according to the following lemma given by Theorem 4 of Xu et al. [24], by (2.9), since $F^{*k} \in \mathcal{L}(\gamma)$ for all $k \geqslant 2$, using (1.2) (or (1.3)), we have $H_2 \in \mathcal{L}(\gamma)$.

Lemma 4. *Let F be a distribution such that $F^{*n} \in \mathcal{L}(\gamma)$ for some $n \geqslant 1$ and some $\gamma \geqslant 0$, and let τ be a random variable satisfying $\mathbf{P}(\tau = k) = p_k$ for all nonnegative integer k and $\sum_{k=0}^{\infty} p_k = 1$. Assume that $\mathbf{P}(\tau \geqslant n) > 0$ and the following condition is satisfied: for any $0 < \varepsilon < 1$, there is a positive integer $M = M(F, \varepsilon)$ such that*

$$\sum_{k=M}^{\infty} p_{k+1} \overline{F^{*k}}(x) \leqslant \varepsilon \overline{F^{*\tau}}(x), \quad x \in [0, \infty),$$

where $\overline{F^{*\tau}}(x) = \sum_{k=0}^{\infty} p_k \overline{F^{*k}}(x)$, $x \in (-\infty, \infty)$. Further, suppose that condition (1.3) *or the condition*

$$\liminf \frac{\overline{F^{*k}}(x-t)}{\overline{F^{*k}}(x)} \geqslant \mathrm{e}^{\gamma t} \quad \textit{for each } t > 0 \textit{ and all } 1 \leqslant k \leqslant n-1$$

is satisfied. Then $F^{*\tau} \in \mathcal{L}(\gamma)$.

For H, the conclusion directly comes from $H = H_1 * H_2$, $H_2 \in \mathcal{L}(\gamma)$, $\overline{H_1}(x) = O(\mathrm{e}^{-\beta x})$ for some selected $\beta > \gamma$, and Lemma 2.1 of Pakes [14].

2.2 Proof for type (ii)

For some $\gamma > 0$, let F be a distribution depending on γ such that

$$\overline{F}(x) = \mathbf{1}_{(-\infty,0)}(x) + \sum_{k=0}^{\infty} \frac{\mathrm{e}^{-\gamma x} + \mathrm{e}^{-\gamma \mathrm{e}^{k+1}}}{2} \mathbf{1}_{[\mathrm{e}^k, \mathrm{e}^{k+1})}(x), \quad x \in (-\infty, \infty).$$

According to the explanations and the reference after Theorem B of Watanabe [20], we know that $F \in \mathcal{OL} \setminus \mathcal{L}(\gamma)$, $m(F) = \infty$, $F^{*2} \in \mathcal{L}(\gamma)$, and $\overline{F^{*2}}(x) \sim \overline{G^{*2}}(x)/4$, where G is a standard exponential distribution with index $\gamma > 0$. From the above facts we have $F \notin \mathcal{OS}$ and (1.3) is satisfied. However, (1.2) does not hold, that is, for all $t > 0$,

$$\liminf_{k\to\infty} \frac{\overline{F}(\mathrm{e}^{k+1} - 2t)}{\overline{F}(\mathrm{e}^{k+1} - t)} = \frac{\mathrm{e}^{2\gamma t} + 1}{\mathrm{e}^{\gamma t} + 1} < \mathrm{e}^{\gamma t}.$$

Further, along the proof line of type (i), by (1.3) we can prove that H, H_2, and $F^{*k} \in \mathcal{L}(\gamma) \setminus \mathcal{OS}$ for all $k \geqslant 3$.

