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Richter's local limit theorem, its refinement, and related results*

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In memoriam Gennadiy P. Chistyakov (May 1, 1945 – December 30, 2022)

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Abstract. We give a detailed exposition of the proof of Richter's local limit theorem in a refined form and establish the stability of the remainder term in this theorem under small perturbations of the underlying distribution (including smoothing). We also discuss related quantitative bounds for characteristic functions and Laplace transforms.

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1 Introduction and formulation of the results

Let $(X_n)_{n\geqslant 1}$ be independent copies of a random variable X with mean $\mathbf{E}X = 0$ and variance $\mathrm{Var}(X) = 1$. Throughout, we assume without mentioning that the normalized sum

$$Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

has a bounded density p_{n_0} for some $n = n_0$, that is, $p_{n_0}(x) \leqslant M$ for all $x \in \mathbb{R}$ with some constant M. Then all Z_n with $n \geqslant 2n_0$ have continuous bounded densities $p_n(x)$. An asymptotic behavior of these densities describing their closeness to the normal density

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}, \quad x \in \mathbb{R},$$

is governed by several local limit theorems. First of all, there is a uniform local limit theorem due to Gnedenko:

$$\sup_x \big|p_n(x) - \varphi(x)\big| \to 0 \quad \text{as } n \to \infty.$$

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Under higher-order moment assumptions, say if $\mathbf{E}|X|^m < \infty$ for an integer $m \geqslant 3$, then this statement may be considerably sharpened in the form of a nonuniform local limit theorem

$$\sup_x \big(1+|x|^m\big)\big|p_n(x) - \varphi_m(x)\big| = o\big(n^{-(m-2)/2}\big), \tag{1.1}$$

where φ_m denotes the Edgeworth correction of φ of order m (see [10, 15, 16]). In various applications, this relation is typically effective in the range $|x| \leqslant \sqrt{c \log n}$, since then the ratio $p_n(x)/\varphi(x)$ remains close to 1 (for a suitable c). For example, (1.1) is crucial in the study of rates in the entropic central limit theorem, rates for Rényi divergences of finite orders, and for the relative Fisher information [5, 6].

As for larger regions, the asymptotic behavior of $p_n(x)$ is governed by the following remarkable theorem due to Richter [17], assuming the finiteness of an exponential moment of the random variable X.

Theorem 1. *Suppose that for some $b > 0$,*

$$\mathbf{E}e^{b|X|} < \infty. \tag{1.2}$$

Then for $x = o(\sqrt{n})$, the densities of Z_n admit the representation

$$\frac{p_n(x)}{\varphi(x)} = \exp\bigg\{\frac{x^3}{\sqrt{n}}\,\lambda\bigg(\frac{x}{\sqrt{n}}\bigg)\bigg\}\bigg(1 + O\bigg(\frac{1+|x|}{\sqrt{n}}\bigg)\bigg), \tag{1.3}$$

where $\lambda(\tau)$ represents an analytic function in some neighborhood of zero.

It was shown by Amosova [1] that condition (1.2) is necessary for the existence of a representation like (1.3) in the region $|x| = o(\sqrt{n})$ with some analytic function λ.

The function λ in (1.3) is representable as a power series, called the Cramér series,

$$\lambda(\tau) = \sum_{k=0}^{\infty} \lambda_k \tau^k, \tag{1.4}$$

which is absolutely convergent in some disc $|\tau| < \tau_0$ of the complex plane. It has appeared in the work by Cramér [8] in a similar representation for the ratio of the tails of distribution functions of Z_n and the standard normal law (see also [9, 13, 14]).

Let us also mention that (1.3) is stated in Richter's work in a slightly different form with $O(|x|/\sqrt{n})$ in the last brackets and for $|x| > 1$. A similar result is proved in the book by Ibragimov and Linnik [12] under the assumption that X has a bounded continuous density.

As a consequence of (1.3), we immediately obtain, for example, that

$$\frac{p_n(x)}{\varphi(x)} \to 1 \quad \text{as } n \to \infty \tag{1.5}$$

uniformly in the region $|x| = o(n^{1/6})$. In the region $c_0 n^{1/6} \leqslant |x| \leqslant c_1 n^{1/2}$, the behavior may be quite different, and to describe it, the appearance of the term $O((1+|x|)/\sqrt{n})$ in (1.3) is undesirable. The purpose of this paper is to give a detailed exposition of the proof of Theorem 1, clarifying the meaning of the leading coefficient in (1.4) and replacing this term with an n-depending quantity. We basically employ the tools of [12] and derive the following refinement.

Theorem 2. *Let the conditions of Theorem 1 be fulfilled, and let $n \geqslant 2n_0$. There is a constant $\tau_0 > 0$ with the following property. With $\tau = x/\sqrt{n}$, for $|\tau| \leqslant \tau_0$, we have*

$$\frac{p_n(x)}{\varphi(x)} = e^{n\tau^3\lambda(\tau) - \mu(\tau)}\big(1 + O\big(n^{-1}(\log n)^3\big)\big), \tag{1.6}$$

where $\mu(\tau)$ is an analytic function in $|\tau| \leqslant \tau_0$ such that $\mu(0) = 0$.

As we will see in Section 5,

$$\lambda(\tau) = \frac{1}{m!}\,\gamma_m \tau^{m-3} + O\big(|\tau|^{m-2}\big),$$
$$\mu(\tau) = \frac{1}{2(m-2)!}\,\gamma_m \tau^{m-2} + O\big(|\tau|^{m-1}\big),$$

where γ_m ($m \geqslant 3$) is the first nonzero cumulant of the random variable X (assuming that it is not normal). Equivalently, m is the smallest positive integer such that $\mathbf{E}X^m \neq \mathbf{E}Z^m$, where Z is a standard normal random variable, in which case

$$\gamma_m = \mathbf{E}X^m - \mathbf{E}Z^m.$$

With this refinement, it should be clear that relation (1.5) holds uniformly over all x in the potentially larger region

$$|x| \leqslant \varepsilon_n n^{1/2-1/m} \quad (\varepsilon_n \to 0).$$

For example, if the distribution of X is symmetric about the origin, then $\gamma_3 = 0$, so that necessarily $m \geqslant 4$.

Another consequence of (1.6), which cannot be obtained on the basis of (1.3), is needed in the study of the central limit theorem (CLT) with respect to the Rényi divergence of infinite order (including the rate of convergence). Let us recall that the Rényi divergence of a finite order $\kappa > 0$ from the distribution of Z_n to the standard normal law is defined by

$$D_\kappa(p_n \,\|\, \varphi) = \frac{1}{\kappa - 1}\log \int_{-\infty}^{\infty} \left(\frac{p_n(x)}{\varphi(x)}\right)^{\kappa} \varphi(x)\,\mathrm{d}x.$$

As a function of κ, it is nondecreasing, representing a strong distance-like quantity. In the range $0 < \kappa < 1$, it is metrically equivalent to the total variation, that is, L^1-distance between p_n and φ. The case $\kappa = 1$ corresponds to the relative entropy (Kullback–Leibler's distance)

$$D(p_n \,\|\, \varphi) = \lim_{\kappa \to 1} D_\kappa(p_n \,\|\, \varphi) = \int_{-\infty}^{\infty} p_n(x) \log \frac{p_n(x)}{\varphi(x)}\,\mathrm{d}x,$$

and another important case $\kappa = 2$ leads to the function of the χ^2-Pearson distance. So far, information-theoretic CLTs of the form $D_\kappa(p_n \,\|\, \varphi) \to 0$ as $n \to \infty$ have been completely characterized in terms of the distribution of X (i.e., in the i.i.d. situation). However, such a statement remains fully open for the limit distance

$$D_\infty(p_n \,\|\, \varphi) = \lim_{\kappa \to \infty} D_\kappa(p_n \,\|\, \varphi) = \sup_x \log \frac{p_n(x)}{\varphi(x)}.$$

Equivalently, the problem is to find conditions under which the related quantity (the Tsallis distance of infinite order)

$$T_\infty(p_n \,\|\, \varphi) = \sup_x \frac{p_n(x) - \varphi(x)}{\varphi(x)}$$

tends to zero for growing n (note that we may not put the absolute values sign, since all p_n may be compactly supported).

As a first natural step towards this variant of the CLT, we consider the problem of the convergence for the restricted Tsallis distance with the above suprema taken over growing intervals $|x| = O(\sqrt{n})$. With this in mind, Theorem 2 allows us to prove the following:

Corollary 1. *Under the conditions of Theorem* 1*, suppose that* m *is even,* $m \geqslant 4$*, and* $\gamma_m < 0$*. There exist constants* $\tau_0 > 0$ *and* $c > 0$ *with the following property. If* $|\tau| \leqslant \tau_0$*,* $\tau = x/\sqrt{n}$*, then*

$$\frac{p_n(x) - \varphi(x)}{\varphi(x)} \leqslant \frac{c(\log n)^3}{n}. \tag{1.7}$$

Here, the condition about cumulants is fulfilled, for example, when the random variable X is strongly sub-Gaussian in the sense that

$$\mathbf{E}e^{tX} \leqslant e^{t^2/2} \quad \text{for all } t \in \mathbb{R} \tag{1.8}$$

(recall that that $\mathbf{E}X^2 = 1$, whereas the condition $\mathbf{E}X = 0$ is necessary). This interesting class of probability distributions is rather rich, and we refer the reader to [7] for discussions and various examples. Our main motivation stemmed from the fact that the strong sub-Gaussianity is necessary for the convergence $D_\infty(p_n \,\|\, \varphi) \to 0$. As we have recently learned, (1.8) had previously appeared under the name "sharp sub-Gaussianity" in the work by Guionnet and Husson [11] for a completely different reason as a condition to have LDPs for the largest eigenvalue of Wigner matrices with the same rate function as in the case of Gaussian entries.

