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Boundary value problems in elastostatics with singular data

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Abstract. We consider the main boundary value problems of linear elastostatics with nonregular data. We prove existence and uniqueness results for bounded and exterior domains of $\mathbb{R}^3$ of class C^k ($k \geqslant 2$). In the case of isotropic body, we prove the results for domains of class $C^{1,\alpha}$ ($\alpha \in (0,1]$) and of class C^1 in the case of the displacement problem.

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1 Introduction

The boundary-value problems of elastostatics for regular domains and data are today a well-defined part of the variational theory for elliptic systems. For instance, let Ω be a bounded domain of $\mathbb{R}^3$, and let $\{\mathcal{S}_1, \mathcal{S}_2\}$ be complementary subsurfaces of $\partial\Omega$. If Ω is of class C^k ($k \geqslant 2$) and

$$\hat{\boldsymbol{u}} \in W^{k-1/q,\,q}(\mathcal{S}_1), \qquad \hat{\boldsymbol{s}} \in W^{k-1-1/q,\,q}(\mathcal{S}_2),$$

$q \in (1, +\infty)$, then it is well known that the classical mixed problem

$$\operatorname{div} \mathsf{C}\,[\nabla \boldsymbol{u}] = \mathbf{0} \quad \text{in } \Omega, \tag{1.1}$$

$$\boldsymbol{u} = \hat{\boldsymbol{u}} \quad \text{on } \mathcal{S}_1, \tag{1.2}$$

$$\tau \boldsymbol{u} + \boldsymbol{s}(\boldsymbol{u}) = \hat{\boldsymbol{s}} \quad \text{on } \mathcal{S}_2 \quad (\tau \geqslant 0), \tag{1.3}$$

has a unique solution $\boldsymbol{u} \in W^{k,q}(\Omega)$, provided that the elasticity tensor C is regular and satisfies natural definiteness assumptions and $\hat{\boldsymbol{s}}$ is in equilibrium for $\mathcal{S}_1 = \emptyset$ and $\tau = 0$; see, for example, [11, Chap. VI] and [1, 3, 6, 12]. Here $\boldsymbol{s}(\boldsymbol{u}) = \mathsf{C}[\nabla \boldsymbol{u}]\boldsymbol{n}$, with $\boldsymbol{n}$ unit normal on $\partial\Omega$, is the traction field on the boundary. For $\mathcal{S}_2 = \emptyset$ (resp. $\mathcal{S}_1 = \emptyset$ and $\tau = 0$) we have the Dirichlet (or displacement) problem (resp., the Neumann, or traction, problem) [8]. For $\mathcal{S}_1 = \emptyset$ and $\tau > 0$, we have the Robin problem.

Clearly, even in view of possible applications, it is quite natural to detect whether the existence and uniqueness for (1.1)–(1.3) still hold under weaker regularity assumptions on the boundary data, for instance, in the presence of concentrated loads. From the existence of a regular solution of a boundary value problem with datum in a space H it follows, by transposition, the existence of a so-called very weak solution (see, e.g., [1,12]), which is defined by a suitable integral equation and corresponds to data in the dual space H'. In this approach the main problem concerns the sense to give to the attainability of the boundary value (see [1, Chap. 6]). An alternative approach to the existence of a solution to the boundary value problem of elastostatics, confined to homogeneous bodies but undoubtedly more sharp and strictly connected to the structure of system (1.1), is based on the classical theory of layer integral equations [11, Chap. VI]. (From a historical point of view, this approach has been the first one treating in full generality, at least for isotropic and homogeneous bodies, the boundary value problems of elastostatics.) Following this technique and assuming the body to be homogeneous and isotropic with boundary of class $C^{1,\alpha}$, $\alpha \in (0,1]$, Cialdea [2] was able to show the existence of a solution to the traction problem with a nonregular datum $\hat{\boldsymbol{s}}$, that is, system (1.1)–(1.3) with $\mathcal{S}_1 = \emptyset$, $\tau = 0$, and $\hat{\boldsymbol{s}}$ a Borel measure on $\partial\Omega$. He proved that (1.1)–(1.3) has a solution expressed by a double layer potential with density in $L^q(\partial\Omega)$ the tractions of which take the boundary value $\hat{\boldsymbol{s}}$ in a well-defined sense.

In this paper, we are concerned just with the problem of establishing the existence and uniqueness of a solution of (1.1)–(1.3) with singular data (the result for the displacement problem, that is, $\mathcal{S}_2 = \emptyset$ in (1.1)–(1.3), was recently proved in [19]). In particular, using the results we recall in Section 2 on the trace operators associated with the elastic layer potentials [17] and the Fredholm alternative, in Sections 3–4, we prove that if Ω is of class C^k ($k \geqslant 2$), C is constant and positive definite (strongly elliptic for $\mathcal{S}_2 = \emptyset$), and

$$\hat{\boldsymbol{u}} \in W^{2-k-1/q,\,q}(\mathcal{S}_1), \qquad \hat{\boldsymbol{s}} \in W^{1-k-1/q,\,q}(\mathcal{S}_2),$$

then (1.1)–(1.3) has a solution $\boldsymbol{u}$ expressed by layer potentials and thus taking the boundary values in a well-defined sense. It is unique in reasonable function classes.

Moreover, we consider the problem

$$\begin{aligned} &\operatorname{div} \mathsf{C}[\nabla \boldsymbol{u}] = \mathbf{0} \quad \text{in } \Omega, \\ &\boldsymbol{u} = \hat{\boldsymbol{u}} \quad \text{on } \mathcal{S}_1, \qquad \tau\boldsymbol{u} + \boldsymbol{s}(\boldsymbol{u}) = \hat{\boldsymbol{s}} \quad \text{on } \mathcal{S}_2, \qquad \lim_{r\to+\infty} \boldsymbol{u}(x) = \mathbf{0} \end{aligned} \tag{1.4}$$

($\tau \geqslant 0$, and r will be defined further) in an exterior domain of $\mathbb{R}^3$ (see, e.g., [13,14]). We show that under the stated hypotheses on C, $\hat{\boldsymbol{u}}$, and $\hat{\boldsymbol{s}}$, (1.4) has a unique solution.

We also consider the particular case of isotropic bodies. In this case, we prove the existence and uniqueness for bounded and exterior domains of class $C^{1,\alpha}$ ($\alpha \in (0,1]$) (Section 5) and of class C^1 for the displacement problem (Section 6).

Notation and classical results

We essentially use the notation of the classical monograph [8]. A domain Ω is an open connected set of $\mathbb{R}^3$. We deal with bounded or exterior domains with connected boundaries, although the results of this paper can be easily extended to more general bounded or exterior domains with compact but not necessarily connected boundaries. A domain Ω is said to be of class C^k ($k \in \mathbb{N}$) (resp., $C^{k,\alpha}$, $k \in \mathbb{N}$, $\alpha \in (0,1]$) if for every $\xi \in \partial\Omega$, there is a neighborhood of ξ (on $\partial\Omega$) that is the graph of a function of class C^k (resp., $C^{k,\alpha}$). We assume that Ω is at least of class C^1. The symbol c is used to denote a positive constant the numerical value of which is unessential to our purposes. We denote by $\boldsymbol{n}$ the unit normal to $\partial\Omega$ exterior (resp., interior) with respect to Ω for a bounded (resp., exterior) domain Ω. We denote by o the origin of the reference frame; we suppose $o \in \Omega$ (resp., $o \in \complement\overline{\Omega}$) for a bounded (resp., exterior) domain Ω. For every $x \in \mathbb{R}^3$, we set $\boldsymbol{x} = x - o$ and $r = |\boldsymbol{x}|$. If Ω is exterior, then we set $\Omega_R = \Omega \cap S_R$, where $S_R = \{x \in \mathbb{R}^3\colon\ r < R\}$. As usual, if $f(x)$ and $g(r) > 0$ are two functions on Ω, by $f = o(g)$ and $f = O(g)$ we mean that $\lim_{r\to+\infty} f(x)/g(r) = 0$ and $|f(x)| \leqslant cg(r)$.

