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Green's function and existence of solutions for a third-order boundary value problem involving integral condition

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Abstract. We prove the existence of at least one nontrivial solution for a third-order boundary value problem with an integral condition under different growth assumptions on the nonlinearity in the equation. The main tool in the proofs is Schauder's fixed point theorem. To compare the applicability of the obtained results, we consider some examples.

MSC: 34B15

Keywords: third-order nonlinear boundary value problems, integral boundary condition, existence of solutions, Green's function, Schauder's fixed point theorem

1 Introduction

We study the boundary value problem consisting of the nonlinear third-order differential equation

$$x''' + f(t,x) = 0, \quad t \in (0,1), \tag{1.1}$$

and the boundary conditions

$$x(0) = 0, \quad x'(0) = 0, \qquad x(1) = \int_0^1 g(s)x(s)\,\mathrm{d}s. \tag{1.2}$$

In what follows we assume that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(t,0) \not\equiv 0$ for $t \in [0,1]$, $g : [0,1] \to \mathbb{R}$ is continuous, and $\int_0^1 s^2 g(s)\,\mathrm{d}s \neq 1$. The assumption $f(t,0) \not\equiv 0$ excludes the possibility of the trivial solution.

By a solution of (1.1)–(1.2) we understand function $x \in C^3[0,1]$ that satisfies differential equation (1.1) for $0 < t < 1$ and boundary conditions (1.2).

The purpose of the paper is giving and comparing the results on the existence of nontrivial solutions to (1.1)–(1.2) under different growth conditions on the nonlinearity f by applying Schauder's fixed point theorem. To obtain these results, we first rewrite problem (1.1)–(1.2) as an equivalent integral equation by constructing the corresponding Green function. The problem then becomes to show that an integral operator has a fixed point in some set of functions.

Among an immense number of papers dealing with nonlinear differential equations subject to a variety of boundary conditions, the author would like to mention some recent achievements. Applying the upper and lower solution method and Schauder's fixed point theorem, the existence of solutions for a third-order three-point boundary value problem was proved in [3]. Using the vector field rotation theory, results on the existence of at least one nontrivial solution to a third-order system with two-point boundary conditions were established in [6]. In [7] the authors, using barrier strip-type conditions, give sufficient conditions for the existence of positive or nonnegative, monotone, convex or concave solutions for a third-order two-point boundary value problem. The existence of multiple positive solutions of nonlinear second-order nonlocal boundary value problems with a nonlinear term having derivative dependence was proved in [15]. In [14] the authors obtain asymptotic formulas for eigenvalues and eigenfunctions of the one-dimensional Sturm–Liouville equation with one classical-type Dirichlet boundary condition and integral-type nonlocal boundary condition. The Sturm–Liouville problem with one classical and another nonlocal two-point boundary condition was investigated in [2].

Much research has been done on boundary value problems with nonlocal and integral conditions in the last decades. Note the papers [1, 4, 5, 10, 11, 12, 16, 17, 19], the authors of which intensively investigated such problems for many years.

The study of the existence of solutions to boundary value problems is often associated with the construction of the corresponding Green functions. Thus Green's functions play an important role in the theory of boundary value problems. A survey of results on Green's functions for stationary problems with nonlocal boundary conditions is presented in [13].

Since our main tool in this paper is Schauder's fixed point theorem, let us state this theorem for the convenience of the reader.

Theorem 1. (See [18].) *Let X be a Banach space, and let $M \subset X$ be a bounded, closed, and convex subset of X. Let $T : M \to M$ be a completely continuous operator. Then T has a fixed point in M.*

Schauder's fixed point theorem is a powerful tool in the study of solvability of boundary value problems. In recent years, by applying Schauder's fixed point theorem many authors have been studied certain boundary value problems; for example, see [3, 4, 8, 9].

Despite the extensive literature on third-order boundary value problems with nonlocal conditions, there are a lot of points to be investigated and improved. Therefore the present paper is an attempt to obtain new results in this field.

The paper contains three sections besides Introduction. In Section 2, we rewrite the main problem as an equivalent integral equation by constructing the corresponding Green function. Also, we give some inequalities for the Green function. In Section 3, we prove our main theorems on the existence of a nontrivial solution to the problem. Finally, in Section 4, we consider some examples to illustrate and compare the applicability of our results.

