DOI 10.1007/s10986-021-09515-y
Lithuanian Mathematical Journal, Vol. 61, No. 2, April, 2021, pp. 145–160

Asymptotic behavior of maxima of independent random variables. Discrete case

Kateryna Akbash [a], **Natalia Doronina** [b], **and Ivan Matsak** [b]

[a] Volodymyr Vynnychenko Central Ukrainian State Pedagogical University,
1, Shevchenko street, Kropyvnytskyi 25006, Ukraine
[b] Taras Shevchenko National University of Kyiv, 2/6, Academician Glushkov avenue, Kyiv 03127, Ukraine
(e-mail: kateryna.akbash@gmail.com; amalphela@gmail.com; ivanmatsak@univ.kiev.ua)

Received April 23, 2020; revised October 2, 2020

Abstract. We study the asymptotic behavior of almost surely extreme values of discrete random variables. We give applications to birth and death processes and processes describing the length of the queue.

MSC: 60K25, 60F15, 60G70

Keywords: extreme values, discrete random variables, almost sure limit theorems

1 Introduction and main results

Let $(\xi_k)_{k\in\mathbb{N}}$ be a sequence of independent copies of a random variable (r.v.) ξ with cumulative distribution function (c.d.f.) $F(x) = \mathbb{P}\{\xi < x\}$, $x \in \mathbb{R}$. For $n \in \mathbb{N}$, put

$$z_n = \max_{1\leqslant i\leqslant n} \xi_i. \tag{1.1}$$

The asymptotic behavior of z_n has been very widely studied (see, e.g., [1, 2, 4, 8, 11, 15, 19]). A detailed bibliography can be found, for example, in the books [10] and [7].

For example, it is known (see [1] and [11]) that for a quite large class of random variables ξ with unbounded support and differentiable c.d.f. F, z_n satisfies a law of the iterated logarithm for the $\limsup$ and a law of the triple logarithm for the $\liminf$. We refer to the recent paper [17], where these laws for continuous r.v.s were significantly strengthened.

We are mainly interested in the asymptotic behavior of z_n in the discrete case, which has been much less studied. It is well known that the asymptotics in continuous and discrete cases can be significantly different (see [2, 16, 18]).

Let ξ be a discrete r.v. with distribution (i, p_i), $i \geqslant 0$; more precisely, let

$$\mathbf{P}(\xi = i) = p_i > 0, \quad \sum_{i=0}^{\infty} p_i = 1.$$

For such a r.v., we denote

$$R(n) = -\ln \mathbf{P}(\xi \geqslant n) = -\ln\biggl(\sum_{i\geqslant n} p_i\biggr), \qquad r(n) = R(n) - R(n-1).$$

Let us define the following functions for sufficiently large $t > 0$:

$$L_0(t) = t, \qquad L_m(t) = \ln L_{m-1}(t), \quad m \in \mathbb{N}.$$

Let $\xi, \xi_1, \xi_2, \dots$ be a sequence of discrete independent identically distributed random variables (i.i.d.r.v.s), and let z_n be defined by (1.1). It was noticed quite a while ago that the asymptotic behavior of z_n in the discrete case is closely related to the sequence

$$a_n = \max\biggl(k \geqslant 0\colon \sum_{i\geqslant k} p_i \geqslant \frac{1}{n}\biggr). \tag{1.2}$$

The case where the r.v. ξ has the Poisson distribution ($p_i = (\lambda^i/i!)\exp(-\lambda)$, $i \geqslant 0$) or in some sense is similar to the Poisson distribution was studied in [16, 18].

In [18] the following theorem is proved.

Theorem A. *Let ξ be a discrete r.v. with distribution (i, p_i), $i \geqslant 0$, let $\beta > 0$ be an arbitrary number, and let a_n be given by* (1.2)*. If the previously defined function $r(n)$ satisfies the condition*

$$r(n) = \beta \ln n + o\bigl(L_2(n)\bigr), \quad n \to \infty, \tag{1.3}$$

then

$$\mathbf{P}\bigl(\exists n_0\colon \forall n \geqslant n_0,\ z_n \in J_n = \{a_n + m,\ m \in I_\beta\}\bigr) = 1, \tag{1.4}$$

$$\forall m \in I_\beta, \quad \mathbf{P}(z_n = a_n + m \ \textit{i.o.}) = 1, \tag{1.5}$$

and

$$a_n = \frac{\ln n}{\beta L_2(n)}\bigl(1 + o(1)\bigr),$$

where $I_\beta = \{-1, 0, 1, \dots, [1 + 1/\beta]\}$, and *"i.o." means "infinitely often".*

For the Poisson distribution with parameter $\lambda > 0$ (in this case, $r(n) = \ln n + o(1)$, $\beta = 1$), Eqs. (1.4) and (1.5) hold when $I_\beta = I_1 = \{-1, 0, 1, 2\}$ and

$$a_n = \frac{\ln n}{L_2(n)}\biggl(1 + \frac{L_3(n) + \ln\lambda + 1 + o(1)}{L_2(n)}\biggr).$$

When the function $r(n)$ increases a bit slower than (1.3), for example, if

$$r(n) = o(\ln n) \quad \text{and} \quad \sum_{n\geqslant 1} \exp\bigl(-\mathrm{e}^{r(n)}\bigr) < \infty, \tag{1.6}$$

then Eqs. (1.4) and (1.5) are still valid for $\beta = 0$ [18].

Note that under the condition

$$r(n) = v_n \ln n, \quad v_n \to \infty, \quad n \to \infty, \tag{1.7}$$

Equalities (1.4) and (1.5) are also true when $\beta = \infty$.

Although not stated in [16, 18], this statement is in fact a simple consequence of these works (see [16, Thms. 1, 2] and [18, Lemma 3]).

Thus, under conditions (1.3), (1.6), and (1.7), we can describe the asymptotics of z_n with sufficient accuracy.

At the same time, for discrete r.v.s, a number of related problems remain open. For example, the geometric distribution, which is important for probability theory and its applications, does not satisfy any of condition (1.3), (1.6), and (1.7), and the natural question arises: does any equation of type (1.4) or (1.5) hold for it?

In this paper, in contrast to [18], we focus on the geometric distribution and random variables with distribution tails decreasing slower than the tails of the geometric distribution. Moreover, we present a discrete variant of some results of [17] and consider some applications.

Let us state the main results of the work.