$\mathfrak{R}$ denotes the set of all (infinitesimal) rigid displacements. The Sobolev space $W^{k,q}(\Omega)$ consists of all $\varphi \in L^1_{\mathrm{loc}}(\Omega)$ such that $\|\varphi\|_{W^{k,q}(\Omega)} = \|\varphi\|_{L^q(\Omega)} + \|\nabla_k\varphi\|_{L^q(\Omega)} < +\infty$; $W^{k,q}_0(\Omega)$ is the completion of $C^\infty_0(\Omega)$ with respect to $\|\varphi\|_{W^{k,q}(\Omega)}$, and $W^{-k,q'}(\Omega)$ is its dual space; $D^{1,2}_0(\Omega)$ denotes the completion of $C^\infty_0(\Omega)$ with respect to $\|\nabla\phi\|_{L^2(\Omega)}$; $W^{k-1/q,\,q}(\partial\Omega)$ is the trace space of $W^{k,q}(\Omega)$, and $W^{1-k-1/q',\,q'}(\partial\Omega)$ is its dual space. By $\int_\Omega^\star f\varphi$ ($\int_{\partial\Omega}^\star f\varphi$) we denote (say) the value of the functional $f \in W^{-k,q'}(\Omega)$ ($f \in W^{-k,q'}(\partial\Omega)$) at $\varphi \in W_0^{k,q}(\Omega)$ ($\varphi \in W^{k,q}(\partial\Omega)$). Of course, if $f\varphi$ is integrable, then $\int \equiv \int^\star$.

If Ω is of class C^k, then since $W^{k-1/q,\,q}(\partial\Omega) \hookrightarrow C^{k-1,\,\mu}(\partial\Omega)$, for $kq > 3$ and $\mu = 1 - 3/q$, we have that $[C^{k-1,\mu}(\partial\Omega)]' \hookrightarrow W^{1-k-1/q',\,q'}(\partial\Omega)$. Then, in particular, $W^{-1,q}(\partial\Omega)$, $q \in (1,2)$ contains the space of all Borel measures on $\partial\Omega$.

Let Ω be a bounded domain, and let $L^q_{\mathrm{div}}(\Omega) = \{\phi \in L^q(\Omega)\colon\ \operatorname{div}\phi \in L^q(\Omega)\}$. Endowed with the norm $\|\phi\|_{L^q_{\mathrm{div}}(\Omega)} = \|\phi\|_{L^q(\Omega)} + \|\operatorname{div}\phi\|_{L^q(\Omega)}$, $L^q_{\mathrm{div}}(\Omega)$ is a Banach space, and for $q \in (1,+\infty)$, the map $\phi \in C^1(\overline{\Omega}) \to \phi\cdot n \in C(\partial\Omega)$ extends to a continuous operator from $L^q_{\mathrm{div}}(\Omega) \to W^{-1/q,q}(\partial\Omega)$, and the following (generalized) divergence theorem holds (see [20, Chap. 1]):

$$\int_\Omega f \operatorname{div}\phi = \int_{\partial\Omega}^\star f\phi\cdot n - \int_\Omega \phi\cdot\nabla f$$

for all $\phi \in L^q_{\mathrm{div}}(\Omega)$ and $f \in W^{1,q'}(\Omega)$.

Let $\mathcal{B}$, $\mathcal{D}$ be two Banach spaces and denote by $\mathcal{B}'$, $\mathcal{D}'$ their dual spaces. A linear continuous map $\mathcal{T} : \mathcal{B} \to \mathcal{D}$ is said to be *Fredholmian* if its range is closed and $\dim\operatorname{Kern}\mathcal{T} = \dim\operatorname{Kern}\mathcal{T}' \in \mathbb{N}_0$, where $\mathcal{T}' : \mathcal{D}' \to \mathcal{B}'$ is the adjoint of $\mathcal{T}$. The classical *Fredholm alternative* [16] states that the equation $a = \mathcal{T}[u]$ has a solution if and only if $\langle\phi', a\rangle = 0$ for all $\phi' \in \operatorname{Kern}\mathcal{T}'$. Moreover, the equation $a' = \mathcal{T}'[u']$ has a solution if and only if $\langle a', \phi\rangle = 0$ for all $\phi \in \operatorname{Kern}\mathcal{T}$.

2 The elastic layer potentials

We refer to [8] for the basics of the theory of linear elastostatics. Recall that the elasticity tensor C in (1.1) is a linear map from $\mathrm{Lin} \to \mathrm{Sym}$ such that $\mathsf{C}[W] = \mathbf{0}$ for all $W \in \mathrm{Skw}$. We suppose C to be symmetric, that is,

$$E\cdot\mathsf{C}[L] = L\cdot\mathsf{C}[E] \quad \forall\, E, L \in \mathrm{Lin}.$$

C is *positive definite* if

$$\pi[E] = E\cdot\mathsf{C}[E] \geqslant |\operatorname{Sym} E|^2 \quad \forall E \in \mathrm{Lin},$$

and *strongly elliptic* if

$$\pi[a\otimes b] > \mathbf{0} \quad \forall a, b \neq \mathbf{0}.$$

Unless otherwise specified, we will suppose C to be at least strongly elliptic.

If the body is isotropic, then C is defined by

$$\mathsf{C}[E] = 2\mu\operatorname{Sym} E + \lambda(\operatorname{tr} E)\mathbf{1} \quad \forall E \in \mathrm{Lin}, \tag{2.1}$$

where λ, μ are the *Lamé moduli*. In such a case, C is positive definite and strongly elliptic if $\mu(3\lambda+2\mu) > 0$ and $\mu(\lambda + 2\mu) > 0$, respectively, and (1.1) writes

$$\mu\Delta u + (\lambda + \mu)\nabla\operatorname{div} u = \mathbf{0}.$$

A *weak solution* of (1.1) is a field $u \in W^{1,q}_{\mathrm{loc}}(\Omega)$ such that

$$\int_\Omega \nabla\phi\cdot\mathsf{C}[\nabla u] = 0 \quad \forall\phi \in C_0^\infty(\Omega).$$

If $q = 2$, then a weak solution is said to be a *variational solution*.

The following *work and energy theorem* holds for a variational solution $\boldsymbol{u} \in W^{1,2}_{\mathrm{loc}}(\overline{\Omega})$ of (1.1) (with $\boldsymbol{u} = o(1)$ for exterior Ω):

$$\int_{\Omega} \pi[\nabla \boldsymbol{u}] = \int_{\partial\Omega}^{\star} \boldsymbol{u} \cdot \boldsymbol{s}(\boldsymbol{u}). \tag{2.2}$$

From (2.2) and the inequality (see [8, p. 105] and [21])

$$\int_{\Omega} |\nabla \boldsymbol{u}|^2 \leqslant \int_{\Omega} \pi[\nabla \boldsymbol{u}] \quad \forall \boldsymbol{u} \in D_0^{1,2}(\Omega)$$

the classical uniqueness results follow: if $\boldsymbol{u}_{|\partial\Omega} = \boldsymbol{0}$, then $\boldsymbol{u} \equiv \boldsymbol{0}$; if C is positive definite and $\int_{\partial\Omega}^{\star} \boldsymbol{u} \cdot \boldsymbol{s}(\boldsymbol{u}) \leqslant 0$, then $\boldsymbol{u} \in \mathfrak{R}$ ($\boldsymbol{u} = \boldsymbol{0}$ for exterior Ω).

As far as the traction problem for bounded domains is concerned, the uniqueness is meant in the class of *normalized displacements*, that is, the fields $\boldsymbol{u}$ satisfying (see [8, p. 186])

$$\int_{\partial\Omega}^{\star} \boldsymbol{u} = \boldsymbol{0}, \qquad \int_{\partial\Omega}^{\star} \boldsymbol{x} \times \boldsymbol{u} = \boldsymbol{0}.$$

Equation (1.1) admits a fundamental solution $\boldsymbol{U}(x-y)$ (see [9, Chap. III]), that is, a regular solution for all $x \neq y$ to

$$\operatorname{div} \mathsf{C}\big[\nabla \boldsymbol{U}(x-y)\big] = \delta(x-y),$$

where δ is the Dirac distribution, expressed by $\boldsymbol{U}(\boldsymbol{z}) = \boldsymbol{\Phi}(\boldsymbol{z})/|\boldsymbol{z}|$ with homogeneous second–order tensor function $\boldsymbol{\Phi}$ of degree zero. If C satisfies (2.1), then $\boldsymbol{\Phi}$ is expressed by (see [8, p. 174])

$$\boldsymbol{\Phi}(\boldsymbol{z}) = \frac{1}{16\pi(1-\nu)}\left\{(3-4\nu)\mathbf{1} + \frac{\boldsymbol{z} \otimes \boldsymbol{z}}{|\boldsymbol{z}|^2}\right\},$$

where $\nu = \lambda/2(\lambda+\mu)$ is the *Poisson ratio*.

