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On the closure under infinitely divisible distribution roots

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Abstract. For some $\gamma > 0$, we show that the distribution class $(\mathcal{L}(\gamma) \cap \mathcal{OS}) \setminus \mathcal{S}(\gamma)$ is not closed under infinitely divisible distribution roots, that is, we provide examples showing that some infinitely divisible distributions belong to this class but their corresponding Lévy distributions do not. To this end, we explore the structural properties of some distribution classes, give a positive conclusion to the Embrechts–Goldie conjecture, and study some properties of a transformation from a heavy-tailed distribution to a light-tailed one.

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1 Introduction and main results

In this paper, unless otherwise stated, we assume that all distributions are supported on $[0, \infty)$.

Let H be an infinitely divisible distribution with the Laplace transform

$$\int_{0-}^{\infty} \mathrm{e}^{-\lambda y}\, H(\mathrm{d}y) = \mathrm{e}^{-a\lambda - \int_0^\infty (1-\mathrm{e}^{-\lambda y})\, \nu(\mathrm{d}y)}, \quad \operatorname{Re} \lambda \geqslant 0,$$

where $a \geqslant 0$, and ν is the Lévy measure satisfying $\mu = \nu(\infty) = \nu((1, \infty)) < \infty$ and $\int_0^1 y\, \nu(\mathrm{d}y) < \infty$, which generates the Lévy distribution

$$F(x) = \frac{\nu(x)}{\mu} \mathbf{1}_{(1,\infty)}(x) = \frac{\nu((1, x])}{\mu} \mathbf{1}_{(1,\infty)}(x), \quad x \in (-\infty, \infty).$$

Then the distribution H admits the representation $H = H_1 * H_2$, the convolution of two distributions H_1 and H_2, where $\overline{H_1}(x) = 1 - H_1(x) = O(\mathrm{e}^{-\beta x})$ for all $\beta > 0$, and

$$H_2(x) = \mathrm{e}^{-\mu} \sum_{k=0}^{\infty} \frac{F^{*k}(x)\mu^k}{k!}, \quad x \in (-\infty, \infty).$$

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See, for example, Feller [11, pp. 450 and 571], Embrechts et al. [10], or Sato [16, Chap. 4]. Here H_2 is called the compound Poisson distribution generated by a Lévy distribution and a certain Poisson distribution as a special compound distribution or compound convolution.

If an infinitely divisible distribution in a certain distribution class can deduce its Lévy distribution belonging to the same class as well, then the class is said to be closed under infinitely divisible distribution roots. On the contrary, if the Lévy distribution in a certain distribution class can deduce its infinitely divisible distribution belonging to the same class as well, then the class is said to be closed under infinitely divisible distribution. For the recent results on the latter, see, for example, Watanabe [22] and Cui et al. [7]. In this paper, we study the former. Clearly, the research of this subject is closely related to distribution classes. Therefore we introduce some notions and notations of the commonly used distribution classes.

Here and thereafter, all limit relations refer to $x \to \infty$ without a special statement. Let two functions g_1 and g_2 be eventually positive. For $i \neq j$, $i, j = 1, 2$, we denote $g_{i,j} = \limsup g_i(x)/g_j(x)$. Then $g_i(x) = O(g_j(x))$ means that $g_{i,j} < \infty$, $g_i(x) \asymp g_j(x)$ means that $g_{i,j} < \infty$, $g_{j,i} < \infty$, $g_i(x) \lesssim g_j(x)$ means that $g_{i,j} \leqslant 1$, $g_1(x) \sim g_2(x)$ means that $g_{1,2} = g_{2,1} = 1$, and $g_i(x) = o(g_j(x))$ means that $g_{i,j} = 0$.

We say that a distribution F belongs to the distribution class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$ if $\overline{F}(x) > 0$ for all x and if

$$\overline{F}(x-t) \sim \mathrm{e}^{\gamma t}\overline{F}(x)$$

for each t. In this definition, if $\gamma > 0$ and the distribution F is lattice, then x and t should be restricted to values in the lattice span; see Bertoin and Doney [1, Def. 1].

Further, if a distribution F belongs to the class $\mathcal{L}(\gamma)$, $m(F) = \int_0^\infty \mathrm{e}^{\gamma y}\, F(\mathrm{d}y) < \infty$, and

$$\overline{F^{*2}}(x) \sim 2m(F)\overline{F}(x),$$

then we say that F belongs to the distribution class $\mathcal{S}(\gamma)$.

In particular, the classes $\mathcal{L} = \mathcal{L}(0)$ and $\mathcal{S} = \mathcal{S}(0)$ are called the long-tailed distribution class and the subexponential distribution class, respectively. Note that the requirement $F \in \mathcal{L}$ is not needed in the definition of the class $\mathcal{S}$. See Chistyakov [4, Lemma 2].

The class $\mathcal{S}(\gamma)$ was introduced by Chistyakov [4] for $\gamma = 0$ and by Chover et al. [5, 6] for $\gamma > 0$. Moreover, the class $\cup_{\gamma \geqslant 0}\mathcal{L}(\gamma)$ is properly contained in the distribution class $\mathcal{OL}$ introduced by Shimura and Watanabe [18] as follows.

We say that a distribution F belongs to the distribution class $\mathcal{OL}$, if for each (or, equivalently, for some) $t \neq 0$,

$$C(F,t) := \limsup \frac{\overline{F}(x-t)}{\overline{F}(x)} < \infty.$$

We also say that a distribution F belongs to the distribution class $\mathcal{OS}$ introduced by Klüppelberg [14] if

$$C^*(F) := \limsup \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} < \infty.$$

The closure under infinitely divisible distribution roots has been proved for the class $\mathcal{S}(\gamma)$; see Embrechts et al. [10] for $\gamma = 0$ and Sgibnev [17], Pakes [15], and Watanabe [20] for $\gamma > 0$. In addition, Watanabe and Yamamuro [23] show that the class $\mathcal{OS}$ is closed under infinitely divisible distribution roots if the Lévy distribution F is infinitely divisible. Thus the following interesting problem naturally arises:

- Does $H \in \mathcal{L}(\gamma)$ imply $F \in \mathcal{L}(\gamma)$?

To clarify this issue, we might firstly answer the following problem:

- Does $H_2 \in \mathcal{L}(\gamma)$ imply $F \in \mathcal{L}(\gamma)$?

For the class $\mathcal{L}(\gamma)$, the latter problem on closure under compound convolution roots is a natural extension of the famous Embrechts–Goldie conjecture on closure of the class $\mathcal{L}(\gamma)$ under convolution roots [8, 9]:

- If $F^{*k} \in \mathcal{L}(\gamma)$ for some (even for all) $k \geqslant 2$, then $F \in \mathcal{L}(\gamma)$.

Thus we call the above-mentioned two problems the generalized Embrechts–Goldie problems.

It is known that the class $\mathcal{S}$ is closed under convolution root; see Embrechts et al. [10]. So does the class $\mathcal{S}(\gamma)$ for some $\gamma > 0$ if the distribution $F \in \mathcal{L}(\gamma)$; see Embrechts and Goldie [9]. However, if $F \notin \mathcal{L}(\gamma)$ for some $\gamma > 0$, then the conclusion does not hold; see Watanabe [21]. Earlier, Shimura and Watanabe [19] showed that there is a distribution F such that $F^{*2} \in \mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$, whereas $F \in \mathcal{OL} \setminus (\cup_{\gamma \geqslant 0} \mathcal{L}(\gamma) \cup \mathcal{OS})$. Therefore the Embrechts–Goldie conjecture for $k = 2$ is not valid.

Moreover, Shimura and Watanabe [18] show that in general the class $\mathcal{OS}$ is not closed under infinitely divisible distribution roots. For the classes $(\mathcal{L} \cap \mathcal{OS}) \setminus \mathcal{S}$ and $\mathcal{L} \setminus \mathcal{OS}$, Xu et al. [24] give a negative answers to the Embrechts–Goldie conjecture for all $k \geqslant 2$ and the above-mentioned two generalized Embrechts–Goldie problems in the case $\gamma = 0$. In this paper, we are mostly interested in the case $\gamma > 0$.

We will further see that the method used by Xu et al. [24] in the case $\gamma = 0$ cannot be directly used in the case $\gamma > 0$. Thus, for the classes $(\mathcal{L}(\gamma) \cap \mathcal{OS}) \setminus \mathcal{S}(\gamma)$, we need to find a new method with more technical details to provide negative answers to the Embrechts–Goldie conjecture in the case of $\gamma > 0$ and $k \geqslant 2$ and the two generalized Embrechts–Goldie problems.

Theorem 1. *Let $\gamma > 0$. There exists a Lévy distribution F such that $F \in \mathcal{OL} \setminus (\mathcal{L}(\gamma) \cup \mathcal{OS})$ with $m(F) < \infty$, whereas the corresponding infinitely divisible distribution H with $m(H) < \infty$ and F^{*k} for all $k \geqslant 2$ are in the class $\big(\mathcal{L}(\gamma) \cap \mathcal{OS}\big) \setminus \mathcal{S}(\gamma)$, and*

$$\begin{aligned}
&\mathrm{e}^{-\mu} m(H_1) \sum_{l=1}^{\infty} \left(\frac{\mu^{2l}}{(2l)!} + \frac{\mu^{2l+1}}{(2l+1)!} \right) \big(m\big(F^{*2}\big)\big)^{l-1} l \\
&\quad \leqslant \liminf \frac{\overline{H}(x)}{\overline{F^{*2}}(x)} \leqslant \limsup \frac{\overline{H}(x)}{\overline{F^{*2}}(x)} \\
&\quad \leqslant \mathrm{e}^{-\mu} m(H_1) \sum_{l=1}^{\infty} \left(\frac{\mu^{2l-1}}{(2l-1)!} + \frac{\mu^{2l}}{(2l)!} \right) \sum_{i=0}^{l-1} \big(m\big(F^{*2}\big)\big)^{i} \big(C^*\big(F^{*2}\big) - m\big(F^{*2}\big)\big)^{l-1-i}. \qquad (1.1)
\end{aligned}$$

These results complement the corresponding results of Embrechts et al. [10], Sgibnev [17], Pakes [15], Watanabe [20], and Watanabe and Yamamuro [23]. To prove these results, we also found some useful methods and interesting conclusions.

We prove Theorem 1 in Section 5. To this end, we give some structural properties of the class $\mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$, as preliminary results, in Sections 2 and 3, which include the closure under random convolution and the convolution roots for the class. In particular, we give a positive answer to the Embrechts–Goldie conjecture under two conditions, which are proved to be necessary in a certain sense; see Theorem 6 and Remark 5. In Section 4, we study a transformation from heavy-tailed distributions to light-tailed distributions, by which we can get many needed distributions.

2 On the closure under random convolution

First, we provide the following lemma, which, together with Remark 1 below, is one of the keys to prove the main results in this section and has its own independent interest.

Lemma 1. *Let* $\gamma \geqslant 0$*, and let* F_1 *and* F_2 *be two distributions such that* $F_1 * F_2 \in \mathcal{L}(\gamma)$*. For every* $i = 1, 2$*, assume that either*

$$\liminf \frac{\overline{F_i}(x-t)}{\overline{F_i}(x)} \geqslant \mathrm{e}^{\gamma t} \quad \text{for each } t > 0 \tag{2.1}$$

or

$$\overline{F_i}(x) = o\big(\overline{F_1 * F_2}(x)\big). \tag{2.2}$$

Then, for the same i*,*

$$\big|\overline{F_i}(x-t) - \mathrm{e}^{\gamma t}\overline{F_i}(x)\big| = o\big(\overline{F_1 * F_2}(x)\big) \quad \text{for each } t > 0. \tag{2.3}$$

Further, let F *be a distribution such that* $F^{*n} \in \mathcal{L}(\gamma)$ *for some* $n \geqslant 2$*. If for every* $k = 1, \dots, n-1$*,*

$$\liminf \frac{\overline{F^{*k}}(x-t)}{\overline{F^{*k}}(x)} \geqslant \mathrm{e}^{\gamma t} \quad \text{for each } t > 0, \tag{2.4}$$

then, for the same k*,*

$$\big|\overline{F^{*k}}(x-t) - \mathrm{e}^{\gamma t}\overline{F^{*k}}(x)\big| = o\big(\overline{F^{*n}}(x)\big) \quad \text{for each } t > 0. \tag{2.5}$$

Proof. Conclusions (2.3) and (2.5) with $\gamma = 0$ follow from Theorem 2.1 (1) in Xu et al. [24]. Clearly, they are valid for $\gamma \geqslant 0$ under condition (2.2), and (2.5) is directly implied by (2.3) with $F_1 = F^{*k}$ and $F_2 = F^{*(n-k)}$ for $k = 1, \dots, n-1$ and $n \geqslant 2$. Therefore we only need to prove (2.3) for $\gamma > 0$ under condition (2.1).