2 Construction and estimation of Green's function

The goal of this section is to rewrite problem (1.1)–(1.2) as an equivalent integral equation. So let us consider the linear equation

$$x''' + h(t) = 0, \quad t \in (0, 1), \tag{2.1}$$

together with boundary conditions (1.2).

Proposition 1. *Let $h : [0, 1] \to \mathbb{R}$ be a continuous function. Then the function defined by*

$$x(t) = \int_0^1 H(t, s)h(s)\,\mathrm{d}s$$

is the unique solution of boundary value problem (2.1), (1.2)*, where*

$$H(t,s) = G(t,s) + \frac{t^2}{1 - \int_0^1 s^2 g(s)\,\mathrm{d}s} \int_0^1 G(\xi,s) g(\xi)\,\mathrm{d}\xi, \tag{2.2}$$

and

$$G(t,s) = \frac{1}{2}\begin{cases} s(1-t)((1-s)t + (t-s)), & 0 \leqslant s \leqslant t \leqslant 1,\\ t^2(1-s)^2, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases} \tag{2.3}$$

Proof. Let $x(t)$ be a solution of problem (2.1)–(1.2). Integrating equation (2.1) thrice, we get

$$x(t) = x(0) + tx'(0) + \frac{1}{2}t^2\,x''(0) - \frac{1}{2}\int_0^t (t-s)^2 h(s)\,\mathrm{d}s,$$

and, in view of boundary conditions (1.2), we obtain

$$x(t) = \frac{1}{2}t^2 x''(0) - \frac{1}{2}\int_0^t (t-s)^2 h(s)\,\mathrm{d}s.$$

Since $x(1) = \int_0^1 g(s)x(s)\,\mathrm{d}s$, it follows that

$$x(1) = \frac{1}{2}x''(0) - \frac{1}{2}\int_0^1 (1-s)^2 h(s)\,\mathrm{d}s = \int_0^1 g(s)x(s)\,\mathrm{d}s$$

or

$$x''(0) = \int_0^1 (1-s)^2 h(s)\,\mathrm{d}s + 2\int_0^1 g(s)x(s)\,\mathrm{d}s.$$

Therefore

$$\begin{aligned} x(t) &= \frac{1}{2}\int_0^1 t^2(1-s)^2 h(s)\,\mathrm{d}s + \int_0^1 t^2 g(s)x(s)\,\mathrm{d}s - \frac{1}{2}\int_0^t (t-s)^2 h(s)\,\mathrm{d}s \\ &= \int_0^1 t^2 g(s)x(s)\,\mathrm{d}s + \frac{1}{2}\int_0^t s(1-t)\big((1-s)t + (t-s)\big)h(s)\,\mathrm{d}s + \frac{1}{2}\int_t^1 t^2(1-s)^2 h(s)\,\mathrm{d}s \\ &= \int_0^1 t^2 g(s)x(s)\,\mathrm{d}s + \int_0^1 G(t,s)h(s)\,\mathrm{d}s. \end{aligned} \tag{2.4}$$

Multiplying both sides of (2.4) by g and integrating over $(0,1)$, we get

$$\int_0^1 g(s)x(s)\,\mathrm{d}s = \int_0^1 \left(s^2 g(s)\int_0^1 g(\xi)x(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s + \int_0^1 \left(g(s)\int_0^1 G(s,\xi)h(\xi)\,\mathrm{d}\xi\right)\mathrm{d}s$$
$$= \int_0^1 s^2 g(s)\,\mathrm{d}s \cdot \int_0^1 g(\xi)x(\xi)\,\mathrm{d}\xi + \int_0^1 \left(g(\xi)\int_0^1 G(\xi,s)h(s)\,\mathrm{d}s\right)\mathrm{d}\xi$$
$$= \int_0^1 s^2 g(s)\,\mathrm{d}s \cdot \int_0^1 g(s)x(s)\,\mathrm{d}s + \int_0^1 \left(\int_0^1 g(\xi)G(\xi,s)\,\mathrm{d}\xi\right)h(s)\,\mathrm{d}s.$$