For all $\boldsymbol{\psi}, \boldsymbol{\phi} \in L^1(\partial\Omega)$, the fields

$$\boldsymbol{v}[\boldsymbol{\psi}](x) = \int_{\partial\Omega} \boldsymbol{U}(x-\zeta)\boldsymbol{\psi}(\zeta)\,\mathrm{d}\sigma_\zeta, \tag{2.3}$$

$$\boldsymbol{w}[\boldsymbol{\varphi}](x) = \int_{\partial\Omega} \mathsf{C}\big[\nabla \boldsymbol{U}(x-\zeta)\big](\boldsymbol{\varphi} \otimes \boldsymbol{n})(\zeta)\,\mathrm{d}\sigma_\zeta \tag{2.4}$$

represent analytical solutions of (1.1) in $\mathbb{R}^3 \setminus \partial\Omega$ and are known as *simple-layer potential* and *double layer potential* with densities $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$, respectively.

The fields (2.3) and (2.4) have the following asymptotic behavior:

$$\nabla_k \boldsymbol{v}[\boldsymbol{\psi}](x) = O\big(r^{-1-k}\big), \qquad \nabla_k \boldsymbol{w}[\boldsymbol{\varphi}](x) = O\big(r^{-2-k}\big),$$

and

$$\int_{\partial\Omega} \boldsymbol{\psi} = \boldsymbol{0} \quad \Longrightarrow \quad \nabla_k \boldsymbol{v}[\boldsymbol{\psi}](x) = O\big(r^{-2-k}\big).$$

We have

$$\|\boldsymbol{v}[\boldsymbol{\psi}]\|_{W^{k,q}(\Omega)} \leqslant c\|\boldsymbol{\psi}\|_{W^{k-1-1/q,\,q}(\partial\Omega)}, \qquad (2.5)$$
$$\|\boldsymbol{w}[\boldsymbol{\varphi}]\|_{W^{k,q}(\Omega)} \leqslant c\|\boldsymbol{\varphi}\|_{W^{k-1/q,\,q}(\partial\Omega)}$$

for some constants c depending only on k, q, and Ω, and the following limits exist for almost all $\xi \in \partial\Omega$ and axis $\boldsymbol{l}$ in a ball tangent (on the side of $\boldsymbol{n}$) to $\partial\Omega$ at ξ:

$$\lim_{\epsilon\to 0^+} \boldsymbol{v}[\boldsymbol{\psi}]\big(\xi \mp \epsilon\boldsymbol{l}(\xi)\big) = \mathcal{S}[\boldsymbol{\psi}](\xi), \qquad (2.6)$$

$$\lim_{\epsilon\to 0^+} \boldsymbol{w}[\boldsymbol{\varphi}]\big(\xi \mp \epsilon\boldsymbol{l}(\xi)\big) = \mathcal{W}^{\pm}[\boldsymbol{\psi}](\xi), \qquad (2.7)$$

$$\lim_{\epsilon\to 0^+} \mathsf{C}\big[\nabla\boldsymbol{v}[\boldsymbol{\psi}]\big]\big(\xi \mp \epsilon\boldsymbol{l}(\xi)\big)\boldsymbol{n}(\xi) = \mathcal{T}^{\pm}[\boldsymbol{\psi}](\xi), \qquad (2.8)$$

$$\lim_{\epsilon\to 0^+} \mathsf{C}\big[\nabla\boldsymbol{w}[\boldsymbol{\varphi}]\big]\big(\xi \mp \epsilon\boldsymbol{l}(\xi)\big)\boldsymbol{n}(\xi) = \mathcal{Z}[\boldsymbol{\varphi}](\xi). \qquad (2.9)$$

Note that (2.6) represents the trace of the simple-layer potential with density $\boldsymbol{\psi}$ and shows that $\boldsymbol{v}[\boldsymbol{\psi}]$ is continuous in $\mathbb{R}^3$; (2.7) represent the traces of the double layer potential on both “faces" of $\partial\Omega$; (2.8) are the traces of the traction field associated with the simple-layer potential on the faces of $\partial\Omega$; (2.9) represents the trace of the traction field associated with the double layer potential and shows that $\mathsf{C}[\nabla\boldsymbol{w}[\boldsymbol{\varphi}]]\boldsymbol{n}$ is continuous in $\mathbb{R}^3$.

The values (2.6)–(2.9) define the linear and continuous operators

$$\mathcal{S} : W^{k-1-1/q,\,q}(\partial\Omega) \to W^{k-1/q,\,q}(\partial\Omega), \qquad (2.10)$$
$$\mathcal{W}^{\pm} : W^{k-1/q,\,q}(\partial\Omega) \to W^{k-1/q,\,q}(\partial\Omega),$$
$$\mathcal{T}^{\pm} : W^{k-1-1/q,\,q}(\partial\Omega) \to W^{k-1-1/q,\,q}(\partial\Omega),$$
$$\mathcal{Z} : W^{k-1-1/q,\,q}(\partial\Omega) \to W^{k-1-1/q,\,q}(\partial\Omega),$$

and the classical jump conditions hold:

$$\boldsymbol{\psi} = \mathcal{T}^{+}[\boldsymbol{\psi}] - \mathcal{T}^{-}[\boldsymbol{\psi}], \qquad (2.11)$$

$$\boldsymbol{\varphi} = \mathcal{W}^{+}[\boldsymbol{\varphi}] - \mathcal{W}^{-}[\boldsymbol{\varphi}]. \qquad (2.12)$$

As observed in [17], (2.10) can be extended to its adjoint operator

$$\mathcal{S}' : W^{1-k-1/q',\,q'}(\partial\Omega) \to W^{2-k-1/q',\,q'}(\partial\Omega),$$

defining the trace of the simple-layer potential with density $\boldsymbol{\psi} \in W^{1-k-1/q',\,q'}(\partial\Omega)$, that is,

$$\boldsymbol{v}[\boldsymbol{\psi}](x) = \int_{\partial\Omega}^{\star} \boldsymbol{U}(x-\zeta)\boldsymbol{\psi}(\zeta)\,\mathrm{d}\sigma_{\zeta}$$

(for the meaning of $\int^{\star}$, see Notation in Section 1), and from (2.5) it follows that

$$\|\boldsymbol{v}[\boldsymbol{\psi}]\|_{W^{2-k,\,q'}(\Omega)} \leqslant c\|\boldsymbol{\psi}\|_{W^{1-k-1/q',\,q'}(\partial\Omega)}. \qquad (2.13)$$

Moreover, $\mathcal{W}^{\pm}$ and $\mathcal{T}^{\mp}$ are adjoint to each other, so that (say)

$$\mathcal{W}^{-}: W^{2-k-1/q',q'}(\partial\Omega) \to W^{2-k-1/q',q'}(\partial\Omega)$$

is the adjoint of $\mathcal{T}^{+}$ and defines the trace of a double layer potential $\boldsymbol{w}[\boldsymbol{\varphi}]$ with density in $W^{2-k-1/q',q'}(\partial\Omega)$:

$$\boldsymbol{w}[\boldsymbol{\varphi}](x) = \overset{\star}{\int\limits_{\partial\Omega}} \mathsf{C}\big[\nabla \boldsymbol{U}(x-\zeta)\big](\boldsymbol{\varphi}\otimes\boldsymbol{n})(\zeta)\,\mathrm{d}\sigma_{\zeta}.$$

Finally, $\mathcal{Z}$ can be extended to its adjoint operator

$$\mathcal{Z}': W^{2-k-1/q',q'}(\partial\Omega) \to W^{1-k-1/q',q'}(\partial\Omega),$$

which defines the trace of the traction field of the double layer potential $\boldsymbol{w}[\boldsymbol{\varphi}]$ with density $\boldsymbol{\varphi} \in W^{2-k-1/q',q'}(\partial\Omega)$.