2.3 Proof for type (iii)

Define $\mathcal{F}_3(0)$ as the class of distributions whose tail functions have the form

$$\begin{aligned}\overline{F_0}(x) = {} & \mathbf{1}_{(-\infty,\, a_0)}(x) \\ & + C\sum_{n=0}^{\infty}\left(\left(\sum_{i=n}^{\infty} a_i^{-\alpha} - a_n^{-\alpha}\left(a_n^{-1}x^s - 1\right)\right)\mathbf{1}_{[a_n^{1/s},\,(2a_n)^{1/s})}(x)\right. \\ & \left. + \sum_{i=n+1}^{\infty} a_i^{-\alpha}\mathbf{1}_{[(2a_n)^{1/s},\, a_{n+1}^{1/s})}(x)\right) \end{aligned} \tag{2.10}$$

with density

$$f_0(x) = Cs\sum_{n=0}^{\infty} x^{s-1}a_n^{-\alpha-1}\mathbf{1}_{[a_n^{1/s},\,(2a_n)^{1/s})}(x), \quad x \in (-\infty, \infty), \tag{2.11}$$

where $s \in (1, 2)$, $\alpha \in ((s-1)/s,\, 1/2)$, and a, r, and the sequence $A = \{a_0, a_n, n \geqslant 1\}$ are defined as for $\mathcal{F}_1(0)$. Further, for some $\gamma > 0$, let the distribution set

$$\mathcal{F}_3(\gamma) = \{F \text{ in (2.2): } F_0 \in \mathcal{F}_3(0)\}.$$

According to the proof of Proposition 2.1 in Xu et al. [22], we have $F_0 \notin \mathcal{OL}$ with expectation $\mu(F_0) = \infty$ and $F_0^{*2} \in \mathcal{L} \setminus \mathcal{OS}$, which implies $F \notin \mathcal{OL}$ and $m(F) = \infty$. Therefore F^{*k} for all $k \geqslant 3$, H_2, and H do not belong to the class $\mathcal{OS}$.

For the remaining conclusions, similarly to the proof for type (i), we only need to show (2.6). To this end, for each integer $n \geqslant 1$, since $W_0(x) = 0$ for $x \in [2(2a_n)^{1/s}, a_{n+1}^{1/s})$, we only need to deal with $W_0(x)$ for x in the following three cases:

(i) $x \in [a_n^{1/s},\, 2(2a_n)^{1/s} - a_n^{3/4s})$,
(ii) $x \in [2(2a_n)^{1/s} - a_n^{3/4s},\, 2(2a_n)^{1/s} - a_n^{s^{-1}-4^{-1}})$, and
(iii) $x \in [2(2a_n)^{1/s} - a_n^{s^{-1}-4^{-1}},\, 2(2a_n)^{1/s})$.

First, for case (i), by (2.10) and

$$\overline{F_0}\left(\frac{x}{2}\right) \geqslant \overline{F_0}\big((2a_n)^{1/s} - 2^{-1}a_n^{3/4s}\big)$$

we have

$$\begin{aligned}
\frac{W_0(x)}{T_0(x)} &\leqslant \frac{2Csa_n^{-\alpha-s^{-1}} \int_{[x/2,\,x]} \overline{F_0}(x-y)\,\mathrm{d}y}{\overline{F_0}\big((2a_n)^{1/s} - 2^{-1}a_n^{3/4s}\big) \int_{[x/2,\,(x+2^{-1}a_n^{3/4s})/2]} \overline{F_0}(y)\,\mathrm{d}y} \\
&\leqslant \frac{8Csa_n^{-\alpha-7\cdot(4s)^{-1}} \int_{[0,\,(2a_n)^{1/s}-2^{-1}a_n^{3/4s}]} \overline{F_0}(y)\,\mathrm{d}y}{\overline{F_0}^2\big((2a_n)^{1/s} - 4^{-1}a_n^{3/4s}\big)} \\
&= O\big(a_n^{-1/4s}\big) \to 0, \quad n \to \infty.
\end{aligned} \tag{2.12}$$