One important issue, which is not addressed in the formulation of Theorem 2, is how we can control the involved constant in the O-remainder term in (1.6). To better quantify this asymptotic representation, we actually prove the following statement using the same analytic functions $\lambda(\tau)$ and $\mu(\tau)$.

Theorem 3. *Assume that* $\mathbf{E}e^{\alpha|X|} \leqslant 2$ $(\alpha > 0)$*. There exist absolute positive constants* C *and* c *such that whenever* $n \geqslant n_1$ *and* $\tau = x/\sqrt{n}$*,* $|\tau| \leqslant \tau_0$*, we have*

$$\frac{p_n(x)}{\varphi(x)} = e^{n\tau^3\lambda(\tau) - \mu(\tau)}\left(1 + B\alpha^{-6}\frac{(\log n)^3}{n}\right),$$

where $|B| \leqslant C$ *and*

$$n_1 = CM^4 n_0^2 \alpha^{-12}, \qquad \tau_0 = \frac{c\alpha^3}{M^2 n_0}.$$

This statement should be useful in applications to smoothed distributions to guarantee that the constant in the remainder term may be chosen to be common for all distributions under consideration.

To make the proofs/arguments more transparent and self-contained, we include a short review of various related results – partly technical, but often interesting in themselves – about maxima of densities, analytic characteristic functions, and log-Laplace transforms. The rest of the paper is organized as follows. In Section 2, we recall basic properties of the maximum of convolved densities and then develop their applications to bounding restricted integrals of powers of characteristic functions (Section 3). In Section 4, we discuss behavior of analytic characteristic functions near the origin. Section 5 is devoted to the so-called saddle points and associated Taylor expansions for the log-Laplace transforms. Here we also analyze the functions $\lambda(\tau)$ and $\mu(\tau)$. Section 6 deals with contour integration needed to establish preliminary representations for $p_n(x)$. Final steps in the proof of Theorem 3 are made in Section 7. The proof of Corollary 1 is postponed to Section 8.

2 Maximum of convolved densities

Convolved densities are known to have improved smoothing properties. First, let us emphasize the following general fact (which explains the condition $n \geqslant 2n_0$ mentioned before Theorem 1).

Proposition 1. *If independent random variables* $\xi_1, \dots, \xi_m$ $(m \geqslant 2)$ *have bounded densities, then the sum* $S_m = \xi_1 + \dots + \xi_m$ *has a bounded uniformly continuous density vanishing at infinity.*

Proof. Denote by q_k the densities of ξ_k and assume that $q_k(x) \leqslant M_k$ for all $x \in \mathbb{R}$ with some constants M_k ($k \leqslant m$). By the Plancherel theorem, for the characteristic functions $g_k(t) = \mathbf{E}e^{it\xi_k}$, we have

$$\int_{-\infty}^{\infty} |g_k(t)|^m \,dt \leqslant \int_{-\infty}^{\infty} |g_k(t)|^2 \,dt = 2\pi \int_{-\infty}^{\infty} q_k(x)^2 \,dx \leqslant 2\pi \int_{-\infty}^{\infty} M_k q_k(x) \,dx$$
$$= 2\pi M_k,$$

where we used the property $|g_k(t)| \leqslant 1$, $t \in \mathbb{R}$. Hence, by Hölder's inequality, the characteristic function $g(t) = g_1(t) \dots g_m(t)$ of S_m is integrable and has the L^1-norm

$$\int_{-\infty}^{\infty} |g(t)| \,dt \leqslant \left(\int_{-\infty}^{\infty} |g_1(t)|^m \,dt\right)^{1/m} \cdots \left(\int_{-\infty}^{\infty} |g_m(t)|^m \,dt\right)^{1/m}$$
$$\leqslant 2\pi (M_1 \cdots M_m)^{1/m} < \infty. \tag{2.1}$$

We may conclude that the random variable S_m has a bounded, uniformly continuous density expressed by the inversion Fourier formula

$$q(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} g(t) \,dt, \quad x \in \mathbb{R}. \tag{2.2}$$

Since g is integrable, it also follows that $q(x) \to 0$ as $|x| \to \infty$ (by the Riemann–Lebesgue lemma). □

Consider the functional

$$M(\xi) = \operatorname*{ess\,sup}_x q(x),$$

where ξ is a random variable with density q (we may put $M(\xi) = \infty$ in all other cases). Since, by (2.2),

$$q(x) \leqslant \frac{1}{2\pi} \int_{-\infty}^{\infty} |g(t)| \,dt$$

for all $x \in \mathbb{R}$, inequality (2.1) also implies that

$$M(S_m) \leqslant \big(M(\xi_1) \cdots M(\xi_m)\big)^{1/m}. \tag{2.3}$$

This shows in particular that $M(\xi)$ may not increase by adding to ξ an independent random variable. However, relation (2.3) does not correctly reflect the behavior of $M(S_m)$ with respect to the growing parameter m, especially in the i.i.d. situation. A more precise statement is described in the following relation, where the geometric mean of maxima is replaced with the harmonic mean.

Proposition 2. *Given independent random variables ξ_k, $1 \leqslant k \leqslant m$, we have*

$$\frac{1}{M(S_m)^2} \geqslant \frac{1}{2} \sum_{k=1}^{m} \frac{1}{M(\xi_k)^2}. \tag{2.4}$$

This bound may be viewed as a counterpart of the entropy power inequality in information theory. It can be obtained by combining Rogozin's maximum-of-density theorem with Ball's bound on the volume of slices of the cube. Namely, it was shown in [18] that if the values $M_k = M(\xi_k)$ are fixed, then $M(S_m)$ is maximized for ξ_k uniformly distributed in the intervals of length $1/M_k$. Of course, in this case, $M(S_m)$ has a rather complicated structure as a function in variables $M_1, \dots, M_m$.

On the other hand, if $T_m = a_1\eta_1 + \dots + a_m\eta_m$, where η_k are independent and uniformly distributed in $(0,1)$, and the coefficients satisfy $a_1^2 + \dots + a_m^2 = 1$, then

$$1 \leqslant M(T_m) \leqslant \sqrt{2}; \tag{2.5}$$

see [2]. In geometric language, this is the same as saying that $1 \leqslant |Q \cap H| \leqslant \sqrt{2}$, where $Q = (0,1)^m$ is the unit cube, H is an arbitrary hyperplane in $\mathbb{R}^m$ passing through the center of the cube, and $|\cdot|$ stands for the $(m-1)$-dimensional volume. To obtain (2.4), put

$$a_k = \frac{1}{aM_k}, \qquad a^2 = \sum_{k=1}^m \frac{1}{M_k^2},$$

so that, by the upper bound in (2.5),

$$M\left(\sum_{k=1}^m \frac{1}{M_k}\,\eta_k\right) = M\left(a\sum_{k=1}^m a_k\,\eta_k\right) = \frac{1}{a}M(T_m) \leqslant \frac{1}{a}\sqrt{2}. \tag{2.6}$$

Since, by [18], $M(S_m)$ does not exceed the first term in (2.6), we get $M(S_m) \leqslant \sqrt{2}/a$, that is, (2.4).

With this argument, this relation is mentioned in [3], where its multidimensional analog is derived by applying the Hausdorff–Young inequality with best constants (due to Beckner and Lieb).

Remark 1. Modulo a universal constant, the left inequality in (2.5) may be extended to a more general setting. Namely, if a random variable ξ has a density q with a finite standard deviation σ, then

$$M(\xi) \geqslant \frac{1}{12\sigma}. \tag{2.7}$$

Here equality is attained for the uniform distribution on arbitrary bounded intervals of the real line. This relation is well known; as an early reference, we can mention Statulevičius [19, p. 651], where (2.7) is stated without proof. Since it is used below, let us include a short argument. For normalization, we may assume that $M(\xi) = 1$ and $\mathbf{E}\xi = 0$. In this case the tail function $H(x) = \mathbf{P}\{|\xi| \geqslant x\}$ has a Lipschitz seminorm at most 2, implying that $H(x) \geqslant 1 - 2x$ for all $x \geqslant 0$. This gives

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 q(x)\,\mathrm{d}x = 2\int_0^{\infty} xH(x)\,\mathrm{d}x \geqslant 2\int_0^{1/2} x(1-2x)\,\mathrm{d}x = \frac{1}{12}.$$

3 L^p-norms of characteristic functions and Orlicz norms

One useful consequence of (2.4) is the next bound on L^{2m}-norms of characteristic functions.