The following results are proved in [17].

Lemma 1. *Let Ω be a bounded or an exterior domain of class C^k ($k \geqslant 2$). The operator $\mathcal{S}$ is Fredholmian, and* $\operatorname{Kern}\mathcal{S} = \operatorname{Kern}\mathcal{S}' = \{\mathbf{0}\}$.

Lemma 2. *Let Ω be a bounded or an exterior domain of class C^k ($k \geqslant 2$). The operators $\mathcal{W}^{\pm}$ and $\mathcal{T}^{\pm}$ are Fredholmian,* $\operatorname{Kern}\mathcal{W}^{+} = \operatorname{Kern}\mathcal{T}^{-} = \{\mathbf{0}\}$, *and*

$$\begin{aligned} \operatorname{Kern}\mathcal{T}^{+} &= \begin{cases} \{\boldsymbol{\psi} : \mathcal{S}[\boldsymbol{\psi}] \in \mathfrak{R}\}, & \Omega \text{ bounded},\\ \{\mathbf{0}\}, & \Omega \text{ exterior}, \end{cases} \\ \operatorname{Kern}\mathcal{W}^{-} &= \begin{cases} \mathfrak{R}, & \Omega \text{ bounded},\\ \{\mathbf{0}\}, & \Omega \text{ exterior}. \end{cases} \end{aligned} \tag{2.14}$$

Lemma 3. *Let Ω be a bounded or an exterior domain of class C^k ($k \geqslant 2$). The operator $\mathcal{Z}$ is Fredholmian, and* $\operatorname{Kern}\mathcal{Z} = \operatorname{Kern}\mathcal{Z}' = \mathfrak{R}$.

3 The Dirichlet–Neumann–Robin problem

First of all, let us recall the following well-known existence theorems for regular data (see [1, 7, 12]).

Theorem 1. *Let Ω be a bounded domain of class C^k ($k \geqslant 2$). If $\hat{\boldsymbol{u}} \in W^{k-1/q,q}(\partial\Omega)$, $q \in (1,+\infty)$, and $\boldsymbol{\phi} \in C_0^{\infty}(\Omega)$, then the displacement problem*

$$\operatorname{div}\mathsf{C}[\nabla\boldsymbol{u}] = \boldsymbol{\phi} \quad \text{in } \Omega, \qquad \boldsymbol{u} = \hat{\boldsymbol{u}} \quad \text{on } \partial\Omega$$

has a unique solution $\boldsymbol{u} \in W^{k,q}(\Omega)$, and

$$\|\boldsymbol{u}\|_{W^{k,q}(\Omega)} \leqslant c\big\{\|\hat{\boldsymbol{u}}\|_{W^{k-1/q,q}(\partial\Omega)} + \|\boldsymbol{\phi}\|_{W^{k-2,q}(\Omega)}\big\}.$$

Theorem 2. *Let Ω be a bounded domain of class C^k ($k \geqslant 2$), and let C be positive definite. If $\hat{\boldsymbol{s}} \in W^{k-1-1/q,q}(\partial\Omega)$, $q \in (1,+\infty)$ satisfies*

$$\int\limits_{\partial\Omega} \boldsymbol{\varrho}\cdot\hat{\boldsymbol{s}} - \int\limits_{\Omega} \boldsymbol{\varrho}\cdot\boldsymbol{\phi} = 0 \quad \forall \boldsymbol{\varrho} \in \mathfrak{R}$$

and $\boldsymbol{\phi} \in C_0^\infty(\Omega)$*, then the traction problem*

$$\operatorname{div} \mathsf{C}[\nabla \boldsymbol{u}] = \boldsymbol{\phi} \quad \textit{in } \Omega, \qquad \boldsymbol{s}(\boldsymbol{u}) = \hat{\boldsymbol{s}} \quad \textit{on } \partial\Omega$$

has a unique normalized solution $\boldsymbol{u} \in W^{k,q}(\Omega)$*, and*

$$\|\boldsymbol{u}\|_{W^{k,q}(\Omega)} \leqslant c\big\{\|\hat{\boldsymbol{s}}\|_{W^{k-1-1/q,\,q}(\partial\Omega)} + \|\boldsymbol{\phi}\|_{W^{k-2,q}(\Omega)}\big\}.$$

By the results on the trace operators associated with the layer potentials recalled in the previous section we can apply the Fredholm alternative to prove the existence and uniqueness for the classical problems of elastostatics with singular data. Let us consider the general boundary-value problem

$$\operatorname{div} \mathsf{C}[\nabla \boldsymbol{u}] = \boldsymbol{0} \quad \text{in } \Omega, \tag{3.1}$$

$$\alpha \boldsymbol{u} + \gamma \boldsymbol{s}(\boldsymbol{u}) = \boldsymbol{a} \quad \text{on } \partial\Omega, \tag{3.2}$$

where α and γ are assigned scalars, not both zero and such that $\alpha\gamma \geqslant 0$. For exterior Ω, we require that

$$\boldsymbol{u}(x) = o(1).$$

Note that for $\gamma = 0$, $\alpha = 0$, and $\alpha\gamma > 0$, we have the Dirichlet, Neumann, and Robin problems, respectively. The following theorem holds. (The result stated by Theorem 3 for the Dirichlet problem was recently proved in [19].)

Theorem 3. *Let* Ω *be a bounded or an exterior domain of class* C^k $(k \geqslant 2)$*, and assume that* C *is positive definite for* $\gamma \neq 0$*. If*

$$\boldsymbol{a} \in W^{h-k-1/q,\,q}(\partial\Omega), \quad h = \begin{cases} 2, & \gamma = 0, \\ 1 & \textit{otherwise,} \end{cases}$$

and

$$\int\limits_{\partial\Omega}^{\star} \boldsymbol{a} \cdot \boldsymbol{\varrho} = 0 \quad \forall \boldsymbol{\varrho} \in \mathfrak{R}$$

for bounded Ω *and* $\alpha = 0$*, then* (3.1)–(3.2) *has a solution expressed by a simple-layer potential with density* $\boldsymbol{\psi} \in W^{1-k-1/q,\,q}(\partial\Omega)$*. It satisfies the estimate*

$$\|\boldsymbol{u}\|_{W^{2-k,q}(\Omega)} \leqslant c\|\boldsymbol{a}\|_{W^{h-k-1/q,\,q}(\partial\Omega)}, \tag{3.3}$$

and for $\gamma \neq 0$*, it is unique in the class of all* $\boldsymbol{u} \in W^{2-k,q}_{\mathrm{loc}}(\Omega)$ *such that*

$$\int\limits_{\Omega}^{\star} \boldsymbol{u} \cdot \boldsymbol{\phi} = \begin{cases} -\frac{1}{\gamma} \int_{\partial\Omega}^{\star} \boldsymbol{a} \cdot \boldsymbol{z}, & \Omega \textit{ bounded}, \\ +\frac{1}{\gamma} \int_{\partial\Omega}^{\star} \boldsymbol{a} \cdot \boldsymbol{z}, & \Omega \textit{ exterior}, \end{cases} \tag{3.4}$$

for all $\boldsymbol{\phi} \in C_0^\infty(\Omega)$*, with* $\boldsymbol{z}$ *solution of*

$$\operatorname{div} \mathsf{C}[\nabla \boldsymbol{z}] = \boldsymbol{\phi} \quad \textit{in } \Omega, \qquad \alpha \boldsymbol{z} + \gamma \boldsymbol{s}(\boldsymbol{z}) = \boldsymbol{0} \quad \textit{on } \partial\Omega,$$

and $\boldsymbol{z} = o(1)$ *if* Ω *is exterior. If* $\gamma = 0$*, then* $\boldsymbol{u}$ *is unique in the class of all* $\boldsymbol{u} \in W^{2-k,q}_{\mathrm{loc}}(\Omega)$ *such that*

