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On the rate of convergence in the global central limit theorem for random sums of uniformly strong mixing random variables

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Abstract. We present upper bounds of the integral $\int_{-\infty}^{\infty} |x|^l |\mathbf{P}\{Z_N < x\} - \Phi(x)|\,\mathrm{d}x$ for $0 \leqslant l \leqslant 1+\delta$, where $0 < \delta \leqslant 1$, $\Phi(x)$ is a standard normal distribution function, and $Z_N = S_N/\sqrt{\mathbf{E}S_N^2}$ is the normalized random sum with $\mathbf{E}S_N^2 > 0$ ($S_N = X_1 + \cdots + X_N$) of centered random variables $X_1, X_2, \dots$ satisfying the uniformly strong mixing condition. The number of summands N is a nonnegative integer-valued random variable independent of $X_1, X_2, \dots$.

MSC: 60F05

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1 Introduction and main results

Let $X_1, X_2, \dots$ be a sequence of real centered random variables (r.v.s). For $a \leqslant b$, we denote by $\mathcal{F}_a^b$ the σ-algebra of events generated by r.v.s $X_a, X_{a+1}, \dots, X_b$. As usual, $\mathbb{R}$ is the real line, $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, and $\mathbf{1}_A$ is the indicator of an event A.

We consider the weak dependence condition defined between the "past" and "future" in terms of the uniformly strong mixing coefficient $\varphi(\tau)$ introduced by Ibragimov (1959): We say that a sequence of r.v.s $X_1, X_2, \dots$ satisfies the uniformly strong mixing (u.s.m.) condition (or the φ-mixing condition) with the u.s.m. coefficient $\varphi(\tau)$ if

$$\varphi(\tau) = \sup_{t\in\mathbb{N}} \sup_{\substack{A\in\mathcal{F}_1^t,\, B\in\mathcal{F}_{t+\tau}^\infty \\ \mathbf{P}(A)>0}} \frac{|\mathbf{P}(AB) - \mathbf{P}(A)\mathbf{P}(B)|}{\mathbf{P}(A)} \underset{\tau\to\infty}{\to} 0 \tag{1.1}$$

(see [4] or [5]).

In what follows, $\Phi(x)$ is the standard normal distribution function. By $C(\cdot)$ with an index or without it we denote a positive finite factor depending only on the quantities indicated in the parentheses (not necessarily the same at different occurrences).

Recall the following result for sums with a fixed number n of random summands satisfying the u.s.m. condition (1.1), which will be used to prove the corresponding results for random sums.

Theorem A. (*See* [15, *Cor*. 3].) *Let a sequence of r.v.s* $X_1, X_2, \dots$ *with* $\mathbf{E}X_i = 0$ *and* $\mathbf{E}|X_i|^{2+\delta} < \infty$, *where* $0 < \delta \leqslant 1$, $i = 1, \dots, n$, *satisfy the u.s.m. condition* (1.1) *with coefficient* $\varphi(\tau) \leqslant K\mathrm{e}^{-\mu\tau}$, *where* $0 < K < \infty$ *and* $\mu > 0$ *are constants. Denote*

$$Z_n = \frac{S_n}{\sqrt{\mathbf{E}S_n^2}}, \qquad S_n = \sum_{i=1}^{n} X_i, \qquad \mathcal{I}_{l,n} = \int_{-\infty}^{\infty} |x|^l \big|\mathbf{P}\{Z_n < x\} - \Phi(x)\big| \mathrm{d}x,$$

$$\lambda_{l,n} = \big|\mathbf{E}|Z_n|^l - \mathbf{E}|Y|^l\big|, \qquad L_{r,n} = \frac{1}{(\mathbf{E}S_n^2)^{r/2}} \sum_{i=1}^{n} \mathbf{E}|X_i|^r,$$

where $\mathbf{E}S_n^2 > 0$, *and* Y *is a standard normal r.v. Then*

$$\mathcal{I}_{l,n} \leqslant C_0 L_{2+\delta,n} \ln^{1+\delta}(1+n) \tag{1.2}$$

if (i) $0 \leqslant l \leqslant 1$ *or* (ii) $1 < l \leqslant 1 + \delta$ *and* $L_{2,n} \leqslant C_*$; *and*

$$\lambda_{l,n} \leqslant C_0 L_{2+\delta,n} \ln^{1+\delta}(1+n)$$

if (i) $1 \leqslant l \leqslant 2$ *or* (ii) $2 < l \leqslant 2 + \delta$ *and* $L_{2,n} \leqslant C_*$.
Here $C_0 = C(K, \mu, l)$ *in cases* (i), *and* $C_0 = C(K, \mu, l, C_*)$ *in cases* (ii).