Then for case (ii), it follows from (2.10) and

$$(2a_n)^{1/s} - 2^{-1}a_n^{3/4s} \leqslant \frac{x}{2} \leqslant y \leqslant (2a_n)^{1/s}$$

that

$$\begin{aligned}
\frac{W_0(x)}{T_0(x)} &\leqslant \frac{\int_{[x/2,\,(2a_n)^{1/s}]} \overline{F_0}(x-y)F_0(\mathrm{d}y)}{\int_{[x/2,\,(2a_n)^{1/s}]} \overline{F_0}(y)\overline{F_0}(x-y)\,\mathrm{d}y} \\
&\leqslant \frac{Csa_n^{-\alpha-(4s)^{-1}}\overline{F_0}\big(x-(2a_n)^{1/s}\big)}{\overline{F_0}\big((2a_n)^{1/s} - 2^{-1}a_n^{s^{-1}-4^{-1}}\big) \int_{[(2a_n)^{1/s}-2^{-1}a_n^{s^{-1}-4^{-1}},\,(2a_n)^{1/s}-4^{-1}a_n^{s^{-1}-4^{-1}}]} \overline{F_0}(y)\,\mathrm{d}y} \\
&\leqslant \frac{Csa_n^{-\alpha-5\cdot(4s)^{-1}+4^{-1}}\overline{F_0}\big((2a_n)^{1/s} - a_n^{3/4s}\big)}{\overline{F_0}^2\big((2a_n)^{1/s} - 4^{-1}a_n^{s^{-1}-4^{-1}}\big)} \\
&= O\big(a_n^{-3\cdot 4^{-1}(2s^{-1}-1)}\big) \to 0, \quad n \to \infty.
\end{aligned} \tag{2.13}$$

Finally, we consider case (iii). From (2.10) and (2.11) we have

$$\begin{aligned}
W_0(x) &= \int\limits_{[x/2,\,x]} \overline{F_0}(x-y)\,F_0(\mathrm{d}y) \leqslant \int\limits_{[x/2,(2a_n)^{1/s}]} \overline{F_0}(x-y)\,F_0(\mathrm{d}y) \\
&= f_0\big((2a_n)^{1/s}\big) \int\limits_{[x/2,\,(2a_n)^{1/s}]} \overline{F_0}(x-y)\,\mathrm{d}y \overset{z=x-y}{=} f_0\big((2a_n)^{1/s}\big) \int\limits_{[x-(2a_n)^{1/s},\,x/2]} \overline{F_0}(z)\,\mathrm{d}z \\
&\leqslant f_0\big((2a_n)^{1/s}\big) \int\limits_{[(2a_n)^{1/s}-a_n^{s^{-1}-4^{-1}},\,(2a_n)^{1/s}]} \overline{F_0}(z)\,\mathrm{d}z \\
&\leqslant f_0\big((2a_n)^{1/s}\big) a_n^{s^{-1}-4^{-1}} \overline{F_0}\big((2a_n)^{1/s} - a_n^{s^{-1}-4^{-1}}\big) \\
&= O\big(a_n^{-2\alpha-1/2}\big)
\end{aligned}$$

and

$$T_0(x) = \int\limits_{[x/2,\,x]} \overline{F_0}(x-y)\overline{F_0}(y)\,\mathrm{d}y \geqslant Ca_n^{-\alpha-1} \int\limits_{[0,\,x/2]} \overline{F_0}(y)\,\mathrm{d}y \geqslant Ca_n^{-\alpha-1} \int\limits_{[a_n^{1/s}/2,\,a_n^{1/s}]} \overline{F_0}(y)\,\mathrm{d}y$$
$$= \frac{Ca_n^{-\alpha-1+1/s}\overline{F_0}(a_n^{1/s})}{2} \sim \frac{C^2 a_n^{-2\alpha-1+1/s}}{2},$$

which implies

$$\frac{W_0(x)}{T_0(x)} = \frac{\int_{[x/2,\,x]} \overline{F_0}(x-y)F_0(\mathrm{d}y)}{\int_{[x/2,\,x]} \overline{F_0}(x-y)\overline{F_0}(y)\,\mathrm{d}y} = O\big(a_n^{2^{-1}-s^{-1}}\big) \to 0, \quad n\to\infty. \tag{2.14}$$

Combining (2.12)–(2.14), we get (2.6).