$$\int_{\Omega}^{\star} \boldsymbol{u} \cdot \boldsymbol{\phi} = \begin{cases} +\frac{1}{\alpha} \int_{\partial\Omega}^{\star} \boldsymbol{a} \cdot \boldsymbol{s}(\boldsymbol{z}), & \Omega \text{ bounded}, \\ -\frac{1}{\alpha} \int_{\partial\Omega}^{\star} \boldsymbol{a} \cdot \boldsymbol{s}(\boldsymbol{z}), & \Omega \text{ exterior}, \end{cases} \tag{3.5}$$

for all $\boldsymbol{\phi} \in C^{\infty}_{0}(\Omega)$ *such that* $\int_{\Omega} \boldsymbol{\varrho} \cdot \boldsymbol{\phi} = 0 \; \forall\, \boldsymbol{\varrho} \in \mathfrak{R}$*, with* $\boldsymbol{z}$ *solution of*

$$\operatorname{div} \mathsf{C}[\nabla \boldsymbol{z}] = \boldsymbol{\phi} \quad \text{in } \Omega, \qquad \boldsymbol{z} = \boldsymbol{0} \quad \text{on } \partial\Omega,$$

and $\boldsymbol{z} = o(1)$ *if* Ω *is exterior.*

Proof. Let Ω be bounded. If $\alpha\gamma = 0$, then by Lemma 1 or Lemma 2 the equation

$$\big(\alpha \mathcal{S}' + \gamma \mathcal{T}^{+}\big)[\boldsymbol{\psi}] = \boldsymbol{a} \tag{3.6}$$

has a solution $\boldsymbol{\psi} \in W^{1-k-1/q,\,q}(\partial\Omega)$, and the field $\boldsymbol{u} = \boldsymbol{v}[\boldsymbol{\psi}]$ is a solution that is C^{∞} in Ω and satisfies (3.2) in the sense of (3.6). Let $\boldsymbol{a}_j$ be a regular sequence on $\partial\Omega$ that converges to $\boldsymbol{a}$ strongly in $W^{h-k-1/q,\,q}(\partial\Omega)$. Let $\alpha = 0$ (say), and let $\boldsymbol{v}[\boldsymbol{\psi}_j]$ be the solution of (3.1)–(3.2) with datum $\boldsymbol{a}_j$. By (2.13) $\boldsymbol{v}[\boldsymbol{\psi}_j]$ converges to $\boldsymbol{v}[\boldsymbol{\psi}]$ strongly in $W^{2-k,q}(\Omega)$. Integration by parts gives

$$\int_{\Omega} \boldsymbol{v}[\boldsymbol{\psi}_j] \cdot \boldsymbol{\phi} = -\frac{1}{\gamma} \int_{\partial\Omega} \boldsymbol{a}_j \cdot \boldsymbol{z}.$$

Hence (3.4) follows by letting $j \to +\infty$. Analogously, for $\gamma = 0$, we obtain (3.5). Estimate (3.3) follows from Theorems 1 and 2 and a duality argument.

If $\alpha\gamma > 0$, then since $\mathcal{S}$ is compact from $W^{1-k-1/q,\,q}(\partial\Omega)$ into itself, $\alpha\mathcal{S} + \gamma\mathcal{T}^{+}$ is a compact perturbation of a Fredholmian operator, and so it enjoys the same property. Set

$$\boldsymbol{\sigma}[\boldsymbol{\varphi}] = \alpha \boldsymbol{v}[\boldsymbol{\varphi}] - \gamma \boldsymbol{w}[\boldsymbol{\varphi}].$$

To prove existence of a solution of (3.6), it is sufficient to show that if $\boldsymbol{\varphi} \in \operatorname{Kern}(\alpha\mathcal{S} - \gamma\mathcal{W}^{-})$, then $\boldsymbol{\varphi} = \boldsymbol{0}$. Now, since $\boldsymbol{\sigma}[\boldsymbol{\varphi}]^{-} = \boldsymbol{0}$, by uniqueness $\boldsymbol{\sigma}[\boldsymbol{\varphi}] = \boldsymbol{0}$ in $\complement\Omega$. By the jump conditions (2.11) and (2.12) we have $\boldsymbol{\sigma}[\boldsymbol{\varphi}]^{+} = -\gamma\boldsymbol{\varphi}$, $\boldsymbol{s}(\boldsymbol{\sigma}[\boldsymbol{\varphi}])^{+} = \alpha\boldsymbol{\varphi}$ so that, integrating by parts,

$$\int_{\Omega} \pi\big[\nabla\boldsymbol{\sigma}[\boldsymbol{\varphi}]\big] = \int_{\partial\Omega} \boldsymbol{\sigma}[\boldsymbol{\varphi}]^{+} \cdot \boldsymbol{s}\big(\boldsymbol{\sigma}[\boldsymbol{\varphi}]\big)^{+} = -\alpha\gamma \int_{\partial\Omega} |\boldsymbol{\varphi}|^{2}.$$

Hence $\boldsymbol{\varphi} = \boldsymbol{0}$. The uniqueness follows from the usual argument. The proof of the existence and uniqueness in exterior domains is analogous to the previous one and so is omitted. □

Note that, choosing $\boldsymbol{\phi} = \operatorname{div} \mathsf{C}[\nabla\boldsymbol{\zeta}]$ in (3.5) (say) with $\boldsymbol{\zeta} \in C^{k}(\overline{\Omega})$ vanishing on $\partial\Omega$, we have[1]

$$\int_{\Omega}^{\star} \boldsymbol{u} \cdot \operatorname{div} \mathsf{C}[\nabla\boldsymbol{\zeta}] = \pm\frac{1}{\alpha} \int_{\partial\Omega} \boldsymbol{a} \cdot \boldsymbol{s}(\boldsymbol{\zeta}) \quad \forall \boldsymbol{\zeta} \in C^{k}(\overline{\Omega}). \tag{3.7}$$

In particular, if $\boldsymbol{\zeta} \in C^{\infty}_{0}(\Omega)$, then $\int_{\Omega}^{\star} \boldsymbol{u} \cdot \operatorname{div} \mathsf{C}[\nabla\boldsymbol{\zeta}] = 0$ for all $\boldsymbol{\zeta} \in C^{\infty}_{0}(\Omega)$, that is, $\boldsymbol{u}$ satisfies (3.1)–(3.2) in the sense of distributions.

[1] Some authors call a field $\boldsymbol{u} \in L^{1}_{\mathrm{loc}}(\Omega)$ satisfying (3.7) a *very weak solution* of (3.1)–(3.2) (see, e.g., [1, 12]).

Remark 1. Similar results to those stated in Theorem 3 have been proved for the Dirichlet problem associated to Stokes and Oseen systems in [18] (also see [15]).

Remark 2. By the well-known interpolation and stability results [10] Theorem 3 ensures that $\alpha\mathcal{S} + \gamma\mathcal{T}^{\pm}$ are Fredholmian. Hence the existence and uniqueness follows for boundary data in the Sobolev–Besov space $W^{s,q}(\partial\Omega)$. In particular, for every $\boldsymbol{a} \in L^q(\partial\Omega)$, (3.1)–(3.2) has a unique very weak solution, which takes the boundary datum in the following sense:

$$\lim_{t\to 0^{\pm}} \big(\alpha\boldsymbol{u} + \gamma\boldsymbol{s}(\boldsymbol{u})\big)\big(\xi - t\boldsymbol{l}(\xi)\big) = \boldsymbol{a}(\xi)$$

for almost all $\xi \in \partial\Omega$ and axis $\boldsymbol{l}$ in a ball tangent to $\partial\Omega$ at ξ.