Proposition 3. *If $g(t)$ is the characteristic function of a random variable ξ, then for any integer $m \geqslant 1$,*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left|g(t)\right|^{2m}\mathrm{d}t \leqslant \frac{1}{\sqrt{m}}M(\xi). \tag{3.1}$$

Proof. We apply Proposition 2 to $2m$ summands $\xi_1, -\xi_1', \dots, \xi_m, -\xi_m'$ assuming that ξ_k, ξ_k' are independent copies of ξ. Introduce the symmetrized random variable $\tilde{S}_m = S_m - S_m'$, where S_m' is an independent copy of S_m. By (2.4), we then get

$$M(\tilde{S}_m) \leqslant \frac{1}{\sqrt{m}} M(\xi).$$

In addition, $\tilde{S}_m$ has the characteristic function $|g(t)|^{2m}$. If $M(\xi)$ is finite, then we may apply Proposition 1 and conclude that $\tilde{S}_m$ has a bounded continuous density $q_m(x)$ vanishing at infinity. Moreover, $q_m(x)$ is maximized at $x = 0$, and its value at this point is described by the inversion formula (2.2), which gives

$$M(\tilde{S}_m) = q_m(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|g(t)\right|^{2m} \mathrm{d}t. \qquad \square$$

Using (2.3), we can obtain a similar relation but without the factor $1/\sqrt{m}$ in (3.1).

When $M(\xi)$ is finite and m is large, this bound may be considerably sharpened asymptotically with respect to m when restricting the integration to the regions $|t| \geqslant \varepsilon > 0$. Before making this precise, first, let us note that since the random variable ξ has a density, we have

$$\delta_g(\varepsilon) = \max_{|t| \geqslant \varepsilon} \left|g(t)\right| < 1 \tag{3.2}$$

for all $\varepsilon > 0$. This holds by the continuity of g, and since $|g(t)| < 1$ for all $t \neq 0$ (which is true for any nonlattice distribution), $g(t)$ tends to zero as $t \to \infty$, by the Riemann–Lebesgue lemma. By the way, this property remains to hold in the more general situation where the m-fold convolution of the distribution of ξ with itself has a density (whereas the distribution of ξ may be not absolutely continuous). Indeed, in that case, (3.2) may be applied to g^m, and it remains to notice that this relation does not depend on m.

Property (3.2) may be quantified using, for example, the following observation due to Statulevičius [19].

Proposition 4. *If a random variable ξ has a bounded density with $M = M(\xi)$ and finite variance $\sigma^2 = \mathrm{Var}(\xi)$, $\sigma > 0$, then its characteristic function g satisfies, for all $\varepsilon > 0$,*

$$\delta_g(\varepsilon) \leqslant \exp\left\{-\frac{\varepsilon^2}{96M^2(2\sigma\varepsilon + \pi)^2}\right\}. \tag{3.3}$$

This relation may be extended to nonbounded densities q, in which case the parameter M should be replaced with quantiles of the random variable $q(\xi)$. The moment condition may also be removed, and instead it is sufficient to deal with quantiles of $|\xi - \xi'|$, where ξ' is an independent copy of ξ; see [4] for details.

Returning to (3.1) and applying (3.3) with $\varepsilon \leqslant 1$, we then have

$$\int_{|t| \geqslant \varepsilon} \left|g(t)\right|^{4m} \mathrm{d}t \leqslant \delta_g(\varepsilon)^{2m} \int_{-\infty}^{\infty} \left|g(t)\right|^{2m} \mathrm{d}t \leqslant \frac{2\pi M}{\sqrt{m}} \exp\left\{-\frac{m\varepsilon^2}{CM^2}\right\}$$

with some absolute constant C. Thus the resulting bound decays asymptotically fast in m.

Let us derive a similar bound in the scheme of independent copies $(X_n)_{n \geqslant 1}$ of the random variable X with $\mathrm{Var}(X) = 1$, assuming that the normalized sum Z_n has a bounded density for $n = n_0$ with $M = M(Z_{n_0})$. Consider the characteristic function $f(t) = \mathbf{E}\mathrm{e}^{\mathrm{i}tX}$. We apply Propositions 3–4 with $\xi = X_1 + \cdots + X_{n_0}$, in which case $g(t) = f^{n_0}(t)$ and $M(\xi) = 1/\sqrt{n_0}\, M$. Then, for any $1 \leqslant m \leqslant n/2n_0$, by (3.1),

$$\int\limits_{|t|\geqslant\varepsilon} \left|f(t)\right|^n \mathrm{d}t = \int\limits_{|t|\geqslant\varepsilon} \left|f(t)\right|^{n-2mn_0} \left|g(t)\right|^{2m} \mathrm{d}t \leqslant \delta_f^{n-2mn_0}(\varepsilon) \int\limits_{-\infty}^{\infty} \left|g(t)\right|^{2m} \mathrm{d}t$$
$$\leqslant \frac{2\pi M}{\sqrt{mn_0}}\, \delta_f^{n-2mn_0}(\varepsilon).$$

If $n \geqslant 4n_0$, then let us choose $m = [n/(4n_0)]$. Then $n - 2mn_0 \geqslant n/2$, whereas $m \geqslant n/(8n_0)$, and we arrive at

$$\int\limits_{|t|\geqslant\varepsilon} \left|f(t)\right|^n \mathrm{d}t \leqslant \frac{4\pi M}{\sqrt{2n}}\, \delta_f^{n/2}(\varepsilon).$$

By (3.3) with $\varepsilon \leqslant 1$, we also have

$$\delta_f^{n_0}(\varepsilon) \leqslant \exp\left\{-\frac{\varepsilon^2}{96M^2(2\sqrt{n_0}+\pi)^2}\right\},$$

which may be simplified to

$$\delta_f(\varepsilon) \leqslant \exp\left\{-\frac{\varepsilon^2}{96(2+\pi)^2 n_0 M^2}\right\}.$$

Combining the two bounds, we may summarize.

Corollary 2. *Let* $\mathrm{Var}(X) = 1$, *and suppose that* Z_n *has a density for* $n = n_0$ *bounded by* M. *Then for all* $0 < \varepsilon \leqslant 1$ *and* $n \geqslant 4n_0$, *the characteristic function* f *of* X *satisfies*

$$\int\limits_{|t|\geqslant\varepsilon} \left|f(t)\right|^n \mathrm{d}t \leqslant \frac{4\pi M}{\sqrt{2n}}\, \exp\left\{-\frac{n\varepsilon^2}{Cn_0M^2}\right\}, \quad C = 5200.$$

4 Behavior of characteristic functions near zero

While the boundedness of the density is important to control integrability properties of powers of the characteristic function of a random variable X, condition (1.2) on the finiteness of an exponential moment of X guarantees that the characteristic function

$$f(z) = \mathbf{E}\mathrm{e}^{\mathrm{i}zX}, \quad z = t + \mathrm{i}y, \quad t, y \in \mathbb{R},$$

is well defined and analytic in the strip $|y| = |\mathrm{Re}(z)| < b$ of the complex plane. Equivalently, we will assume throughout that for some $\alpha > 0$,

$$\mathbf{E}\mathrm{e}^{\alpha|X|} \leqslant 2. \tag{4.1}$$

This parameter is more convenient to quantify the behavior of $f(z)$ near zero.

For example, using $x\mathrm{e}^{-x} \leqslant \mathrm{e}^{-1}$ ($x \geqslant 0$) and assuming that $|y| \leqslant \alpha/2$, we then have

$$\left|f'(z)\right| = \left|\mathbf{E}X\mathrm{e}^{\mathrm{i}zX}\right| \leqslant \mathbf{E}|X|\mathrm{e}^{|yX|} \leqslant \mathbf{E}|X|\mathrm{e}^{\alpha|X|/2} = \mathbf{E}|X|\mathrm{e}^{-\alpha|X|/2}\mathrm{e}^{\alpha|X|} \leqslant \frac{4}{\alpha\mathrm{e}}.$$

Hence $|f(z) - 1| \leqslant 4/(\alpha\mathrm{e})|z|$ (since $f(0) = 1$). Thus we obtain the following:

Lemma 1. *For all complex numbers z in the disc $|z| \leqslant \alpha/2$,*

$$\big|f'(z)\big| \leqslant \frac{4}{\alpha \mathrm{e}}, \qquad \big|f(z) - 1\big| \leqslant \frac{2}{\mathrm{e}}.$$

This allows us to consider the log-Laplace transform

$$K(z) = \log \mathbf{E}\mathrm{e}^{zX} = \log f(-\mathrm{i}z)$$

as an analytic function in the disc $|z| \leqslant \alpha/2$. Since it has the derivative $K'(z) = -\mathrm{i}f'(-\mathrm{i}z)/f(-\mathrm{i}z)$, from Lemma 1 we get that in this disc,

$$\big|K'(z)\big| \leqslant \frac{6}{\alpha}, \qquad \big|K(z)\big| \leqslant 3. \tag{4.2}$$

We may also bound the derivatives of all orders.