Firstly, for $i = 1$ and $j = 2$ or $i = 2$ and $j = 1$, we prove that

$$\lim_{v\to\infty} \limsup \frac{\int_{0-}^{v} \overline{F_i}(x-t-y) - \mathrm{e}^{\gamma t}\overline{F_i}(x-y)\, F_j(\mathrm{d}y)}{\overline{F_1 * F_2}(x)} = 0 \tag{2.6}$$

and

$$\lim_{v\to\infty} \liminf \frac{\int_{0-}^{v} \overline{F_i}(x-t-y) - \mathrm{e}^{\gamma t}\overline{F_i}(x-y)\, F_j(\mathrm{d}y)}{\overline{F_1 * F_2}(x)} = 0. \tag{2.7}$$

For some $v > 0$ and all $x > v + t$, using integration by parts to the integrals $\int_{x-t-v}^{x-t} \overline{F_1}(x-t-y)\, F_2(\mathrm{d}y)$ and $\int_{x-t-v}^{x} \overline{F_1}(x-y)\, F_2(\mathrm{d}y)$ in the second equation below, we have

$$\begin{aligned}
\overline{F_1 * F_2}(x-t) &= \overline{F_2}(x-t) + \Bigg(\int_{0-}^{x-t-v} + \int_{x-t-v}^{x-t}\Bigg)\overline{F_1}(x-t-y)\, F_2(\mathrm{d}y) \\
&= \mathrm{e}^{\gamma t}\overline{F_1 * F_2}(x) + \overline{F_2}(x-t) + \int_{0-}^{x-t-v} \big(\overline{F_1}(x-t-y) - \mathrm{e}^{\gamma t}\overline{F_1}(x-y)\big)\, F_2(\mathrm{d}y) \\
&\quad + \int_{x-t-v}^{x-t} \overline{F_1}(x-t-y)\, F_2(\mathrm{d}y) - \mathrm{e}^{\gamma t}\overline{F_2}(x) - \mathrm{e}^{\gamma t}\int_{x-t-v}^{x} \overline{F_1}(x-y)\, F_2(\mathrm{d}y)
\end{aligned}$$

$$\begin{aligned}
&= \mathrm{e}^{\gamma t}\overline{F_1 * F_2}(x) + \int\limits_{0-}^{v} \big(\overline{F_2}(x-t-y) - \mathrm{e}^{\gamma t}\overline{F_2}(x-y)\big)\, F_1(\mathrm{d}y) \\
&\quad - \mathrm{e}^{\gamma t}\int\limits_{v}^{v+t} \overline{F_2}(x-y)\, F_1(\mathrm{d}y) + \int\limits_{0-}^{x-t-v} \big(\overline{F_1}(x-t-y) - \mathrm{e}^{\gamma t}\overline{F_1}(x-y)\big)\, F_2(\mathrm{d}y) \\
&\quad + \big(\overline{F_1}(v) - \mathrm{e}^{\gamma t}\overline{F_1}(v+t)\big)\overline{F_2}(x-t-v) \\
&\geqslant \mathrm{e}^{\gamma t}\overline{F_1 * F_2}(x) + \int\limits_{0-}^{v} \big(\overline{F_2}(x-t-y) - \mathrm{e}^{\gamma t}\overline{F_2}(x-y)\big)\, F_1(\mathrm{d}y) \\
&\quad - (\mathrm{e}^{\gamma t}-1)\overline{F_2}(x-v-t)\overline{F_1}(v) + \int\limits_{0-}^{x-t-v} \big(\overline{F_1}(x-t-y) - \mathrm{e}^{\gamma t}\overline{F_1}(x-y)\big)\, F_2(\mathrm{d}y) \\
&= I_1(x) + I_2(x,v) - I_3(x,v) + I_4(x,v). \qquad (2.8)
\end{aligned}$$

Now we deal with $I_3(x,v)$. By $F_1 * F_2 \in \mathcal{L}(\gamma)$ and (2.1), using Fatou's lemma, we have

$$\begin{aligned}
\frac{\overline{F_1 * F_2}(x)}{\overline{F_2}(x-v-t)\overline{F_1}(v)} &\sim \frac{\overline{F_1 * F_2}(x-v-t)\mathrm{e}^{-\gamma(v+t)}}{\overline{F_2}(x-v-t)\overline{F_1}(v)} \gtrsim \frac{\sum_{k=1}^{[v/t]}\int_{v-kt}^{v-(k-1)t}\mathrm{e}^{\gamma y}\,F_1(\mathrm{d}y)}{\mathrm{e}^{\gamma(v+t)}\overline{F_1}(v)} \\
&\geqslant \frac{\sum_{k=1}^{[v/t]}\mathrm{e}^{\gamma(v-kt)}\big(\overline{F_1}(v-kt) - \overline{F_1}(v-kt+t)\big)}{\mathrm{e}^{\gamma(v+t)}\overline{F_1}(v)} \\
&= \sum_{k=1}^{[v/t]}\mathrm{e}^{-\gamma(k+1)t}\frac{\overline{F_1}(v-kt)}{\overline{F_1}(v-kt+t)-1}\frac{\overline{F_1}(v-kt+t)}{\overline{F_1}(v)} \\
&\gtrsim \sum_{k=1}^{\infty}\mathrm{e}^{-2\gamma t}(\mathrm{e}^{\gamma t}-1) = \infty \quad \text{as } v \to \infty.
\end{aligned}$$

Thus

$$\lim_{v\to\infty}\limsup \frac{I_3(x,v)}{\overline{F_1 * F_2}(x)} = 0. \qquad (2.9)$$

Then we deal with $I_2(x,v)$ and $I_4(x,v)$. According to condition (2.1), we know that for any $\epsilon > 0$, there is a constant $v_0 = v_0(F_1, F_2, \epsilon) > 0$ such that for $v \geqslant v_0$ and $x \geqslant 2v + t$,

$$\begin{aligned}
I_2(x,v) \wedge I_4(x,v) &\geqslant \Bigg(-\epsilon\int\limits_{0-}^{x}\overline{F_2}(x-y)\,F_1(\mathrm{d}y)\Bigg) \wedge \Bigg(-\epsilon\int\limits_{0-}^{x}\overline{F_1}(x-y)\,F_2(\mathrm{d}y)\Bigg) \\
&\geqslant -\epsilon\overline{F_1 * F_2}(x), \qquad (2.10)
\end{aligned}$$

where $a \wedge b = \min\{a,b\}$. Thus, on the one hand,

$$\lim_{v\to\infty}\liminf \frac{I_2(x,v)}{\overline{F_1 * F_2}(x)} \geqslant -\epsilon. \qquad (2.11)$$

On the other hand, by (2.8), $F_1 * F_2 \in \mathcal{L}(\gamma)$, (2.9), and (2.10) we have

$$\lim_{v\to\infty} \limsup \frac{I_2(x,v)}{\overline{F_1 * F_2}(x)} \leqslant \lim_{v\to\infty} \limsup \frac{\overline{F_1 * F_2}(x-t) - I_1(x,v) + I_3(x,v) - I_4(x,v)}{\overline{F_1 * F_2}(x)} \leqslant \epsilon. \tag{2.12}$$

According to (2.11), (2.12), and the arbitrariness of ϵ, we get (2.6) and (2.7) with $i = 2$ and $j = 1$. Similarly, we have (2.6) and (2.7) for $i = 1$ and $j = 2$.

Secondly, we only need to prove (2.3) with $i = 1$. To this end, we first prove that

$$\liminf \frac{\overline{F_1}(x-t) - \mathrm{e}^{\gamma t}\overline{F_1}(x)}{\overline{F_1 * F_2}(x)} = 0. \tag{2.13}$$

Assume that, on the contrary, by (2.1) there exists a constant $C = C(F_1, F_2, t, \gamma) > 0$ such that

$$\liminf \frac{\overline{F_1}(x-t) - \mathrm{e}^{\gamma t}\overline{F_1}(x)}{\overline{F_1 * F_2}(x)} \geqslant C. \tag{2.14}$$

Then (2.7), (2.14), Fatou's lemma, and $F_1 * F_2 \in \mathcal{L}(\gamma)$ lead to the following contradiction:

$$\begin{aligned} 0 &= \lim_{v\to\infty} \liminf \frac{\int_{0-}^{v} (\overline{F}_1(x-t-y) - \mathrm{e}^{\gamma t}\overline{F_1}(x-y))\, F_2(\mathrm{d}y)}{\overline{F_1 * F_2}(x)} \\ &\geqslant \lim_{v\to\infty} \int_{0-}^{v} \liminf \frac{\overline{F_1}(x-t-y) - \mathrm{e}^{\gamma t}\overline{F_1}(x-y)}{\overline{F_1 * F_2}(x-y)} \liminf \frac{\overline{F_1 * F_2}(x-y)}{\overline{F_1 * F_2}(x)}\, F_2(\mathrm{d}y) \\ &\geqslant C \int_{0-}^{\infty} \mathrm{e}^{\gamma y}\, F_2(\mathrm{d}y) > 0. \end{aligned}$$

Therefore (2.13) holds. Next, we prove that

$$\limsup \frac{\overline{F_1}(x-t) - \mathrm{e}^{\gamma t}\overline{F_1}(x)}{\overline{F_1 * F_2}(x)} \leqslant 0. \tag{2.15}$$

By (2.6), (2.13), and $F_1 * F_2 \in \mathcal{L}(\gamma)$, using Fatou's lemma, we have

$$\begin{aligned} 0 &= \lim_{v\to\infty} \limsup \left(\int_{0-}^{t} + \int_{t}^{2t} + \int_{2t}^{v} \right) \frac{\overline{F_1}(x-2t-y) - \mathrm{e}^{2\gamma t}\overline{F_1}(x-y))}{\overline{F_1 * F_2}(x)}\, F_2(\mathrm{d}y) \\ &\geqslant \limsup \frac{\int_t^{2t} (\overline{F_1}(x-2t-y) - \mathrm{e}^{\gamma y}\overline{F_1}(x-2t))\, F_2(\mathrm{d}y)}{\overline{F_1 * F_2}(x)} \\ &\quad + \frac{\int_t^{2t} \liminf \mathrm{e}^{\gamma y}(\overline{F_1}((x-y)-(2t-y)) - \mathrm{e}^{\gamma(2t-y)}\overline{F_1}(x-y))\, F_2(\mathrm{d}y)}{\overline{F_1 * F_2}(x)} \end{aligned}$$

$$+\lim_{v\to\infty}\left(\int_{0-}^{t}+\int_{2t}^{v}\right)\mathrm{e}^{\gamma y}\liminf\frac{\overline{F_1}((x-y)-2t)-\mathrm{e}^{2\gamma t}\overline{F_1}(x-y)}{\overline{F_1*F_2}(x-y)}F_2(\mathrm{d}y)$$

$$=\limsup\int_{t}^{2t}\frac{\overline{F_1}(x-2t-y)-\mathrm{e}^{\gamma y}\overline{F_1}(x-2t))\,F_2(\mathrm{d}y)}{\overline{F_1*F_2}(x)}$$

$$\geqslant\liminf\frac{\int_t^{2t}(\overline{F_1}((x-3t)-(y-t))-\mathrm{e}^{\gamma(y-t)}\overline{F_1}(x-3t))}{\overline{F_1*F_2}(x-3t)}F_2(\mathrm{d}y)$$

$$+\limsup\frac{\int_t^{2t}\mathrm{e}^{\gamma(y-t)}(\overline{F_1}((x-2t)-t)-\mathrm{e}^{\gamma t}\overline{F_1}(x-2t))}{\overline{F_1*F_2}(x-2t)}F_2(\mathrm{d}y)$$

$$\geqslant\mathrm{e}^{-\gamma t}\int_{t}^{2t}\mathrm{e}^{\gamma y}F_2(\mathrm{d}y)\limsup\frac{\overline{F_1}\big((x-2t)-t\big)-\mathrm{e}^{\gamma t}\overline{F_1}(x-2t)}{\overline{F_1*F_2}(x-2t)}. \tag{2.16}$$

Thus (2.15) follows from (2.16). Combining (2.15) with (2.13) yields (2.3) with $i=1$. □

Remark 1.

(i) If $\gamma=0$ or $F_i\in\mathcal{L}(\gamma)$, $i=1,2$, for some $\gamma>0$, then condition (2.1) is obviously satisfied. In addition, there are many distributions $F_i\notin\mathcal{L}(\gamma)$, $i=1,2$, such that (2.1) holds and $F_1*F_2\in\mathcal{L}(\gamma)$; see, for example, Proposition 1.

(ii) On the one hand, according to Proposition 1, condition (2.1) does not imply condition (2.2). On the other hand, according to Theorem 3.1 of Shimura and Watanabe [19], condition (2.2) does not imply condition (2.1).

Now we give a result on closure of the class $\mathcal{L}(\gamma)$ under convolution.

Theorem 2. *For any $n\geqslant 2$, let F_i, $1\leqslant i\leqslant n$, be distributions, where $F_n\in\mathcal{L}(\gamma)$ for some $\gamma\geqslant 0$. Further, assume that* (2.1) *and the following condition are satisfied*:

$$\left|\overline{F_i}(x-t)-\mathrm{e}^{\gamma t}\overline{F_i}(x)\right|=o\big(\overline{F_n}(x)\big)\quad\textit{for each } t>0 \textit{ and } 1\leqslant i\leqslant n-1. \tag{2.17}$$

Then $F_{i_1}*\cdots*F_{i_l}*F_n\in\mathcal{L}(\gamma)$ for all $1\leqslant l\leqslant n-1$ and $1\leqslant i_1<\cdots<i_l\leqslant n-1$.

Proof. We first prove that $F_{j_1}*F_n\in\mathcal{L}(\gamma)$. For each $t>0$ and any $0<\epsilon<1$, since $F_n\in\mathcal{L}(\gamma)$, there is a constant $v_0=v_0(F,t)>2t$ large enough such that for all $v\geqslant v_0$,

$$\left|\overline{F_n}(v)-\mathrm{e}^{\gamma t}\overline{F_n}(v+t)\right|\leqslant\epsilon\overline{F_n}(v+t). \tag{2.18}$$

Thus

$$\int_{0-}^{x-t-v}\left|\overline{F}_n(x-t-y)-\mathrm{e}^{\gamma t}\overline{F}_n(x-y)\right|F_{j_1}(\mathrm{d}y)$$

$$<\epsilon\int_{0-}^{x-t-v}\overline{F}_n(x-t-y)\,F_{j_1}(\mathrm{d}y)\leqslant\epsilon\overline{F_{j_1}*F_n}(x-t). \tag{2.19}$$

Further, when $x \geqslant 2v+t$, by the third equation in (2.8) with $F_1 = F_n$ and $F_2 = F_{j_1}$, integration by parts of the integral $\int_v^{v+t} \mathrm{e}^{\gamma t}\overline{F}_{j_1}(x-y)\,F_n(\mathrm{d}y)$, (2.19), and (2.18) we have

$$\begin{aligned}
\overline{F_{j_1} * F_n}(x-t) &= \mathrm{e}^{\gamma t}\overline{F_{j_1} * F_n}(x) + \int_{0-}^{v} \big(\overline{F}_{j_1}(x-t-y) - \mathrm{e}^{\gamma t}\overline{F}_{j_1}(x-y)\big)\,F_n(\mathrm{d}y) \\
&\quad + \big(\overline{F_n}(v) - \mathrm{e}^{\gamma t}\overline{F_n}(v+t)\big)\overline{F}_{j_1}(x-t-v) - \mathrm{e}^{\gamma t}\int_{v}^{v+t} \overline{F}_{j_1}(x-y)\,F_n(\mathrm{d}y) \\
&\quad + \int_{0-}^{x-t-v} \big(\overline{F}_n(x-t-y) - \mathrm{e}^{\gamma t}\overline{F}_n(x-y)\big)\,F_{j_1}(\mathrm{d}y) \\
&\leqslant \mathrm{e}^{\gamma t}\overline{F_{j_1} * F_n}(x) + \int_{0-}^{v} \big(\overline{F}_{j_1}(x-t-y) - \mathrm{e}^{\gamma t}\overline{F}_{j_1}(x-y)\big)\,F_n(\mathrm{d}y) \\
&\quad + \epsilon\overline{F_{j_1} * F_n}(x-t) + \big(\overline{F_n}(v) - \mathrm{e}^{\gamma t}\overline{F_n}(v+t)\big)\overline{F}_{j_1}(x-t-v) \\
&\quad - \mathrm{e}^{\gamma t}\overline{F}_{j_1}(x-v)\big(\overline{F_n}(v) - \overline{F_n}(v+t)\big) \\
&\leqslant \mathrm{e}^{\gamma t}\overline{F_{j_1} * F_n}(x) + \int_{0-}^{v} \big(\overline{F}_{j_1}(x-t-y) - \mathrm{e}^{\gamma t}\overline{F}_{j_1}(x-y)\big)\,F_n(\mathrm{d}y) \\
&\quad + \epsilon\overline{F_{j_1} * F_n}(x-t) + \big(\overline{F_n}(v) - \mathrm{e}^{\gamma t}\overline{F_n}(v+t)\big)\overline{F}_{j_1}(x-t-v) \\
&\quad + \mathrm{e}^{\gamma t}\overline{F}_{j_1}(x-v)\overline{F_n}(v+t) \\
&\leqslant \mathrm{e}^{\gamma t}\overline{F_{j_1} * F_n}(x) + \int_{0-}^{v} \big(\overline{F}_{j_1}(x-t-y) - \mathrm{e}^{\gamma t}\overline{F}_{j_1}(x-y)\big)\,F_n(\mathrm{d}y) \\
&\quad + \epsilon\overline{F_{j_1} * F_n}(x-t) + (\epsilon + \mathrm{e}^{\gamma t})\overline{F_n}(v+t)\overline{F}_{j_1}(x-t-v) \qquad (2.20)
\end{aligned}$$

and

$$\begin{aligned}
\overline{F_{j_1} * F_n}(x-t) &\geqslant \mathrm{e}^{\gamma t}\overline{F_{j_1} * F_n}(x) + \int_{0-}^{v} \big(\overline{F}_{j_1}(x-t-y) - \mathrm{e}^{\gamma t}\overline{F}_{j_1}(x-y)\big)\,F_n(\mathrm{d}y) \\
&\quad - \epsilon\overline{F_{j_1} * F_n}(x-t) - (\epsilon + \mathrm{e}^{\gamma t})\overline{F_n}(v)\overline{F}_{j_1}(x-t-v). \qquad (2.21)
\end{aligned}$$