4 Mixed problems

Once a regular boundary-value problem is solved, we can construct the associated Green function. Consider, for instance, the Dirichlet problem in a bounded domain

$$\operatorname{div} \mathsf{C}[\nabla\boldsymbol{u}] = \boldsymbol{0} \quad \text{in } \Omega, \qquad \boldsymbol{u} = \hat{\boldsymbol{u}} \quad \text{on } \partial\Omega. \tag{4.1}$$

For every $y \in \Omega$, the equations

$$\begin{aligned} \operatorname{div}_x \mathsf{C}\big[\nabla\boldsymbol{A}(x,y)\big] &= \boldsymbol{0} \quad \text{in } \Omega, \\ \boldsymbol{A}(\xi,y) - \boldsymbol{U}(x-\xi) &= \boldsymbol{0} \quad \text{on } \partial\Omega \end{aligned}$$

have a unique regular solution, and the field $\boldsymbol{G}(x,y) = \boldsymbol{U}(x-y) - \boldsymbol{A}(x,y)$ defines the Green function of the Dirichlet problem. The very weak solution of (4.1) can be written as

$$\int\limits_{\partial\Omega}^{\star} \mathsf{C}[\nabla\boldsymbol{G}](x,\zeta)(\hat{\boldsymbol{u}}\otimes\boldsymbol{n})(\zeta)\,\mathrm{d}\sigma_\zeta. \tag{4.2}$$

Beside its intrinsic interest, (4.2) is also useful to deal with mixed problems as the Dirichlet–Neumann–Robin problem (1.1)–(1.3) in a bounded domain we treat in details as a sample. To this end, we follow [11, p. 606]. Let us look for a solution of (1.1)–(1.3) expressed by

$$\begin{aligned} \boldsymbol{u}(x) &= -\int\limits_{\mathcal{S}_2}^{\star} \boldsymbol{G}(x,\zeta)\boldsymbol{\psi}(\zeta)\,\mathrm{d}\sigma_\zeta + \int\limits_{\mathcal{S}_1}^{\star} \mathsf{C}[\nabla\boldsymbol{G}](x,\zeta)(\hat{\boldsymbol{u}}\otimes\boldsymbol{n})(\zeta)\,\mathrm{d}\sigma_\zeta \\ &= \tilde{\boldsymbol{v}}[\boldsymbol{\psi}](x) + \tilde{\boldsymbol{w}}[\hat{\boldsymbol{u}}](x). \end{aligned} \tag{4.3}$$

Since by construction $\boldsymbol{u}$ satisfies (1.2), we have to find $\boldsymbol{\psi}$ such that, on $\mathcal{S}_2$,

$$\tau\tilde{\boldsymbol{v}}[\boldsymbol{\psi}] + \boldsymbol{s}\big(\tilde{\boldsymbol{v}}[\boldsymbol{\psi}]\big) = \hat{\boldsymbol{s}} - \tau\tilde{\boldsymbol{w}}[\hat{\boldsymbol{u}}] - \boldsymbol{s}\big(\tilde{\boldsymbol{w}}[\hat{\boldsymbol{u}}]\big). \tag{4.4}$$

By the regularity properties of the Green function over regular boundaries (4.4) is a Fredholm equation of index zero. Thus, to show that (4.4) is uniquely solvable, it is sufficient that the homogeneous equation has only the trivial solution. If $\tau\tilde{\boldsymbol{v}}[\boldsymbol{\psi}] + \boldsymbol{s}(\tilde{\boldsymbol{v}}[\boldsymbol{\psi}]) = \boldsymbol{0}$ on $\mathcal{S}_2$, then $\tilde{\boldsymbol{v}}[\boldsymbol{\psi}]$ is a regular solution of

$$\operatorname{div} \mathsf{C}[\nabla\boldsymbol{u}] = \boldsymbol{0} \quad \text{in } \Omega, \qquad \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \mathcal{S}_1, \qquad \tau\boldsymbol{u} + \boldsymbol{s}(\boldsymbol{u}) = \boldsymbol{0} \quad \text{on } \mathcal{S}_2.$$

By uniqueness $\tilde{\boldsymbol{v}}[\boldsymbol{\psi}] = \mathbf{0}$ in Ω. Since $\tilde{\boldsymbol{v}}[\boldsymbol{\psi}]$ is continuous through $\partial\Omega$, again by uniqueness $\tilde{\boldsymbol{v}}[\boldsymbol{\psi}] = \mathbf{0}$ in $\complement\Omega$, so that (2.11) implies $\boldsymbol{\psi} = \mathbf{0}$.

Moreover, the usual argument shows that $\boldsymbol{u}$ satisfies

$$\int_{\Omega}^{\star} \boldsymbol{u}\cdot\boldsymbol{\phi} = \int_{\mathcal{S}_1} \hat{\boldsymbol{u}}\cdot\boldsymbol{s}(\boldsymbol{z}) - \int_{\mathcal{S}_2} \hat{\boldsymbol{s}}\cdot\boldsymbol{z} \tag{4.5}$$

for all $\boldsymbol{\phi} \in C^\infty(\Omega)$, with $\boldsymbol{z}$ solution of

$$\operatorname{div}\mathsf{C}[\nabla\boldsymbol{z}] = \boldsymbol{\phi} \quad \text{in } \Omega, \qquad \boldsymbol{z} = \mathbf{0} \quad \text{on } \mathcal{S}_1, \qquad \tau\boldsymbol{z} + \boldsymbol{s}(\boldsymbol{z}) = \mathbf{0} \quad \text{on } \mathcal{S}_2. \tag{4.6}$$

Therefore, calling a *very weak solution* of (1.1)–(1.3) a field $\boldsymbol{u} \in W^{2-k,q}(\Omega)$ that satisfies (4.5) for all $\boldsymbol{\phi} \in C^\infty(\Omega)$ with $\boldsymbol{z}$ the solution of (4.6), we can state the following:

Theorem 4. *Let Ω be a bounded domain of class C^k ($k \geqslant 2$). If $\hat{\boldsymbol{u}} \in W^{2-k-1/q,\,q}(\mathcal{S}_1)$ and $\hat{\boldsymbol{s}} \in W^{1-k-1/q,\,q}(\mathcal{S}_2)$, then* (1.1)–(1.3) *has a unique very weak solution*

$$\boldsymbol{u} \in W^{2-k,q}(\Omega) \cap C^\infty(\Omega),$$

expressed by (4.3) *for some $\boldsymbol{\psi} \in W^{1-k-1/q,\,q}(\mathcal{S}_2)$, and*

$$\|\boldsymbol{u}\|_{W^{2-k,\,q}(\Omega)} \leqslant c\bigl\{\|\hat{\boldsymbol{u}}\|_{W^{2-k-1/q,\,q}(\mathcal{S}_1)} + \|\hat{\boldsymbol{s}}\|_{W^{1-k-1/q,\,q}(\mathcal{S}_2)}\bigr\}.$$

Analogous problems in exterior domains (like (1.4)) can be treated by the same method.

Remark 3. Taking into account that, for every $x \in \Omega$,

$$\bigl|\nabla_m\boldsymbol{v}[\boldsymbol{\psi}]\bigr|(x) = \Biggl|\int_{\partial\Omega}^{\star} \bigl[\nabla_m\boldsymbol{U}(x-\zeta)\bigr]\boldsymbol{\psi}(\zeta)\,\mathrm{d}\sigma_\zeta\Biggr| \leqslant c(x,m,\Omega,\mathsf{C})\|\boldsymbol{\psi}\|_{W^{1-k-1/q,\,q}(\partial\Omega)},$$

we see that for all Ω'' such that $\overline{\Omega''} \subset \Omega$, there is a positive constant c depending only on Ω'', Ω, and C such that

$$\|\boldsymbol{u}\|_{W^{m,q}(\Omega'')} \leqslant c\bigl\{\|\hat{\boldsymbol{u}}\|_{W^{2-k-1/q,\,q}(\mathcal{S}_1)} + \|\hat{\boldsymbol{s}}\|_{W^{1-k-1/q,\,q}(\mathcal{S}_2)}\bigr\}.$$

5 The Dirichlet–Neumann–Robin problem in domains of class $C^{1,\alpha}$

Let C be expressed by (2.1). If Ω is of class $C^{1,\alpha}$ for some $\alpha \in (0,1]$, then classical results of V. Kupradze and S. Mikhlin ensure that the operators

$$\mathcal{W}^\pm : L^q(\partial\Omega) \to L^q(\partial\Omega)$$

are Fredholmian for all $q \in (1,+\infty)$ and $\operatorname{Kern}\mathcal{W}^\pm, \operatorname{Kern}\mathcal{T}^\pm \subset C^{0,\alpha}(\partial\Omega)$ (see [11, Chap. VI]). Thus, proceeding as for Lemma 2 in [17], we see that $\operatorname{Kern}\mathcal{W}^\pm$ and $\operatorname{Kern}\mathcal{T}^\pm$ are expressed by (2.14). Hence from the same argument used in the proof of Theorem 3 the following existence results easily follow.