Theorem 1. *Let $(\xi_k)_{k\in\mathbb{N}}$ be a sequence of independent copies of a discrete random variable ξ with distribution (i, p_i), $i \geqslant 0$, and let a_n be defined by* (1.2). *Suppose that for each fixed m,*

$$\lim_{n\to\infty} \frac{r(n+m)}{r(n)} = 1, \tag{1.8}$$

$$\exists C_0 < \infty\colon \quad \forall n \geqslant 1,\ \ r(n) \leqslant C_0. \tag{1.9}$$

(i) *If*

$$\sum_{n\geqslant 1} r^2(n) = \infty, \tag{1.10}$$

then for any integer m,

$$\mathbf{P}(z_n = a_n + m \ \textit{i.o.}) = \mathbf{P}(z_n = \xi_n = a_n + m \ \textit{i.o.}) = 1. \tag{1.11}$$

(ii) *If condition* (1.10) *does not hold, then for any integer $m > 0$,*

$$\mathbf{P}\big(z_n = \xi_n \in (a_n - m,\, a_n + m) \ \textit{i.o.}\big) = \mathbf{P}\big(\xi_n \in (a_n - m,\, a_n + m) \ \textit{i.o.}\big) = 0. \tag{1.12}$$

Note that Eqs. (1.11) and (1.12) describe the asymptotics of z_n at the moments of "high jumps", that is, $\xi_n \geqslant z_{n-1}$. Such "high jumps" seem to be the most interesting for applications. We do not know whether Eq. (1.12) is true for all z_n if condition (1.10) is not satisfied.

To state the following result, we introduce some necessary notation. We extend the sequence $(r(n))$ to the function $r : (0,\infty) \to \mathbb{R}$ by setting $r(x) = r(\lceil x \rceil)$, where $\lceil x \rceil$ is the least integer $\geqslant x$.

Let $R(x) = \int_0^x r(y)\,\mathrm{d}y$. The function R is a piecewise linear extension of the sequence $R(n)$.

Given a function $H : \mathbb{R} \to \mathbb{R}$, we denote by H^{-1} its generalized inverse defined by

$$H^{-1}(y) = \inf\big\{x \in \mathbb{R}\colon\ H(x) \geqslant y\big\}, \quad y \in \mathbb{R}.$$

Put

$$\alpha_m(t) = \sum_{i=1}^{m} L_i(t), \qquad a_m(t) = R^{-1}\big(\alpha_m(t)\big), \qquad d(n) = R^{-1}\big(L_1(n) - L_3(n)\big).$$

Theorem 2. *Let $(\xi_k)_{k\in\mathbb{N}}$ be a sequence of independent copies of a discrete random variable ξ with distribution (i, p_i), $i \geqslant 0$, and let $m \geqslant 1$ be some fixed integer. Let the following condition be satisfied*:

$$\lim_{t\to\infty} \frac{r(tx)}{r(t)} = x^{\rho}, \quad \rho > -1, \quad \forall x > 0. \tag{1.13}$$

Then

$$\mathbf{P}\left(\limsup_{n\to\infty}\frac{r(a_1(n))(z_n+\theta_n-a_m(n))}{L_{m+1}(n)}=1\right)=1, \tag{1.14}$$

$$\mathbf{P}\left(\liminf_{n\to\infty}\frac{L_2(n)r(a_1(n))(z_n+\theta_n-d(n))}{2L_3(n)}=-1\right)=1, \tag{1.15}$$

where θ_n *is a r.v. such that* $0\leqslant\theta_n\leqslant 1$.

Corollary 1.

(i) *If condition* (1.9) *holds, then the value* θ_n *in formula* (1.14) *can be omitted.*
(ii) *Similarly,* θ_n *can be omitted in formula* (1.15) *if* $L_2(n)r(a_1(n))/L_3(n)\to 0$, $n\to\infty$.
(iii) *If* $L_2(n)r(a_1(n))/L_3(n)\to\infty$, $n\to\infty$, *then*

$$\mathbf{P}\left(\liminf_{n\to\infty}\big(z_n-d(n)\big)=\kappa\right)=1, \tag{1.16}$$

where $\kappa\in[-1,0]$. *Here and further, by* κ *we denote a nonrandom constant, not necessarily the same in different parts of the paper.*

In the statements mentioned, we assumed that the functions $F(x)$ and $r(x)$ are known exactly. Unfortunately, in many important practical cases, they are not. More often, only their asymptotics is known as $x\to\infty$. Consider one such example.

Proposition 1. *Let* $(\xi_k)_{k\in\mathbb{N}}$ *be a sequence of independent copies of a discrete random variable* ξ *with distribution* (i,p_i), $i\geqslant 0$, *and let* $m\geqslant 1$ *be some fixed integer.*

Suppose that for some $\gamma>0$ *and* $C_1<\infty$, *we have the asymptotic relation*

$$R(n)=\gamma n+C_1+o(1). \tag{1.17}$$

Then for any integer m, *Eq.* (1.11) *holds with* $a_n=[(\ln n-C_1+o(1))/\gamma]$, *and*

$$\mathbf{P}\left(\limsup_{n\to\infty}\frac{\gamma z_n-\alpha_m(n)}{L_{m+1}(n)}=1\right)=1, \tag{1.18}$$

$$\mathbf{P}\left(\liminf_{n\to\infty}\frac{z_n-(L_1(n)-L_3(n))}{\gamma}=\kappa\right)=1, \tag{1.19}$$

where $\kappa\in[-1-C_1/\gamma,\ -C_1/\gamma]$.

At the end of the paper, we consider examples of application of the obtained results on asymptotics of extreme values of birth and death processes and processes in queuing systems (QSs).

Such problems were studied in many works [2, 3, 9, 13, 21, 24]. However, it was mainly the case of weak convergence.

2 Proof of Theorem 1

Let us start with auxiliary lemmas. For a sequence of discrete i.i.d.r.v.s ξ, $\xi_1,\xi_2,\dots$ with distribution (i,p_i), $i\geqslant 0$, we construct random events A_n, A'_n as follows:

$$A_n=\{\xi_n=z_n=a_n+m\},\qquad A'_n=\big\{\xi_n\in[a_n-m,\ a_n+m)\big\}, \tag{2.1}$$

where a_n is defined by formula (1.2), and m is some integer.