Using (2.17), we have

$$\begin{aligned}
&\int_{0-}^{v} \big|\overline{F}_{j_1}(x-t-y) - \mathrm{e}^{\gamma t}\overline{F}_{j_1}(x-y)\big|\,F_n(\mathrm{d}y) \\
&\quad = o\Bigg(\int_{0-}^{v} \overline{F_n}(x-y)\,F_n(\mathrm{d}y)\Bigg) = o\Bigg(\int_{0-}^{v} \mathrm{e}^{\gamma y}\,F_n(\mathrm{d}y)\overline{F_n}(x)\Bigg) = o\big(\overline{F_{j_1} * F_n}(x)\big). \qquad (2.22)
\end{aligned}$$

By Fatou's lemma, (2.1), and $F_n \in \mathcal{L}(\gamma)$ we have

$$\begin{aligned}
&\lim_{v\to\infty} \liminf_{x\to\infty} \frac{\overline{F_{j_1} * F_n}(x)}{\overline{F_n}(v)\overline{F}_{j_1}(x-t-v)} \\
&\geqslant \liminf_{v\to\infty} \liminf \frac{\int_{v+t}^{2v} \frac{\overline{F_{j_1}}(x-y)}{\overline{F}_{j_1}(x-t-v)}\, F_n(\mathrm{d}y)}{\overline{F_n}(v)} = \liminf_{v\to\infty} \frac{\int_{v+t}^{2v} \mathrm{e}^{\gamma y}\, F_n(\mathrm{d}y)}{\mathrm{e}^{\gamma(v+t)}\overline{F_n}(v)} \\
&\geqslant \lim_{v\to\infty} \frac{\sum_{k=1}^{[vt^{-1}]-1} \mathrm{e}^{\gamma(v+kt)}\overline{F_n}(v+kt) - \overline{F_n}(v+kt+t)}{\mathrm{e}^{\gamma(v+t)}\overline{F_n}(v)} \\
&= \mathrm{e}^{-\gamma t} \lim_{v\to\infty} \sum_{k=1}^{[vt^{-1}-1]} \mathrm{e}^{\gamma kt}\big(\mathrm{e}^{-\gamma kt} - \mathrm{e}^{-\gamma(k+1)t}\big) \\
&= \mathrm{e}^{-\gamma t} \lim_{v\to\infty} \big[vt^{-1}-1\big]\big(1-\mathrm{e}^{-\gamma t}\big) = \infty. \qquad (2.23)
\end{aligned}$$

Clearly, we also have

$$\lim_{v\to\infty} \liminf \frac{\overline{F_{j_1} * F_n}(x)}{\overline{F_n}(v+t)\overline{F}_{j_1}(x-t-v)} = \infty. \qquad (2.24)$$

Combining (2.19)–(2.24) and the arbitrariness of ϵ, we get that $F_{j_1} * F_n \in \mathcal{L}(\gamma)$.

Finally, using induction, if $F_{j_1} * \cdots * F_{j_{l-1}} * F_n \in \mathcal{L}(\gamma)$ for some $2 \leqslant l \leqslant n-1$, then $F_{j_1} * \cdots * F_{j_l} * F_n \in \mathcal{L}(\gamma)$ by (2.17) and

$$\big|\overline{F_{j_l}}(x-t) - \mathrm{e}^{\gamma t}\overline{F_{j_l}}(x)\big| = o\big(\overline{F_n}(x)\big) = o\big(\overline{F_{j_1} * \cdots * F_{j_{l-1}} * F_n}(x)\big)$$

for each $t > 0$. $\square$

Remark 2. For $n = 2$ and some $\gamma \geqslant 0$, Theorem 3 of Embrechts and Goldie [8] states the following result on the closure of the class $\mathcal{L}(\gamma)$ under convolution. Let $F_2 \in \mathcal{L}(\gamma)$. Then $F = F_1 * F_2 \in \mathcal{L}(\gamma)$ if either (i) $F_1 \in \mathcal{L}(\gamma)$ or (ii) $\overline{F_1}(x) = o(\overline{F_2}(x))$. For the latter, more generally, if $F_n \in \mathcal{L}(\gamma)$ for some $n \geqslant 2$ and

$$\overline{F_i}(x) = o\big(\overline{F_n}(x)\big) \quad \text{for each } 1 \leqslant i \leqslant n-1, \qquad (2.25)$$

then $F_1 * \cdots * F_n \in \mathcal{L}(\gamma)$. Clearly, (2.17) is implied by (2.25) and $F_n \in \mathcal{L}(\gamma)$. Therefore the theorem is a slight extension of Theorem 3 in Embrechts and Goldie [8]. However, as Remark 1(ii) points out, the inverse implication not always holds.

Based on Theorem 2, we obtain the following result.

Corollary 1. *For some $n \geqslant 2$, $F^{*k} \in \mathcal{L}(\gamma)$ for all $k \geqslant n+1$ if $F^{*n} \in \mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$ and either of the following two cases is true*:

(i)
$$\liminf \frac{\overline{F}(x-t)}{\overline{F}(x)} \geqslant \mathrm{e}^{\gamma t} \quad \textit{for each } t > 0, \qquad (2.26)$$

and

$$\big|\overline{F}(x-t) - \mathrm{e}^{\gamma t}\overline{F}(x)\big| = o\big(\overline{F^{*n}}(x)\big) \quad \textit{for each } t > 0; \qquad (2.27)$$

(ii)
$$\overline{F}(x) = o\big(\overline{F^{*2}}(x)\big). \qquad (2.28)$$

Proof. Using the method of induction, we only need to prove that $F^{*(n+1)} \in \mathcal{L}(\gamma)$.

In case (i), since $F^{*n} \in \mathcal{L}(\gamma)$, it follows that for each $t > 0$,

$$\liminf \frac{\overline{F^{*n}}(x-t)}{\overline{F^{*n}}(x)} \geqslant \mathrm{e}^{\gamma t} \quad \text{and} \quad \left|\overline{F^{*n}}(x-t) - \mathrm{e}^{\gamma t}\overline{F^{*n}}(x)\right| = o\big(\overline{F^{*n}}(x)\big).$$

Further, by (2.26), (2.27), and Theorem 2 we have that $F^{*(n+1)} = F * F^{*n} \in \mathcal{L}(\gamma)$.

In case (ii), we first prove the following result: if condition (2.28) is satisfied, then

$$\overline{F^{*k}}(x) = o\big(\overline{F^{*(k+1)}}(x)\big) \quad \text{for all } k \geqslant 1. \tag{2.29}$$

Indeed, according to (2.28) and the induction hypothesis, for any $\epsilon > 0$, there is a constant $x_k > 0$ such that

$$\overline{F}(x) \leqslant \varepsilon \overline{F^{*2}}(x) \quad \text{and} \quad \overline{F^{*(k-1)}}(x) \leqslant \varepsilon \overline{F^{*k}}(x) \quad \text{for } x \geqslant x_k.$$

An integration by parts of the integral $\int_{x-x_k}^{x} \mathrm{e}^{\gamma t}\overline{F^{*(k-1)}}(x-y)\,F(\mathrm{d}y)$ yields

$$\begin{aligned}
\overline{F^{*k}}(x) &= \overline{F}(x) + \left(\int_{0-}^{x-x_k} + \int_{x-x_k}^{x}\right)\overline{F^{*(k-1)}}(x-y)\,F(\mathrm{d}y) \\
&= \int_{0-}^{x-x_k} \overline{F^{*(k-1)}}(x-y)\,F(\mathrm{d}y) + \overline{F^{*(k-1)}}(x_k)\overline{F}(x-x_k) + \int_{0-}^{x_k} \overline{F}(x-y)\,F^{*(k-1)}(\mathrm{d}y) \\
&\leqslant \varepsilon \int_{0-}^{x-x_k} \overline{F^{*k}}(x-y)\,F(\mathrm{d}y) + \overline{F^{*(k-1)}}(x_k)\overline{F}(x-x_k) + \varepsilon \int_{0-}^{x_k} \overline{F^{*2}}(x-y)\,F^{*(k-1)}(\mathrm{d}y) \\
&\leqslant \varepsilon \int_{0-}^{x-x_k} \overline{F^{*k}}(x-y)\,F(\mathrm{d}y) + \varepsilon\overline{F^{*2}}(x) + \varepsilon \int_{x-x_k-}^{x} \overline{F^{*(k-1)}}(x-y)\,F^{*2}(\mathrm{d}y) \\
&\leqslant 2\varepsilon\overline{F^{*(k+1)}}(x).
\end{aligned}$$

By the arbitrariness of ϵ we immediately obtain (2.29).

In the following, we continue to prove Corollary 1 in case (ii). By (2.29) we have $\overline{F}(x) = o(\overline{F^{*n}}(x))$. Furthermore, by $F^{*n} \in \mathcal{L}(\gamma)$ and Remark 2, $F^{*(n+1)} \in \mathcal{L}(\gamma)$. □

Remark 3.

(i) According to Lemma 1, we can conclude that condition (2.27) is implied by $F^{*n} \in \mathcal{L}(\gamma)$, (2.26), and the following condition:

$$\liminf \frac{\overline{F^{*(n-1)}}(x-t)}{\overline{F^{*(n-1)}}(x)} \geqslant \mathrm{e}^{\gamma t} \quad \text{for each } t > 0. \tag{2.30}$$

Specifically, when $n = 2$, (i) is just the same as (2.26). In addition, if condition (2.26) is replaced by condition (2.4), then condition (2.27) can be canceled.

(ii) Condition (2.26) has been used in Lemma 7 and Theorem 7 of Foss and Korshunov [12]. On the one hand, some distributions that do not satisfy conditions (2.26) and (2.28) are given in Example 1 in Foss and Korshunov [12], Proposition 3.2 and Remark 4.1 in Chen et al. [2], and Theorem 1.1 in Watanabe [21]. On the other hand, many distributions simultaneously satisfy the two conditions or satisfy one of these two conditions; see, for example, Remark 1(ii).

Finally, we give three results on the closure under random convolution. In the following, let τ be a random variable with distribution G satisfying $G(\{k\}) = \mathbf{P}(\tau = k) = p_k$ for all nonnegative integers k and $\sum_{k=0}^{\infty} p_k = 1$. Then the distribution $F^{*\tau} = \sum_{k=0}^{\infty} p_k F^{*k}$ is called a compound convolution generated by distributions F and G.

Theorem 3. *For some $n \geqslant 2$, let F be a distribution such that $F^{*n} \in \mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$. Assume that condition* (2.4) *or condition* (2.28) *is satisfied. Further, suppose that $p_k = 0$ for all $k > n$ and $p_n > 0$. Then $F^{*\tau} = \sum_{k=1}^{n} p_i F^{*k} = H(n) \in \mathcal{L}(\gamma)$.*

Proof. According to Lemma 1, by $F^{*n} \in \mathcal{L}(\gamma)$ and (2.4) or (2.28), we conclude that (2.5) holds. Further, by (2.5) and $p_n > 0$ we have that for all $t > 0$,

$$\frac{|\overline{H(n)}(x-t) - \mathrm{e}^{\gamma t}\overline{H(n)}(x)|}{\overline{H(n)}(x)} \leqslant \frac{\sum_{k=1}^{n} p_k |\overline{F^{*k}}(x-t) - \mathrm{e}^{\gamma t}\overline{F^{*k}}(x)}{\overline{F^{*n}}(x) p_n} \to 0,$$

so that $H(n) \in \mathcal{L}(\gamma)$. □

Theorem 4. *Let F be a distribution such that $F^{*n} \in \mathcal{L}(\gamma)$ for some $n \geqslant 1$ and some $\gamma \geqslant 0$. Assume that $P(\tau \geqslant n) > 0$ and the following condition is satisfied: for any $0 < \varepsilon < 1$, there is a positive integer $M = M(F, \varepsilon)$ such that*

$$\sum_{k=M}^{\infty} p_{k+1} \overline{F^{*k}}(x) \leqslant \varepsilon \overline{F^{*\tau}}(x) \quad \text{for all } x \geqslant 0. \tag{2.31}$$

Further, suppose that condition (2.4) *or* (2.28) *is satisfied. Then the random convolution $F^{*\tau} \in \mathcal{L}(\gamma)$, and*

$$\liminf \frac{\overline{F^{*\tau}}(x)}{\overline{F^{*n}}(x)} \geqslant \sum_{l=1}^{\infty} l \sum_{k=nl}^{n(l+1)-1} p_k \big(m\big(F^{*n}\big)\big)^{l-1}. \tag{2.32}$$

In particular, if $m(F^{*n}) = \infty$, then $\overline{F^{*n}}(x) = o(\overline{F^{*\tau}}(x))$.