This yields that

$$\int_0^1 g(s)x(s)\,\mathrm{d}s = \frac{\int_0^1(\int_0^1 g(\xi)G(\xi,s)\,\mathrm{d}\xi)h(s)\,\mathrm{d}s}{1-\int_0^1 s^2 g(s)\,\mathrm{d}s}$$

and

$$x(t) = \int_0^1 G(t,s)h(s)\,\mathrm{d}s + \frac{t^2}{1-\int_0^1 s^2 g(s)\,\mathrm{d}s}\cdot\int_0^1\left(\int_0^1 G(\xi,s)g(\xi)\,\mathrm{d}\xi\right)h(s)\,\mathrm{d}s. \qquad \square$$

Hence the boundary value problem (1.1)–(1.2) is equivalent to the integral equation

$$x(t) = \int_0^1 H(t,s)f\big(s,x(s)\big)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 1, \tag{2.5}$$

in the sense that x is a solution of (1.1)–(1.2) iff it is a solution of (2.5). Here $H(t,s)$ denotes Green's function for the problem $x''' = 0$ with boundary conditions (1.2) and is explicitly given by (2.2).

Remark 1. In the proof of Proposition 1, Green's function H was constructed directly. In [10] the reader can find another way of constructing Green's functions for such types of problems.

Now we prove some inequalities for the functions G and H.

Proposition 2. *For all* $(t,s) \in [0,1]\times[0,1] = \Omega$ *and* $G(t,s)$ *given by* (2.3)*, we have*

$$0 \leqslant G(t,s) \leqslant \frac{-11+5\sqrt{5}}{4}. \tag{2.6}$$

Proof. The first part of inequality (2.6) is obvious. For $0 \leqslant s \leqslant t \leqslant 1$, we have

$$\max_{\Omega} G(t,s) = \max_{\Omega}\left(\frac{1}{2}s(1-t)\big((1-s)t+(t-s)\big)\right) = \frac{-11+5\sqrt{5}}{4},$$

and for $0 \leqslant t \leqslant s \leqslant 1$,

$$\max_{\Omega} G(t,s) = \max_{\Omega}\left(\frac{1}{2}t^2(1-s)^2\right) = \frac{1}{32}. \qquad \square$$

Proposition 3. *The function* $G(t,s)$ *in* (2.3) *satisfies*

$$\int_0^1 G(t,s)\,\mathrm{d}s \leqslant \frac{2}{81} \quad \textit{for all } t \in [0,1].$$

Proof. Let $t \in [0,1]$ and consider

$$\begin{aligned}\int_0^1 G(t,s)\,\mathrm{d}s &= \frac{1}{2}\int_0^t s(1-t)\big((1-s)t+(t-s)\big)\,\mathrm{d}s + \frac{1}{2}\int_t^1 t^2(1-s)^2\,\mathrm{d}s \\ &= \frac{1}{6}(1-t)t^2 \leqslant \max_{t\in[0,1]}\left(\frac{1}{6}(1-t)t^2\right) = \frac{2}{81}. \qquad \square\end{aligned}$$

Proposition 4. *For all* $(t,s) \in [0,1]\times[0,1]$, *we have*

$$\big|H(t,s)\big| \leqslant \frac{-11+5\sqrt{5}}{4}\cdot h_0 \tag{2.7}$$

for $H(t,s)$ *given by* (2.2), *where* $h_0 = 1 + \int_0^1 |g(\xi)|\,\mathrm{d}\xi/(|1-\int_0^1 s^2 g(s)\,\mathrm{d}s|)$.