Lemma 2. *For all complex numbers z in the disc $|z| \leqslant \alpha/4$,*

$$\big|K^{(k)}(z)\big| \leqslant 3k! \left(\frac{4}{\alpha}\right)^k, \quad k = 1, 2, \dots . \tag{4.3}$$

Moreover, if $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$, then

$$\big|K'''(z)\big| \leqslant \frac{8}{\alpha^3}, \quad |z| \leqslant \frac{\alpha}{16}. \tag{4.4}$$

As a consequence,

$$\big|K''(z) - 1\big| \leqslant \frac{1}{2}, \quad |z| \leqslant \frac{\alpha^3}{16}. \tag{4.5}$$

Thus these derivatives have at most a factorial growth in absolute value with respect to the growing parameter k. For the particular orders $k = 2$ and $k = 3$, and under our moment assumptions, the bound (4.3) may be refined in a smaller disc according to (4.4)–(4.5).

Proof. To obtain (4.3), we may apply Cauchy's formula

$$K^{(k)}(z) = \frac{k!}{2\pi \mathrm{i}} \int\limits_{|w-z|=r} \frac{K(w)}{w^{k+1}}\, \mathrm{d}w$$

with $r = \alpha/4$ together with the second bound in (4.2).

Turning to the refined bounds, note that in terms of the Laplace transform $L(z) = \mathbf{E}\mathrm{e}^{zX}$, we have $K' = L'/L$ and

$$K''' = \frac{L'''}{L} - 3\frac{L''L'}{L^2} + 2\frac{L'^3}{L^3}. \tag{4.6}$$

For $x \geqslant 0$ and $p = 1, 2, 3$, we use the elementary inequality $x^p \mathrm{e}^{-x} \leqslant (p/\mathrm{e})^p$. Suppose that $|z| \leqslant (1-c)\alpha$ with $1/2 \leqslant c < 1$. Since $L^{(p)}(z) = \mathbf{E}X^p \mathrm{e}^{zX}$, we then have

$$\big|L^{(p)}(z)\big| \leqslant \mathbf{E}|X|^p \mathrm{e}^{(1-c)\alpha|X|}.$$

Hence, by (4.1),

$$\left|L^{(p)}(z)\right| \leqslant \mathbf{E}|X|^p \mathrm{e}^{-c\alpha|X|}\mathrm{e}^{\alpha|X|} \leqslant 2\left(\frac{p}{c\alpha\mathrm{e}}\right)^p.$$

In particular,

$$\left|L'(z)\right| \leqslant \frac{2}{c\alpha\mathrm{e}}, \qquad \left|L''(z)\right| \leqslant \frac{8}{(c\alpha\mathrm{e})^2}, \qquad \left|L'''(z)\right| \leqslant \frac{54}{(c\alpha\mathrm{e})^3}, \tag{4.7}$$

so

$$\left|L(z)-1\right| \leqslant \frac{2}{c\alpha\mathrm{e}}|z| \leqslant \frac{2(1-c)}{c\mathrm{e}}, \qquad \left|L(z)\right| \geqslant 1 - \frac{2(1-c)}{c\mathrm{e}}.$$

Putting $q^{-1} = 1 - 2(1-c)/(c\mathrm{e})$, from (4.6) it follows that

$$\left|K'''(z)\right| \leqslant (c\alpha\mathrm{e})^{-3}\left(54q + 48q^2 + 16q^3\right).$$

Choosing $c = 15/16$, we have $q = (1 - 2/(15\mathrm{e}))^{-1} < 1.06$, and the last expression becomes smaller than $8\alpha^{-3}$. Hence

$$\left|K'''(z)\right| \leqslant \frac{8}{\alpha^3}, \qquad \left|K''(z) - 1\right| \leqslant \frac{8}{\alpha^3}|z|$$

for $|z| \leqslant \alpha/16$, where we used $K''(0) = 1$. The last inequality readily implies (4.5). □

We will now show that $|f(z)|$ is bounded away from 1 in a certain region near zero.

Lemma 3. *Let $\mathbf{E}X = 0$ and $\mathbf{E}X^2 = 1$. For all complex numbers $z = t + \mathrm{i}y$ with $|t| \leqslant \alpha^3/8$ and $|y| \leqslant |t|/2$, we have $|f(z)| \leqslant \mathrm{e}^{-t^2/5}$.*

Proof. Using $f'(0) = 0$ and $f''(0) = -1$, we may start with the integral Taylor formula

$$f(z) = 1 - \frac{1}{2}z^2 + \frac{1}{2}z^3 \int_0^1 f'''(sz)(1-s)^2\,\mathrm{d}s.$$

This equality is needed in the disc $|z| \leqslant r$ of radius $r = \sqrt{5/4}\,\alpha^3/8$. By the triangle inequality, we then have

$$\left|f(z)\right| \leqslant \left|1 - \frac{1}{2}z^2\right| + \frac{A}{6}|z|^3, \quad A = \max_{|z|\leqslant r}\left|f'''(z)\right|. \tag{4.8}$$

First, let us check that

$$\left|1 - \frac{1}{2}z^2\right| \leqslant 1 - \frac{1}{3}t^2, \tag{4.9}$$

which actually holds in the larger region $|y| \leqslant |t|/2$, $|t| \leqslant 1/4$. In this case, $t^2 - y^2 \geqslant 3t^2/4$ and $|ty| \leqslant t^2/2$, implying

$$\begin{aligned}\left|1 - \frac{1}{2}z^2\right|^2 &= 1 - \left(t^2 - y^2\right) + \frac{1}{4}\left(t^2 - y^2\right)^2 + (ty)^2 \\ &\leqslant 1 - \frac{3}{4}t^2 + \frac{1}{2}t^4 \leqslant \left(1 - \frac{1}{3}t^2\right)^2,\end{aligned}$$

where we used $|t| \leqslant 1/4$ on the last step. Thus (4.9) follows.

Turning to the maximum in (4.8), we may apply the last bound in (4.7) valid for $|z| \leqslant (1-c)\alpha$. Hence we have the constraint $(1-c)\alpha \geqslant r$, which is fulfilled for the choice $c = 1 - \sqrt{5/4}/8$ (due to $\alpha < 1$; see Lemma 4). In this case, we get

$$\big|f'''(z)\big| \leqslant \frac{54}{(c\alpha \mathrm{e})^3} < \frac{4.3}{\alpha^3}.$$

For $z = t + \mathrm{i}y$, $|y| \leqslant |t|/2$, we have $|z|^3 \leqslant (5/4)^{3/2}|t|^3$, and (4.8)–(4.9) therefore give

$$\big|f(z)\big| \leqslant 1 - \frac{1}{3}t^2 + \frac{4.3}{6\alpha^3}\left(\frac{5}{4}\right)^{3/2}|t|^3 \leqslant 1 - \frac{1}{3}t^2 + \frac{4.3}{48}\left(\frac{5}{4}\right)^{3/2}t^2 \leqslant 1 - \frac{1}{5}t^2. \qquad \square$$

Finally, let us make a few remarks about the relationship between conditions (1.2) and (4.1). When the random variable X has a finite exponential moment and α is optimal, then (4.1) becomes an equality. In this case the quantity $1/\alpha$ represents the Orlicz norm of X generated by the Young function $\psi(x) = \mathrm{e}^{|x|} - 1$, $x \in \mathbb{R}$:

$$\|X\|_\psi = \inf\left\{\lambda > 0\colon\ \mathbf{E}\,\psi\left(\frac{X}{\lambda}\right) \leqslant 1\right\}.$$

If $\mathbf{E}X^2 = 1$, then the parameter α may not be large, since the L^2-norm is dominated by the L^ψ-norm. More precisely, using $x^2\mathrm{e}^{-x} \leqslant 4\mathrm{e}^{-2}$ $(x \geqslant 0)$, we have

$$\alpha^2 = \mathbf{E}(\alpha X)^2 \leqslant 4\mathrm{e}^{-2}\mathbf{E}\mathrm{e}^{\alpha|X|} = 8\mathrm{e}^{-2},$$

implying $\alpha \leqslant 2\mathrm{e}^{-1}\sqrt{2} < 1.05$. In fact, this bound may be sharpened.

Lemma 4. *If* $\mathbf{E}X^2 = 1$ *and* (4.1) *holds, then* $\alpha < 1$.

Proof. We may assume that $X \geqslant 0$, and then we need to show that $\mathbf{E}\mathrm{e}^X > 2$. It is easy to check that $x + x^3/6 \geqslant ax^2$ for all $x \geqslant 0$ with the optimal constant $a = 2/\sqrt{6}$. Since $\mathbf{E}X^k \geqslant (\mathbf{E}X^2)^{k/2} = 1$ for $k \geqslant 2$, we get

$$\begin{aligned}\mathbf{E}\mathrm{e}^X &= 1 + \frac{1}{2}\,\mathbf{E}X^2 + \mathbf{E}\left(X + \frac{1}{6}\,X^3\right) + \sum_{k=4}^{\infty}\frac{1}{k!}\,\mathbf{E}X^k \geqslant \frac{3}{2} + a + \sum_{k=4}^{\infty}\frac{1}{k!}\\ &= \mathrm{e} - \frac{7}{6} + \frac{2}{\sqrt{6}} > 2.36. \qquad \square\end{aligned}$$

Note that if we start with a more general condition $B = \mathbf{E}\mathrm{e}^{b|X|} < \infty$ as in Theorem 1, then (4.1) is fulfilled for a certain constant $\alpha > 0$. Indeed, if $B \leqslant 2$, then we may take $\alpha = b$. Otherwise,

$$\mathbf{E}\mathrm{e}^{\varepsilon b|X|} \leqslant \left(\mathbf{E}\mathrm{e}^{b|X|}\right)^\varepsilon \leqslant B^\varepsilon = 2$$

for $\varepsilon = 1/\log_2(B)$. Hence $\alpha = \varepsilon b = b/\log_2(B)$ works as well. The two cases may be united by taking

$$\alpha = \frac{b}{\log_2(\max(B,2))}.$$

5 Saddle point and Taylor expansions

Assume that $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$, and $\mathbf{E}\mathrm{e}^{\alpha|X|} \leqslant 2$ $(\alpha > 0)$. Since the log-Laplace transform $K(z) = \log \mathbf{E}\mathrm{e}^{zX}$ was defined as an analytic function in the disc $|z| \leqslant \alpha/2$ of the complex plane, it may be expanded as an absolutely convergent power series

$$K(z) = \frac{1}{2}z^2 + \sum_{k=3}^{\infty} \frac{\gamma_k}{k!} z^k.$$

Here the coefficients $\gamma_k = K^{k)}(0)$ are called the cumulants of X. Every γ_k represents a certain polynomial in moments of X up to order k. In particular, $\gamma_3 = \mathbf{E}X^3$ and $\gamma_4 = \mathbf{E}X^4 - 3$.