Proof. Using (2.31) and (2.4) or (2.28), combined with Theorem 3 of the paper and the proof of Proposition 6.1 in [23], we conclude that $F^{*\tau} \in \mathcal{L}(\gamma)$. Finally, the Fatou lemma implies (2.32). □

Theorem 5. *Let F be a distribution such that $F^{*n} \in \mathcal{L}(\gamma) \cap \mathcal{OS}$ for some $n \geqslant 1$ and some $\gamma \geqslant 0$. Let τ be a random variable as in Theorem* 4 *such that for some $\varepsilon_0 > 0$,*

$$\sum_{l=1}^{\infty} \left(\sum_{k=(l-1)n+1}^{ln} p_k \right) \big(C^*\big(F^{*n}\big) - m\big(F^{*n}\big) + \varepsilon_0\big)^l < \infty.$$

Further, suppose that condition (2.4) *or* (2.28) *is satisfied. Then $F^{*\tau} \in \mathcal{L}(\gamma) \cap \mathcal{OS}$, and*

$$\limsup \frac{\overline{F^{*\tau}}(x)}{\overline{F^{*n}}(x)} \leqslant \sum_{l=1}^{\infty} \left(\sum_{k=(l-1)n+1}^{ln} p_k \right) \sum_{i=0}^{l-1} \big(m\big(F^{*n}\big)\big)^i \big(C^*\big(F^{*n}\big) - m\big(F^{*n}\big)\big)^{l-1-i}.$$

The proof of the theorem is similar to that of Theorem 2.2(2b) in [24], and we omit its details.

Remark 4.

(i) Theorems 4 and 5 in the case of $n = 1$ are due to Lemma 4 and Corollary 1 of Yu and Wang [26], which slightly improves Proposition 6.1 in Watanabe and Yamamuro [23]. However, when $n \geqslant 2$, the two theorems do not require the condition $F \in \mathcal{L}(\gamma)$ in the corresponding results of Yu and Wang [26]. As mentioned earlier, there exists a distribution F satisfying condition (2.4) or (2.28), and thus by the two theorems, $F^{*\tau} \in \mathcal{L}(\gamma)$, but $F \notin \mathcal{L}(\gamma)$; see, for example, Proposition 1. Therefore the two theorems are an extension and improvement of Lemma 6 in Watanabe and Yamamuro [26] and Proposition 6.1 in Yu and Wang [23] for the case $\gamma > 0$.

(ii) Specifically, if $p_k > 0$ for k large enough and $p_{k+1}/p_k \to 0$ as $k \to \infty$, then condition (2.31) is satisfied for any distribution F. For example, $p_k = \mathrm{e}^{-\lambda}\lambda^k/k!$ for all nonnegative integers k, where λ is a positive constant.

In addition, for distribution F, if Kesten's inequality holds, that is, there are two positive constants C and α such that

$$\overline{F^{*k}}(x) \leqslant C\mathrm{e}^{\alpha k}\overline{F}(x) \quad \text{for all } k \geqslant 1 \text{ and } x \geqslant 0, \tag{2.33}$$

and if $\sum_{k=1}^{\infty} p_k \mathrm{e}^{\alpha k} < \infty$, then condition (2.31) is satisfied. For example, $p_k = qp^k$ for all nonnegative integers k, where p and q are two positive constants such that $p + q = 1$ and $p\mathrm{e}^{\alpha} < 1$. Using the method applied in the proof of Lemma 5 in Watanabe and Yamamuro [26], we can show that (2.33) holds for F satisfying $F^{*n} \in \mathcal{L}(\gamma) \cap \mathcal{OS}$ for some $n \geqslant 2$ and $\gamma \geqslant 0$.

3 On closure under convolution roots

In this section, we give a positive answer to the Embrechts–Goldie conjecture, which also plays a role in the proof of Theorem 1. To this end, we first give two lemmas.

Lemma 2. *Let F be a distribution such that $m(F) < \infty$ and $F^{*2} \in \mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$. Then $F \in \mathcal{OS}$ if and only if*

$$a(F) = \limsup_{k\to\infty} \limsup \frac{\int_k^{x-k} \overline{F}(x-y)\, F(\mathrm{d}y)}{\overline{F^{*2}}(x)} < 1. \tag{3.1}$$

Proof. For sufficiency, assume that (3.1) holds. Then we have to prove that $F \in \mathcal{OS}$. For any $0 < \varepsilon < (1 - a(F))/4$, there is an integer $k_0 = k_0(F, \varepsilon) \geqslant 1$ such that

$$\limsup \frac{\int_k^{x-k} \overline{F}(x-y)\, F(\mathrm{d}y)}{\overline{F^{*2}}(x)} < a(F) + \varepsilon$$

and $\overline{F}(k)\mathrm{e}^{\gamma k} < \varepsilon$ for all $k \geqslant k_0$. Thus by $F^{*2} \in \mathcal{L}(\gamma)$ and the first equation in (2.8) we have

$$\begin{aligned}\liminf \frac{2\int_{0-}^{k} \overline{F}(x-y)\, F(\mathrm{d}y)}{\overline{F^{*2}}(x)} &= \liminf \biggl(1 - \frac{\int_k^{x-k} \overline{F}(x-y)\, F(\mathrm{d}y) + \overline{F}(x-k)\overline{F}(k)}{\overline{F^{*2}}(x)}\biggr) \\ &\geqslant 1 - \bigl(a(F) + \varepsilon\bigr) - \overline{F}(k)\mathrm{e}^{\gamma k} > \frac{1 - a(F)}{2} > 0. \end{aligned} \tag{3.2}$$

Still by $F^{*2} \in \mathcal{L}(\gamma)$ we have the following inequality:

$$\liminf \frac{2\int_{0-}^{k} \overline{F}(x-y)\, F(\mathrm{d}y)}{\overline{F^{*2}}(x)} \leqslant \liminf \frac{2\overline{F}(x-k)F(k)\overline{F^{*2}}(x-k)}{\overline{F^{*2}}(x)\overline{F^{*2}}(x-k)} \leqslant \frac{2F(k)\mathrm{e}^{\gamma k}}{C^*(F)}. \tag{3.3}$$

By (3.2) and (3.3) we have $C^*(F) < \infty$, and thus $F \in \mathcal{OS}$.

For necessity, assume that $F \in \mathcal{OS}$. Then we have to prove $a(F) < 1$. Assume that, on the contrary, $a(F) = 1$. Then from the inequality

$$\begin{aligned} 1 &\geqslant \liminf_{k\to\infty} \liminf \frac{\int_{0-}^{k} \overline{F}(x-y)\,F(\mathrm{d}y)}{\overline{F^{*2}}(x)} + \limsup_{k\to\infty} \limsup \frac{\int_{k}^{x-k} \overline{F}(x-y)\,F(\mathrm{d}y)}{\overline{F^{*2}}(x)} \\ &\geqslant \liminf_{k\to\infty} \liminf \frac{\overline{F}(x)F(k)}{\overline{F^{*2}}(x)} + 1 = \frac{1}{C^*(F)} + 1 \end{aligned}$$

we get $C^*(F) = \infty$, which is contradictory to the fact that $F \in \mathcal{OS}$. □

Lemma 3. *Let F be a distribution such that $F^{*2} \in \mathcal{L}(\gamma) \cap \mathcal{OS}$ for some $\gamma > 0$. Further, if condition* (2.26) *is satisfied and*

$$C^*\big(F^{*2}\big) < 6m\big(F^{*2}\big) = 6\big(m(F)\big)^2, \tag{3.4}$$

then $F \in \mathcal{OS}$.

Proof. According to Lemma 7 in [12], by condition (2.26) we have that

$$\liminf \frac{\overline{F^{*2}}(x)}{\overline{F}(x)} \geqslant 2m(F). \tag{3.5}$$

Now we estimate $a(F)$ in (3.1). By (3.5) and integration by parts, for any $0 < \delta < 1$, we have

$$\begin{aligned} a(F) &\leqslant \limsup_{k\to\infty} \limsup \frac{\int_{k}^{x-k} \overline{F^{*2}}(x-y)\,F(\mathrm{d}y)}{(1+\delta)m(F)\overline{F^{*2}}(x)} \\ &\leqslant \limsup_{k\to\infty} \limsup \frac{\overline{F^{*2}}(x-k)\overline{F^{*2}}(k) + \int_{k}^{x-k} \overline{F^{*2}}(x-y)F^{*2}(\mathrm{d}y)}{(1+\delta)^2 m^2(F)\overline{F^{*2}}(x)} \\ &= \frac{C^*(F^{*2}) - 2m(F^{*2})}{(1+\delta)^2 m^2(F)} \to \frac{C^*(F^{*2}) - 2m(F^{*2})}{4m^2(F)} \quad \text{as } \delta \uparrow 1. \end{aligned}$$

Thus, according to Lemma 2, by condition (3.4) we get $F \in \mathcal{OS}$. □

Theorem 6.

(i) *Let F be a distribution such that $F \in \mathcal{OS}$ and $F^{*2} \in \mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$. Further, assume that condition* (2.26) *or* (2.28) *is satisfied. Then $F \in \mathcal{L}(\gamma)$.*

(ii) *Let F be a distribution satisfying $m(F) < \infty$,* (3.4)*, and condition* (2.26) *for some $\gamma \geqslant 0$. If $F^{*2} \in \mathcal{L}(\gamma) \cap \mathcal{OS}$, then $F \in \mathcal{L}(\gamma) \cap \mathcal{OS}$.*

Proof. (i) Since $F \in \mathcal{OS}$, we have $C^*(F) < \infty$. So, according to Lemma 1, by $F^{*2} \in \mathcal{L}(\gamma)$ and condition (2.26) or condition (2.28), we have,

$$\frac{|\overline{F}(x-t) - \mathrm{e}^{\gamma t}\overline{F}(x)}{\overline{F}(x)} \leqslant \frac{C^*(F)|\overline{F}(x-t) - \mathrm{e}^{\gamma t}\overline{F}(x)|}{\overline{F^{*2}}(x)} \to 0,$$

that is $F \in \mathcal{L}(\gamma)$.

(ii) According to Lemma 3, we have $F \in \mathcal{OS}$. Further, by $F \in \mathcal{OS}$, $F^{*2} \in \mathcal{L}(\gamma)$ and condition (2.26), according to statement (i) of the theorem, we get that $F \in \mathcal{L}(\gamma)$. □

Remark 5.

(i) Condition (3.4) may be improved. However, it is necessary in a certain sense; see Xu et al. [24, Thm. 2.2(1)]. There $F \in \mathcal{OL} \setminus (\mathcal{L} \cup \mathcal{OS})$ and $F^{*2} \in \mathcal{L} \cap \mathcal{OS}$, whereas condition (3.4) does not hold, since otherwise, $F \in \mathcal{L} \cap \mathcal{OS}$ by Theorem 6.

A similar example for $\gamma > 0$ can be seen in Proposition 1.

Clearly, if $F^{*2} \in \mathcal{S}(\gamma)$, then condition (3.4) is naturally satisfied. Therefore the condition $C^*(F^{*2}) - 2m(F^{*2}) < 4(m(F))^2$ indicates that the distribution F^{*2} cannot be too far away from the class $\mathcal{S}(\gamma)$.

(ii) Clearly, condition (2.26) is necessary for $F \in \mathcal{L}(\gamma)$. The condition cannot follow from condition (3.4). In Watanabe [21], there is a distribution F such that $F^{*2} \in \mathcal{S}(\gamma)$ for some $\gamma > 0$, and thus condition (3.4) holds, whereas both $F \in \mathcal{OS} \setminus \mathcal{L}(\gamma)$ and condition (2.26) do not hold.

Corollary 2. *Let F be a distribution such that $F \in \mathcal{OS} \setminus \mathcal{L}(\gamma)$ for some $\gamma \geqslant 0$. Further, if condition* (i) *is satisfied for each integer $n \geqslant 2$, then $F^{*k} \notin \mathcal{L}(\gamma)$ for all integers $k \geqslant 2$.*

Proof. We assume that there exists an integer $n \geqslant 2$ such that $F^{*n} \in \mathcal{L}(\gamma)$ and $F^{*i} \notin \mathcal{L}(\gamma)$ for all $1 \leqslant i < n$. Under condition (i), by $F^{*n} \in \mathcal{L}(\gamma)$ and Corollary 1 we have $F^{*i} \in \mathcal{L}(\gamma)$ for all $i \geqslant n$. Hence $F^{*2(n-1)} \in \mathcal{L}(\gamma)$, but $F^{*(n-1)} \notin \mathcal{L}(\gamma)$. From (i) of Theorem 6 we have $F^{*(n-1)} \notin \mathcal{OS}$, which is contradictory to $F \in \mathcal{OS}$. □

4 A transformation between distributions

To prove Theorem 1, we need to construct some appropriate light-tailed distributions, some of which come from associated heavy-tailed distributions defined by a certain transformation. For some constant $\gamma > 0$ and distribution F_0, we define the distribution F_γ of the form

$$\overline{F_\gamma}(x) = \mathbf{1}_{(-\infty,0)}(x) + \mathrm{e}^{-\gamma x}\overline{F_0}(x)\mathbf{1}_{[0,\infty)}(x) \quad \text{for } x \in (-\infty, \infty). \tag{4.1}$$

We might as well call F_γ the γ-transformation of F_0. Clearly, F_γ is light-tailed. In this way, we can characterize the light-tailed distribution F_γ through the corresponding heavy-tailed distribution F_0 with some good properties; see, for example, Klüppelberg [13] and Xu et al. [25]. Here we give some new properties of the distribution F_γ.

Lemma 4. *Let F_0, F_{10}, F_{20} be distributions, and let F_γ, $F_{1\gamma}$, $F_{2\gamma}$ be their γ-transformations with $\gamma > 0$. Then the following conclusions hold*:

(i) $\overline{F_\gamma}(x-t)/\overline{F_\gamma}(x) \geqslant \mathrm{e}^{\gamma t}$ *for each $t > 0$ and all $x \geqslant t$.*

(ii) *For each $t > 0$ and $i = 1, 2$, if*

$$\left|\overline{F_{i0}}(x-t) - \overline{F_{i0}}(x)\right| = o\bigl(\overline{F_{10} * F_{20}}(x)\bigr), \tag{4.2}$$

then

$$\left|\overline{F_{i\gamma}}(x-t) - \mathrm{e}^{\gamma t}\overline{F_{i\gamma}}(x)\right| = o\bigl(\overline{F_{1\gamma} * F_{2\gamma}}(x)\bigr).$$

(iii) *For $i \neq j$ and $i, j = 1, 2$, if $\mu(F_{j0}) = \int_0^\infty \overline{F_{j0}}(y)\,\mathrm{d}y = \infty$, then*

$$\overline{F_{i\gamma}}(x) = o\bigl(\overline{F_{1\gamma} * F_{2\gamma}}(x)\bigr). \tag{4.3}$$

Particularly, if $F_{10} = F_{20} = F_0$, then $F_{1\gamma} = F_{2\gamma} = F_\gamma \notin \mathcal{OS}$, and

$$\overline{F_\gamma^{*2}}(x) \sim \gamma \mathrm{e}^{-\gamma x} \int_{0-}^{x} \overline{F_0}(x-y)\overline{F_0}(y)\,\mathrm{d}y = \gamma \int_{0-}^{x} \overline{F_\gamma}(x-y)\overline{F_\gamma}(y)\,\mathrm{d}y. \tag{4.4}$$

Proof. We only prove (ii) and (iii).