Proof. For $(t,s) \in [0,1]\times[0,1]$, we have

$$\begin{aligned}\big|H(t,s)\big| &= \left|G(t,s) + \frac{t^2}{1-\int_0^1 s^2 g(s)\,\mathrm{d}s}\int_0^1 G(\xi,s)g(\xi)\,\mathrm{d}\xi\right| \\ &\leqslant \big|G(t,s)\big| + \frac{t^2}{|1-\int_0^1 s^2 g(s)\,\mathrm{d}s|}\int_0^1 \big|G(\xi,s)\big|\big|g(\xi)\big|\,\mathrm{d}\xi \\ &\leqslant \frac{-11+5\sqrt{5}}{4}\left(1+\frac{\int_0^1 |g(\xi)|\,\mathrm{d}\xi}{|1-\int_0^1 s^2 g(s)\,\mathrm{d}s|}\right). \qquad \square\end{aligned}$$

Proposition 5. *The function* $H(t,s)$ *in* (2.2) *satisfies*

$$\int_0^1 \big|H(t,s)\big|\,\mathrm{d}s \leqslant \frac{2}{81}\cdot h_0 \quad \textit{for all } t \in [0,1]. \tag{2.8}$$

Proof. For $t \in [0,1]$, we have

$$\begin{aligned}\int_0^1 \big|H(t,s)\big|\,\mathrm{d}s &= \int_0^1 \left|G(t,s) + \frac{t^2}{1-\int_0^1 s^2 g(s)\,\mathrm{d}s}\int_0^1 G(\xi,s)g(\xi)\,\mathrm{d}\xi\right|\mathrm{d}s \\ &\leqslant \int_0^1 \big|G(t,s)\big|\,\mathrm{d}s + \frac{t^2}{|1-\int_0^1 s^2 g(s)\,\mathrm{d}s|}\int_0^1\left(\int_0^1 \big|G(\xi,s)g(\xi)\big|\,\mathrm{d}\xi\right)\mathrm{d}s\end{aligned}$$

$$= \int_0^1 \left|G(t,s)\right| \mathrm{d}s + \frac{t^2}{|1 - \int_0^1 s^2 g(s)\,\mathrm{d}s|} \int_0^1 \left|g(\xi)\right| \cdot \left(\int_0^1 \left|G(\xi,s)\right| \mathrm{d}s \right) \mathrm{d}\xi$$
$$\leqslant \frac{2}{81}\left(1 + \frac{\int_0^1 |g(\xi)|\,\mathrm{d}\xi}{|1 - \int_0^1 s^2 g(s)\,\mathrm{d}s|}\right). \qquad \square$$

3 Existence of solutions

In this section, we prove our main results on the existence of at least one nontrivial solution to problem (1.1)–(1.2) by applying Schauder's fixed point theorem. For our constructions, consider the Banach space $C[0,1]$ endowed with the norm

$$\|x\| = \max_{0 \leqslant t \leqslant 1} \left|x(t)\right|, \quad x \in C[0,1].$$

Theorem 2. *Let $q : [0,1] \to [0,+\infty)$ be a continuous function such that $q \not\equiv 0$ and*

$$\int_0^1 q(s)\,\mathrm{d}s = q_0 < +\infty.$$

Suppose that there exist a continuous function $F : \mathbb{R} \to [0,+\infty)$ and a constant $r > 0$ such that

$$\left|f(t,x)\right| \leqslant q(t)F(x) \quad \textit{for all } (t,x) \in [0,1] \times \mathbb{R}$$

and

$$\max_{|x| \leqslant r} F(x) \leqslant \frac{r}{\frac{-11+5\sqrt{5}}{4} h_0 q_0}.$$

Then problem (1.1)–(1.2) *has at least one nontrivial solution such that $|x(t)| \leqslant r$ for all $t \in [0,1]$.*

Proof. Let $B = \{x \in C[0,1]\colon \|x\| \leqslant r\}$ and define the operator $T : B \to C[0,1]$ by

$$(Tx)(t) = \int_0^1 H(t,s) f\bigl(s, x(s)\bigr)\,\mathrm{d}s, \quad t \in [0,1].$$

Since the boundary value problem (1.1)–(1.2) is equivalent to the integral equation (2.5), we need to prove that the operator has a fixed point. To establish the existence of a fixed point for T, we show that the conditions of Theorem 1 hold. A standard application of the Arzelà–Ascoli theorem guarantees that T is completely continuous. We only need that $T(B) \subset B$.