Theorem 5. *Let Ω be a bounded or an exterior domain of class $C^{1,\alpha}$ ($\alpha \in (0,1]$). If $\hat{\boldsymbol{s}} \in L^q(\partial\Omega)$ is in equilibrium for Ω bounded and $\alpha = 0$, then the problem*

$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\operatorname{div}\boldsymbol{u} = \mathbf{0} \quad \textit{in } \Omega, \qquad \alpha\boldsymbol{u} + \gamma\boldsymbol{s}(\boldsymbol{u}) = \hat{\boldsymbol{s}} \quad \textit{on } \partial\Omega$$

(α, γ not both zero and such that $\alpha\gamma \geqslant 0$) with $\boldsymbol{u} = o(1)$ if Ω is exterior, has a unique solution expressed by a simple-layer potential with density in $L^q(\partial\Omega)$, and

$$\left\|(\nabla \boldsymbol{u})^*\right\|_{L^q(\partial\Omega)} \leqslant c\|\hat{\boldsymbol{s}}\|_{L^q(\partial\Omega)},$$

where $f^* = \operatorname{ess\,sup} f$.

From (2.11) it follows

$$\|\boldsymbol{\psi}\|_{L^q(\partial\Omega)} \leqslant \left\|\mathcal{T}^+[\boldsymbol{\psi}]\right\|_{L^q(\partial\Omega)} + \left\|\mathcal{T}^-[\boldsymbol{\psi}]\right\|_{L^q(\partial\Omega)} \leqslant c\left\|\mathcal{S}[\boldsymbol{\psi}]\right\|_{W^{1,q}(\partial\Omega)}.$$

Therefore, proceeding as for Lemma 1 in [17], we see that the operator $\mathcal{S} : L^q(\partial\Omega) \to W^{1,q}(\partial\Omega)$ is Fredholmian and $\operatorname{Kern}\mathcal{S} = \{\mathbf{0}\}$. Hence we have the following:

Theorem 6. *Let Ω be a bounded or an exterior domain of class $C^{1,\alpha}$ ($\alpha \in (0,1]$). If $\hat{\boldsymbol{u}} \in L^q(\partial\Omega)$, then the problem*

$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla \operatorname{div}\boldsymbol{u} = \mathbf{0} \quad \text{in } \Omega, \qquad \boldsymbol{u} = \hat{\boldsymbol{u}} \quad \text{on } \partial\Omega,$$

with $\boldsymbol{u} = o(1)$ if Ω is exterior, has a unique solution expressed by a simple-layer potential with density $\boldsymbol{\psi} \in W^{-1,q}(\partial\Omega)$, and

$$\|\boldsymbol{u}^*\|_{L^q(\partial\Omega)} \leqslant c\|\hat{\boldsymbol{u}}\|_{L^q(\partial\Omega)}.$$

Moreover, if $\hat{\boldsymbol{u}} \in W^{1,q}(\partial\Omega)$, then $\boldsymbol{\psi} \in L^q(\partial\Omega)$, and

$$\left\|(\nabla \boldsymbol{u})^*\right\|_{L^q(\partial\Omega)} \leqslant c\|\hat{\boldsymbol{u}}\|_{W^{1,q}(\partial\Omega)}.$$

From (2.12) it follows

$$\|\boldsymbol{\varphi}\|_{W^{1,q}(\partial\Omega)} \leqslant \left\|\mathcal{W}^+[\boldsymbol{\varphi}]\right\|_{W^{1,q}(\partial\Omega)} + \left\|\mathcal{W}^-[\boldsymbol{\varphi}]\right\|_{W^{1,q}(\partial\Omega)} \leqslant c\left\|\mathcal{Z}[\boldsymbol{\varphi}]\right\|_{L^q(\partial\Omega)}.$$

Proceeding as for Lemma 3 in [17], we see that the operator $\mathcal{Z} : W^{1,q}(\partial\Omega) \to L^q(\partial\Omega)$ is Fredholmian and $\operatorname{Kern}\mathcal{Z} = \mathfrak{R}$. Let $\mathfrak{C} = \{\boldsymbol{\psi}\colon \mathcal{S}[\boldsymbol{\psi}] \in \mathfrak{R}\}$.

Theorem 7. *Let Ω be a bounded or an exterior domain of class $C^{1,\alpha}$ ($\alpha \in (0,1]$). If $\mu(3\lambda+2\mu) > 0$ and $\hat{\boldsymbol{s}} \in W^{-1,q}(\partial\Omega)$ is in equilibrium for bounded Ω, then the problem*

$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla \operatorname{div}\boldsymbol{u} = \mathbf{0} \quad \text{in } \Omega, \qquad \boldsymbol{s}(\boldsymbol{u}) = \hat{\boldsymbol{s}} \quad \text{on } \partial\Omega,$$

with $\boldsymbol{u} = o(1)$ if Ω is exterior, has a unique solution expressed by

$$\boldsymbol{u} = \begin{cases} \boldsymbol{w}[\boldsymbol{\varphi}], & \Omega \text{ bounded}, \\ \boldsymbol{w}[\boldsymbol{\varphi}] + \boldsymbol{v}[\boldsymbol{\psi}], & \Omega \text{ exterior}, \end{cases}$$

for some $\boldsymbol{\varphi} \in L^q(\partial\Omega)$ and $\boldsymbol{\psi} \in \mathfrak{C}$. Moreover, if $\hat{\boldsymbol{s}} \in L^q(\partial\Omega)$, then $\boldsymbol{\varphi} \in W^{1,q}(\partial\Omega)$.

Proof. Let Ω be exterior. Consider the equation

$$\mathcal{Z}[\boldsymbol{\varphi}] = \hat{\boldsymbol{s}} - \mathcal{T}^-[\boldsymbol{\psi}] \tag{5.1}$$

with $\boldsymbol{\psi} \in \mathfrak{C}$. Of course, (5.1) has a solution if and only if

$$\overset{\star}{\int\limits_{\partial\Omega}} (\hat{\boldsymbol{s}} - \mathcal{T}^-[\boldsymbol{\psi}]) \cdot \boldsymbol{\varrho} = 0 \quad \forall \boldsymbol{\varrho} \in \mathfrak{R}.$$

This is equivalent to say that the homogeneous system

$$\int_{\partial\Omega} \mathcal{T}^-[\boldsymbol{\psi}] \cdot \boldsymbol{\varrho} = 0 \quad \forall \boldsymbol{\varrho} \in \mathfrak{R}$$

has only the null solution $\boldsymbol{\psi} = \mathbf{0}$. To show this, choose $\boldsymbol{\varrho}$ such that $\mathcal{S}[\boldsymbol{\psi}] = \boldsymbol{\varrho}$. Then, integrating by parts, we have

$$\int_{\Omega} \pi\big[\nabla \boldsymbol{v}[\boldsymbol{\psi}]\big] = \int_{\partial\Omega} \mathcal{S}[\boldsymbol{\psi}] \cdot \mathcal{T}^-[\boldsymbol{\psi}] = 0.$$

Hence it follows that $\boldsymbol{\psi} = \mathbf{0}$. The proof for bounded Ω is immediate. □

Remark 4. Since for $q \in (1, 2)$, $W^{-1,q}(\partial\Omega)$ contains the space of Borel measures on $\partial\Omega$, Theorem 7 extends, in particular, the existence theorem of [2].

6 The displacement problem in domains of class C^1

In this section, we are concerned with the problem

$$\mu\Delta\boldsymbol{u} + (\lambda + \mu)\nabla \operatorname{div} \boldsymbol{u} = \mathbf{0} \quad \text{in } \Omega, \tag{6.1}$$

$$\boldsymbol{u} = \hat{\boldsymbol{u}} \quad \text{on } \partial\Omega, \tag{6.2}$$

for bounded Ω.