Recall that to prove Theorem A, we used the powerful and general direct Stein method introduced in [13] for estimating the rate of convergence of sums of weakly dependent r.v.s to the normal distribution.

In this paper, we are interested in estimates of the quantities

$$\mathcal{I}_l = \int_{-\infty}^{\infty} |x|^l \big|\mathbf{P}\{Z_N < x\} - \Phi(x)\big| \,\mathrm{d}x, \quad \lambda_l = \big|\mathbf{E}|Z_N|^l - \mathbf{E}|Y|^l\big|,$$

where

$$Z_N = \frac{S_N}{\sqrt{\mathbf{E}S_N^2}}, \quad S_N = \sum_{i=1}^{N} X_i, \quad S_0 = 0,$$

assuming that $\mathbf{E}S_N^2 > 0$, the number of summands N is a nonnegative integer-valued r.v. independent of X_1, $X_2, \dots$, the centered summands $X_1, X_2, \dots$ satisfy the u.s.m. condition (1.1), and Y is a standard normal r.v.

There are not many results on the convergence rate in the central limit theorem for random sums with weakly dependent summands. Strictly stationary sequences satisfying the u.s.m. condition (1.1), assuming that the number of summands and summands are dependent, were considered in [9]. Similar results for strictly stationary sequences of martingales have been obtained in [8]. A stationary sequence of m-dependent r.v.s, assuming that the number of summands and summands are independent, was investigated in the recent paper [10]. Without the convergence rate, the asymptotic normality of random sums of stationary m-dependent r.v.s was investigated in [12], in the recent paper [6], and that of martingales in [11].

For a wide range of various well-known and less common methods for estimation of the accuracy of probabilistic approximations, we refer to [2] and references therein. For the results on the convergence rate in the central limit theorem for weakly dependent random variables, we refer to [1, 7, 14] and references therein.

However, the author has not found any published results on the upper bounds of the quantities $\mathcal{I}_l$ and λ_l, $l \geqslant 0$, for random sums with summands satisfying the u.s.m. condition (1.1). Note that for independent summands $X_1, X_2, \dots$, the corresponding upper estimates of $\mathcal{I}_l$ for $0 \leqslant l \leqslant 1 + \delta$, where $0 < \delta \leqslant 1$ (and λ_l for $1 \leqslant l \leqslant 2 + \delta$) have been obtained in the recent paper [16].

To investigate the asymptotic normality and the convergence rate for random sums of independent (as well as dependent) summands, we use, as usual, the additional r.v.s

$$A_N=\sum_{i=1}^{N}\mathbf{E}X_i,\qquad B_N^2=\sum_{i=1}^{N}\mathbf{V}X_i,\qquad l_{r,N}=\sum_{i=1}^{N}\mathbf{E}|X_i|^r.$$

In [17], seemingly for the first time, we introduced the additional r.v.s

$$\kappa_N^2=\sum_{i=1}^{N}\sum_{j=1}^{N}\operatorname{cov}(X_i,X_j)=B_N^2+2\sum_{1\leqslant i<j\leqslant N}\operatorname{cov}(X_i,X_j),$$

which are very useful for investigating the asymptotics of the normality and the convergence rate for random sums of dependent (including weakly dependent) summands. Here $\operatorname{cov}(\xi,\eta)=\mathbf{E}\xi\eta-\mathbf{E}\xi\mathbf{E}\eta$ is the covariance of real r.v.s ξ and η. Moreover, we assume that $\sum_{i=1}^{0}(\cdot)=0$.

The main results of this paper are Theorems 1–3.