(ii) For $i \neq j$ and $i, j = 1, 2$, according to (4.1),

$$\begin{aligned}
&\overline{F_{1\gamma} * F_{2\gamma}}(x) \\
&\quad = \overline{F_{i\gamma}}(x) + \int_{0-}^{x} \overline{F_{j\gamma}}(x-y)\, F_{i\gamma}(\mathrm{d}y) = \mathrm{e}^{-\gamma x}\Biggl(\overline{F_{10} * F_{20}}(x) + \gamma \int_{0-}^{x} \overline{F_{j0}}(x-y)\overline{F}_{i0}(y)\,\mathrm{d}y\Biggr) \\
&\quad = \mathrm{e}^{-\gamma x} S_0(x). \qquad (4.5)
\end{aligned}$$

Further, by (4.2) we have, for $1 \leqslant i \neq j \leqslant 2$,

$$\frac{|\overline{F}_{i\gamma}(x-t) - \mathrm{e}^{\gamma t}\overline{F}_{i\gamma}(x)|}{\overline{F_{1\gamma} * F_{2\gamma}}(x)} \leqslant \frac{\mathrm{e}^{\gamma t}|\overline{F}_{i0}(x-t) - \overline{F}_{i0}(x)|}{\overline{F_{10} * F_{20}}(x)} \to 0.$$

(iii) According to (4.5), we have that for $i \neq j$,

$$\overline{F_{i\gamma} * F_{j\gamma}}(x) \geqslant \mathrm{e}^{-\gamma x}\gamma\overline{F_{i0}}(x)\int_{0-}^{x}\overline{F_{j0}}(x-y)\,\mathrm{d}y = \gamma\overline{F_{i\gamma}}(x)\int_{0-}^{x}\overline{F_{j0}}(y)\,\mathrm{d}y.$$

Therefore since $\mu(F_{j0}) = \infty$, (4.3) holds. Particularly, $F_\gamma \notin \mathcal{OS}$, and (4.4) is proved. □

The following result plays a key role in proof of Theorem 1 and has its own independent value.

Theorem 7. *Let F_{i0} be an absolutely continuous distribution with density f_{i0} such that*

$$F_{i0}(x-t,\, x] = \overline{F_{i0}}(x-t) - \overline{F_{i0}}(x) = O\bigl(f_{i0}(x-t) + f_{i0}(x)\bigr) \quad \textit{for each } t > 0 \qquad (4.6)$$

and

$$\int_{x/2-}^{x} \overline{F}_{i0}(x-y)\, F_{j0}(\mathrm{d}y) = o\Biggl(\int_{x/2-}^{x} \overline{F}_{i0}(x-y)\overline{F}_{j0}(y)\,\mathrm{d}y\Biggr) \qquad (4.7)$$

for $i \neq j$ and $i, j = 1, 2$. Then for each $\gamma > 0$, $F_{1\gamma} * F_{2\gamma} \in \mathcal{L}(\gamma)$.

Proof. From (4.5) we know that $F_{1\gamma} * F_{2\gamma} \in \mathcal{L}(\gamma)$ is equivalent to the condition that the function S_0 in (4.5) belongs to the long-tailed function class

$$\mathcal{L}_d = \bigl\{f \text{ on } (-\infty, \infty)\colon\ f(x) > 0 \text{ for } x \text{ large enough, and } f(x-t) \sim f(x) \text{ for each } t > 0\bigr\}.$$

To this end, on the one hand, for each $t > 0$ and x large enough, by (4.5) we have

$$\begin{aligned}
&S_0(x-t) - S_0(x) \\
&\quad = \overline{F_{10} * F_{20}}(x-t) - \overline{F_{10} * F_{20}}(x) \\
&\qquad + \gamma \int_{0}^{(x-t)/2} \overline{F_{20}}(x-t-y)\overline{F_{10}}(y)\,\mathrm{d}y - \gamma \int_{0}^{x/2} \overline{F_{20}}(x-y)\overline{F_{10}}(y)\,\mathrm{d}y \\
&\qquad + \gamma \int_{0}^{(x-t)/2} \overline{F_{10}}(x-t-y)\overline{F_{20}}(y)\,\mathrm{d}y - \gamma \int_{0}^{x/2} \overline{F_{10}}(x-y)\overline{F_{20}}(y)\,\mathrm{d}y
\end{aligned}$$

$$\geqslant \gamma \int_0^{(x-t)/2} \big(\overline{F_{20}}(x-t-y) - \overline{F_{20}}(x-y)\big)\overline{F_{10}}(y)\,\mathrm{d}y - \gamma \int_{(x-t)/2}^{x/2} \overline{F_{20}}(x-y)\overline{F_{10}}(y)\,\mathrm{d}y$$

$$+ \gamma \int_0^{(x-t)/2} \big(\overline{F_{10}}(x-t-y) - \overline{F_{10}}(x-y)\big)\overline{F_{20}}(y)\,\mathrm{d}y - \gamma \int_{(x-t)/2}^{x/2} \overline{F_{10}}(x-y)\overline{F_{20}}(y)\,\mathrm{d}y$$

$$\geqslant -\gamma \int_{(x-t)/2}^{(x+t)/2} \overline{F_{20}}(x-y)\overline{F_{10}}(y)\,\mathrm{d}y - \gamma t \overline{F_{10}}\left(\frac{x-t}{2}\right)\overline{F_{20}}\left(\frac{x-t}{2}\right)$$

$$\geqslant -\gamma t \left(\int_{(x-t)/2}^{x-t} \overline{F}_{20}(x-t-y)\,F_{10}(\mathrm{d}y) + \overline{F}_{20}\left(\frac{x-t}{2}\right)\overline{F}_{10}(x-t)\right).$$

Thus by

$$S_0(x-t) \geqslant \gamma \int_{(x-t)/2}^{x-t} \overline{F}_{20}(x-t-y)\overline{F}_{10}(y)\,\mathrm{d}y \geqslant \gamma \overline{F}_{20}\left(\frac{x-t}{2}\right)\overline{F}_{10}(x-t)\frac{x-t}{2}$$

and (4.7) we get that

$$\liminf \frac{S_0(x-t) - S_0(x)}{S_0(x-t)} \geqslant 0. \tag{4.8}$$

On the other hand, by (4.6),

$$\overline{F_{10} * F_{20}}(x-t) - \overline{F_{10} * F_{20}}(x)$$

$$= \sum_{i=1}^{2} \big(\overline{F_{i0}}(x-t) - \overline{F_{i0}}(x)\big) + \int_{(x-t)/2}^{x-t} \overline{F}_{20}(x-t-y)\,F_{10}(\mathrm{d}y)$$

$$+ \int_{(x-t)/2}^{x-t} \overline{F}_{10}(x-t-y)\,F_{20}(\mathrm{d}y) - \overline{F_{10}}\left(\frac{x-t}{2}\right)\overline{F_{20}}\left(\frac{x-t}{2}\right) + \overline{F_{10}}\left(\frac{x}{2}\right)\overline{F_{20}}\left(\frac{x}{2}\right)$$

$$- \int_{x/2}^{x} \overline{F}_{20}(x-y)\,F_{10}(\mathrm{d}y) - \int_{x/2}^{x} \overline{F}_{10}(x-y)\,F_{20}(\mathrm{d}y)$$

$$\leqslant \sum_{i=1}^{2} \big(\overline{F_{i0}}(x-t) - \overline{F_{i0}}(x)\big) + \int_{(x-t)/2}^{x-t} \overline{F}_{20}(x-t-y)\,F_{10}(\mathrm{d}y)$$

$$+ \int_{(x-t)/2}^{x-t} \overline{F}_{10}(x-t-y)\,F_{20}(\mathrm{d}y),$$

and

$$\int_0^{(x-t)/2} \big(f_{i0}(x-t-y)+f_{i0}(x-y)\big)\overline{F_{j0}}(y)\,\mathrm{d}y$$
$$= \int_{(x-t)/2}^{x-t} \overline{F_{j0}}(x-t-y)\,F_{i0}(\mathrm{d}y) + \int_{(x+t)/2}^{x} \overline{F_{j0}}(x-y)\,F_{i0}(\mathrm{d}y)$$

for $i \neq j$ and $i,j=1,2$, we have

$$\begin{aligned}
&S_0(x-t)-S_0(x)\\
&\quad\leqslant \overline{F_{10}*F_{20}}(x-t)-\overline{F_{10}*F_{20}}(x)+\gamma\int_0^{(x-t)/2}\big(\overline{F_{20}}(x-t-y)-\overline{F_{20}}(x-y)\big)\overline{F_{10}}(y)\,\mathrm{d}y\\
&\qquad+\gamma\int_0^{(x-t)/2}\big(\overline{F_{10}}(x-t-y)-\overline{F_{10}}(x-y)\big)\overline{F_{20}}(y)\,\mathrm{d}y\\
&\quad\leqslant \sum_{i=1}^{2}\big(\overline{F_{i0}}(x-t)-\overline{F_{i0}}(x)\big)+\int_{(x-t)/2}^{x-t}\overline{F}_{20}(x-t-y)\,F_{10}(\mathrm{d}y)+\int_{(x-t)/2}^{x-t}\overline{F}_{10}(x-t-y)\,F_{20}(\mathrm{d}y)\\
&\qquad+\sum_{1\leqslant i\neq j\leqslant 2}O\bigg(\int_0^{(x-t)/2}\big(f_{i0}(x-t-y)+f_{i0}(x-y)\big)\,\overline{F_{j0}}(y)\,\mathrm{d}y\bigg)\\
&\quad=\sum_{1\leqslant i\neq j\leqslant 2}O\bigg(\int_{x-t}^{x}\overline{F}_{i0}(x-y)\,F_{j0}(\mathrm{d}y)+\int_{(x-t)/2}^{x-t}\overline{F}_{i0}(x-t-y)\,F_{j0}(\mathrm{d}y)+\int_{(x+t)/2}^{x}\overline{F_{i0}}(x-y)\,F_{j0}(\mathrm{d}y)\bigg)\\
&\quad=\sum_{1\leqslant i\neq j\leqslant 2}O\bigg(\int_{(x-t)/2}^{x}\overline{F}_{i0}(x-y)\,F_{j0}(\mathrm{d}y)+\int_{(x-t)/2}^{x-t}\overline{F}_{i0}(x-t-y)\,F_{j0}(\mathrm{d}y)\bigg)\\
&\quad=o\big(S_0(x)+S_0(x-t)\big).
\end{aligned}\tag{4.9}$$

From (4.8) we have that there is a positive constant $x_0=x_0(t)$ such that $S_0(x)\leqslant 2S_0(x-t)$ for all $x\geqslant x_0$. Thus by (4.9) we have

$$\limsup \frac{S_0(x-t)-S_0(x)}{S_0(x-t)} \leqslant 0. \tag{4.10}$$

Combining this with (4.8) and (4.10), we get that $S_0\in\mathcal{L}_d$. □

Remark 6. Here both $F_{1\gamma}$ and $F_{2\gamma}$ are not required to belong to the class $\mathcal{L}(\gamma)$, which is substantially different from Theorem 3 in Embrechts and Goldie [8], Theorem 1.1(1b) in Xu et al. [24], and Theorem 2 of the paper. For example, when $F_{10}=F_{20}=F_0$, there is a distribution $F_0\notin\mathcal{L}$ satisfying (4.6) and (4.7); see distributions in the classes $\mathcal{F}_1(0)$ below. Thus $F_{1\gamma}=F_{2\gamma}=F_\gamma\notin\mathcal{L}(\gamma)$, whereas $F_\gamma^{*2}\in\mathcal{L}(\gamma)$ by Theorem 7.

5 Proof of Theorem 1

Firstly, we construct a new light-tailed distribution class. Let F_0 be a distribution such that

$$\overline{F_0}(x) = \mathbf{1}_{(-\infty,a_0)}(x) + C\sum_{n=0}^{\infty}\left(\left(\sum_{i=n}^{\infty}\frac{1}{a_i^{\alpha}} - \frac{x-a_n}{a_n^{\alpha+1}}\right)\mathbf{1}_{[a_n,2a_n)}(x) + \left(\sum_{i=n+1}^{\infty}\frac{1}{a_i^{\alpha}}\right)\mathbf{1}_{[2a_n,a_{n+1})}(x)\right) \tag{5.1}$$

with density $f_0(x) = C\sum_{n=0}^{\infty} a_n^{-\alpha-1}\mathbf{1}_{[a_n,2a_n)}(x)$ for all x, where $C = (\sum_{n=0}^{\infty} a_n^{-\alpha})^{-1}$, $\alpha \in (3/2, (\sqrt{5}+1/2)$, $r = 1+1/\alpha$, $a^r > \max\{1, 8a\}$, and $a_n = a^{r^n}$ for all nonnegative integers.

Let $\mathcal{F}_1(0)$ be the class consisting of two-parametric heavy-tailed distributions $F_0 = F_0(\alpha, a)$ defined by (5.1). Then we define a new distribution class.

DEFINITION 1. We say that the distribution F_γ with the density f_γ for some constant $\gamma > 0$ belongs to the class $\mathcal{F}_1(\gamma)$ if the corresponding distribution F_0 in (4.1) belongs to the class $\mathcal{F}_1(0)$.

Secondly, we study the properties of distributions in the class $\mathcal{F}_1(\gamma)$. For simplicity, in the following text, we replace F_γ and f_γ by F and f, respectively.