Lemma 1. *Let random events* A_n *be given by* (2.1)*, and let* m *be an arbitrary fixed integer. If, under Theorem* 1*, the function* $r(n)$ *satisfies Eq.* (1.10)*, then*

$$\sum_{n=1}^{\infty} \mathbf{P}(A_n) = \infty. \tag{2.2}$$

Proof. In Lemma 3 of [18] the following lower bounds for the series in (2.2) are obtained:

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbf{P}(A_n) &= \sum_{n=1}^{\infty} \mathbf{P}(\xi = a_n + m)\bigl(1 - \mathbf{P}(\xi \geqslant a_n + m + 1)\bigr)^{n-1} \\ &\geqslant \sum_{k=0}^{\infty} \mathbf{P}(\xi = k + m)\bigl(1 - \mathbf{P}(\xi \geqslant k + m + 1)\bigr)^{\exp(R(k+1))} \sum_{n:\ a_n = k} 1, \end{aligned} \tag{2.3}$$

$$\sum_{n:\ a_n = k} 1 = \exp\bigl(R(k+1)\bigr)\bigl(1 - \exp\bigl(-r(k+1)\bigr)\bigr) + \theta_k, \tag{2.4}$$

where $|\theta_k| \leqslant 1$, and

$$\begin{aligned}\mathbf{P}(\xi = k + m) &= \exp\bigl(-R(k+m)\bigr) - \exp\bigl(-R(k+m+1)\bigr) \\ &= \exp\bigl(-R(k+m)\bigr)\bigl(1 - \exp\bigl(-r(k+m+1)\bigr)\bigr). \end{aligned} \tag{2.5}$$

Obviously,

$$\sum_{k=0}^{\infty} \mathbf{P}(\xi = k + m)\bigl(1 - \mathbf{P}(\xi \geqslant k + m + 1)\bigr)^{\exp(R(k+1))} |\theta_k| \leqslant 1. \tag{2.6}$$

Putting (2.3)–(2.6) together, we get

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbf{P}(A_n) \geqslant \sum_{k=0}^{\infty} &\exp\bigl(-R(k+m) + R(k+1)\bigr)\bigl(1 - \exp\bigl(-r(k+1)\bigr)\bigr) \\ &\times \bigl(1 - \exp\bigl(-r(k+m+1)\bigr)\bigr)\bigl(1 - \exp\bigl(-R(k+m+1)\bigr)\bigr)^{\exp(R(k+1))} \\ &- 1. \end{aligned} \tag{2.7}$$

Further, note that if condition (1.8) of Theorem 1 is satisfied, then only two cases are possible:

(a) There exist $\delta > 0$ and subsequence (k_i) such as $r(k_i) \geqslant \delta$, $r(k_i + 1) \geqslant \delta, \ \ldots, \ r(k_i + m + 1) \geqslant \delta$;
(b) $r(k) \to 0$ as $k \to \infty$.

Let us start with case (a). By estimate (2.7) we have

$$\begin{aligned}\sum_{n=1}^{\infty} \mathbf{P}(A_n) \geqslant \sum_{i} &\exp\bigl(-|m|C_0\bigr)\bigl(1 - \exp(-\delta)\bigr)^2 \\ &\times \bigl(1 - \exp\bigl(-R(k_i + m + 1)\bigr)\bigr)^{\exp(R(k_i+1))} \\ &- 1. \end{aligned} \tag{2.8}$$

As it is well known,

$$\left(1 - \frac{1}{x}\right)^x \uparrow \frac{1}{e} \quad \text{as } x \uparrow \infty. \tag{2.9}$$

Suppose that $m \geqslant 0$. Then $R(k+m+1) \geqslant R(k+1)$, and the series on the right-hand side of inequality (2.8) diverges, and therefore the series on the left-hand side also diverges.

Let $m < 0$. As mentioned before, inequality (2.8) also holds. Given the inequality

$$\big|R(k+1) - R(k+m+1)\big| \leqslant |m|C_0 \tag{2.10}$$

and asymptotic relation (2.9) obtained for sufficiently large k and $\epsilon \leqslant 0.1$, we have

$$\big(1-\exp\big(-R(k+m+1)\big)\big)^{\exp(R(k+1))} \geqslant \left(\frac{1}{e}-\epsilon\right)^{\exp(mC_0)},$$

that is, again, the series in the right- and left-hand sides of (2.8) diverge.

Let us turn to case (b). Again, we use estimate (2.7) and the elementary inequality $1-\exp(-x) \geqslant x - x^2/2$, $x \geqslant 0$. Then for sufficiently large k, the corresponding item in the sum on the right-hand side is estimated from below by the value

$$\exp\big(-R(k+m)+R(k+1)\big)\left(r(k+m+1)-\frac{r(k+m+1)^2}{2}\right)$$
$$\times\left(r(k+1)-\frac{r(k+1)^2}{2}\right)\big(1-\exp\big(-R(k+m+1)\big)\big)^{\exp(R(k+1))}.$$

Since $r(k) \to 0$ as $k \to \infty$, for any fixed integer m, we have

$$-R(k+m)+R(k+1) \to 0 \quad \text{and} \quad \big(1-\exp\big(-R(k+m+1)\big)\big)^{\exp(R(k+1))} \to \frac{1}{e}.$$

If we add condition (1.10), then we easily to see that the series (2.2) diverges. □

Lemma 2. *Let random events A'_n be given by Eq.* (2.1)*, let m be an arbitrary fixed integer, and let the conditions of Theorem* 1 *hold except* (1.10)*, that is,*

$$\sum_{n\geqslant 1} r^2(n) < \infty. \tag{2.11}$$

Then

$$\sum_{n=1}^{\infty} \mathbf{P}(A'_n) < \infty. \tag{2.12}$$

Proof. Just as in Lemma 1, we have the equality

$$\sum_{n=1}^{\infty}\mathbf{P}(A'_n) = \sum_{n=1}^{\infty}\mathbf{P}\big(\xi_n \in [a_n-m,\, a_n+m)\big) = \sum_{k=0}^{\infty}\mathbf{P}\big(\xi\in[k-m,\, k+m)\big)\sum_{n\colon a_n=k} 1$$
$$= \sum_{j=-m}^{m}\sum_{k=0}^{\infty}\mathbf{P}(\xi = k+j)\sum_{n\colon a_n=k} 1.$$

Clearly, to prove inequality (2.12), it suffices to establish the boundedness of the sums

$$S_j = \sum_{k=0}^{\infty}\mathbf{P}(\xi=k+j)\sum_{n\colon a_n=k} 1, \quad j\in[-m,m]. \tag{2.13}$$

Under condition (1.8), we have $r(k+m) = r(k)(1+o(1))$, $k \to \infty$.
Therefore

$$\begin{aligned}\mathbf{P}(\xi = k+j) &= \exp\bigl(-R(k+j)\bigr) - \exp\bigl(-R(k+j+1)\bigr) \\ &= \exp\bigl(-R(k+j)\bigr) r(k)\bigl(1+o(1)\bigr). \end{aligned} \tag{2.14}$$