Similarly,

$$K'(z) = z + \sum_{k=2}^{\infty} \frac{\gamma_{k+1}}{k!} z^k.$$

The next object is important for contour integration.

DEFINITION 1. Given $\tau \in \mathbb{C}$, a saddle point is a solution $z_0 = z_0(\tau)$ of the equation

$$K'(z) = \tau. \tag{5.1}$$

Thus a saddle point is the solution of

$$\tau = z + \sum_{k=2}^{\infty} \frac{\gamma_{k+1}}{k!} z^k. \tag{5.2}$$

Proposition 5. *In the disc $|\tau| \leqslant \alpha^3/32$, Eq.* (5.1) *has a unique solution $z_0(\tau)$. Moreover, it represents an injective analytic function satisfying $z_0'(0) = 1$ and*

$$\left|z_0(\tau)\right| \leqslant 2\tau \leqslant \frac{\alpha^3}{16}, \quad |\tau| \leqslant \frac{\alpha^3}{32}. \tag{5.3}$$

Proof. Let us use (5.2) as the definition of the analytic function $\tau = K'(z)$. If τ is sufficiently small, say $|\tau| \leqslant \tau_0$, then this equality may be inverted as a power series in τ,

$$z = z_0(\tau) = \tau - \frac{\gamma_3}{2}\tau^2 + \frac{3\gamma_3^2 - \gamma_4}{6}\tau^3 + \cdots.$$

Let us indicate an explicit expression for τ_0 in the form of a positive function of α.

By Lemma 2 (see (4.4)),

$$\left|\tau'(z) - 1\right| \leqslant \frac{8}{\alpha^3}|z|, \quad |z| \leqslant \frac{\alpha}{16}. \tag{5.4}$$

We may use this relation for $|z| \leqslant \alpha^3/16$, since $\alpha < 1$ by Lemma 4. Given two points z_1 and z_2 in the disc $|z| \leqslant \alpha^3/16$, define the path $z_t = (1-t)z_1 + tz_2$ connecting these points. We have

$$\tau(z_2) - \tau(z_1) = (z_2 - z_1)\left(1 + \int_0^1 \left(\tau'(z_t) - 1\right) \mathrm{d}t\right), \tag{5.5}$$

implying

$$\left|\tau(z_2) - \tau(z_1)\right| \geqslant |z_2 - z_1| \left(1 - \int_0^1 \left|\tau'(z_t) - 1\right| \mathrm{d}t\right).$$

Since $|z_t| \leqslant \alpha^3/16$, it follows from (5.4) that

$$\left|\tau(z_2) - \tau(z_1)\right| \geqslant \frac{1}{2}|z_2 - z_1|.$$

As a consequence, the map $z \to \tau(z)$ is injective in the disc $|z| \leqslant \alpha^3/16$. In addition, since $\tau(0) = 0$, we have

$$\left|\tau(z)\right| \geqslant \frac{1}{2}|z|. \tag{5.6}$$

Therefore the image of the circle $|z| = \alpha^3/16$ under this map represents a closed curve on the complex plane outside the circle $|\tau| = \alpha^3/32$. Since the image of the disc $|z| \leqslant \alpha^3/16$ under τ is a connected set, while $\tau(0) = 0$, this set must contain the disc $|\tau| \leqslant \alpha^3/32$. Thus the inverse map $z_0(\tau) = \tau^{-1}$ is well defined and represents a holomorphic injective function in $|\tau| \leqslant \alpha^3/32$ satisfying (5.3) by (5.6) and $z_0'(0) = 1$ by (5.4). Hence we may take $\tau_0 = \alpha^3/32$.

In addition, $z_0(\tau)$ takes real values for real τ. Indeed, since all cumulants are real numbers, $\tau(z)$ is real for real z, and so is the inverse function z_0. Also, by (5.5),

$$\tau = z_0(\tau)\left(1 + \int_0^1 \left(\tau'\big(tz_0(\tau)\big) - 1\right) \mathrm{d}t\right),$$

which shows that $z_0(\tau) > 0$ as long as $0 < \tau \leqslant \alpha^3/32$ (since the expression under the integral sign is a real-valued function whose absolute value does not exceed $1/2$). □

It is natural to determine the leading term in the Taylor expansion for $z_0(\tau)$ when expanding this function as a power series in τ. Assuming that X is not normal, let γ_m ($m \geqslant 3$) be the first nonzero cumulant of X. Then, as $|z| \to 0$,

$$K(z) = \frac{1}{2}z^2 + \frac{\gamma_m}{m!}z^m + O\big(|z|^{m+1}\big),$$

so that

$$K'(z) = z + \frac{\gamma_m}{(m-1)!}z^{m-1} + O\big(|z|^m\big) \tag{5.7}$$

and

$$K''(z) = 1 + \frac{\gamma_m}{(m-2)!}z^{m-2} + O\big(|z|^{m-1}\big). \tag{5.8}$$

Since $z_0(\tau) = \tau + O(|\tau|^2)$ as $\tau \to 0$ (see Proposition 5), from (5.7) we get

$$\begin{aligned}\tau = K'\big(z_0(\tau)\big) &= z_0(\tau) + \frac{\gamma_m}{(m-1)!}z_0(\tau)^{m-1} + O\big(\left|z_0(\tau)\right|^m\big) \\ &= z_0(\tau) + \frac{\gamma_m}{(m-1)!}\tau^{m-1} + O\big(|\tau|^m\big).\end{aligned}$$

Therefore

$$z_0(\tau) = \tau - \frac{\gamma_m}{(m-1)!}\tau^{m-1} + O\bigl(|\tau|^m\bigr). \tag{5.9}$$

Next, let us write down the Taylor expansion around the point $z_0 = z_0(\tau)$:

$$K(z) - \tau z = K(z_0) - \tau z_0 + \sum_{k=2}^{\infty} \frac{\rho_k}{k!}(z - z_0)^k, \quad \rho_k = K^{(k)}(z_0). \tag{5.10}$$

Here we used the property that the function $K(z) - \tau z$ has the derivative $K'(z_0) - \tau = 0$ at the saddle point $z = z_0$. Thus the linear term in (5.10) corresponding to $k = 1$ is vanishing. As for the free term corresponding to $k = 0$, we have

$$K(z_0) = \sum_{k=2}^{\infty} \frac{\gamma_k}{k!} z_0^k \quad \text{and} \quad \tau z_0 = z_0 K'(z_0) = \sum_{k=2}^{\infty} \frac{\gamma_k}{(k-1)!} z_0^k.$$

Hence

$$K(z_0) - \tau z_0 = -\sum_{k=2}^{\infty} \frac{k-1}{k!} \gamma_k z_0^k = -\frac{1}{2} z_0^2 - \frac{1}{3}\gamma_3 z_0^3 + \cdots . \tag{5.11}$$

Using (5.9) and (5.11), we actually have

$$\begin{aligned} K(z_0) - \tau z_0 = -\frac{1}{2} z_0^2 - \frac{m-1}{m!}\gamma_m z_0^m + \cdots &= -\frac{1}{2}\left(\tau - \frac{\gamma_m}{(m-1)!}\tau^{m-1} + O\bigl(|\tau|^m\bigr)\right)^2 \\ &- \frac{m-1}{m!}\gamma_m\left(\tau - \frac{\gamma_m}{(m-1)!}\tau^{m-1} + O\bigl(|\tau|^m\bigr)\right)^m + \cdots , \end{aligned}$$

which is simplified to

$$K(z_0) - \tau z_0 = -\frac{1}{2}\tau^2 + \frac{1}{m!}\gamma_m \tau^m + O\bigl(|\tau|^{m+1}\bigr) - \frac{1}{2}\tau^2 + \tau^3 \lambda(\tau). \tag{5.12}$$

Thus, applying Proposition 5 and recalling that $K(z)$ is analytic in $|z| \leqslant \alpha/2$ (Lemma 1), we obtain the following:

Proposition 6. *The function*

$$\lambda(\tau) = \frac{1}{\tau^3}\left(K\bigl(z_0(\tau)\bigr) - \tau z_0(\tau) + \frac{1}{2}\tau^2\right)$$

is well defined and analytic in the disc $|\tau| \leqslant \alpha^3/32$. Moreover, as $\tau \to 0$,

$$\lambda(\tau) = \frac{1}{m!}\gamma_m \tau^{m-3} + O\bigl(|\tau|^{m-2}\bigr). \tag{5.13}$$

DEFINITION 2. Being an analytic function, $\lambda(\tau)$ is represented as a power series in the disc $|\tau| \leqslant \alpha^3/32$. It is called Cramér's series.