Taking into account the identity $\Delta\boldsymbol{u} = \nabla(\operatorname{div}\boldsymbol{u}) - \operatorname{curl}\operatorname{curl}\boldsymbol{u}$, we can write (6.1) as

$$(\mu + \kappa)\Delta\boldsymbol{u} + (\lambda + \mu - \kappa)\nabla(\operatorname{div}\boldsymbol{u}) - \kappa \operatorname{curl}\operatorname{curl}\boldsymbol{u} = \mathbf{0}$$

for every $\kappa \in \mathbb{R}$. In particular, choosing $\kappa = \mu(\lambda + \mu)/(\lambda + 3\mu)$ and calling

$$\boldsymbol{S}_0(\boldsymbol{u}) = \frac{2\mu(\lambda + 2\mu)}{\lambda + 3\mu}\nabla\boldsymbol{u} + \frac{(\lambda + \mu)(\lambda + 2\mu)}{\lambda + 3\mu}(\operatorname{div}\boldsymbol{u})\mathbf{1} + 2\frac{\mu(\lambda + \mu)}{\lambda + 3\mu}\tilde{\nabla}\boldsymbol{u}$$

the *pseudostress* field and

$$\boldsymbol{s}_0(\boldsymbol{u}) = \boldsymbol{S}_0(\boldsymbol{u})\boldsymbol{n}$$

the *pseudotraction* field, we obtain the Somigliana-type formula

$$\boldsymbol{u}(x) = \boldsymbol{v}\big[\boldsymbol{s}_0(\boldsymbol{u})\big](x) + \boldsymbol{w}_0[\boldsymbol{u}](x),$$

where $\boldsymbol{v}[\boldsymbol{s}_0(\boldsymbol{u})]$ is the simple-layer potential with density $\boldsymbol{s}_0(\boldsymbol{u})$, and

$$\boldsymbol{w}_0[\boldsymbol{u}](x) = \int_{\partial\Omega} \boldsymbol{S}_0\big(\boldsymbol{U}(x - \zeta)\big)(\boldsymbol{u} \otimes \boldsymbol{n})(\zeta)\,\mathrm{d}\sigma_\zeta.$$

Clearly, for every density $\boldsymbol{\varphi} \in L^1(\partial\Omega)$, the *double-layer* field $\boldsymbol{w}_0[\boldsymbol{\varphi}]$ is an analytical solution of (6.1) in $\mathbb{R}^3 \setminus \partial\Omega$. Moreover, a direct computation shows that [5, 11]

$$\boldsymbol{w}_0[\boldsymbol{\varphi}] = \int_{\partial\Omega} \frac{[(x - \zeta) \cdot \boldsymbol{n}(\zeta)](\zeta)}{|x - \zeta|^3} \left\{\lambda'\boldsymbol{\varphi}(\zeta) + \mu'\frac{[(x - \zeta) \cdot \boldsymbol{\varphi}(\zeta)](x - \zeta)}{|x - \zeta|^5}\right\} \mathrm{d}\sigma_\zeta,$$

and if Ω is of class C^1, then the trace of $\boldsymbol{w}_0[\boldsymbol{\varphi}]$ exists on both “faces” of $\partial\Omega$:

$$\boldsymbol{w}_0[\boldsymbol{\varphi}](x) \to \mathcal{W}_0^{\pm}[\boldsymbol{\varphi}](\xi) = \pm\frac{1}{2}\boldsymbol{\varphi}(\xi) + \int_{\partial\Omega} \boldsymbol{S}_0\big(\boldsymbol{U}(\xi-\zeta)\big)(\boldsymbol{\varphi}\otimes\boldsymbol{n})(\zeta)\,\mathrm{d}\sigma_\zeta$$
$$= \left(\pm\frac{1}{2}\mathcal{I} + \mathcal{K}\right)[\boldsymbol{\varphi}](\xi)$$

for almost all $\xi \in \partial\Omega$ with compact $\mathcal{K} : L^q(\partial\Omega) \to L^q(\partial\Omega)$, and the conjugates of $\mathcal{W}_0^{\pm}$ give the trace on $\partial\Omega$ of the pseudotractions of the simple-layer potential $\boldsymbol{v}[\boldsymbol{\psi}]$:

$$\mathcal{T}_0^{\pm}[\boldsymbol{\psi}] = \boldsymbol{s}_0\big(\boldsymbol{v}[\boldsymbol{\psi}]\big)^{\pm} = \pm\frac{1}{2}\boldsymbol{\psi}(\xi) - \mathcal{K}'[\boldsymbol{\psi}],$$

where $\mathcal{K}'$ is the adjoint of $\mathcal{K}$.

Using the results of [4], we are able to prove the following:

Lemma 4. *Let Ω be a bounded domain of class C^1. If $\mu(\lambda+2\mu) > 0$, then the operator $\mathcal{W}_0^{+}$ from $L^q(\partial\Omega)$ into itself and from $W^{1,q}(\partial\Omega)$ into itself, $q \in (1,+\infty)$, is Fredholmian, and* $\mathrm{Kern}\,\mathcal{W}_0^{+} = \mathrm{Kern}\,\mathcal{T}_0^{-} = \{\mathbf{0}\}$.

Proof. The Fredholm property of $\mathcal{W}_0^{+}$ follows from the results of [4]. If $\boldsymbol{\psi} \in \mathrm{Kern}\,\mathcal{T}_0^{-}$, then $\boldsymbol{\psi} \in L^q(\partial\Omega)$ for all $q \in (1,+\infty)$ (see [4] p. 182), so that we can integrate by parts to see that $\boldsymbol{v}[\boldsymbol{\psi}]$ is zero in $\complement\Omega$. By uniqueness $\boldsymbol{v}[\boldsymbol{\psi}] = \mathbf{0}$ in Ω, so that $\boldsymbol{\psi} = \mathcal{T}_0^{+}[\boldsymbol{\psi}] - \mathcal{T}_0^{-}[\boldsymbol{\psi}] = \mathbf{0}$ and $\mathrm{Kern}\,\mathcal{T}_0^{-} = \{\mathbf{0}\} = \mathrm{Kern}\,\mathcal{W}_0^{+}$. □

The existence and uniqueness of a solution of (6.1)–(6.2) is now a simple consequence of Fredholm's alternative.

Theorem 8. *Let Ω be a bounded domain of class C^1, and let $\mu(\lambda+2\mu) > 0$. If $\hat{\boldsymbol{u}} \in L^q(\partial\Omega)$, $q \in (1,+\infty)$, then* (6.1)–(6.2) *has a solution expressed by*

$$\boldsymbol{u} = \boldsymbol{w}_0[\boldsymbol{\varphi}]$$

for some $\boldsymbol{\varphi} \in L^q(\partial\Omega)$. The solution is unique in the class of all fields $\boldsymbol{u} \in L^1_{\mathrm{loc}}(\Omega)$ that satisfy the relation

$$\int_{\Omega} \boldsymbol{u}\cdot\boldsymbol{\phi} = \int_{\partial\Omega} \hat{\boldsymbol{u}}\cdot\boldsymbol{s}_0(\boldsymbol{z})$$

for all $\boldsymbol{\phi} \in C_0^{\infty}(\Omega)$, with $\boldsymbol{z}$ the solution of

$$\mu\Delta\boldsymbol{z} + (\lambda+\mu)\nabla\,\mathrm{div}\,\boldsymbol{z} = \boldsymbol{\phi} \quad \text{in } \Omega, \qquad \boldsymbol{z} = \mathbf{0} \quad \text{on } \partial\Omega.$$

If $\hat{\boldsymbol{u}} \in W^{1,q}(\partial\Omega)$, then $\boldsymbol{\varphi} \in W^{1,q}(\partial\Omega)$.

Remark 5. By this method we can also consider the displacement problem (6.1)–(6.2) in exterior domains of class C^1 with the condition $\boldsymbol{u} = o(1)$. If Ω is of class C^2, the theorem also holds in the borderline case $\hat{\boldsymbol{u}} \in L^1(\partial\Omega)$ [5].

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