Therefore we complete the proof of Theorem 1.

3 Some remarks

In this section, we make some explanations for conditions (2.3) and (2.4) in Lemma 3, the Embrechts–Goldie conjecture, the weakly tail equivalence of some distributions in types (i)–(iii), and the influence of convolution on distribution shape.

3.1 On conditions (2.3) and (2.4)

Proposition 1. *Conditions* (2.3) *and* (2.4) *in Lemma* 3 *do not imply each other.*

Proof. On one hand, let X be a random variable with distribution F_0 in Example 3.3 of Xu et al. [23] such that

$$\overline{F_0}(x) = \mathbf{1}_{(-\infty,\,0)}(x) + \big(x_1^{-1}\big(x_1^{-\alpha}-1\big)x+1\big)\mathbf{1}_{[0,\,x_1)}(x)$$
$$+ \sum_{n=1}^{\infty}\big(\big(x_n^{-\alpha} + \big(x_n^{-\alpha-2} - x_n^{-\alpha-1}\big)(x-x_n)\big)\mathbf{1}_{[x_n,\,2x_n)}(x) + x_n^{-\alpha-1}\mathbf{1}_{[2x_n,\,x_{n+1})}(x)\big),$$

$x\in(-\infty,\infty)$, where $\alpha\in(5,\infty)$, $x_1>4^{\alpha}$, and $x_{n+1}=x_n^{1+\alpha^{-1}}$, $n\geqslant 1$. According to Example 3.3 of Xu et al. [23], we already know that $F_0\notin\mathcal{L}$ and $EX^2<\infty$. Further, we denote the density of X by f_0. Then for all $t>0$ and $n\geqslant 1$, when $x\in[x_n,\,2x_n+t)$,

$$\overline{F_0}(x-t) - \overline{F_0}(x) \leqslant tx_n^{-\alpha-1} = O\big(f_0(x-t)+f_0(x)\big),$$

and when $x\in[2x_n+t,\,x_{n+1})$,

$$\overline{F_0}(x-t) - \overline{F_0}(x) = f_0(x-t)+f_0(x) = 0.$$

Hence condition (2.3) holds for the distribution F_0. Also, let

$$W_0(x) = \int\limits_{[x/2,\,x]} \overline{F_0}(x-y)\,F_0(\mathrm{d}y) \quad\text{and}\quad T_0(x) = \int\limits_{[x/2,\,x]} \overline{F_0}(x-y)\overline{F_0}(y)\,\mathrm{d}y.$$

From

$$W_0(2x_n) = \left(x_n^{-\alpha-1} - x_n^{-\alpha-2}\right) \int\limits_{[x_n,\, 2x_n]} \overline{F_0}(2x_n - y)\,\mathrm{d}y = \left(x_n^{-\alpha-1} - x_n^{-\alpha-2}\right) \int\limits_{[0,\, x_n]} \overline{F_0}(z)\,\mathrm{d}z$$

and

$$\begin{aligned} T_0(2x_n) &= \int\limits_{[x_n,\, 2x_n]} \overline{F_0}(2x_n - y)\overline{F_0}(y)\,\mathrm{d}y \\ &= \int\limits_{[x_n,\, 2x_n]} \overline{F_0}(2x_n - y)\left(\left(x_n^{-\alpha} + \left(x_n^{-\alpha-2} - x_n^{-\alpha-1}\right)(y - x_n)\right)\right)\mathrm{d}y \\ &= \int\limits_{[0,\, x_n]} \overline{F_0}(z)\left(x_n^{-\alpha-1} + \left(x_n^{-\alpha-1} - x_n^{-\alpha-2}\right)z\right)\mathrm{d}z \\ &\leqslant x_n^{-\alpha-1} \int\limits_{[0,\, x_n]} \overline{F_0}(z)(1 + z)\,\mathrm{d}z \end{aligned}$$