It follows that $\lambda(\tau)$ is bounded for small τ, but we will need to quantify this property in terms of the parameter α. Recall that $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$, and $\mathbf{E}e^{\alpha|X|} \leqslant 2$ $(\alpha > 0)$.

Proposition 7. *We have*

$$\big|\lambda(\tau)\big| \leqslant 700\alpha^{-3}, \quad |\tau| \leqslant \frac{\alpha^3}{64}. \tag{5.14}$$

Proof. Proposition 5 allows us to apply Cauchy's formula, which yields

$$z_0''(\tau) = \frac{4}{2\pi \mathrm{i}} \int\limits_{|\xi-\tau|=r} \frac{z_0(\xi)}{(\xi-\tau)^3}\, \mathrm{d}\xi$$

with $r = \alpha^3/64$. Moreover, by (5.3), the latter implies

$$\big|z_0''(\tau)\big| \leqslant \frac{4}{r^2} \max_{|\xi-\tau|=r} \big|z_0(\xi)\big| \leqslant \frac{4}{r^2}\cdot\frac{\alpha^3}{16} = 2^{12}\alpha^{-3}. \tag{5.15}$$

Next, we note that, by Definition 1 of the saddle point (see (5.1)), the function

$$\psi(\tau) = K\big(z_0(\tau)\big) - \tau z_0(\tau) + \frac{1}{2}\tau^2$$

has the first three derivatives

$$\psi'(\tau) = K'\big(z_0(\tau)\big)z_0'(\tau) - z_0(\tau) - \tau z_0'(\tau) + \tau = \tau - z_0(\tau),$$
$$\psi''(\tau) = 1 - z_0'(\tau), \qquad \psi'''(\tau) = -z_0''(\tau).$$

Since $\psi(0) = \psi'(0) = \psi''(0) = 0$, we may apply the Taylor integral formula together with (5.15) to conclude that

$$\big|\psi(\tau)\big| \leqslant \frac{|\tau|^3}{6} \max_{|\xi|\leqslant|\tau|} \big|\psi'''(\xi)\big| \leqslant \frac{|\tau|^3}{6} \max_{|\xi|\leqslant|\tau|} \big|z_0''(\xi)\big| \leqslant \frac{1}{6}\cdot 2^{12}\,\alpha^{-3}.$$

As $\psi(\tau) = \tau^3\lambda(\tau)$, relation (5.14) follows. □

Let us now introduce another analytic function which appears in representation (1.6) of Theorem 2.

Proposition 8. *The function*

$$\mu(\tau) = \frac{1}{2}\log K''\big(z_0(\tau)\big)$$

is well defined and analytic in the disc $|\tau| \leqslant \alpha^3/32$. Moreover, as $\tau \to 0$,

$$\mu(\tau) = \frac{1}{2(m-2)!}\gamma_m \tau^{m-2} + O\big(|\tau|^{m-1}\big). \tag{5.16}$$

Proof. By (5.3), $|z_0(\tau)| \leqslant \alpha^3/16$. Hence, by Lemma 2, $K''(z_0(\tau))$ takes values in the disc with center at 1 of radius $1/2$. Thus the principal value of $\log K''(z_0(\tau))$ is well defined and represents an analytic function in $|\tau| \leqslant \alpha^3/32$. Moreover, by (5.8)–(5.9),

$$K''\big(z_0(\tau)\big) = 1 + \frac{\gamma_m}{(m-2)!} z_0(\tau)^{m-2} + O\big(\big|z_0(\tau)\big|^{m-1}\big)$$
$$= 1 + \frac{\gamma_m}{(m-2)!}\tau^{m-2} + O\big(|\tau|^{m-1}\big).$$

Taking the logarithm of this expression, we arrive at (5.16). □

Let us also mention that the function $K(z)$ is convex and has a positive second derivative on the real line, more precisely, on the interval where it is finite. Hence $\mu(\tau)$ is real-valued for real τ.

6 Contour integration

Let $(X_n)_{n\geqslant 1}$ be independent copies of a random variable X with $\mathbf{E}X = 0$, $\mathrm{Var}(X) = 1$, and characteristic function $f(t) = \mathbf{E}\mathrm{e}^{\mathrm{i}tX}$. We now consider the normalized sum

$$Z_n = \frac{X_1 + \cdots + X_n}{\sqrt{n}}$$

assuming that $M = M(Z_{n_0})$ is finite. As already discussed in Section 2, in this case, all Z_n with $n \geqslant 2n_0$ have continuous bounded densities expressed by the inversion formula

$$p_n(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx} f_n(t)\,\mathrm{d}t, \quad x \in \mathbb{R},$$

where

$$f_n(t) = f^n\biggl(\frac{t}{\sqrt{n}}\biggr)$$

denotes the characteristic functions of Z_n. Equivalently,

$$p_n(x) = \frac{\sqrt{n}}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}tx\sqrt{n}} f^n(t)\,\mathrm{d}t. \tag{6.1}$$

Using contour integration, we can cast this formula in a different form involving the log-Laplace transform $K(z) = \log \mathbf{E}\mathrm{e}^{zX}$ and the saddle point $z_0 = z_0(\tau)$ for the real value $\tau = x/\sqrt{n}$. This is a preliminary step towards Theorems 2 and 3.

As before, let $\mathbf{E}\mathrm{e}^{\alpha|X|} \leqslant 2$ with parameter $\alpha > 0$.

Lemma 5. *Let $n \geqslant 4n_0$. If $0 < \varepsilon \leqslant \alpha^3/16$ and $|\tau| \leqslant \varepsilon/2$, then*

$$p_n(x) = \frac{\sqrt{n}}{2\pi}\int_{-\varepsilon}^{\varepsilon} \exp\bigl\{n\bigl(K(z_0+\mathrm{i}t) - \tau(z_0+\mathrm{i}t)\bigr)\bigr\}\,\mathrm{d}t + \theta R_n \tag{6.2}$$

with $|\theta| \leqslant 1$ and

$$R_n = 5M\exp\biggl\{-\frac{n\varepsilon^2}{Cn_0M^2}\biggr\}, \quad C = 5200. \tag{6.3}$$

Proof. Applying Corollary 2, we get from (6.1) that for any $\varepsilon \in (0,1]$,

$$\biggl|p_n(x) - \frac{\sqrt{n}}{2\pi}\int_{-\varepsilon}^{\varepsilon} \mathrm{e}^{-\mathrm{i}tx\sqrt{n}} f^n(t)\,\mathrm{d}t\biggr| \leqslant 4M\exp\biggl\{-\frac{n\varepsilon^2}{Cn_0M^2}\biggr\} \tag{6.4}$$

with $C = 5200$.

Due to the assumption on ε, we may apply Lemma 3, which gives

$$\big|f(\pm\varepsilon + \mathrm{i}y)\big| \leqslant \mathrm{e}^{-\varepsilon^2/5} \quad \text{whenever } |y| \leqslant \frac{\varepsilon}{2}. \tag{6.5}$$

Assuming for definiteness that $x \geqslant 0$, we take the rectangle contour

$$L = L_1 + L_2 + L_3 + L_4$$

with segment parts

$$L_1 = [-\varepsilon,\, \varepsilon], \qquad L_2 = [\varepsilon,\, \varepsilon - \mathrm{i}h],$$
$$L_3 = [\varepsilon - \mathrm{i}h,\, -\varepsilon - \mathrm{i}h], \qquad L_4 = [-\varepsilon - \mathrm{i}h,\, -\varepsilon],$$

where $h > 0$ is chosen to satisfy $h \leqslant \varepsilon/2$. With this choice, the complex numbers $z = t + \mathrm{i}y$ with $|t| \leqslant \varepsilon$ and $|y| \leqslant h$ lie in the domain of the definition of $K(z)$. Then, by Cauchy's theorem,

$$\int_{L_1} \mathrm{e}^{-\mathrm{i}zx\sqrt{n}} f^n(z)\,\mathrm{d}z + \int_{L_2} \mathrm{e}^{-\mathrm{i}zx\sqrt{n}} f^n(z)\,\mathrm{d}z,$$
$$\int_{L_3} \mathrm{e}^{-\mathrm{i}zx\sqrt{n}} f^n(z)\,\mathrm{d}z + \int_{L_4} \mathrm{e}^{-\mathrm{i}zx\sqrt{n}} f^n(z)\,\mathrm{d}z = 0.$$