For all $t \in [0,1]$ and all $x \in B$, we have

$$\left|(Tx)(t)\right| \leqslant \int_0^1 \left|H(t,s)\right|\left|f\bigl(s,x(s)\bigr)\right| \mathrm{d}s \leqslant \int_0^1 \left|H(t,s)\right| q(s) F\bigl(x(s)\bigr)\,\mathrm{d}s$$
$$\leqslant \frac{-11+5\sqrt{5}}{4} \cdot h_0 \cdot \frac{r}{\frac{-11+5\sqrt{5}}{4} \cdot h_0 \cdot q_0} \int_0^1 q(s)\,\mathrm{d}s = r,$$

where we have used inequality (2.7). Thus $\|Tx\| \leqslant r$, and so $T(B) \subset B$. $\square$

Remark 2. The constant q_0 in Theorem 2 can be replaced by the constant $q_m = \max_{0\leqslant t\leqslant 1} q(t)$.

Theorem 3. *Suppose that there exist constants $\alpha > 0$, $\beta > 0$, and $r > 0$ such that*

$$\bigl|f(t,x)\bigr| \leqslant \alpha|x| + \beta \quad \textit{for all } (t,x) \in [0,1]\times[-r,r] = \Omega_1.$$

If

$$2\alpha h_0 < 81 \quad \textit{and} \quad r \geqslant \frac{2\beta h_0}{81 - 2\alpha h_0},$$

then problem (1.1)–(1.2) *has at least one nontrivial solution such that $|x(t)| \leqslant r$ for all $t \in [0,1]$.*

Proof. Let set B and operator T be as in the proof of the previous theorem. Therefore we need to show that $T(B) \subset B$.

For all $t \in [0,1]$ and $x \in B$, we have

$$\begin{aligned}
\bigl|(Tx)(t)\bigr| &\leqslant \int_0^1 \bigl|H(t,s)\bigr|\bigl|f\bigl(s,x(s)\bigr)\bigr|\,\mathrm{d}s \leqslant \max_{\Omega_1}\bigl|f(t,x)\bigr| \cdot \int_0^1 \bigl|H(t,s)\bigr|\,\mathrm{d}s \\
&\leqslant \bigl(\alpha\|x\| + \beta\bigr)\frac{2}{81}h_0 = \alpha\frac{2}{81}h_0\|x\| + \beta\frac{2}{81}h_0 \\
&\leqslant \alpha\frac{2}{81}h_0 r + r\biggl(1 - \alpha\frac{2}{81}h_0\biggr) = r,
\end{aligned}$$

where, we have used inequality (2.8). Thus $\|Tx\| \leqslant r$, and so $T(B) \subset B$. $\square$

4 Examples

In this section, we consider some examples to illustrate and compare the applicability of the obtained results.

Example 1. Consider the problem

$$x''' + t\bigl(34 + x^2\bigr) = 0, \quad t \in (0,1), \tag{4.1}$$

$$x(0) = 0, \quad x'(0) = 0, \qquad x(1) = \int_0^1 \bigl(1 + s^3\bigr)x(s)\,\mathrm{d}s. \tag{4.2}$$

The function $f(t,x) = t(34 + x^2)$ is continuous for $(t,x) \in [0,1]\times\mathbb{R}$, and $f(t,0) = 34t \not\equiv 0$ for $t \in [0,1]$. The function $g(t) = 1 + t^3$ is continuous for $t \in [0,1]$, $h_0 = 7/2$, and

$$\int_0^1 s^2 g(s)\,\mathrm{d}s = \int_0^1 s^2\bigl(1 + s^3\bigr)\,\mathrm{d}s = \frac{1}{2} \neq 1.$$

It is easy to see that for all $(t,x) \in [0,1]\times\mathbb{R}$,

$$\bigl|f(t,x)\bigr| = \bigl|t\bigl(34 + x^2\bigr)\bigr| \leqslant t\bigl(34 + x^2\bigr).$$

Thus $q(t) = t \geqslant 0$ for $t \in [0,1]$, and $\int_0^1 q(s)\,\mathrm{d}s = 1/2 = q_0$; $F(x) = 34 + x^2 > 0$ for $x \in \mathbb{R}$. Choose $r = 4$ and find $\max_{|x|\leqslant 4} F(x) = 34 + 4^2 = 50$. Therefore we have

$$\max_{|x|\leqslant r} F(x) = 50 \leqslant 50.7 \approx \frac{r}{\frac{-11+5\sqrt{5}}{4} h_0 q_0}.$$

Hence by Theorem 2 problem (4.1)–(4.2) has at least one nontrivial solution $x(t)$ such that $|x(t)| \leqslant 4$ for all $t \in [0,1]$.