Further, we put estimates (2.14) and (2.4) into formula (2.13):

$$S_j \leqslant \sum_{k=0}^{\infty} \exp\bigl(R(k+1) - R(k+j)\bigr) r(k)^2 \bigl(1+o(1)\bigr) + \sum_{k=0}^{\infty} \mathbf{P}(\xi = k+j)|\theta_k|.$$

Obviously, in the last estimate the second sum on the right is less that 1. Hence, taking into account condition (1.9) of Theorem 1 and condition (2.11), we obtain the boundedness of the values S_j. $\square$

We proceed directly to the proof of Theorem 1. For the sequence of random events (A_n) defined by (2.1), we introduce the following notation:

$$S_n' = \sum_{j=1}^{n} \mathbf{P}(A_j), \qquad S_n'' = \sum_{1 \leqslant j < l \leqslant n} \mathbf{P}(A_j \cap A_l).$$

Firstly, we show that under the conditions of Theorem 1, there exists a constant $K < \infty$ such that

$$\limsup_{n\to\infty} \frac{S_n''}{{S_n'}^2} \leqslant K. \tag{2.15}$$

To this end, we use the simple equality

$$S_n'' = \sum_{1 \leqslant j < l \leqslant n} \mathbf{P}(A_j)\mathbf{P}(A_l) C_{j,l}, \tag{2.16}$$

where $C_{j,l} = (\mathbf{P}(\xi_i \leqslant a_l + m, i = 1, \dots, j))^{-1}$ (see [18, Lemma 3]).

Let us estimate the value of $C_{j,l}$ from above. Suppose that $a_l = k$. Then $l < \exp(R(k+1))$, and

$$C_{j,l} \leqslant \bigl(1 - \exp\bigl(-R(k+m+1)\bigr)\bigr)^{-l} \leqslant \bigl(1 - \exp\bigl(-R(k+m+1)\bigr)\bigr)^{-\exp(R(k+1))}.$$

Note that for $x > 1$, the function $(1 - 1/x)^{-x}$ decreases as x increases. From this and from inequality (2.10) we obtain

$$C_{j,l} \leqslant C_2 = \bigl(1 - \exp\bigl(-R(1)\bigr)\bigr)^{-\exp(R(1)+|m|C_0)}.$$

Substituting the last estimate into (2.16), we have

$$S_n'' \leqslant C_2 \sum_{1 \leqslant j < l \leqslant n} \mathbf{P}(A_j)\mathbf{P}(A_l).$$

Finally, by Lemma 1 $S_n' \to \infty$ as $n \to \infty$. Then

$$S_n'^2 = 2 \sum_{1 \leqslant j < l \leqslant n} \mathbf{P}(A_j)\mathbf{P}(A_l) + O(S_n').$$

The last estimates together give inequality (2.15) at $K = C_2/2$.

It remains to rewrite inequality (2.15) as follows:

$$\limsup_{n\to\infty}\frac{\sum_{j=1}^{n}\sum_{l=1}^{n}\mathbf{P}(A_j\cap A_l)}{(\sum_{j=1}^{n}\mathbf{P}(A_j))^2}\leqslant\limsup_{n\to\infty}\frac{2S_n^{''}+S_n'}{{S_n'}^2}\leqslant 2K. \tag{2.17}$$

The generalized Borel–Cantelli lemma (see [22, Chap. 6, Sec. 26]) allows us to derive the following inequality from estimates (2.2) and (2.17):

$$\mathbf{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}A_k\right)\geqslant\frac{1}{2K}.$$

Hence from the Hewitt–Savage zero–one law we have

$$\mathbf{P}\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}A_k\right)=1,$$

that is, Eq. (1.11) of the Theorem 1 is proved.

Item (ii) of Theorem 1 follows directly from Lemma 2 and the Borel–Cantelli lemma.

3 Proof of Theorem 2

DEFINITION 1. We say that a function $H:\mathbb{R}\to\mathbb{R}$ satisfies condition $(\mathbb{U}_1)$ if the following holds:

(i) $\lim_{x\to+\infty}H(x)=+\infty$;
(ii) the function H is strictly increasing for $x\in(x_0,\infty)$, where $x_0:=\inf\{x\in\mathbb{R}\colon\ H(x)>0\}$;
(iii) there exists $\rho>-1$ such that the derivative H' is regularly varying at $+\infty$ with index ρ.

The notion of regularly varying functions is well known; see [5, 20].

The main results of [17] on the asymptotic behavior of extreme values of a continuous r.v. obtained just in the case where the function $H(x)=-\ln(1-F(x))$ satisfies condition $(\mathbb{U}_1)$.

If the function r satisfies condition (1.13), then it is regularly varying at $+\infty$ with index ρ, denoted $r\in RV_\rho$. Then also $R\in RV_{\rho+1}$, $R^{-1}\in RV_{1/(\rho+1)}$ [5, Prop. 1.5.8, Thm. 1.5.12], and $h\in RV_{-\rho/(\rho+1)}$. Moreover,

$$R^{-1}(x)=\int_0^x h(y)\,\mathrm{d}y,\quad h(y)=\frac{1}{r(R^{-1}(y))}. \tag{3.1}$$

Consider a r.v. ξ^c with distribution function $F(x)=1-\exp(-R(x))$, $x>0$, $F(0)=0$. Let $(\xi_k^c)_{k\in\mathbb{N}}$ be a sequence of independent copies of a r.v. ξ^c, and let

$$z_n^c=\max_{1\leqslant i\leqslant n}\xi_i^c.$$

First, we show that under the conditions of the Theorem 2, we have the following asymptotic equalities:

$$\limsup_{n\to\infty}\frac{r(a_m(n))(z_n^c-a_m(n))}{L_{m+1}(n)}=1\quad\text{a.s. }\forall m\in\mathbb{N}, \tag{3.2}$$

$$\liminf_{n\to\infty}\frac{L_2(n)r(d(n))(z_n^c-d(n))}{2L_3(n)}=-1\quad\text{a.s.} \tag{3.3}$$

If the function R would satisfy condition ($\mathbb{U}_1$), then Eqs. (3.2) and (3.3) would a simple consequence of Theorems 1 and 2 in [17]. Unfortunately, the function R is not differentiable in a countable set of points. Therefore we have to slightly modify the corresponding proof from [17].

Let us establish Eq. (3.2). Let τ^{e} be a standard exponentially distributed r.v., that is, $\mathbf{P}(\tau^{\mathrm{e}} < x) = 1 - \exp(-x)$. Let $(\tau_k^e)_{k\in\mathbb{N}}$ be a sequence of independent copies of a r.v. τ^{e}, and let

$$z_n^{\mathrm{e}} = \max_{1\leqslant k\leqslant n} \tau_k^{\mathrm{e}}.$$

Without loss of generality, we can assume that

$$z_n - a_m(n) = R^{-1}\bigl(z_n^{\mathrm{e}}\bigr) - R^{-1}\bigl(\alpha_m(n)\bigr) \tag{3.4}$$

(see proof of Theorem 1 in [17]).