Note that in the lower half-plane $z = t - \mathrm{i}y$, $0 \leqslant y \leqslant h$, we have $|\mathrm{e}^{-\mathrm{i}zx\sqrt{n}}| = \mathrm{e}^{-yx\sqrt{n}} \leqslant 1$. Moreover, $|f(z)|$ is bounded away from 1 on L_2 and L_4 according to (6.5), which gives

$$\Big|\int_{L_2}\Big| + \Big|\int_{L_4}\Big| \leqslant \varepsilon \mathrm{e}^{-n\varepsilon^2/5} \leqslant \frac{1}{16}\mathrm{e}^{-n\varepsilon^2/5}.$$

To simplify, note that

$$4M \exp\Big\{-\frac{n\varepsilon^2}{Cn_0M^2}\Big\} + \frac{1}{16}\mathrm{e}^{-n\varepsilon^2/5} \leqslant R_n,$$

where we used $M \geqslant 1/12$ (see Remark 1). Combining this bound with (6.4), we arrive at

$$p_n(x) = \frac{\sqrt{n}}{2\pi}\int_{L_3} \mathrm{e}^{-\mathrm{i}zx\sqrt{n}} f^n(z)\,\mathrm{d}z + \theta R_n = \frac{\sqrt{n}}{2\pi}\int_{-\varepsilon}^{\varepsilon} \mathrm{e}^{-\mathrm{i}(t-\mathrm{i}h)x\sqrt{n}} f^n(t - \mathrm{i}h)\,\mathrm{d}t + \theta R_n.$$

Using the log-Laplace transform, let us rewrite the above as a contour integral

$$p_n(x) = \frac{\sqrt{n}}{2\pi\mathrm{i}}\int_{h-\mathrm{i}\varepsilon}^{h+\mathrm{i}\varepsilon} \exp\big\{n(K(z) - \tau z)\big\}\,\mathrm{d}z + \theta R_n$$

with $\tau = x/\sqrt{n}$ and apply it with $h = z_0 = z_0(\tau)$. Due to the requirement $0 \leqslant \tau \leqslant \varepsilon/2$, we have $0 \leqslant \tau \leqslant \alpha^3/32$ and $0 \leqslant z_0 \leqslant \alpha^3/16$ according to (5.3), so that Proposition 5 is applicable. After the change of variable, we thus obtain (6.2)–(6.3). $\square$

As a next step, let us show that, at the expense of a small error, the integration in (6.2) may be restricted to the interval $|t| \leqslant t_n$ with

$$t_n = n^{-1/2}\sqrt{8\log n}, \quad n \geqslant 4n_0.$$

This can be achieved under stronger conditions such as

$$|\tau| \leqslant \frac{\varepsilon}{80\, M^2 n_0}, \quad 0 \leqslant \varepsilon \leqslant \frac{\alpha^3}{80}. \tag{6.6}$$

Indeed, using (5.10) and (5.12) in the representation (6.2), we may rewrite (6.2) as

$$p_n(x) = \frac{\sqrt{n}}{2\pi}\int_{-\varepsilon}^{\varepsilon} \exp\left\{ n\sum_{k=2}^{\infty} \rho_k \frac{(\mathrm{i}t)^k}{k!} \right\} \mathrm{d}t\, \mathrm{e}^{n(-\tau^2/2+\tau^3\lambda(\tau))} + \theta R_n,$$

where $\tau = x\sqrt{n}$ and $\rho_k = K^{(k)}(z_0)$. Equivalently,

$$\frac{p_n(x)}{\varphi(x)} = \frac{\sqrt{n}}{\sqrt{2\pi}}\mathrm{e}^{n\tau^3\lambda(\tau)} \int_{-\varepsilon}^{\varepsilon} \exp\left\{ n\sum_{k=2}^{\infty} \rho_k \frac{(\mathrm{i}t)^k}{k!} \right\} \mathrm{d}t + \theta R_n \mathrm{e}^{x^2/2}. \tag{6.7}$$

Here the new remainder term

$$R_n \mathrm{e}^{x^2/2} = 5M \exp\left\{ -\frac{\varepsilon^2 n}{CM^2 n_0} + \tau^2 n \right\}, \quad C = 5200,$$

is still exponentially small with respect to n due to the first condition in (6.6), which strengthens the assumption $|\tau| \leqslant \varepsilon/2$ in Lemma 5 (recall that $M \geqslant 1/12$). In this case the expression in the exponent will be of order $-cn\varepsilon^2/(n_0 M^2)$ up to an absolute constant $c > 0$. Hence (6.7) yields

$$\frac{p_n(x)}{\varphi(x)} = \frac{\sqrt{n}}{\sqrt{2\pi}}\mathrm{e}^{n\tau^3\lambda(\tau)} \int_{-\varepsilon}^{\varepsilon} \exp\left\{ n\sum_{k=2}^{\infty} \rho_k \frac{(\mathrm{i}t)^k}{k!} \right\} \mathrm{d}t + \theta R_n, \tag{6.8}$$

where

$$R_n = 5M \exp\left\{ -\frac{c\varepsilon^2 n}{M^2 n_0} \right\}. \tag{6.9}$$

Now, by Lemma 2,

$$\frac{1}{2} \leqslant \rho_2 \leqslant \frac{3}{2}, \qquad |\rho_k| \leqslant 3k! \left(\frac{4}{\alpha}\right)^k \quad (k \geqslant 3). \tag{6.10}$$

It follows that

$$\begin{aligned} \mathrm{Re}\left(\sum_{k=2}^{\infty} \rho_k \frac{(\mathrm{i}t)^k}{k!} \right) &= -\rho_2 \frac{t^2}{2} + \sum_{k=2}^{\infty} (-1)^k \rho_{2k} \frac{t^{2k}}{(2k)!} \leqslant -\frac{1}{4}t^2 + 3\sum_{k=2}^{\infty} \left(\frac{4t}{\alpha}\right)^{2k} \\ &= -\frac{1}{4}t^2 + 3\big(16t^2\big) \sum_{k=2}^{\infty} \frac{(4t)^{2k-2}}{\alpha^{2k}} \leqslant -\frac{1}{8}t^2, \end{aligned}$$

where we used $\alpha < 1$ and $|t| \leqslant \varepsilon \leqslant \alpha^3/80$ so as to bound the last sum according to the second assumption in (6.6). Hence, when restricted to $|t| \geqslant t_n$, the absolute value of the integral in (6.8) does not exceed

$$2\int_{t_n}^{\infty} \mathrm{e}^{-nt^2/8}\,\mathrm{d}t = \frac{4}{\sqrt{n}}\int_{t_n\sqrt{n}/2}^{\infty} \mathrm{e}^{-s^2/2}\,\mathrm{d}s < \frac{2\sqrt{2\pi}}{\sqrt{n}}\mathrm{e}^{-nt_n^2/8} = \frac{2\sqrt{2\pi}}{n^{3/2}}.$$

As a result, assuming the conditions (6.6),

$$\frac{p_n(x)}{\varphi(x)} = \frac{\sqrt{n}}{\sqrt{2\pi}}\mathrm{e}^{n\tau^3\lambda(\tau)}\int_{|t|\leqslant t_n'} \exp\bigg\{n\sum_{k=2}^{\infty}\rho_k\frac{(\mathrm{i}t)^k}{k!}\bigg\}\,\mathrm{d}t + 2\theta_1 n^{-1}\mathrm{e}^{n\tau^3\lambda(\tau)} + \theta_2 R_n, \tag{6.11}$$

where $t_n' = \min(t_n, \varepsilon)$, $|\theta_j| \leqslant 1$, and where R_n is now defined in (6.9).

7 Proof of Theorem 3

As a final step, we need to explore an asymptotic behavior of the integral in (6.11), where we recall that $\rho_k = K^{(k)}(z_0)$, $z_0 = z_0(\tau)$ being the saddle point for $\tau = x/\sqrt{n}$. In view of the conditions in (6.6), we choose

$$\varepsilon = \frac{\alpha^3}{80}, \qquad \tau_0 = \frac{\varepsilon}{80M^2 n_0} = \frac{c_0\alpha^3}{M^2 n_0}$$

with $c_0 = 1/6400$. Note that with this choice the definition (6.9) becomes

$$R_n = 5M\exp\bigg\{-\frac{c_1\alpha^6 n}{M^2 n_0}\bigg\}, \tag{7.1}$$

where $c_1 > 0$ is an absolute constant. Suppose that $|\tau| \leqslant c\tau_0$ with a constant $0 < c \leqslant 1$ to be chosen later on.