Theorem 1. *Let a sequence of real r.v.s $X_1,X_2,\ldots$ with $\mathbf{E}X_i=0$ and $\mathbf{E}|X_i|^{2+\delta}<\infty$, where $0<\delta\leqslant 1$, $i=1,2,\ldots$, satisfy the u.s.m. condition* (1.1) *with coefficient $\varphi(\tau)\leqslant K\mathrm{e}^{-\mu\tau}$, where $0<K<\infty$ and $\mu>0$ are constants. Let N be a nonnegative integer-valued r.v. independent of $X_1,X_2,\ldots$. Then*

$$\mathcal{I}_l\leqslant C_1\frac{\mathbf{E}l_{2+\delta,N}\ln^{1+\delta}(1+N)}{(\mathbf{E}\kappa_N^2)^{(2+\delta)/2}}+C_2\frac{\mathbf{E}|\kappa_N^2-\mathbf{E}\kappa_N^2|}{\mathbf{E}\kappa_N^2},\tag{1.3}$$

for $0\leqslant l\leqslant 1$; if, in addition, $L_{2,k}\leqslant C_*$ for $k=1,2,\ldots$, then

$$\mathcal{I}_l\leqslant C_3\frac{\mathbf{E}l_{l+1,N}}{(\mathbf{E}\kappa_N^2)^{(l+1)/2}}+C_4\frac{\mathbf{E}l_{2+\delta,N}\ln^{1+\delta}(1+N)}{(\mathbf{E}\kappa_N^2)^{(2+\delta)/2}}+C_5\frac{\mathbf{E}|\kappa_N^{l+1}-(\mathbf{E}\kappa_N^2)^{(l+1)/2}|}{(\mathbf{E}\kappa_N^2)^{(l+1)/2}}\tag{1.4}$$

for $1<l\leqslant 1+\delta$. Here the factors $C_1=C_1(l,\delta,C_0)$, $C_2=C_2(l)$, $C_3=C(K,\mu,l,C_*)$, $C_4=C(l,\delta,C_0)$, and $C_5=C(K,\mu,l,C_*)$, where C_0 is taken from Theorem A.

In particular, if the summands are identically distributed with zero mixed moments, then we have the following result.

Corollary 1. *Let $X,X_1,X_2,\ldots$ be real identically distributed r.v.s with $\mathbf{E}X=0$, $0<\sigma^2=\mathbf{E}X^2$, $\beta_{2+\delta}=\mathbf{E}|X|^{2+\delta}<\infty$, where $0<\delta\leqslant 1$, and $\mathbf{E}X_iX_j=0$, $1\leqslant i\neq j\leqslant\infty$, satisfy the u.s.m. condition* (1.1) *with coefficient $\varphi(\tau)\leqslant K\mathrm{e}^{-\mu\tau}$, where $0<K<\infty$ and $\mu>0$ are constants. Let N be a nonnegative integer-valued r.v. with $\mathbf{E}N>0$, independent of $X_1,X_2,\ldots$. Then $Z_N=S_N/(\sigma\sqrt{\mathbf{E}N})$,*

$$\mathcal{I}_l\leqslant C_1\frac{\beta_{2+\delta}}{\sigma^{2+\delta}}\frac{\mathbf{E}N\ln^{1+\delta}(1+N)}{(\mathbf{E}N)^{(2+\delta)/2}}+C_2\frac{\mathbf{E}|N-\mathbf{E}N|}{\mathbf{E}N}\tag{1.5}$$

for $0\leqslant l\leqslant 1$ with $\mathbf{E}N\ln^{1+\delta}(1+N)<\infty$, and

$$\mathcal{I}_l\leqslant C_3\frac{\beta_{l+1}}{\sigma^{l+1}}\frac{1}{(\mathbf{E}N)^{(l-1)/2}}+C_4\frac{\beta_{2+\delta}}{\sigma^{2+\delta}}\frac{\mathbf{E}N\ln^{1+\delta}(1+N)}{(\mathbf{E}N)^{(2+\delta)/2}}+C_5\frac{\mathbf{E}|N^{(l+1)/2}-(\mathbf{E}N)^{(l+1)/2}|}{(\mathbf{E}N)^{(l+1)/2}}$$

for $1<l\leqslant 1+\delta$ with $\mathbf{E}N^{(l+1)/2}<\infty$.

By λ_l we denote the absolute value of the difference between the absolute moments of the random sum Z_N and the standard normal r.v. Y,

$$\lambda_l = \left|\mathbf{E}|Z_N|^l - \mathbf{E}|Y|^l\right|.$$

The estimates of λ_l follow from the estimates of $\mathcal{I}_l$ of Theorem 1. Namely, we have the following result.