The following equality was obtained in Lemma 2 in [17]:

$$\limsup_{n\to\infty} \frac{z_n^{\mathrm{e}} - \alpha_m(n)}{L_{m+1}(n)} = 1 \quad \text{a.s.} \tag{3.5}$$

Furthermore, we assume that

$$z_n^{\mathrm{e}}(n) \geqslant \alpha_m(n) \tag{3.6}$$

(since $R^{-1}(x)$ is an increasing function, taking into account (3.5), it suffices to choose only those n for which (3.6) holds).

We fix an arbitrary sufficiently small $\epsilon > 0$ and introduce the following notation:

$$h_n^- = \inf_{\alpha_m(n)\leqslant t\leqslant z_n^{\mathrm{e}}} h(t), \qquad h_n^+ = \sup_{\alpha_m(n)\leqslant t\leqslant z_n^{\mathrm{e}}} h(t),$$

$$\zeta_n^- = \sup\bigl(t \leqslant z_n^{\mathrm{e}}\colon\ h(t) \leqslant h_n^-(1+\epsilon)\bigr), \qquad \zeta_n^+ = \sup\bigl(t \leqslant z_n^{\mathrm{e}}\colon\ h(t) \geqslant h_n^+(1-\epsilon)\bigr).$$

Then by (3.1) we obtain

$$h_n^-\bigl(z_n^{\mathrm{e}} - \alpha_m(n)\bigr) \leqslant R^{-1}\bigl(z_n^{\mathrm{e}}\bigr) - R^{-1}\bigl(\alpha_m(n)\bigr) \leqslant h_n^+\bigl(z_n^{\mathrm{e}} - \alpha_m(n)\bigr).$$

The functions $r(t)$ and $h(t)$ after construction are continuous from the left. Therefore

$$h\bigl(\zeta_n^-\bigr) \leqslant h_n^-(1+\epsilon), \qquad h\bigl(\zeta_n^+\bigr) \geqslant h_n^+(1-\epsilon),$$

and thus

$$\begin{aligned}\frac{1}{1+\epsilon} h\bigl(\zeta_n^-\bigr)\bigl(z_n^{\mathrm{e}} - \alpha_m(n)\bigr) &\leqslant R^{-1}\bigl(z_n^{\mathrm{e}}\bigr) - R^{-1}\bigl(\alpha_m(n)\bigr)\\ &\leqslant \frac{1}{1-\epsilon} h\bigl(\zeta_n^+\bigr)\bigl(z_n^{\mathrm{e}} - \alpha_m(n)\bigr).\end{aligned}$$

Keeping in mind the equality $h(\alpha_m(n)) = 1/r(a_m(n))$ and Eq. (3.4), we can rewrite the last inequality as

$$\begin{aligned}\frac{1}{1+\epsilon}\frac{h(\zeta_n^-)}{h(\alpha_m(n))}\frac{z_n^{\mathrm{e}} - \alpha_m(n)}{L_{m+1}(n)} &\leqslant \frac{r(a_m(n))(z_n - a_m(n))}{L_{m+1}(n)}\\ &\leqslant \frac{1}{1-\epsilon}\frac{h(\zeta_n^+)}{h(\alpha_m(n))}\frac{z_n^{\mathrm{e}} - \alpha_m(n)}{L_{m+1}(n)}.\end{aligned} \tag{3.7}$$

It is known (see [10, Chap. 4, Ex. 4.3.3]) that

$$\frac{z_n^{\mathrm{e}}}{\ln n} \to 1 \quad \text{and} \quad \frac{z_n^{\mathrm{e}}}{\alpha_m(n)} \to 1 \quad \text{a.s.}$$

as $n \to \infty$. However, $\zeta_n^-, \zeta_n^+ \in (\alpha_m(n), z_n^{\mathrm{e}})$, and therefore

$$\frac{\zeta_n^-}{\alpha_m(n)} \to 1, \qquad \frac{\zeta_n^+}{\alpha_m(n)} \to 1.$$

From this we obtain that

$$\frac{h(\zeta_n^-)}{h(\alpha_m(n))} \to 1, \qquad \frac{h(\zeta_n^+)}{h(\alpha_m(n))} \to 1 \tag{3.8}$$

as $n \to \infty$ (see similar conversions in [17]).

Putting together relations (3.5), (3.7), and (3.8), we obtain

$$\frac{1}{1+\epsilon} \leqslant \limsup_{n\to\infty} \frac{r(a_m(n))(z_n - a_m(n))}{L_{m+1}(n)} \leqslant \frac{1}{1-\epsilon} \quad \text{a.s.} \tag{3.9}$$

Estimates (3.9) are satisfied for any $\epsilon > 0$, from which we obtain Eq. (3.2). Similarly, based on Lemma 4 in [17], we can prove Eq. (3.3).

It remains to make the transformation from (3.2) and (3.3) to equalities (1.14) and (1.15) in Theorem 2.

Further, note that for $k \in \mathbb{N}$, the random events

$$\{\xi^c < k\} \quad \text{and} \quad \{[\xi^c] < k\}$$

are equivalent, that is,

$$\mathbf{P}([\xi^c] < k) = \mathbf{P}(\xi^c < k) = 1 - \exp(R(k)).$$

Thus r.v.s $[\xi^c]$ and ξ are identically distributed. The same is true for r.v.s $[z_n^c]$ and z_n. If we denote $\theta_n^c = z_n^c - [z_n^c]$, then $z_n^c - \theta_n^c$ and z_n have the same asymptotic behavior at infinity. Hence by (3.2) and (3.3) we obtain that there exist θ_n, $0 \leqslant \theta_n \leqslant 1$, such that for every fixed $m \in \mathbb{N}$,

$$\limsup_{n\to\infty} \frac{r(a_m(n))(z_n + \theta_n - a_m(n))}{L_{m+1}(n)} = 1 \quad \text{a.s.,} \tag{3.10}$$

$$\liminf_{n\to\infty} \frac{L_2(n) r(d(n))(z_n + \theta_n - d(n))}{2L_3(n)} = -1 \quad \text{a.s.} \tag{3.11}$$

Since the functions $r(x)$ and $R^{-1}(x)$ are regularly varying at infinity, we have the implication

$$\frac{\alpha_m(n))}{\alpha_1(n))} \to 1 \quad \Longrightarrow \quad \frac{r(a_m(n))}{r(a_1(n))} \to 1, \quad n \to \infty$$

(see [6]).