we have

$$\frac{W_0(2x_n)}{T_0(2x_n)} \geqslant \frac{\int_{[0,x_n]} \overline{F_0}(y)\,\mathrm{d}y}{\int_{[0,x_n]} \overline{F_0}(y)(1+y)\,\mathrm{d}y} \to \frac{\mathbf{E}X}{\mathbf{E}X + \mathbf{E}X^2/2} > 0 \quad \text{as } n \to \infty,$$

which implies that F_0 does not satisfy condition (2.4).

On the other hand, let F_1 be the distribution such that

$$\overline{F_1}(x) = \mathbf{1}_{(-\infty,1)}(x) + \sum_{n=1}^{\infty} \left(\frac{\mathbf{1}_{[2n-1,2n)}(x)}{2x - 2n + 1} + \frac{\mathbf{1}_{[2n,2n+1)}(x)}{2n + 1} \right)$$

with density

$$f_1(x) = 2\sum_{n=1}^{\infty} \frac{\mathbf{1}_{[2n-1,2n)}(x)}{(2x - 2n + 1)^2}, \quad x \in (-\infty, \infty).$$

Since

$$\overline{F_1}\left(2n - \frac{3}{2}\right) - \overline{F_1}\left(2n + \frac{1}{2}\right) = \frac{1}{4n^2 - 1} \quad \text{and} \quad f_1\left(2n - \frac{3}{2}\right) = f_1\left(2n + \frac{1}{2}\right) = 0$$

for all $n \geqslant 1$, F_1 does not satisfy condition (2.3). Further, it is also easy to verify that $f_1(x) = o(\overline{F_1}(x))$, and hence $F_1 \in \mathcal{L}$ and satisfies condition (2.4). However, we need a distribution that does not belong to $\mathcal{L}$. To this end, let $y_0 \geqslant 0$ and $a > 1$ be two constants such that $a\overline{F_1}(y_0) \leqslant 1$. We define the distribution F_0 in terms of F_1 by

$$\overline{F_0}(x) = \overline{F_1}(x)\mathbf{1}_{(-\infty,x_1)}(x) + \sum_{i=1}^{\infty} \left(\overline{F_1}(x_i)\mathbf{1}_{[x_i,y_i)}(x) + \overline{F_1}(x)\mathbf{1}_{[y_i,x_{i+1})}(x)\right), \quad x \in (-\infty, \infty),$$

where $\{x_i,\, i \geqslant 1\}$ and $\{y_i,\, i \geqslant 1\}$ are two sequences of positive constants satisfying

$$x_i < y_i < x_{i+1}, \quad \overline{F_1}(x_i) = a\overline{F}_1(y_i), \quad y_i - x_i \to \infty, \quad \text{and} \quad x_{i+1} - y_i \to \infty \quad \text{as } i \to \infty.$$

It is easy to see that

$$\overline{F_1}(x) \leqslant \overline{F_0}(x) \leqslant a\overline{F_1}(x) \quad \text{and} \quad \lim_{n\to\infty} \frac{\overline{F_0}(y_n - 1)}{\overline{F_0}(y_n)} = a > 1.$$

Thus the distribution F_0 does not belong to class $\mathcal{L}$ and still satisfies condition (2.4), whereas F_0 does not satisfy condition (2.3) as F_1. $\square$

3.2 On the Embrechts–Goldie conjecture

It is well known that the class $\mathcal{S}$ is closed under convolution roots; see Embrechts et al. [10, Thm. 2]. Further, Embrechts and Goldie [8, 9] put forward a famous conjecture:

$$\text{If } F^{*k} \in \mathcal{L}(\gamma) \text{ for some (even for all) } k \geqslant 2 \text{ and } \gamma \geqslant 0, \text{ then } F \in \mathcal{L}(\gamma).$$

However, the class $\mathcal{S}(\gamma)$ for some $\gamma > 0$ is not closed under the convolution roots in general; see Watanabe [20].