Theorem 2. *Let the conditions of Theorem* 1 *hold. Then*

$$\lambda_l \leqslant C_6 \frac{\mathbf{E} l_{2+\delta,N} \ln^{1+\delta}(1+N)}{(\mathbf{E}\kappa_N^2)^{(2+\delta)/2}} + C_7 \frac{\mathbf{E}|\kappa_N^2 - \mathbf{E}\kappa_N^2|}{\mathbf{E}\kappa_N^2}$$

for $1 \leqslant 1 \leqslant 2$*; if, in addition,* $L_{2,k} \leqslant C_*$ *for* $k = 1, 2, \dots,$ *then*

$$\lambda_l \leqslant C_8 \frac{\mathbf{E} l_{l,N}}{(\mathbf{E}\kappa_N^2)^{l/2}} + C_9 \frac{\mathbf{E} l_{2+\delta,N} \ln^{1+\delta}(1+N)}{(\mathbf{E}\kappa_N^2)^{(2+\delta)/2}} + C_{10} \frac{\mathbf{E}|\kappa_N^l - (\mathbf{E}\kappa_N^2)^{l/2}|}{(\mathbf{E}\kappa_N^2)^{l/2}}$$

for $2 < l \leqslant 2 + \delta$.

Here the factors $C_i = lC_{i-5}$, $i = 6, 7, 8, 9, 10$, *where* C_1, C_2, C_3, C_4, *and* C_5 *are taken from Theorem* 1.

If, in addition, the summands are identically distributed with zero mixed moments, then from Theorem 2 we obtain the following result.

Corollary 2. *Let* $X, X_1, X_2, \dots$ *be real identically distributed r.v.s with* $\mathbf{E}X = 0$, $0 < \sigma^2 = \mathbf{E}X^2$, $\beta_{2+\delta} = \mathbf{E}|X|^{2+\delta} < \infty$, *where* $0 < \delta \leqslant 1$, *and* $\mathbf{E}X_iX_j = 0$, $1 \leqslant i \neq j \leqslant \infty$, *satisfy the u.s.m. condition* (1.1) *with coefficient* $\varphi(\tau) \leqslant K\mathrm{e}^{-\mu\tau}$, *where* $0 < K < \infty$ *and* $\mu > 0$ *are constants. Let* N *be a nonnegative integer-valued r.v. with* $\mathbf{E}N > 0$, *independent of* $X_1, X_2, \dots$. *Then* $Z_N = S_N/(\sigma\sqrt{\mathbf{E}N})$,

$$\lambda_l \leqslant C_6 \frac{\beta_{2+\delta}}{\sigma^{2+\delta}} \frac{\mathbf{E}N \ln^{1+\delta}(1+N)}{(\mathbf{E}N)^{(2+\delta)/2}} + C_7 \frac{\mathbf{E}|N - \mathbf{E}N|}{\mathbf{E}N} \tag{1.6}$$

for $1 \leqslant l \leqslant 2$ *with* $\mathbf{E}N \ln^{1+\delta}(1+N) < \infty$, *and*

$$\lambda_l \leqslant C_8 \frac{\beta_l}{\sigma^l} \frac{1}{(\mathbf{E}N)^{(l-2)/2}} + C_9 \frac{\beta_{2+\delta}}{\sigma^{2+\delta}} \frac{\mathbf{E}N \ln^{1+\delta}(1+N)}{(\mathbf{E}N)^{(2+\delta)/2}} + C_{10} \frac{\mathbf{E}|N^{l/2} - (\mathbf{E}N)^{l/2}|}{(\mathbf{E}N)^{l/2}}$$

for $2 < l \leqslant 2 + \delta$ *with* $\mathbf{E}N^{l/2} < \infty$.

To present the results for three concrete random indices N, we recall the definition of the τ-shifted $\mathcal{L}$ distribution (τ-shifted Poisson distribution, τ-shifted binomial distribution, τ-shifted negative binomial distribution, and so on), which was first introduced in [16]. We write $\xi \sim \mathcal{L}$ if the distribution of a r.v. ξ is $\mathcal{L}$.

DEFINITION 1. We say that a discrete r.v. N is distributed by the τ-shifted $\mathcal{L}$ distribution ($\tau \geqslant 0$) (for short, $N - \tau \sim \mathcal{L}$) or that N is a τ-shifted r.v. if for any discrete r.v. $\xi \sim \mathcal{L}$ taking values x_k with probabilities p_k,

$$\mathbf{P}\{N = x_k + \tau\} = \mathbf{P}\{\xi = x_k\} = p_k.$$

In particular, the 0-shifted $\mathcal{L}$ distribution coincides with the $\mathcal{L}$ distribution.