In the same way, we get

$$\frac{r(d(n))}{r(a_1(n))} \to 1, \quad n \to \infty,$$

which, together with Eqs. (3.10) and (3.11), completes the proof of Theorem 2.

Proof of Corollary 1. Here only item (iii) needs some explanation. It is simply deduced from Theorem 2. Indeed, put

$$\chi_n = \frac{L_2(n)r(a_1(n))}{L_3(n)}.$$

Then by Eq. (1.15), for all $\epsilon > 0$, we have

$$\mathbf{P}\left(\exists (n_i)\colon\ z_{n_i} - d(n_i) + \theta_{n_i} \leqslant \frac{-1+\epsilon}{\chi_{n_i}}\ \text{i.o.}\right) = 1,$$
$$\mathbf{P}\left(\exists n_0\colon\ \forall n \geqslant n_0,\ z_n - d(n) + \theta_n \geqslant \frac{-1-\epsilon}{\chi_n}\right) = 1.$$

Since ϵ is an arbitrary positive number and $\chi_n \to \infty$ as $n \to \infty$, from the last relations it follows that

$$\liminf_{n\to\infty} \big(z_n - d(n)\big) = \kappa \in [-1, 0] \quad \text{a.s.} \tag{3.12}$$

Moreover, by the Hewitt–Savage zero–one law, κ is a degenerate r.v., that is, (1.16) holds. □

Remark 1. Since z_n is an integer r.v., then the following relations seem of interest:

$$\mathbf{P}\big(z_n - [d_n] \in \{-1, 0, 1\}\ \text{i.o.}\big) = 1, \qquad \mathbf{P}\big(z_n - [d_n] < -1\ \text{i.o.}\big) = 0,$$

which we obtain from Eq. (3.12).

4 Proof of Proposition 1

From condition (1.17) we obtain $r(n) = \gamma + o(1)$. Thus condition (1.8) is satisfied. Accordingly, by Theorem 1 we have Eq. (1.11). The formula for $a(n)$ simply follows from definition (1.2):

$$a_n = \max\left(k \geqslant 0\colon\ \exp\big(-R(k)\big) \geqslant \frac{1}{n}\right) = \max\left(k \geqslant 0\colon\ k \leqslant \frac{\ln n - C_1 + o(1)}{\gamma}\right)$$
$$= \left[\frac{\ln n - C_1 + o(1)}{\gamma}\right].$$

We obtain relations (1.18) and (1.19) from Theorem 2, since its conditions are also satisfied. It only remains to find the asymptotic behavior of the function $R^{-1}(x)$.

An anonymous reviewer has somewhat refined the interval for κ compared to the original version. Here is his reasoning. If R denotes the piecewise linear extension of the sequence $(R(n))$, then

$$R(x) = R\big([x]\big) - r\big([x]\big)\big([x] - x\big) = \gamma[x] + C_1 + o(1) - \big(\gamma + o(1)\big)\big([x] - x\big)$$
$$= \gamma x + C_1 + O(1)$$

as $x \to \infty$. Therefore, denoting $x_u = R^{-1}(u)$, we get, as $u \to \infty$,

$$u = R(x_u) = \gamma x_u + C_1 + o(1), \quad x_u = \frac{u - C_1 - o(1)}{\gamma} = \frac{u}{\gamma} - \frac{C_1}{\gamma} + o(1).$$

Hence $d(n) = (L_1(n) - L_3)/\gamma - C_1/\gamma + o(1)$, and (1.19) holds with $\kappa \in [-1 - C_1/\gamma,\ -C_1/\gamma]$.

5 Examples

Let us consider some examples of application of Theorems 1 and 2 and Proposition 1.

Example 1 [*Geometric distribution*]. Let $0 < q < 1$, and let

$$\mathbf{P}(\xi = k) = p_k = q(1-q)^k, \quad k \geqslant 0.$$

Then

$$\mathbf{P}(\xi \geqslant k) = (1-q)^k = \exp(-\gamma k), \quad \gamma = \ln \frac{1}{1-q},$$

that is, $R(k) = \gamma k$, $r(k) = \gamma$.

It is clear that conditions (1.9) and (1.10) of Theorem 1 hold, and therefore r.v.s z_n satisfy Eq. (1.11). Moreover, via formula (1.2) we find $a_n = [(1/\gamma) \ln n]$.

Similarly, the conditions of the Theorem 2 hold. By Corollary 1 we obtain

$$\limsup_{n\to\infty} \frac{\gamma z_n - \alpha_m(n)}{L_{m+1}(n)} = 1, \quad \liminf_{n\to\infty} \frac{z_n - (L_1(n) - L_3(n))}{\gamma} = \kappa \quad \text{a.s.,}$$

where $\kappa \in [-1, 0]$.

In fact, the last equation can be refined. Namely, based on the results of [15], we can prove that the geometric distribution satisfies the following:

$$\mathbf{P}\left(z_n \leqslant \left[\frac{L_1(n) - L_3(n)}{\gamma} \right] \text{ i.o.} \right) = 1.$$

From this and Remark 1 we have

$$\mathbf{P}\left(z_n - \left[\frac{L_1(n) - L_3(n)}{\gamma} \right] \in \{-1, 0\} \text{ i.o.} \right) = 1,$$

$$\mathbf{P}\left(z_n - \left[\frac{L_1(n) - L_3(n)}{\gamma} \right] < -1 \text{ i.o.} \right) = 0.$$

Example 2 [*Queuing system* $M/M/m$]. Let us now consider a queuing system with m servers, $1 \leqslant m < \infty$, and customers that arrive according to the Poisson process with intensity λ, service times being independent copies of a random variable η with exponential distribution

$$\mathbf{P}(\eta \leqslant x) = 1 - \exp(-\mu x), \quad x \geqslant 0.$$

In the standard notation, this queuing system has the type $M/M/m$; see [12, 14].