Proposition 1. *If $F \in \mathcal{F}_1(\gamma)$ for some $\gamma > 0$, then*:

(i) *Condition* (2.26) *is satisfied, but condition* (2.28) *is not.*

(ii) *$F \in \mathcal{OL} \setminus (\mathcal{L}(\gamma) \cup \mathcal{OS})$ with $m(F) < \infty$, whereas $F^{*k} \in (\mathcal{L}(\gamma) \cap \mathcal{OS}) \setminus \mathcal{S}(\gamma)$ for all $k \geqslant 2$ and $F_0^{*k} \in \mathcal{OS} \setminus \mathcal{L}$ for all $k \geqslant 1$. Further, if condition* (2.31) *is satisfied with some nonnegative integer-valued random variable τ, then $F^{*\tau} \in (\mathcal{L}(\gamma) \cap \mathcal{OS}) \setminus \mathcal{S}(\gamma)$.*

Proof. (i) According to Lemma 4(i), condition (2.26) holds for F. However, condition (2.28) does not hold. Indeed, by (5.1)

$$\overline{F_0}(4a_n - y) = \overline{F_0}(2a_n) = \overline{F_0}(4a_n) \quad \text{for } y \in [0, 2a_n].$$

Then by (4.5), $F_0 \in \mathcal{OS}$, and $\mu(F_0) < \infty$ we have that, for n large enough,

$$\begin{aligned}\overline{F^{*2}}(4a_n) &= \mathrm{e}^{-4\gamma a_n}\left(\overline{F_0^{*2}}(4a_n) + 2\gamma\int_0^{2a_n}\overline{F_0}(4a_n - y)\overline{F_0}(y)\,\mathrm{d}y\right)\\ &\leqslant 2\big(C^*(F_0) + 2\gamma\mu(F_0)\big)\mathrm{e}^{-4\gamma a_n}\overline{F_0}(4a_n) = o\big(\overline{F}(4a_n)\big).\end{aligned}$$

(ii) Clearly, for $x \in [a_n, a_{n+1})$, since $r = 1 + \alpha^{-1}$ and $a_{n+1} = a_n^r$, $n \geqslant 0$, we have

$$\sup_{2a_{n-1}\leqslant y\leqslant a_{n+1}} f_0(y) = a_n^{-\alpha-1} = a_{n+1}^{-\alpha} \leqslant \overline{F_0}(a_{n+1}) \leqslant \overline{F_0}(x) = O\big(x^{-\alpha}\big). \tag{5.2}$$

Further, since $\alpha > 3/2$, we have $\mu(F_0) < \infty$, and thus $m(F) < \infty$. Then from

$$\lim_{n\to\infty}\frac{\overline{F_0}(2a_n - 1)}{\overline{F_0}(2a_n)} = 2 > 1$$

we conclude that $F_0 \notin \mathcal{L}$, and thus $F \notin \mathcal{L}(\gamma)$ for each $\gamma > 0$. In addition, by (5.1) and (5.2), for all $n \geqslant 0$ and $x \in [a_n, a_{n+1})$, we have

$$W_0(x) = \int_{x/2}^{x} \overline{F_0}(x-y)F_0(\mathrm{d}y) \leqslant \sup_{y\in[a_n/2,a_{n+1})} f_0(y) \int_{x/2}^{x} \overline{F}_0(x-y)\,\mathrm{d}y \leqslant \mu(F_0)\overline{F}_0(x),$$

and thus $F_0 \in \mathcal{OS}$. Further, according to Proposition 2.6 of [18], $F_0^{*k} \in \mathcal{OS}$ for all $k \geqslant 1$. Then according to Corollary 2, by $F_0 \notin \mathcal{L}$ we have that $F_0^{*k} \notin \mathcal{L}$ for all $k \geqslant 1$.

Now we show that $F^{*2} \in \mathcal{L}(\gamma)$. To this end, according to Theorem 7, since (4.6) holds, we just need to prove (4.7) with $F_{10} = F_{20} = F_0$, that is, $W_0(x) = o(T_0(x))$, where $T_0(x) = \int_{x/2}^{x} \overline{F_0}(x-y)\overline{F_0}(y)\,\mathrm{d}y$. Since $W_0(x) = 0$ for $x \in [4a_n, a_{n+1})$, $n \geqslant 0$, we only need to deal with $W_0(x)$ for x in the following two cases: $x \in [a_n, 3a_n)$ and $x \in [3a_n, 4a_n)$.

When $x \in [a_n, 3a_n)$, by $x/2 + a_n/4 < \min\{x, 7a_n/4\}$ we have

$$T_0(x) \geqslant \int_{x/2}^{x/2+a_n/4} \overline{F_0}(x-y)\overline{F_0}(y)\,\mathrm{d}y \geqslant \overline{F_0}\left(\frac{x}{2}\right)\overline{F_0}\left(\frac{x}{2}+\frac{a_n}{4}\right)\frac{a_n}{4} \geqslant \overline{F_0}\left(\frac{3a_n}{2}\right)\overline{F_0}\left(\frac{7a_n}{4}\right)\frac{a_n}{4}.$$

Thus by $\overline{F_0}(3a_n/2) \asymp a_n^{-\alpha} \asymp \overline{F_0}(7a_n/4)$ and $\alpha < (\sqrt{5}+1)/2 < 2$ we have that

$$W_0(x) = Ca_n^{-\alpha-1} \int_{\max\{a_n,x/2\}}^{\min\{x,2a_n\}} \overline{F_0}(x-y)\,\mathrm{d}y \leqslant \frac{4Ca_n^{-\alpha-2}\mu(F_0)T_0(x)}{\overline{F_0}(\frac{3a_n}{2})\overline{F_0}(\frac{7a_n}{4})} = o\bigl(T_0(x)\bigr). \tag{5.3}$$

When $x \in [3a_n, 4a_n)$, since

$$\int_{x/2}^{x} \overline{F_0}(x-y)\,\mathrm{d}y \leqslant \mu(F_0) < \infty, \qquad \int_{a_n}^{2a_n} \overline{F_0}(x-y)\,\mathrm{d}y \leqslant \overline{F_0}(a_n)a_n \to 0,$$

and $\overline{F_0}(y) \asymp a_n^{-\alpha-1}$ for $y \in [3a_n, 4a_n)$, we have

$$W_0(x) \leqslant Ca_n^{-\alpha-1} \int_{a_n}^{2a_n} \overline{F_0}(x-y)\,\mathrm{d}y = o\left(a_n^{-\alpha-1} \int_{x/2}^{x} \overline{F_0}(x-y)\,\mathrm{d}y\right) = o\bigl(T_0(x)\bigr). \tag{5.4}$$

Combining (5.3) and (5.4), we conclude that $F^{*2} \in \mathcal{L}(\gamma)$.

Further, according to Lemma 1, Corollary 1(i), and Theorem 4, we have that F^{*k} for all $k \geqslant 2$ and $F^{*\tau}$ satisfying (2.31) belong to the class $\mathcal{L}(\gamma)$.

In addition, according to Theorem 6(i), by $F^{*2} \in \mathcal{L}(\gamma)$ and $F \notin \mathcal{L}(\gamma)$ we have $F \notin \mathcal{OS}$.

In the following, we prove that $F^{*2}$and thus F^{*k} for all $k \geqslant 2$ belong to the class $\mathcal{OS}$. Since $\mu(F^{*2}) = 2\mu(F) \leqslant 2\mu(F_0) < \infty$, we can define an integrated tail distribution $(F^{*2})_I$ of F^{*2} such that

$$\overline{(F^{*2})_I}(x) = \mathbf{1}_{(-\infty,0)}(x) + \frac{\int_x^\infty \overline{F^{*2}}(y)\,\mathrm{d}y\,\mathbf{1}_{[0,\infty)}(x)}{\mu(F^{*2})}, \quad x \in (-\infty, \infty).$$

According to the Karamata theorem, from $F^{*2} \in \mathcal{L}(\gamma)$ we have that $\overline{F^{*2}}(x) \sim \mu(F^{*2})\overline{(F^{*2})_I}(x)$. Thus there are two constants $0 < K_1, K_2 < \infty$ such that

$$\sup_{x \geqslant 0} \frac{\overline{F^{*2}}(x)}{\overline{(F^{*2})_I}(x)} = K_1 \quad \text{and} \quad \inf_{x \geqslant 0} \frac{\overline{F^{*2}}(x)}{\overline{(F^{*2})_I}(x)} = K_2.$$

Integrating by parts, using the continuity of F^{*2}, we have

$$\begin{aligned}\overline{F^{*4}}(x) &= \int_0^x \overline{F^{*2}}(x-y)\, F^{*2}(\mathrm{d}y) + \overline{F^{*2}}(x) \leqslant K_1 \int_0^x \overline{(F^{*2})_I}(x-y)\, F^{*2}(\mathrm{d}y) + \overline{F^{*2}}(x) \\ &\leqslant K_1 \int_0^x \overline{F^{*2}}(x-y)\, (F^{*2})_I(\mathrm{d}y) + K_1 \overline{(F^{*2})_I}(x) + \overline{F^{*2}}(x) \\ &\leqslant K_1 \big(\mu(F^{*2})\big)^{-1} \int_0^x \overline{F^{*2}}(x-y)\, \overline{F^{*2}}(y)\,\mathrm{d}y + \left(1 + \frac{K_1}{K_2}\right)\overline{F^{*2}}(x).\end{aligned}$$

Similarly,

$$\overline{F^{*4}}(x) \geqslant K_2 \big(\mu(F^{*2})\big)^{-1} \int_0^x \overline{F^{*2}}(x-y)\overline{F^{*2}}(y)\,\mathrm{d}y - K_2 \overline{F^{*2}}(x).$$

Therefore $F^{*2} \in \mathcal{OS}$ is equivalent to

$$\int_{x/2}^x \overline{F^{*2}}(x-y)\overline{F^{*2}}(y)\,\mathrm{d}y = O\big(\overline{F^{*2}}(x)\big). \tag{5.5}$$

Further, from (4.5), $F_0 \in \mathcal{OS}$, and $\mu(F_0) < \infty$ we have

$$\begin{aligned}\overline{F^{*2}}(x) &= \mathrm{e}^{-\gamma x}\big(\overline{F_0^{*2}}(x) + 2\gamma T_0(x)\big) = O\big(\mathrm{e}^{-\gamma x}\big(\overline{F_0}(x) + 2\gamma T_0(x)\big)\big) \\ &= O\big(\mathrm{e}^{-\gamma x} T_0(x)\big),\end{aligned}$$

which implies $\overline{F^{*2}}(x) \asymp \mathrm{e}^{-\gamma x} T_0(x)$. Hence, to prove (5.5), we only need to prove that

$$R_0(x) = \int_{x/2}^x T_0(x-y) T_0(y)\,\mathrm{d}y = O\big(T_0(x)\big). \tag{5.6}$$

To prove (5.6), we need to study the properties of $T_0(x)$. For each $x \geqslant 4a_0$, there is a nonnegative integer n such that $x \in [4a_n, 4a_{n+1})$. By (5.1) and $\mu(F_0) < \infty$ we have

$$\begin{aligned}\overline{F_0}(x) \int_{x/2}^x \overline{F_0}(x-y)\,\mathrm{d}y \leqslant T_0(x) \leqslant \mu(F_0)\overline{F_0}\left(\frac{x}{2}\right) = O\big(\overline{F_0}(2a_n)\big) \\ = O\big(a_{n+1}^{-\alpha}\big) = O\big(x^{-\alpha}\big).\end{aligned} \tag{5.7}$$

Thus by $\alpha > 3/2$ and (5.7) we have $\int_0^\infty T_0(y)\,\mathrm{d}y < \infty$. In addition, for any $4a_0 < x_1 < x$,

$$T_0(x) \leqslant \left(\int_{x_1/2}^{x_1} + \int_{x_1}^{x}\right)\overline{F_0}(x-y)\overline{F_0}(y)\,\mathrm{d}y \leqslant T_0(x_1) + \overline{F_0}(x_1)\mu(F_0)$$
$$\leqslant T_0(x_1)\left(1 + \frac{\mu(F_0)}{\int_0^{x_1/2}\overline{F_0}(y)\,\mathrm{d}y}\right), \tag{5.8}$$

that is, $T_0(x)$ is almost decreasing. We also need the following properties of $T(x)$, the proof of which is given in Appendix.

Lemma 5.

(i) *When* $x \in [4a_{n-1}, 2a_n - 2a_n^{2-\alpha}]$, $T_0(x) \asymp \overline{F_0}(x)$.

(ii) *When* $x \in [3a_n/2, 2a_n - 2a_{n-1}) \cup [2a_n + 6a_{n-1}, 4a_n - 4a_{n-1}]$, $T_0(x - 4a_{n-1}) = O(T_0(x))$; *when* $x \in [2a_n - 2a_{n-1}, 2a_n + 6a_{n-1})$, $T_0(x - 4a_{n-1}) = O(a_n^{\alpha-2+\alpha/(1+\alpha)}T_0(x))$.

(iii) *When* $x \in [3a_n/2, 4a_n - 4a_{n-1})$, $T_0(x - a_n^{2-\alpha}) = O(T_0(x))$.

Now, we continue to prove (5.6) in the proof of Proposition 1 for the following four cases: $x \in [a_n, 3a_n/2)$, $x \in [3a_n/2, 3a_n)$, $x \in [3a_n, 4a_n - 4a_{n-1})$, and $x \in [4a_n - 4a_{n-1}, a_{n+1})$.

For the case $x \in [a_n, 3a_n/2)$, by (5.8), $\mu(T_0) < \infty$, (5.7), and (A.1) we have

$$R_0(x) = O\left(T_0\left(\frac{a_n}{2}\right)\int_{x/2}^{x} T_0(x-y)\,\mathrm{d}y\right) = O\big(a_n^{-\alpha}\big) = O\big(\overline{F_0}(x)\big) = O\big(T_0(x)\big). \tag{5.9}$$

For the case $x \in [3a_n/2, 3a_n)$, by (A.1) and (5.8) we have that

$$a_n^{1-2\alpha} = O\left(\int_{3a_n/2}^{7a_n/4}\overline{F_0}(3a_n - y)\overline{F_0}(y)\,\mathrm{d}y\right) = O\big(T_0(3a_n)\big) = O\big(T_0(x)\big). \tag{5.10}$$

Then by (5.7), (5.8), (ii), Lemma 5(iii), (5.10), and $\alpha < 2$ we get

$$R_0(x) = \left(\int_{x/2}^{x-4a_{n-1}} + \int_{x-4a_{n-1}}^{x-a_n^{2-\alpha}} + \int_{x-a_n^{2-\alpha}}^{x}\right)T_0(x-y)T_0(y)\,\mathrm{d}y$$
$$= O\left(T_0\left(\frac{3a_n}{4}\right)T_0(4a_{n-1})a_n + T_0(x-4a_{n-1})\int_{a_n^{2-\alpha}}^{4a_{n-1}}T_0(y)\,\mathrm{d}y + T_0\big(x - a_n^{2-\alpha}\big)\int_0^{a_n^{2-\alpha}}T_0(y)\,\mathrm{d}y\right)$$
$$= O\left(\left(\overline{F_0}^2(2a_{n-1})a_n + a_n^{\alpha-2+\alpha/(1+\alpha)}\int_{a_n^{2-\alpha}}^{4a_{n-1}}y^{-\alpha}\,\mathrm{d}y + 1\right)T_0(x)\right)$$
$$= O\big(a_n^{1-2\alpha} + \big(a_n^{\alpha(\alpha-2)} + 1\big)T_0(x)\big) = O\big(T_0(x)\big). \tag{5.11}$$

For the case $x \in [3a_n, 4a_n - 4a_{n-1})$, by $a_n^{2-\alpha} < a_{n-1}$ we have $x - y \in [x - 2a_n + 2a_n^{2-\alpha},\, x/2) \subset [4a_{n-1},\, 2a_n - 2a_n^{2-\alpha})$ for $y \in [x/2,\, 2a_n - 2a_n^{2-\alpha})$. Thus, according to Lemma 5(i) and (ii), by (A.1), (5.7), and $\alpha > 3/2$,

$$R_0(x) = \left(\int_{x/2}^{2a_n-2a_n^{2-\alpha}} + \int_{2a_n-2a_n^{2-\alpha}}^{x-4a_{n-1}} + \int_{x-4a_{n-1}}^{x}\right) T_0(x-y)T_0(y)\,\mathrm{d}y$$
$$= O\left(\int_{x/2}^{2a_n-2a_n^{2-\alpha}} \overline{F_0}(x-y)\overline{F_0}(y)\,\mathrm{d}y + a_n T_0\big(2a_n - 2a_n^{2-\alpha}\big)T_0(4a_{n-1}) + T_0(x-4a_{n-1})\int_0^{4a_{n-1}} T_0(y)\,\mathrm{d}y\right)$$
$$= O\big(T_0(x) + a_n\overline{F_0}\big(2a_n - 2a_n^{2-\alpha}\big)\overline{F_0}(4a_{n-1}) + T_0(x)\big)$$
$$= O\big(T_0(x) + a_n^{3-2\alpha}a_n^{-1-\alpha}\big) = O\big(T_0(x) + a_n^{3-2\alpha}\overline{F_0}(x)\big) = O\big(T_0(x)\big). \tag{5.12}$$