DEFINITION 2. We say that a r.v. N is distributed by the τ-shifted Poisson distribution with parameters $\tau \in \mathbb{N}_0$ and $\lambda > 0$ (for short, $N - \tau \sim \mathcal{P}(\lambda)$) if

$$\mathbf{P}\{N = k + \tau\} = \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}, \quad k = 0, 1, 2, \dots .$$

DEFINITION 3. We say that a r.v. N is distributed by the τ-shifted binomial distribution with parameters $\tau \in \mathbb{N}_0$, $n \in \mathbb{N}$, and $0 < p < 1$ (for short, $N - \tau \sim \mathcal{B}(n,p)$) if

$$\mathbf{P}\{N = k + \tau\} = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \dots, n.$$

DEFINITION 4. We say that a r.v. N is distributed by the τ-shifted negative binomial distribution with parameters $\tau \in \mathbb{N}_0$, $r \in \mathbb{N}$, and $0 < p < 1$ (for short, $N - \tau \sim \mathcal{NB}(r,p)$ if

$$\mathbf{P}\{N = k + \tau\} = \binom{k-1}{r-1} p^r (1-p)^{k-r}, \quad k = r, r+1, \dots .$$

Now, we present the following statement for three presented τ-shifted $\mathcal{L}$ distributions.

Theorem 3. *Let $X, X_1, X_2, \dots$ be real identically distributed r.v.s with $\mathbf{E}X = 0$, $0 < \sigma^2 = \mathbf{E}X^2$, $\beta_{2+\delta} = \mathbf{E}|X|^{2+\delta} < \infty$, where $0 < \delta \leqslant 1$, and $\mathbf{E}X_iX_j = 0$, $1 \leqslant i \neq j \leqslant \infty$, satisfy the u.s.m. condition* (1.1) *with coefficient $\varphi(\tau) \leqslant K\mathrm{e}^{-\mu\tau}$, where $0 < K < \infty$ and $\mu > 0$ are constants. Let N be a nonnegative integer-valued r.v. independent of $X_1, X_2, \dots$. Then $Z_N = S_N/(\sigma\sqrt{\mathbf{E}N})$, and*:

1. *If $N - \tau \sim \mathcal{P}(\lambda)$ with $\tau \in \mathbb{N}_0$ and $\lambda > 0$, then*

$$\mathcal{I}_l \leqslant C_{11}\frac{\ln^{1+\delta}(1+\tau+\lambda)}{(\tau+\lambda)^{\delta/2}} \tag{1.7}$$

for $0 \leqslant l \leqslant 1$, and

$$\lambda_l \leqslant C_{12}\frac{\ln^{1+\delta}(1+\tau+\lambda)}{(\tau+\lambda)^{\delta/2}} \tag{1.8}$$

for $1 \leqslant l \leqslant 2$.

2. *If $N - \tau \sim \mathcal{B}(n,p)$ with $\tau \in \mathbb{N}_0$, $n \in \mathbb{N}$, and $0 < p < 1$, then*

$$\mathcal{I}_l \leqslant C_{13}\frac{\ln^{1+\delta}(1+\tau+np)}{(\tau+np)^{\delta/2}} \tag{1.9}$$

for $0 \leqslant l \leqslant 1$, and

$$\lambda_l \leqslant C_{14}\frac{\ln^{1+\delta}(1+\tau+np)}{(\tau+np)^{\delta/2}} \tag{1.10}$$

for $1 \leqslant l \leqslant 2$.

3. *If $N - \tau \sim \mathcal{NB}(r,p)$ with $\tau \in \mathbb{N}_0$, $r \in \mathbb{N}$, and $0 < p < 1$, then*

$$\mathcal{I}_l \leqslant C_{15}\frac{\ln^{1+\delta}(1+\tau+r/p)}{(\tau p+r)^{\delta/2}} \tag{1.11}$$

for $0 \leqslant l \leqslant 1$, *and*

$$\lambda_l \leqslant C_{16} \frac{\ln^{1+\delta}(1+\tau+r/p)}{(\tau p+r)^{\delta/2}} \tag{1.12}$$

for $1 \leqslant l \leqslant 2$.

Here $C_i = C_i(K,\mu,\sigma,\beta_{2+\delta},\delta)$, $i = 11,\dots,16$.

Since the 0-shifted $\mathcal{L}$ distribution coincides with the $\mathcal{L}$ distribution, taking $\tau = 0$ in Theorem 3, we obtain the corresponding estimates of $\mathcal{I}_l$ and λ_l for Poisson, binomial, and negative binomial random sums.