We impose the following assumption on the parameters λ and μ ensuring the existence of the stationary regime: $\rho := \lambda/(m\mu) < 1$. For $t \geqslant 0$, let $Q(t)$ denote the length of the queue at time t, that is, the total number of customers in service or pending. Set

$$\bar{Q}(t) = \sup_{0 \leqslant s < t} Q(s), \quad t \geqslant 0.$$

Let us introduce the regeneration moments (S_k) for the process Q: $S_0 := 0$ and, for $i \in \mathbb{N}$, S_i is the arrival time of a new customer after the ith busy period. Let T_i be the duration of the ith regeneration cycle, and let $\bar{Q}(T_1)$

be the maximum length of the queue in the first regeneration cycle. It is well known that $a_T = \mathbf{E}T_1 = 1/(\lambda p_0)$ and

$$p_0 = \left(\sum_{k=0}^{m} \frac{(m\rho)^k}{k!} + \frac{\rho^m m^m}{m!(\frac{1}{\rho} - 1)} \right)^{-1}.$$

Put

$$\mathbf{P}\big(\bar{Q}(T_1) \geqslant n\big) = \exp\big(-R(n)\big). \tag{5.1}$$

In recent paper [9] the authors established that the sequence $(R(n))$ in (5.1) satisfies conditions (1.17) with

$$\gamma = \ln \frac{1}{\rho}, \qquad C_1 = \ln \frac{\rho\, m!}{m^m(1-\rho)}. \tag{5.2}$$

Based on these equalities in [9], it was found that $\bar{Q}(t)$ satisfies a law of the iterated logarithm for the $\limsup$ and a law of the triple logarithm for the $\liminf$.

Here we will strengthen this result as follows:

$$\limsup_{t\to\infty} \frac{\gamma \bar{Q}(t) - \alpha_k(t)}{L_{k+1}(t)} = 1 \quad \text{a.s. } \forall k \geqslant 1, \tag{5.3}$$

and

$$\liminf_{t\to\infty} \left(\bar{Q}(t) - \frac{1}{\gamma}\big(L_1(t) - L_3(t)\big) \right) = \kappa \quad \text{a.s.}, \tag{5.4}$$

where $\kappa \in [-1 - (C_1 + \ln a_T)/\gamma,\ -(C_1 + \ln a_T)/\gamma]$.

Indeed, denote by N the counting process for the sequence (S_k), that is,

$$N(t) = \max\{k \geqslant 0\colon\ S_k \leqslant t\}, \quad t \geqslant 0.$$

By the strong law of large numbers for N, we have $\lim_{t\to\infty} N(t)/t = 1/a_T$ a.s., whence, as $t \to \infty$,

$$\ln N(t) = \ln \frac{t}{a_T} + o(1) \quad \text{a.s.} \tag{5.5}$$

Put $Z_n = \bar{Q}(S_n)$. From Proposition 1 it follows that for r.v. Z_n, Eqs. (1.18) and (1.19) hold with γ and C_1 defined in (5.2), that is,

$$\limsup_{n\to\infty} \frac{\gamma Z_n - \alpha_k(n)}{L_{k+1}(n)} = 1, \quad \liminf_{t\to\infty} \left(Z_n - \frac{1}{\gamma}\big(L_1(n) - L_3(n)\big) \right) = \kappa \quad \text{a.s.},$$

where $\kappa \in [-1 - C_1/\gamma,\ -C_1/\gamma]$.

The procedure of the transition from here to (5.3) and (5.4) is known and is based on estimate (5.5) and the following inequalities:

$$Z_{N(t)} \leqslant \bar{Q}(t) \leqslant Z_{N(t)+1} \quad \text{a.s.}$$

(see, e.g., [9]).

Further, we consider the r.v.s

$$\bar{Q}_n = \sup_{0\leqslant k\leqslant n} Q(t_k), \quad n \geqslant 0,$$

where $t_0 = 0, t_1, t_2, \ldots$ are the moments of receipt of applications in the system.

We easily see that

$$\lim_{n\to\infty}\frac{N(t_n)}{n}=\lim_{n\to\infty}\frac{N(t_n)}{t_n}\frac{t_n}{n}=\frac{1}{\lambda a_T}=p_0\quad\text{a.s.}$$

Repeating the observations mentioned from Proposition 1, we obtain

$$\limsup_{n\to\infty}\frac{\gamma\bar{Q}_n-\alpha_k(n)}{L_{k+1}(n)}=1,\quad \liminf_{n\to\infty}\left(\bar{Q}_n-\frac{L_1(n)-L_3(n)}{\gamma}\right)=\kappa\quad\text{a.s.,}$$

where $\kappa\in[-1-(C_1+\ln p_0)/\gamma,\ -(C_1+\ln p_0)/\gamma]$.

Example 3 [*Birth and death processes*]. Let $X=(X(t))_{t\geqslant 0}$ be a birth and death process with parameters

$$\begin{aligned}&\lambda_n=\lambda v_n+A,\quad \mu_n=\mu v_n+B,\quad n=1,2,\dots,\\&\lambda_0=A,\qquad \mu_0=0,\qquad \lambda,\mu,v_n,A,B>0,\end{aligned}\tag{5.6}$$

that is, $(X(t))_{t\geqslant 0}$ is a time-homogeneous Markov process such that, for $t\geqslant 0$, given $\{X(t)=n\}$, the probability of transition to state $n+1$ over a small period of time δ is $(\lambda v_n+A)\delta+o(\delta)$, and the probability of transition to $n-1$ is $(\mu v_n+B)\delta+o(\delta)$, $n=1,2,3,\dots$. The parameter a can be interpreted as the infinitesimal intensity of population growth due to immigration, and B characterizes the intensity of population decline due to emigration.

In the case $v_n=n$ the birth-death process X is usually called the process with linear grow (see [14, Chap. 7, Sec. 6]).

We assume that $X(0)=0$, $v_n\uparrow\infty$ as $n\uparrow\infty$,

$$\sum_{n\geqslant 1}\frac{1}{v_n}<\infty,\quad\text{and}\quad \rho:=\frac{\lambda}{\mu}<1.\tag{5.7}$$

Put

$$\theta_0=1,\qquad \theta_k=\prod_{i=1}^{k}\frac{\lambda_{i-1}}{\mu_i},\quad k\in\mathbb{N}.$$

Under condition (5.7), there exists a stationary regime, that is, $\lim_{t\to\infty}\mathbf{P}(X(t)=k)=p_k$ with $p_k=\theta_k p_0$, $k=0,1,2,3,\dots$, where $p_0=1/(\sum_{k=0}^{\infty}\theta_k)$.

Further, X is a regenerative process with regeneration moments (S_k), where $S_0=0$ and S_i, $i\in\mathbb{N}$, is the moment of ith return to state 0. It is known that $a_T=\mathbf{E}T_k=1/(Ap_0)$, where $T_k=S_k-S_{k-1}$ is the duration of the kth regeneration cycle; see [24].

Put

$$\bar{X}(t)=\sup_{0\leqslant s<t}X(s),\quad t\geqslant 0,$$

and

$$q(n):=\mathbf{P}\big(\bar{X}(T_1)\geqslant n\big)=\exp\big(-R(n)\big).$$

It is known (see [3] or [24, Eq. (34)]) that $q(n)=1/\sum_{k=0}^{n-1}\alpha_k$, where $\alpha_0=1$ and $\alpha_k=\prod_{i=1}^{k}\mu_i/\lambda_i$ for $k\in\mathbb{N}$.