The explanations and the reference after Theorem B of Watanabe [20] show that there is a distribution F such that $F^{*2} \in \mathcal{L}(\gamma) \setminus \mathcal{OS}$ for some $\gamma \geqslant 0$, whereas $F \in \mathcal{OL} \setminus (\cup_{\gamma\geqslant 0}\mathcal{L}(\gamma) \cup \mathcal{OS})$, or see the proof for type (ii) of Theorem 1 of this paper. By the distributions in type (i), Theorem 1 of this paper also implies the above conclusion, whereas the corresponding distributions have different properties.

Previously, we have seen that there is a distribution $F \in \mathcal{OL} \setminus (\cup_{\gamma\geqslant 0}\mathcal{L}(\gamma) \cup \mathcal{OS})$ such that $F^{*2} \in (\mathcal{L}(\gamma) \cap \mathcal{OS}) \setminus \mathcal{S}(\gamma)$; see Xu et al. [22, Thm. 2.2] for $\gamma = 0$ and Xu et al. [24, Thm. 1] for $\gamma > 0$. Further, Proposition 2.1 of Xu et al. [22] for $\gamma = 0$ and Theorem 1 of this paper for $\gamma > 0$ present some distributions that do not even belong to the class $\mathcal{OL}$, but their convolutions belong to the class $\mathcal{L}(\gamma) \setminus \mathcal{OS}$.

Therefore the Embrechts–Goldie conjecture with $k = 2$ has been refuted for the class $\mathcal{L}(\gamma)$ and its subclasses, that is, they are not closed under convolution roots for each $\gamma \geqslant 0$. More precisely, we have the following conclusion.

Proposition 2. *None of the classes $\mathcal{S}(\gamma)$ for some $\gamma > 0$, $(\mathcal{L}(\gamma) \cap \mathcal{OS}) \setminus \mathcal{S}(\gamma)$ for some $\gamma \geqslant 0$, and $\mathcal{L}(\gamma) \setminus \mathcal{OS}$ for some $\gamma \geqslant 0$ are closed under convolution roots, where the corresponding distributions belong to the class $\mathcal{OL} \setminus \big(\mathcal{L}(\gamma) \cup \mathcal{OS}\big)$ or $\mathcal{OL}^c$.*

In addition, we briefly review some positive conclusions of the Embrechts–Goldie conjecture for the classes in Proposition 2.

The class $\mathcal{S}(\gamma)$ for some $\gamma > 0$ is closed under the convolutional root if the distribution $F \in \mathcal{L}(\gamma)$ (see Embrechts and Goldie [9, Thm.2.10]), or F is an infinitely divisible distribution (see Watanabe [19, Thm. 1.1]). More related conclusions can be found in Pakes [14, 15] and Watanabe [20].

Recently, Theorem 6 of Cui et al. [7] showed that the class $\mathcal{L}(\gamma) \cap \mathcal{OS}$ for some $\gamma > 0$ is also closed under the convolutional root under some technical conditions.

Finally, we refer to Watanabe [20] and Cui et al. [6] for some related work involving local distribution classes.

3.3 On weak tail equivalence

Let F_1 and F_2 be two distributions. If $\overline{F_1}(x) \asymp \overline{F_2}(x)$, then we say that F_1 and F_2 are weakly tail equivalent.

Clearly, the distribution in the proof for type (ii) of Theorem 1 is weakly tail equivalent to the standard exponential distribution with parameter γ. However, for the distributions in the subclass $\mathcal{F}_i(\gamma)$, $i = 1, 3$, there is a different conclusion.