For the case $x \in [4a_n - 4a_{n-1}, a_{n+1})$, we first deal with $T_0(4a_n - 4a_{n-1})$. By $\mu(T_0) < \infty$, (A.1), and $-1-\alpha+3\alpha/(1+\alpha) < 0$ we have that

$$T_0(4a_n - 4a_{n-1})$$
$$= \left(\int_{2a_n-2a_{n-1}}^{2a_n} + \int_{2a_n}^{4a_n-4a_{n-1}}\right)\overline{F_0}(4a_n - 4a_{n-1} - y)\overline{F_0}(y)\,\mathrm{d}y$$
$$\asymp a_n^{-1-\alpha}\int_{2a_n-2a_{n-1}}^{2a_n} \overline{F_0}(4a_n - 4a_{n-1} - y)(1 + 2a_n - y)\,\mathrm{d}y + \overline{F_0}(2a_n)$$
$$\leqslant a_n^{-1-\alpha}a_{n-1}\left(\int_{2a_n-4a_{n-1}}^{2a_n} + \int_{2a_n}^{2a_n+a_{n-1}}\right)\overline{F_0}(y)\,\mathrm{d}y + \overline{F_0}(a_{n+1})$$
$$\asymp \overline{F_0}(a_{n+1})a_{n-1}\left(a_n^{-1-\alpha}\int_{2a_n-4a_{n-1}}^{2a_n} (1 + 2a_n - y)\,\mathrm{d}y + \overline{F_0}(a_{n+1})a_{n-1}\right) + \overline{F_0}(a_{n+1})$$
$$\sim \overline{F_0}(a_{n+1})a_n^{-1-\alpha+3\alpha/(1+\alpha)} + \overline{F_0}(a_{n+1}) = O\big(T_0(x)\big).$$

Then by (5.8), Lemma 5, $\mu(T_0) < \infty$, (5.7), and (A.1) we have

$$R_0(x) \leqslant \left(\int_{2a_n-2a_{n-1}}^{2a_n-2a_n^{2-\alpha}} + \int_{2a_n-2a_n^{2-\alpha}}^{4a_n-8a_{n-1}} + \int_{4a_n-8a_{n-1}}^{x}\right) T_0(x-y)T_0(y)\,\mathrm{d}y$$
$$= O\big(a_{n-1}T_0^2(2a_n - 2a_{n-1}) + T_0\big(2a_n - 2a_n^{2-\alpha}\big)T_0(4a_{n-1})a_n + T_0(4a_n - 8a_{n-1})\big)$$
$$= O\big(a_{n-1}\overline{F_0}^2(2a_n - 2a_{n-1}) + \overline{F_0}\big(2a_n - 2a_n^{2-\alpha}\big)a_n T_0(4a_n - 8a_{n-1}) + T_0(4a_n - 4a_{n-1})\big)$$
$$= O\big(\big(a_n^{-\alpha-1+3\alpha/(\alpha+1)}\overline{F_0}(a_{n+1}) + a_n^{2-2\alpha}T_0(x) + T_0(x)\big) = O\big(T_0(x)\big). \tag{5.13}$$

Combining this with (5.9)–(5.13), we conclude that (5.6) holds, that is, $F^{*2} \in \mathcal{OS}$.

Finally, we prove that $F^{*k} \notin \mathcal{S}(\gamma)$ for all $k \geqslant 2$. On the one hand, successively by

$$f^{\otimes 2}(x) = 2\mathrm{e}^{-\gamma x} \int_{x/2}^{x} \bigl(\gamma \overline{F_0}(x-y) + f_0(x-y)\bigr)\bigl(\gamma \overline{F_0}(y) + f_0(y)\bigr)\,\mathrm{d}y \geqslant 2\gamma^2 \mathrm{e}^{-\gamma x} T_0(x),$$

(4.5), (5.7), (A.1), and (A.2) we have

$$\begin{aligned}
&\int_{a_n}^{2a_n} \overline{F^{*2}}(2a_n - y)\, F^{*2}(\mathrm{d}y) \\
&\quad \geqslant 2\gamma^2 \int_{a_n}^{2a_n} \overline{F^{*2}}(2a_n - y)\mathrm{e}^{-\gamma y} T_0(y)\,\mathrm{d}y \asymp \mathrm{e}^{-\gamma 2a_n} \int_{a_n}^{2a_n} T_0(2a_n - y) T_0(y)\,\mathrm{d}y \\
&\quad \geqslant \left(\int_0^{a_n} \overline{F_0}(y)\,\mathrm{d}y \right)^2 \mathrm{e}^{-\gamma 2a_n} \int_{a_n}^{2a_n} \overline{F_0}(2a_n - y)\overline{F_0}(y)\,\mathrm{d}y \\
&\quad \asymp \mathrm{e}^{-\gamma 2a_n} a_n^{-1-\alpha} \int_{a_n}^{2a_n} \overline{F_0}(2a_n - y)(1 + 2a_n - y)\,\mathrm{d}y \\
&\quad = \mathrm{e}^{-\gamma 2a_n} a_n^{-1-\alpha} \int_0^{a_n} \overline{F_0}(y)(1 + y)\,\mathrm{d}y \sim \mathrm{e}^{-\gamma 2a_n} a_n^{1-2\alpha}.
\end{aligned}$$

On the other hand, by (4.5) we have

$$\overline{F^{*2}}(2a_n) \asymp \mathrm{e}^{-\gamma 2a_n} T_0(2a_n) = \mathrm{e}^{-\gamma 2a_n} \int_{a_n}^{2a_n} \overline{F_0}(2a_n - y)\overline{F_0}(y)\,\mathrm{d}y \asymp \mathrm{e}^{-\gamma 2a_n} a_n^{1-2\alpha}.$$

Combining the above two facts and

$$\overline{F^{*4}}(2a_n) \sim 2\overline{F^{*2}}(2a_n) + 2 \int_{a_n}^{2a_n} \overline{F^{*2}}(2a_n - y)\, F^{*2}(\mathrm{d}y),$$

we have $F^{*2} \notin \mathcal{S}(\gamma)$. Then since $F^{*k} \in \mathcal{L}(\gamma) \cap \mathcal{OS}$ for each $k \geqslant 2$, $\overline{F^{*2}}(x) \asymp \overline{F^{*k}}(x)$ for each $k \geqslant 3$, and by Lemma 2.6 in [20] and $F \notin \mathcal{S}(\gamma)$ we get that $F^{*k} \notin \mathcal{S}(\gamma)$ for all $k \geqslant 2$.

So far, we have completed the proof of Proposition 1. □

Proof of Theorem 1. Consider any distribution $F \in \mathcal{F}_1(\gamma)$. According to Proposition 1 and Remark 4(ii), we have $H_2 = F^{\tau} \in (\mathcal{L}(\gamma) \cap \mathcal{OS}) \setminus \mathcal{S}(\gamma)$ with $m(H_2) < \infty$. Since $\overline{H_1}(x) = O(\mathrm{e}^{-\beta x})$ for each $\beta > 0$, we have that $\overline{H_1}(x) = o(\overline{H_2}(x))$. Further, according to Lemma 2.1 of Pakes [15] or Theorem 1.1(i) of Cheng et al. [3], the infinitely divisible distribution

$$\overline{H}(x) = \overline{H_1 * H_2}(x) \sim m(H_1)\overline{H_2}(x). \tag{5.14}$$

Therefore $H \in \bigl(\mathcal{L}(\gamma) \cap \mathcal{OS}\bigr) \setminus \mathcal{S}(\gamma)$ with $m(H) < \infty$.

Further, by (5.14), according to Proposition 1, Theorem 4, and Theorem 5, (1.1) holds. □

Appendix

Here we prove Lemma 5.

(i) When $x \in [4a_{n-1}, 2a_n - 2a_n^{2-\alpha}]$, by (5.7) we only need to prove $T_0(x) = O(\overline{F_0}(x))$. The conclusion follows from the following (A.3) and (A.4). To this end, we first give two facts. For $x \in [a_n, a_{n+1}]$, by (5.1) and $\sum_{i=n+2}^{\infty} a_i^{-\alpha} \asymp a_{n+2}^{-\alpha} = a_{n+1}^{-\alpha-1}$ we have that

$$\begin{aligned}\overline{F}_0(x) &= C\Biggl(\sum_{i=n+2}^{\infty} a_i^{-\alpha} + a_n^{-\alpha-1}(1+2a_n-x)\mathbf{1}_{[a_n,2a_n)}(x) + a_n^{-\alpha-1}\mathbf{1}_{[2a_n,a_{n+1}]}(x)\Biggr)\\ &\sim C\bigl(a_n^{-\alpha-1}(1+2a_n-x)\mathbf{1}_{[a_n,2a_n)}(x) + a_n^{-\alpha-1}\mathbf{1}_{[2a_n,a_{n+1}]}(x)\bigr)\\ &\geqslant 2Ca_n^{1-2\alpha}\mathbf{1}_{[a_n,\,2a_n-2a_n^{2-\alpha}]}(x),\end{aligned} \tag{A.1}$$

and by the generalized l'Hôspital rule and (5.7) we get that

$$\limsup \frac{\int_0^x \overline{F_0}(y)y\,\mathrm{d}y}{x^{2-\alpha}} \leqslant \limsup \frac{\overline{F_0}(x)x^{\alpha}}{2-\alpha} < \infty, \tag{A.2}$$

that is, $\int_0^x \overline{F_0}(y)y\,\mathrm{d}y = O(x^{2-\alpha})$.

To deal with $T_0(x)$, we further split $[4a_{n-1}, 2a_n - 2a_n^{2-\alpha}] = [4a_{n-1}, 3a_n/2) \cup [3a_n/2,\, 2a_n - 2a_n^{2-\alpha})$. For $x \in [4a_{n-1}, 3a_n/2)$ and n large enough, by the second expression in (A.1) we have

$$\begin{aligned}T_0(x) &\leqslant \overline{F_0}\biggl(\frac{x}{2}\biggr)\int_{x/2}^{x} \overline{F_0}(x-y)\,\mathrm{d}y = O\bigl(\overline{F_0}(2a_{n-1})\bigr) = O\bigl(\overline{F_0}(a_n)\bigr)\\ &= O\biggl(\overline{F_0}\biggl(\frac{3a_n}{2}\biggr)\biggr) = O\bigl(\overline{F_0}(x)\bigr).\end{aligned} \tag{A.3}$$

For $x \in [3a_n/2,\, 2a_n - 2a_n^{2-\alpha})$ and n large enough, by (A.1) and (A.2) we have

$$\begin{aligned}T_0(x) &= \biggl(\int_{x/2}^{a_n} + \int_{a_n}^{x}\biggr)\overline{F_0}(x-y)\overline{F_0}(y)\,\mathrm{d}y\\ &= O\biggl(\overline{F_0}\biggl(\frac{a_n}{2}\biggr)\overline{F_0}\biggl(\frac{3a_n}{4}\biggr)a_n + \int_{a_n}^{x}\overline{F_0}(x-y)a_n^{-\alpha-1}\bigl((1+2a_n-x)+(x-y)\bigr)\,\mathrm{d}y\biggr)\\ &= O\biggl(a_n^{1-2\alpha} + \overline{F_0}(x)\mu(F_0) + a_n^{-\alpha-1}\int_0^{a_n}\overline{F_0}(y)y\,\mathrm{d}y\biggr)\\ &= O\bigl(\overline{F_0}(x) + a_n^{1-2\alpha}\bigr) = O\bigl(\overline{F_0}(x)\bigr).\end{aligned} \tag{A.4}$$

(ii) When $x \in [3a_n/2,\, 2a_n - 2a_{n-1}]$ and n is large enough, $x - 4a_{n-1} - y \in [a_n,\, 2a_n - 6a_{n-1}]$ for $y \in [0,\, x - 4a_{n-1} - a_n]$. Further, because $\alpha \geqslant 3/2$,

$$\frac{a^{2-\alpha}}{a_{n-1}} = a_n^{2-\alpha-\alpha/(1+\alpha)} = a_n^{1-\alpha+1/(1+\alpha)} \leqslant a_n^{1-3/2+2/5} \to 0,$$

and thus $x \in [3a_n/2,\, 2a_n - 2a_{n-1}] \subset [a_n,\, 2a_n - 2a_n^{2-\alpha}]$. Therefore by (A.1), (A.2), and $-1-\alpha+\alpha/(1+\alpha) < 1-2\alpha$ we have

$$\begin{aligned}
&T_0(x-4a_{n-1}) \\
&\quad= \left(\int\limits_{(x-4a_{n-1})/2}^{a_n} + \int\limits_{a_n}^{x-4a_{n-1}}\right)\overline{F_0}(x-4a_{n-1}-y)\overline{F_0}(y)\,\mathrm{d}y \\
&\quad= O\left(\overline{F_0}\left(\frac{a_n}{2}-4a_{n-1}\right)\overline{F_0}\left(\frac{3a_n}{4}-2a_{n-1}\right)a_n + \int\limits_0^{x-4a_{n-1}-a_n}\overline{F_0}(y)\overline{F_0}(x-4a_{n-1}-y)\,\mathrm{d}y\right) \\
&\quad= O\left(a_n^{1-2\alpha} + a_n^{-1-\alpha}\int\limits_0^{x-4a_{n-1}-a_n}\overline{F_0}(y)\big((1+2a_n-x)+4a_{n-1}+y\big)\,\mathrm{d}y\right) \\
&\quad= O\left(a_n^{1-2\alpha} + \overline{F_0}(x)\mu(F_0) + 4a_{n-1}a_n^{-1-\alpha}\mu(F_0) + a_n^{-1-\alpha}\int\limits_0^{a_n}\overline{F_0}(y)y\,\mathrm{d}y\right) \\
&\quad= O\big(a_n^{1-2\alpha} + \overline{F_0}(x) + a_n^{-1-\alpha+\alpha/(1+\alpha)} + a_n^{1-2\alpha}\big) = O\big(\overline{F_0}(x)\big) = O\big(T_0(x)\big).
\end{aligned} \tag{A.5}$$