Further, we write α_k in the following form:

$$\alpha_k=\frac{\beta_k}{\rho^k},\quad \beta_k=\prod_{i=1}^{k}(1+\delta_i),\quad \delta_i=\frac{B/\mu-A/\lambda}{v_i+A/\lambda}.$$

As it is known from the analysis, if the series of condition (5.7) converges, then there exists

$$\lim_{k\to\infty}\beta_k=\beta^*=\prod_{i=1}^{\infty}(1+\delta_i) \tag{5.8}$$

(see [23, Chap. 1, Sec. 4]).

To estimate the value $q(n)$, we need the following:

Lemma 3. *For arbitrary* $p>1$ *and* β_k *that satisfies equality* (5.8)*,*

$$\Lambda_n:=\sum_{k=1}^{n}p^k\beta_k=\beta^*\frac{p^{n+1}}{p-1}\bigl(1+o(1)\bigr),\quad n\to\infty.$$

Proof. By the Stolz–Cesàro theorem we have

$$\lim_{n\to\infty}\frac{\Lambda_n}{p^{n+1}}=\lim_{n\to\infty}\frac{\Lambda_n-\Lambda_{n-1}}{p^{n+1}-p^n}=\lim_{n\to\infty}\frac{p^n\beta_n}{p^{n+1}-p^n}=\frac{\beta^*}{p-1}.$$

The proof is complete. □

The estimate follows directly from Lemma 3:

$$q(n)=\frac{1-\rho}{\rho\beta^*}\rho^n\bigl(1+o(1)\bigr),$$

that is,

$$R(n)=-\ln q(n)=\gamma n+C_1+o(1),$$

where

$$\gamma=\ln\frac{1}{\rho},\quad C_1=\ln\frac{\rho\beta^*}{1-\rho}. \tag{5.9}$$

It remains to apply Proposition 1 and repeat the reasoning from the previous example. Thus, for birth and death processes with parameters defined in (5.6), we obtain

$$\limsup_{t\to\infty}\frac{\gamma\bar{X}(t)-\alpha_k(t)}{L_{k+1}(t)}=1,\quad \liminf_{t\to\infty}\left(\bar{X}(t)-\frac{1}{\gamma}\bigl(L_1(t)-L_3(t)\bigr)\right)=\kappa\quad\text{a.s.},$$

where $\kappa\in[-1-(C_1+\ln a_T)/\gamma,\ -(C_1+\ln a_T)/\gamma]$, and γ and C_1 are given by Eqs. (5.9).

Acknowledgment. We thank the anonymous referees for a number of useful suggestions and helpful comments.

References

1. K.S. Akbash and I.K. Matsak, One improvement of the law of iterated logarithm for the max-scheme, *Ukr. Mat. Zh.*, **64**:1132–1137, 2012 (in Ukrainian). English transl.: *Ukr. Math. J.*, **64**:1290–1296, 2013.
2. C.W. Anderson, Extreme value theory for a class of discrete distribution with application to some stochastic processes, *J. Appl. Probab.*, **7**:99–113, 1970.
3. S. Asmussen, Extreme value theory for queues via cycle maxima, *Extremes*, **1**:137–168, 1998.

4. O. Barndorff-Nielsen, On the limit behaviour of extreme order statistics, *Ann. Math. Stat.*, **34**:992–1002, 1963.
5. N.H. Bingham, C.M. Goldie, and J.L. Teugels, *Regular Variation*, Cambridge Univ. Press, Cambridge, 1989.
6. V.V. Buldygin, O.I. Klesov, and J.G. Steinebach, On some properties of asymptotic quasi-inverse functions and their applications, *Theory Probab. Math. Stat.*, **70**:9–25, 2004.
7. L. de Haan and A. Ferreira, *Extreme Value Theory: An Introduction*, Springer, Berlin, 2006.
8. L. de Haan and A. Hordijk, The rate of growth of sample maxima, *Ann. Math. Stat.*, **43**:1185–1196, 1972.
9. B.V. Dovgay and I.K. Matsak, The asymptotic behavior of extreme values of queue lengths in $M/M/m$ systems, *Kibern. Sist. Anal.*, **55**(2):171–179, 2019 (in Ukrainian).
10. J. Galambos, *The Asymptotic Theory of Extreme Order Statistics*, John Wiley & Sons, New York, 1978.
11. P. Glasserman and S.G. Kou, Limits of first passage times to rare sets in regenerative processes, *Anal. Appl. Probab.*, **5**:424–445, 1995.
12. B.V. Gnedenko and I.N. Kovalenko, *Introduction to Queueing Theory,* 2nd ed., Math. Modeling, Boston, Mass., Vol. 5, Birkhäuser, Boston, MA, 1989.
13. D.L. Iglehart, Extreme values in the $GI/G/1$ gueue, *Ann. Math. Stat.*, **43**:627–635, 1972.
14. S. Karlin, *A First Course in Stochastic Processes*, Academic Press, New York, 1968.
15. M.J. Klass, The Robbins–Siegmund criterion for partial maxima, *Ann. Probab.*, **13**:1369–1370, 1985.
16. I.K. Matsak, Asymptotic behaviour of random variables extreme values. Discrete case, *Ukr. Mat. Zh.*, **68**:945–956, 2016 (in Ukrainian).
17. I.K. Matsak, Asymptotic behavior of maxima of independent random variables, *Lith. Math. J.*, **59**(2):185–197, 2019.
18. I.K. Matsak, Limit theorem for extreme values of discrete random variables and applications, *Teor. Ĭmovirn. Mat. Stat.*, **101**:189–202, 2019 (in Ukrainian).
19. J. Pickands, An iterated logarithm law for the maximum in a stationary Gaussian sequence, *Z. Wahrscheinlichkeitstheor. Verw. Geb.*, **12**:344–355, 1969.
20. E. Seneta, *Regularly Varying Functions*, Springer, Berlin, Heidelberg, New York, 1976.
21. R.F. Serfozo, Extreme values of birth and death processes and queues, *Stochastic Processes Appl.*, **27**:291–306, 1988.
22. F. Spitzer, *Principles of Random Walk*, Grad. Texts Math., Vol. 34, Springer, New York, 1964.
23. E.C. Titchmarsh, *The Theory of Functions*, Oxford Univ. Press, Oxford, 1939.
24. O.K. Zakusylo and I.K. Matsak, On extreme values of some regenerative processes, *Teor. Ĭmovirn. Mat. Stat.*, **97**: 58–71, 2017 (in Ukrainian).