Proposition 3. *For $i = 1, 3$, if the distribution $F \in \mathcal{F}_i(\gamma)$ with some $\gamma > 0$, then F is not weakly tail equivalent to any distribution in $\mathcal{L}(\gamma)$.*

Proof. For $F \in \mathcal{F}_1(\gamma)$ with the corresponding distribution $F_0 \in \mathcal{F}_1(0)$, according to the proof of Theorem 1(i), we have $F_0, F \in \mathcal{OL}$ and

$$\limsup_{t\to\infty} C(F_0,t) = \limsup_{t\to\infty} \limsup \frac{\overline{F_0}(x-t)}{\overline{F_0}(x)} = \infty.$$

Thus, according to Lemma 2.3 of Xu et al. [23], the distribution F_0 is also not weakly tail equivalent to any distribution in $\mathcal{L}$. Therefore the distribution F is not weakly tail equivalent to any distribution in $\mathcal{L}(\gamma)$.

For $F \in \mathcal{F}_3(\gamma)$, according to Theorem 1(iii), $F \notin \mathcal{OL}$, and thus F is not weakly tail equivalent to any distribution in $\mathcal{L}(\gamma)$. □

3.4 On the normalization of distribution shape

From Theorem 1 and Proposition 1 we find a surprising phenomenon that for some $\gamma > 0$, the properties of F_0 and F in the proofs for types (i), (ii), and (iii) are very unusual, but their convolutions F_0^{*k} and F^{*k} for all integers $k \geqslant 2$, the compound Poisson distribution H_2, and the infinitely divisible distribution H may still have decent ones. One of the reasons is that F_0 has a strange shape, that is, the distribution has no mass in some intervals, and the length of such an interval tends to infinity. However, the following proposition shows that convolution can normalize the shape of these strange distributions, so that F_0^{*k}, F^{*k}, $k \geqslant 2$, H_2, and H have better properties.

Proposition 4. *Let F be a distribution with density f. If there is a sequence*

$$\{b_n, a_n\colon\ b_1 = 0,\ b_n < a_n < b_{n+1},\ n \geqslant 1,\ a_n \uparrow \infty \text{ as } n \to \infty\}$$

such that $f(x) > 0$ for $x \in [b_n, a_n]$, $n \geqslant 1$, and $f(x) = 0$ otherwise, then for each pair of positive integers n and m, there is a constant $c = \min\{2^{-1}a_1,\ b_{n+1} - a_n\}$ such that

$$f^{\otimes m}(x) > 0 \quad \text{for } x \in [b_n,\ a_n + mc].$$

Proof. For each integer $n \geqslant 1$ and any two constants $a_n < x_1 < x_2 < a_n + c$, since

$$0 \leqslant x_1 - y < x_2 - y \leqslant 2c, \quad y \in [a_n - c,\ a_n],$$

then $f(x_1 - y)f(x_2 - y) > 0$. Further, according to Fatou's lemma, the density of F^{*2}

$$f_2(x_2) = \lim_{x_1\uparrow x_2} \frac{\overline{F^{*2}}(x_1) - \overline{F^{*2}}(x_2)}{x_2 - x_1} \geqslant \int\limits_{[a_n-c,a_n]} \liminf_{x_1\uparrow x_2} \frac{\overline{F}(x_1 - y) - \overline{F}(x_2 - y)}{x_2 - x_1} f(y)\,\mathrm{d}y$$
$$= \int\limits_{[a_n-c,\,a_n]} f(x_2 - y)f(y)\,\mathrm{d}y > 0,$$

that is, $f_2(x) > 0$ for $x \in [a_n,\ a_n + c]$. Considering $f(x) > 0$ for $x \in [0, b_1] \cup [b_n, a_n]$, we get $f_2(x) > 0$ for $x \in [b_n,\ a_n + c]$. Finally, the proposition holds by induction. □