Example 2. Consider the problem

$$x''' + \bigl(t^2+1\bigr)(x+\sqrt{2})^3 = 0, \quad t \in (0,1), \tag{4.3}$$

$$x(0) = 0, \quad x'(0) = 0, \qquad x(1) = \int_0^1 s^2 x(s)\,\mathrm{d}s. \tag{4.4}$$

The function $f(t,x) = (t^2+1)(x+\sqrt{2})^3$ is continuous for $(t,x) \in [0,1]\times\mathbb{R}$, and $f(t,0) = 2\sqrt{2}(t^2+1) \not\equiv 0$ for $t \in [0,1]$. The function $g(t) = t^2$ is continuous for $t \in [0,1]$, $h_0 = 17/12$, and

$$\int_0^1 s^2 g(s)\,\mathrm{d}s = \int_0^1 s^4\,\mathrm{d}s = \frac{1}{5} \neq 1.$$

For all $(t,x) \in [0,1] \times \mathbb{R}$, we have

$$\bigl|f(t,x)\bigr| = \bigl|\bigl(t^2+1\bigr)\bigl(x+\sqrt{2}\bigr)^3\bigr| \leqslant \bigl(t^2+1\bigr)|x+\sqrt{2}|^3.$$

Therefore $q(t) = t^2+1 > 0$ for $t \in [0,1]$, and $\int_0^1 q(s)\,\mathrm{d}s = 4/3 = q_0$; $F(x) = |x+\sqrt{2}|^3 \geqslant 0$ for $x \in \mathbb{R}$.

For all $r > 0$, we have

$$\max_{|x|\leqslant r} F(x) = (r+\sqrt{2})^3 > \frac{36r}{17(-11+5\sqrt{5})} = \frac{r}{\frac{-11+5\sqrt{5}}{4} h_0 q_0}.$$

Thus we cannot use Theorem 2 to establish the existence of solutions for problem (4.3)–(4.4), whereas Theorem 3 is applicable in this case.

Let $r = 1$. For all $(t,x) \in [0,1] \times [-1,1]$, we have

$$\bigl|f(t,x)\bigr| = \bigl|\bigl(t^2+1\bigr)(x+\sqrt{2})^3\bigr| \leqslant 10\sqrt{2}|x| + 10\sqrt{2}.$$

So $\alpha = \beta = 10\sqrt{2}$, and

$$\alpha\frac{2}{81}h_0 = \frac{85\sqrt{2}}{243} < 1, \qquad \frac{2\beta h_0}{81 - 2\alpha h_0} = \frac{85\sqrt{2}}{243 - 85\sqrt{2}} \leqslant 1.$$

Therefore by Theorem 3 problem (4.3)–(4.4) has at least one nontrivial solution $x(t)$ such that $|x(t)| \leqslant 1$ for all $t \in [0,1]$.

References

1. K. Bingelė, A. Bankauskienė, and A. Štikonas, Spectrum curves for a discrete Sturm–Liouville problem with one integral boundary condition, *Nonlinear Anal. Model. Control.*, **24**(5):755–774, 2019, https://doi.org/10.15388/NA.2019.5.5.

2. K. Bingelė, A. Bankauskienė, and A. Štikonas, Investigation of spectrum curves for a Sturm–Liouville problem with two-point nonlocal boundary conditions, *Math. Model. Anal.*, **25**(1):53–70, 2020, https://doi.org/10.3846/mma.2020.10787.

3. N. Bouteraa and S. Benaicha, Existence of solution for third-order three-point boundary value problem, *Mathematica*, **60**(83):21–31, 2018, https://doi.org/10.24193/mathcluj.2018.1.03.