The integrand in (6.11) may be written as

$$u_n(t) = \exp\bigg\{-n\rho_2\frac{t^2}{2} + n\rho_3\frac{(\mathrm{i}t)^3}{6} + nv(t)\bigg\}$$

with

$$v(t) = \sum_{k=4}^{\infty}\rho_k\frac{(\mathrm{i}t)^k}{k!}.$$

First, assume that $n \geqslant n_1 = \max(4n_0, \varepsilon^{-4})$, which insures that $t_n' = t_n$ (since $\varepsilon < 1/80$). As $|t| \leqslant t_n$, from (6.10) it follows that

$$nv(t) = O\big(nt^4\big) = B\alpha^{-4}\frac{(\log n)^2}{n}. \tag{7.2}$$

Here and below, B denotes a quantity, perhaps different in different places, bounded by an absolute constant. With this convention, since $n \geqslant (80/\alpha^3)^4$, we also have $nv(t) = B$, and by (6.10) with $k = 3$,

$$n\rho_3 t^3 = B\alpha^{-3}\frac{(\log n)^{3/2}}{\sqrt{n}} = B. \tag{7.3}$$

So we may use the Taylor expansion $\mathrm{e}^x = 1 + x + O(x^2)$ in a bounded interval $|x| \leqslant B$ with

$$x = n\rho_3 \frac{(\mathrm{i}t)^3}{6} + nv(t).$$

From (7.2)–(7.3), using again $n \geqslant B\alpha^{-12}$ together with $\alpha < 1$, we have

$$(nv(t))^2 = B\alpha^{-8} \frac{1}{n} \frac{(\log n)^4}{n} = \frac{B}{n}$$

and

$$\big(n\rho_3 t^3\big) \cdot \big(nv(t)\big) = B\alpha^{-7} \frac{1}{n} \frac{(\log n)^{7/2}}{\sqrt{n}} = \frac{B\alpha^{-2}}{n}.$$

Since

$$\big(n\rho_3 t^3\big)^2 = B\alpha^{-6} \frac{(\log n)^3}{n},$$

which dominates (7.2) and the previous two expressions, we obtain that

$$\begin{aligned} u_n(t) &= \mathrm{e}^{-n\rho_2 t^2/2 + x} = \mathrm{e}^{-n\rho_2 t^2/2}\big(1 + x + Bx^2\big) \\ &= \mathrm{e}^{-n\rho_2 t^2/2}\bigg(1 + n\rho_3 \frac{(\mathrm{i}t)^3}{6} + B\alpha^{-6} \frac{(\log n)^3}{n}\bigg). \end{aligned}$$

Hence

$$\int\limits_{|t| \leqslant t_n} u_n(t)\,\mathrm{d}t = \bigg(1 + B\alpha^{-6} \frac{(\log n)^3}{n}\bigg) \int\limits_{|t| \leqslant t_n} \mathrm{e}^{-n\rho_2 t^2/2}\,\mathrm{d}t,$$

and (6.11) is simplified to

$$\begin{aligned} \frac{p_n(x)}{\varphi(x)} = \frac{\sqrt{n}}{\sqrt{2\pi}} \mathrm{e}^{n\tau^3 \lambda(\tau)} \bigg(1 + B\alpha^{-6} \frac{(\log n)^3}{n}\bigg) \int\limits_{|t| \leqslant t_n} \mathrm{e}^{-n\rho_2 t^2/2}\,\mathrm{d}t \\ + 2\theta_1 n^{-1} \mathrm{e}^{n\tau^3 \lambda(\tau)} + \theta_2 R_n. \end{aligned} \tag{7.4}$$

Next, we may extend the integration in (7.4) to the whole real line at the expense of an error not exceeding

$$\begin{aligned} \int\limits_{|t| > t_n} \mathrm{e}^{-n\rho_2 t^2/2}\,\mathrm{d}t &= \frac{2}{\sqrt{\rho_2 n}} \int\limits_{t_n\sqrt{\rho_2 n}}^{\infty} \mathrm{e}^{-s^2/2}\,\mathrm{d}s < \frac{\sqrt{2\pi}}{\sqrt{\rho_2 n}} \mathrm{e}^{-n\rho_2 t_n^2/2} \\ &\leqslant \frac{2\sqrt{\pi}}{\sqrt{n}} \mathrm{e}^{-nt_n^2/4} = \frac{2\sqrt{\pi}}{\sqrt{n}} n^{-2}, \end{aligned}$$

where we used $\rho_2 \geqslant 1/2$. The latter bound is dominated by $B\alpha^{-6}(\log n)^3/n$, and since the integral over the whole real line is equal to $\sqrt{2\pi}/\sqrt{\rho_2 n}$, we obtain from (7.4) a simpler representation

$$\frac{p_n(x)}{\varphi(x)} = \frac{1}{\sqrt{\rho_2}} \mathrm{e}^{n\tau^3 \lambda(\tau)} \bigg(1 + B\alpha^{-6} \frac{(\log n)^3}{n}\bigg) + Bn^{-1} \mathrm{e}^{n\tau^3 \lambda(\tau)} + BR_n.$$

Here the first remainder term may be absorbed in the brackets, so that this formula is further simplified to

$$\frac{p_n(x)}{\varphi(x)} = \frac{1}{\sqrt{\rho_2}} e^{n\tau^3\lambda(\tau)} \bigg(1 + B\alpha^{-6}\frac{(\log n)^3}{n} + Be^{-n\tau^3\lambda(\tau)}R_n\bigg). \tag{7.5}$$

Moreover, we may eliminate the factor $e^{-n\tau^3\lambda(\tau)}$ in front of R_n by choosing a smaller value of c_1 in (7.1) for a proper absolute constant c appearing in the assumption $|\tau| \leqslant c\tau_0$. To this end, it is sufficient to require that

$$\big|\psi(\tau)\big| \leqslant \frac{c_1\alpha^6}{2M^2 n_0}, \quad \psi(\tau) = \tau^3\lambda(\tau).$$

Recall that, by Proposition 7, $|\psi(\tau)| \leqslant 700\alpha^{-3}|\tau|^3$ in the interval $|\tau| \leqslant \alpha^3/64$ (which is larger than $|\tau| \leqslant \tau_0$). Hence the above bound holds as long as

$$|\tau| \leqslant \frac{c_1^{1/3}}{(1400M^2 n_0)^{1/3}}\alpha^3. \tag{7.6}$$

Since $M \geqslant 1/12$, we have $(M^2 n_0)^{1/3} \leqslant 12^{4/3}M^2 n_0$. Hence (7.6) may be strengthened to $|\tau| \leqslant c\tau_0$ with a suitable constant $c > 0$. Under this condition, from (7.5) we thus get

$$\frac{p_n(x)}{\varphi(x)} = \frac{1}{\sqrt{\rho_2}} e^{n\tau^3\lambda(\tau)} \bigg(1 + B\alpha^{-6}\frac{(\log n)^3}{n} + BR_n\bigg), \tag{7.7}$$

where R_n is still defined as in (7.1) with a new constant c_1.

It is now useful to note that the last error term in this representation is dominated by the second last one for sufficiently large n. Indeed, using $e^{-y} \leqslant 2/y^2$ $(y > 0)$, we have

$$R_n \leqslant 2\bigg(\frac{M^2 n_0}{c_1\alpha^6 n}\bigg)^2 \leqslant \alpha^{-6}\frac{1}{n},$$

where the last inequality holds for $n \geqslant CM^4 n_0^2\alpha^{-12}$ with an absolute constant $C > 0$. This condition is slightly stronger than $n \geqslant n_1$, which was assumed before. As a result, (7.7) then yields

$$\frac{p_n(x)}{\varphi(x)} = \frac{1}{\sqrt{\rho_2}} e^{n\tau^3\lambda(\tau)} \bigg(1 + B\alpha^{-6}\frac{(\log n)^3}{n}\bigg).$$

It remains to recall Proposition 8, according to which

$$\rho_2^{-1/2} = K''\big(z_0(\tau)\big)^{-1/2} = e^{-\mu(\tau)},$$

and then we arrive at the statement in Theorem 3 (which is a refinement of Theorem 2).

Let us also note that the case $2n_0 \leqslant n < n_1$ is not interesting, since then $|x| \leqslant \tau_0 n_1$, and (1.6) holds by choosing a suitable constant in O in (1.6).

8 Proof of Corollary 1

Starting from (5.13) and (5.16), we have

$$n\tau^3\lambda(\tau) - \mu(\tau) = \frac{n}{m!}\gamma_m\tau^m + O\bigl(|\tau|^{m+1}\bigr) - \frac{1}{2(m-2)!}\gamma_m\tau^{m-2} + O\bigl(|\tau|^{m-1}\bigr)$$
$$= \frac{\gamma_m}{m!}\tau^{m-2}\Lambda(\tau),$$

where

$$\Lambda(\tau) = n\tau^2 - \frac{m(m-1)}{2} + nO\bigl(\tau^3\bigr) + O(\tau) \geqslant \frac{1}{2}\biggl[n\tau^2 - \frac{m(m-1)}{2}\biggr],$$

which is bounded away from zero if $|\tau| \leqslant \tau_1$ for some constant $\tau_1 > 0$ and $n\tau^2 = x^2 \geqslant m^2$. In this case, (1.6) immediately yields the desired relation (1.7).

In the remaining bounded interval $|x| \leqslant m$, this argument does not work, and it is better to employ the Chebyshev–Edgeworth expansion for the correction $\varphi_m(x)$ in (1.1) (which depends on n as well). In terms of the first nonzero cumulant, (1.1) may be written more accurately as

$$p_n(x) = \varphi(x) + \frac{\gamma_m}{m!}H_m(x)\varphi(x)n^{-(m-2)/2} + \frac{1}{1+|x|^m}o\bigl(n^{-(m-2)/2}\bigr),$$

where $H_m(x)$ denotes the Chebyshev–Hermite polynomial of degree m. As a consequence, for any constant $x_0 > 0$,

$$\sup_{|x|\leqslant x_0} \frac{|p_n(x) - \varphi(x)|}{\varphi(x)} = O\bigl(n^{-(m-2)/2}\bigr),$$

which is stronger than (1.7), since m is even ($m \geqslant 4$).

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