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References

1. J. Bertoin and R.A. Doney, Some asymptotic results for transient random walks, *Adv. Appl. Probab.*, **28**:207–226, 1996.
2. W. Chen, C. Yu, and Y. Wang, Some discussions on the local distribution classes, *Stat. Probab. Lett.*, **83**:654–661, 2013.
3. V.P. Chistyakov, A theorem on sums of independent positive random variables and its application to branching processes, *Theory Probab. Appl.*, **9**:640–648, 1964.
4. J. Chover, P. Ney, and S. Wainger, Degeneracy properties of subcritical branching processes, *Ann. Probab.*, **1**:663–673, 1973.
5. J. Chover, P. Ney, and S. Wainger, Functions of probability measures, *J. Anal. Math.*, **26**:255–302, 1973.
6. Z. Cui, Y. Wang, and H. Xu, Local closure under infinitely divisible distribution roots and Esscher transform, *Mathematics*, **10**(21):4128, 2022.
7. Z. Cui, Y. Wang, and H. Xu, Some positive conclusions related to the Embrechts–Goldie conjecture, *Sib. Math. J.*, **63**(1):216–231, 2022.
8. P. Embrechts and C.M. Goldie, On closure and factorization properties of subexponential tails, *J. Aust. Math. Soc., Ser. A*, **29**:243–256, 1980.
9. P. Embrechts and C.M. Goldie, On convolution tails, *Stochastic Processes Appl.*, **13**:263–278, 1982.
10. P. Embrechts, C.M. Goldie, and N. Veraverbeke, Subexponentiality and infinite divisibility, *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, **49**:335–347, 1979.
11. W. Feller, *An Introduction to Probability Theory and Its Applications, Vol. 2,* 2nd ed., John Wiley & Sons, New York, 1971.
12. S. Foss and D. Korshunov, Lower limits and equivalences for convolution tails, *Ann. Probab.*, **1**:366–383, 2007.
13. C. Klüppelberg, Asymptotic ordering of distribution functions and convolution semigroups, *Semigroup Forum*, **40**: 77–92, 1990.
14. A.G. Pakes, Convolution equivalence and infinite divisibility, *J. Appl. Probab.*, **41**:407–424, 2004.
15. A.G. Pakes, Convolution equivalence and infinite divisibility: Corrections and corollaries, *J. Appl. Probab.*, **41**:295–305, 2007.
16. K. Sato, *Lévy processes and infinitely divisible distributions*, Camb. Stud. Adv. Math., Vol. 68, Cambridge Univ. Press, Cambridge, 1999.
17. M.S. Sgibnev, Asymptotics of infinite divisibility on **R**, *Sib. Math. J.*, **31**:115–119, 1990.
18. T. Shimura and T. Watanabe, Infinite divisibility and generalized subexponentiality, *Bernoulli*, **11**:445–469, 2005.
19. T. Watanabe, Convolution equivalence and distributions of random sums, *Probab. Theory Relat. Fields*, **142**:367–397, 2008.
20. T. Watanabe, The Wiener condition and the conjectures of Embrechts and Goldie, *Ann. Probab.*, **47**:1221–1239, 2019.
21. T. Watanabe and K. Yamamuro, Ratio of the tail of an infinitely divisible distribution on the line to that of its Lévy measure, *Electron. J. Probab.*, **15**:44–74, 2010.
22. H. Xu, S. Foss, and Y. Wang, Convolution and convolution-root properties of long-tailed distributions, *Extremes*, **18**: 605–628, 2015.
23. H. Xu, M. Scheutzow, Y. Wang, and Z. Cui, On the structure of a class of distributions obeying the principle of a single big jump, *Probab. Math. Stat.*, **36**(1):121–135, 2016.
24. H. Xu, Y. Wang, D. Cheng, and C. Yu, On the closure under infinitely divisible distribution roots, *Lith. Math. J.*, **62**(2):259–287, 2022.