4. J.R. Graef and T. Moussaoui, A class of nth-order BVPs with nonlocal conditions, *Comput. Math. Appl.*, **58**(8):1662–1671, 2009, https://doi.org/10.1016/j.camwa.2009.07.009.

5. J.R. Graef and J.R.L. Webb, Third order boundary value problems with nonlocal boundary conditions, *Nonlinear Anal. Theory Methods Appl.*, **71**(5-6):1542–1551, 2009, https://doi.org/10.1016/j.na.2008.12.047.

6. A. Gritsans and F. Sadyrbaev, A two-point boundary value problem for third order asymptotically linear systems, *Electron. J. Qual. Theory Differ. Equ.*, **28**:1–24, 2019, https://doi.org/10.14232/ejqtde.2019.1.28.

7. P.S. Kelevedjiev and T.Z. Todorov, Existence of solutions of nonlinear third-order two-point boundary value problems, *Electron. J. Qual. Theory Differ. Equ.*, **23**:1–15, 2019, https://doi.org/10.14232/ejqtde.2019.1.23.

8. R. Ma, Nonlinear periodic boundary value problems with sign-changing Green's function, *Nonlinear Anal. Theory Methods Appl.*, **74**(5):1714–1720, 2011, https://doi.org/10.1016/j.na.2010.10.043.

9. J. Rodriguez and P. Taylor, Multipoint boundary value problems for nonlinear ordinary differential equations, *Nonlinear Anal. Theory Methods Appl.*, **68**(11):3465–3474, 2008, https://doi.org/10.1016/j.na.2007.03.038.

10. S. Roman and A. Štikonas, Third-order linear differential equation with three additional conditions and formula for Green's function, *Lith. Math. J.*, **50**(4):426–446, 2010, https://doi.org/10.1007/s10986-010-9097-x.

11. M. Sapagovas, V. Griškonienė, and O. Štikonienė, Application of M-matrices theory to numerical investigation of a nonlinear elliptic equation with an integral condition, *Nonlinear Anal. Model. Control*, **22**(4):489–504, 2017, https://doi.org/10.15388/NA.2017.4.5.

12. M. Sapagovas, O. Štikonienė, K. Jakubėlienė, and R. Čiupaila, Finite difference method for boundary value problem for nonlinear elliptic equation with nonlocal conditions, *Bound. Value Probl.*, p. 94, 2019, https://doi.org/10.1186/s13661-019-1202-4.

13. A. Štikonas, A survey on stationary problems, Green's functions and spectrum of Sturm–Liouville problem with nonlocal boundary conditions, *Nonlinear Anal. Model. Control.*, **19**(3):301–334, 2014, https://doi.org/10.15388/NA.2014.3.1.

14. A. Štikonas and E. Şen, Asymptotic analysis of Sturm–Liouville problem with nonlocal integral-type boundary condition, *Nonlinear Anal. Model. Control*, **26**(5):969–991, 2021, https://doi.org/10.15388/namc.2021.26.24299.

15. J.R.L. Webb, Non-local second-order boundary value problems with derivative-dependent nonlinearity, *Philos. Trans. R. Soc. Lond., A, Math. Phys. Eng. Sci.*, **379**(2191):20190383, 2021, https://doi.org/10.1098/rsta.2019.0383.

16. J.R.L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems involving integral conditions, *NoDEA, Nonlinear Differ. Equ. Appl.*, **15**(1–2):45–67, 2008, https://doi.org/10.1007/s00030-007-4067-7.

17. J.R.L. Webb and G. Infante, Non-local boundary value problems of arbitrary order, *J. Lond. Math. Soc., II Ser.*, **79**(1):238–258, 2009, https://doi.org/10.1112/jlms/jdn066.

18. E. Zeidler, *Nonlinear Functional Analysis and Its Applications I. Fixed-Point Theorems*, Springer, New York, 1986.

19. J. Zhao, P. Wang, and W. Ge, Existence and nonexistence of positive solutions for a class of third order BVP with integral boundary conditions in Banach spaces, *Commun. Nonlinear Sci. Numer. Simul.*, **16**(1):402–413, 2011, https://doi.org/10.1016/j.cnsns.2009.10.011.