When $x \in [2a_n - 2a_{n-1},\, 2a_n + 6a_{n-1})$, by (A.1) and (A.2) we have that

$$\begin{aligned}
&T_0(2a_n-6a_{n-1}) \\
&\quad= \left(\int\limits_{a_n-3a_{n-1}}^{a_n} + \int\limits_{a_n}^{2a_n-6a_{n-1}}\right)\overline{F_0}(2a_n-6a_{n-1}-y)\overline{F_0}(y)\,\mathrm{d}y \\
&\quad\asymp \overline{F_0}(a_n)\int\limits_{a_n-6a_{n-1}}^{a_n-3a_{n-1}}\overline{F_0}(y)\,\mathrm{d}y + a_n^{-1-\alpha}\int\limits_{a_n}^{2a_n-6a_{n-1}}\overline{F_0}(2a_n-6a_{n-1}-y)(1+2a_n-y)\,\mathrm{d}y \\
&\quad\asymp \overline{F_0}^2(a_n)a_{n-1} + a_n^{-1-\alpha}\int\limits_0^{a_n-6a_{n-1}}\overline{F_0}(y)(1+6a_{n-1}+y)\,\mathrm{d}y \\
&\quad\asymp \overline{F_0}^2(a_n)a_{n-1} + a_n^{-1-\alpha} + a_n^{-1-\alpha}a_{n-1} + a_n^{-1-\alpha}\int\limits_0^{a_n-6a_{n-1}}\overline{F_0}(y)y\,\mathrm{d}y \\
&\quad\asymp a_n^{-2\alpha+\alpha/(1+\alpha)} + a_n^{-1-\alpha} + a_n^{-1-\alpha+\alpha/(1+\alpha)} + a_n^{1-2\alpha} \sim a_n^{-1-\alpha+\alpha/(1+\alpha)}
\end{aligned}$$

and

$$\begin{aligned}
&T_0(2a_n+6a_{n-1}) \\
&\quad= \left(\int\limits_{a_n+3a_{n-1}}^{2a_n} + \int\limits_{2a_n}^{2a_n+6a_{n-1}}\right)\overline{F_0}(2a_n+6a_{n-1}-y)\overline{F_0}(y)\,\mathrm{d}y \\
&\quad\sim Ca_n^{-1-\alpha}\int\limits_{a_n+3a_{n-1}}^{2a_n}\overline{F_0}(2a_n+6a_{n-1}-y)(1+2a_n-y)\,\mathrm{d}y + \overline{F_0}(2a_n)\mu(F_0)
\end{aligned}$$

$$
\begin{aligned}
&= Ca_n^{-1-\alpha}\left(\int_{a_n+3a_{n-1}}^{2a_n} \overline{F_0}(2a_n+6a_{n-1}-y)(2a_n-y)\,\mathrm{d}y + 1 + \mu(F_0)\right)\\
&= Ca_n^{-1-\alpha}\left(\int_{6a_{n-1}}^{a_n} \overline{F_0}(y)(y-6a_{n-1})\,\mathrm{d}y + \int_{a_n}^{a_n+3a_{n-1}} \overline{F_0}(y)(y-6a_{n-1})\,\mathrm{d}y + 1 + \mu(F_0)\right)\\
&\sim Ca_n^{-1-\alpha}\left(\overline{F_0}(a_n)\int_{6a_{n-1}}^{a_n} (y-6a_{n-1})\,\mathrm{d}y\right.\\
&\quad\left. + Ca_n^{-1-\alpha}\int_{a_n}^{a_n+3a_{n-1}} (1+2a_n-y)(y-6a_{n-1})\,\mathrm{d}y + 1 + \mu(F_0)\right)\\
&\asymp a_n^{1-2\alpha} + a_n^{-2\alpha+\alpha/(1+\alpha)} + a_n^{-1-\alpha} \sim a_n^{1-2\alpha}.
\end{aligned}
$$

Then since $x-4a_{n-1} \in [2a_n-6a_{n-1}, 2a_n+2a_{n-1}]$ for $x \in [2a_n-2a_{n-1}, 2a_n+6a_{n-1}]$, we have

$$
\begin{aligned}
T_0(x-4a_{n-1}) &= \frac{T_0(x-4a_{n-1})T_0(x)}{T_0(x)} = O\left(\frac{T_0(2a_n-6a_{n-1})T_0(x)}{T_0(2a_n+6a_{n-1})}\right)\\
&= O\big(a_n^{\alpha-2+\alpha/(1+\alpha)}T_0(x)\big).
\end{aligned}
\tag{A.6}
$$

When $x \in [2a_n+6a_{n-1},\, 4a_n-4a_{n-1}]$, successively by (A.1), $[(x-4a_{n-1})/2,\, 2a_n) \subset [a_n, 2a_n)$ and $y-4a_{n-1} \in [2a_n,\, 4a_n-8a_{n-1})$ for $y \in [2a_n+4a_{n-1},\, x)$, $x/2 < 2a_n+4a_{n-1}$, $x-2a_n < (x-4a_{n-1})/2$, and $1+2a_n-x+4a_{n-1}+y \leqslant 1+4a_{n-1}$ for $y \in [x-4a_{n-1}-2a_n,\, x-2a_n]$, and $x-y \in [a_n, 2a_n)$ for $y \in [x-2a_n,\, x/2)$. Therefore we have

$$
\begin{aligned}
&T_0(x-4a_{n-1}) \qquad\qquad\qquad\qquad (\text{A.7})\\
&\quad= \left(\int_{(x-4a_{n-1})/2}^{2a_n} + \int_{2a_n}^{x-4a_{n-1}}\right)\overline{F_0}(x-4a_{n-1}-y)\overline{F_0}(y)\,\mathrm{d}y\\
&\quad\sim \int_{(x-4a_{n-1})/2}^{2a_n} \overline{F_0}(x-4a_{n-1}-y)Ca_n^{-\alpha-1}(1+2a_n-y)\,\mathrm{d}y\\
&\qquad + \int_{2a_n+4a_{n-1}}^{x} \overline{F_0}(x-y)\overline{F_0}(y-4a_{n-1})\,\mathrm{d}y\\
&\quad= \int_{x-4a_{n-1}-2a_n}^{(x-4a_{n-1})/2} \overline{F_0}(y)Ca_n^{-\alpha-1}(1+2a_n-x+4a_{n-1}+y)\,\mathrm{d}y + \int_{2a_n+4a_{n-1}}^{x} \overline{F_0}(x-y)\overline{F_0}(y)\,\mathrm{d}y\\
&\quad\leqslant \left(\int_{x-4a_{n-1}-2a_n}^{x-2a_n} + \int_{x-2a_n}^{(x-4a_{n-1})/2}\right)\overline{F_0}(y)Ca_n^{-\alpha-1}(1+2a_n-x+4a_{n-1}+y)\,\mathrm{d}y + T_0(x)
\end{aligned}
$$

$$= \int_{x-4a_{n-1}-2a_n}^{x-2a_n} \overline{F_0}(y) a_n^{-\alpha-1}(1+4a_{n-1})\,\mathrm{d}y + \int_{x-2a_n}^{(x-4a_{n-1})/2} \overline{F_0}(y) a_n^{-\alpha-1} 4a_{n-1}\,\mathrm{d}y$$
$$+ \int_{x+a_{n-1}-2a_n}^{(x-4a_{n-1})/2} \overline{F_0}(y) a_n^{-\alpha-1}(1+2a_n-x+y)\,\mathrm{d}y + T_0(x)$$
$$= O\Biggl(a_n^{-\alpha-1}a_{n-1} \int_{x-4a_{n-1}-2a_n}^{(x-4a_{n-1})/2} \overline{F_0}(y)\,\mathrm{d}y + \int_{x-2a_n}^{x/2} \overline{F_0}(y)\overline{F_0}(x-y)\,\mathrm{d}y + T_0(x)\Biggr)$$
$$= O\Biggl(a_n^{-\alpha-1}a_{n-1}^2 \overline{F_0}(x-4a_{n-1}-2a_n) + \int_{x/2}^{2a_n} \overline{F_0}(x-y)\overline{F_0}(y)\,\mathrm{d}y + T_0(x)\Biggr)$$
$$= O\bigl(a_n^{-\alpha-1}a_{n-1}^2 \overline{F_0}(x-4a_{n-1}-2a_n) + T_0(x)\bigr). \tag{A.8}$$

Next, we deal with $\overline{F_0}(x-4a_{n-1}-2a_n)$ in two cases $x \in [2a_n+6a_{n-1},\, 3a_n+4a_{n-1})$ and $x \in [3a_n+4a_{n-1},\, 4a_n-4a_{n-1})$. For $x \in [2a_n+6a_{n-1},\, 3a_n+4a_{n-1})$, that is, $x-4a_{n-1}-2a_n \in [2a_{n-1}, a_n)$, by (A.1) we have that

$$\overline{F_0}(x-4a_{n-1}-2a_n) = \overline{F_0}(a_n) \asymp \overline{F_0}(a_n+6a_{n-1}) \leqslant \overline{F_0}(x+2a_{n-1}-2a_n).$$

For $x \in [3a_n+4a_{n-1},\, 4a_n-4a_{n-1})$, that is, $x-4a_{n-1}-2a_n \in [a_n,\, 2a_n-8a_{n-1})$ and $x+2a_{n-1}-2a_n \in [a_n+6a_{n-1},\, 2a_n-2a_{n-1})$, by (A.1) we have that

$$\begin{aligned}\overline{F_0}(x-4a_{n-1}-2a_n) &\asymp a_n^{-\alpha-1}(1+4a_{n-1}+4a_n-x) \asymp a_n^{-\alpha-1}(1-2a_{n-1}+4a_n-x)\\ &\asymp \overline{F_0}(x+2a_{n-1}-2a_n).\end{aligned}$$

Thus, for $x \in [2a_n+6a_{n-1},\, 4a_n-4a_{n-1})$, successively by $a_{n-1} \leqslant 1+2a_n-x+y$ and $x-y \in [2a_n-2a_{n-1},\, 2a_n-a_{n-1}]$ for $y \in [x+a_{n-1}-2a_n,\, x+2a_{n-1}-2a_n)$, $x+2a_{n-1}-2a_n > x/2$, and $x+2a_{n-1}-2a_n < x$ we have

$$\begin{aligned}T_0(x-4a_{n-1}) &= O\bigl(a_n^{-\alpha-1}a_{n-1}^2\overline{F_0}(x+2a_{n-1}-2a_n) + T_0(x)\bigr)\\
&= O\Biggl(\int_{x+a_{n-1}-2a_n}^{x+2a_{n-1}-2a_n} \overline{F_0}(y)a_n^{-\alpha-1}(1+2a_n-x+y)\,\mathrm{d}y + T_0(x)\Biggr)\\
&= O\Biggl(\int_{x+a_{n-1}-2a_n}^{x+2a_{n-1}-2a_n} \overline{F_0}(y)\overline{F_0}(x-y)\,\mathrm{d}y + T_0(x)\Biggr)\\
&= O\Biggl(\int_{x/2}^{x} \overline{F_0}(y)\overline{F_0}(x-y)\,\mathrm{d}y + T_0(x)\Biggr) = O\bigl(T_0(x)\bigr).\end{aligned}$$

Therefore conclusion (ii) follows from (A.5)–(A.8).

(iii) To deal with $T_0(x-a_n^{2-\alpha})$, we further split $[3a_n/2,\, 4a_n-4a_{n-1}) = [3a_n/2,\, 2a_n-2a_{n-1}) \cup [2a_n-2a_{n-1},\, 2a_n-2a_n^{2-\alpha}) \cup [2a_n-2a_n^{2-\alpha},\, 2a_n+6a_{n-1}) \cup [2a_n+6a_{n-1},\, 4a_n-4a_{n-1}]$.

For $x \in [3a_n/2,\ 2a_n - 2a_{n-1}) \cup [2a_n + 6a_{n-1},\ 4a_n - 4a_{n-1}]$, by (5.8), $a_n^{2-\alpha} = o(a_{n-1})$, and (ii) we have

$$T_0\big(x - a_n^{2-\alpha}\big) = O\big(T_0(x - 4a_{n-1})\big) = O\big(T_0(x)\big).$$

For $x \in [2a_n - 2a_{n-1},\ 2a_n - 2a_n^{2-\alpha})$, by $x - a_n^{2-\alpha} \in [2a_n - 2a_{n-1} - a_n^{2-\alpha},\ 2a_n - 2a_n^{2-\alpha} - a_n^{2-\alpha}) \subset [4a_{n-1},\ 2a_n - 2a_n^{2-\alpha})$, (i), $a_n^{2-\alpha} = o(a_{n-1})$, and (A.1) we have

$$T_0\big(x - a_n^{2-\alpha}\big) \asymp \overline{F_0}\big(x - a_n^{2-\alpha}\big) = O\big(a_n^{1-2\alpha} + \overline{F_0}(x)\big) = O\big(\overline{F_0}(x)\big) = O\big(T_0(x)\big).$$

For $x \in [2a_n - 2a_n^{2-\alpha},\ 2a_n + 6a_{n-1})$, by (5.8), $x - a_n^{2-\alpha} \geqslant 2a_n - 3a_n^{2-\alpha}$, (A.1), and (A.2) we have that

$$\begin{aligned}
T_0\big(x - a_n^{2-\alpha}\big) &= O\big(T_0\big(2a_n - 3a_n^{2-\alpha}\big)\big) \\
&= O\left(\left(\int\limits_{a_n - 3a_n^{2-\alpha}/2}^{a_n} + \int\limits_{a_n}^{2a_n - 3a_n^{2-\alpha}}\right)\overline{F_0}\big(2a_n - 3a_n^{2-\alpha} - y\big)\overline{F_0}(y)\,\mathrm{d}y\right) \\
&= O\left(\overline{F_0}^2(a_n)a_n^{2-\alpha} + a_n^{-1-\alpha}\int\limits_{a_n}^{2a_n - 3a_n^{2-\alpha}} \overline{F_0}\big(2a_n - 3a_n^{2-\alpha} - y\big)(1 + 2a_n - y)\,\mathrm{d}y\right) \\
&= O\left(a_n^{2-3\alpha} + a_n^{-1-\alpha}\int\limits_0^{a_n - 3a_n^{2-\alpha}} \overline{F_0}(y)\big(1 + 3a_n^{2-\alpha} + y\big)\,\mathrm{d}y\right) \\
&= O\left(a_n^{2-3\alpha} + a_n^{1-2\alpha} + a_n^{-1-\alpha}\int\limits_0^{a_n - 3a_n^{2-\alpha}} \overline{F_0}(y)y\,\mathrm{d}y\right) = O\big(a_n^{1-2\alpha}\big).
\end{aligned}$$

In addition, we know that $T_0(2a_n + 6a_{n-1}) = O(T_0(x))$; see proof in (ii). Therefore

$$T_0\big(x - a_n^{2-\alpha}\big) = \frac{T_0(x - a_n^{2-\alpha})T_0(x)}{T_0(x)} = O\left(\frac{T_0(2a_n - 3a_n^{2-\alpha})T_0(x)}{T_0(2a_n + 6a_{n-1})}\right) = O\big(T_0(x)\big).$$

Based on the above results, conclusion (iii) holds.

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