2 Auxiliary results

To prove the main results, we use some lemmas, which were also useful for estimating $\mathcal{I}_l$ and λ_l for $l \geqslant 0$ for random sums of independent summands (see [16]) and for estimating the uniform metrics $\sup_{x\in\mathbb{R}} |\mathbf{P}\{Z_N < x\} - \Phi(x)|$ for random sums of summands satisfying the u.s.m. condition (1.1) (see [17]). Here we present them without proofs.

In the following lemma, we give a relationship between the first moments of the random sum S_N and the corresponding moment characteristics of r.v.s A_N and κ_N^2.

Lemma 1. (*See* [17].) *Let* $X_1, X_2, \dots$ *be* (*arbitrarily dependent, not necessarily identically distributed*) *r.v.s with* $\mathbf{E}X_i^2 < \infty$, *and let* N *be a nonnegative integer-valued r.v. independent of* $X_1, X_2, \dots$. *Denote*

$$S_N = \sum_{i=1}^N X_i, \qquad A_N = \sum_{i=1}^N \mathbf{E}X_i, \qquad B_N^2 = \sum_{i=1}^N \mathbf{V}X_i, \qquad \kappa_N^2 = \sum_{i=1}^N \sum_{j=1}^N \operatorname{cov}(X_i, X_j).$$

Then

$$\mathbf{E}S_N = \mathbf{E}A_N, \qquad \mathbf{E}S_N^2 = \mathbf{E}\kappa_N^2 + \mathbf{E}A_N^2,$$
$$\mathbf{V}S_N = \mathbf{E}\kappa_N^2 + \mathbf{V}A_N. \tag{2.1}$$

If r.v.s $X_1, X_2, \dots$ *are independent, then*

$$\mathbf{E}S_N^2 = \mathbf{E}B_N^2 + \mathbf{E}A_N^2, \qquad \mathbf{V}S_N = \mathbf{E}B_N^2 + \mathbf{V}A_N.$$

In Lemma 2, we present the upper estimates of the second moment $\mathbf{E}S_N^2$ and variance $\mathbf{V}S_N$ of the random sum S_N with summands satisfying the u.s.m. condition.

Lemma 2. (*See* [17].) *Let a sequence of real r.v.s* $X_1, X_2, \dots$ *satisfy the u.s.m. condition* (1.1) *with* $\sum_{\tau=1}^\infty \varphi^{1/2}(\tau) < \infty$, *and let* N *be a nonnegative integer-valued r.v. independent of* $X_1, X_2, \dots$. *Let the notation of Lemma* 1 *hold, and let*

$$b_N^2 = \sum_{i=1}^N \mathbf{E}X_i^2, \qquad C_\infty\bigl(\varphi^{1/2}\bigr) = 1 + 4\sum_{\tau=1}^\infty \varphi^{1/2}(\tau).$$

Then

$$\mathbf{E}S_N^2 \leqslant C_\infty\bigl(\varphi^{1/2}\bigr)\mathbf{E}b_N^2 + \mathbf{E}A_N^2, \tag{2.2}$$
$$\mathbf{V}S_N \leqslant C_\infty\bigl(\varphi^{1/2}\bigr)\mathbf{E}b_N^2 + \mathbf{V}A_N.$$

We recall that to obtain inequality (2.2) while estimating $\mathbf{E}S_k^2$ with a fixed number $k = 1, 2, \dots$ of summands, it suffices to use inequality (1.3) in [5, p. 363, Lemma 1.1] or the inequality in [3, p. 278, Thm. A.6].

To transfer estimate (1.2) for random sums, we need Lemmas 3, 4, and 5.

Lemma 3. (*See* [16, *Lemma* 1].) *Let ξ and η be real r.v.s with $\mathbf{E}|\xi|^{l+1} < \infty$ and $\mathbf{E}|\eta|^{l+1} < \infty$ for some $l \geqslant 0$, respectively. Then*

$$\int_{-\infty}^{\infty} |x|^l \big|\mathbf{P}\{\xi < x\} - \mathbf{P}\{\eta < x\}\big| \,\mathrm{d}x \leqslant \frac{\mathbf{E}|\xi|^{l+1} + \mathbf{E}|\eta|^{l+1}}{l+1}.$$

In particular, if $\eta = Y$ is a standard normal r.v., then, for $l \geqslant 0$,

$$\int_{-\infty}^{\infty} |x|^l \big|\mathbf{P}\{\xi < x\} - \Phi(x)\big| \,\mathrm{d}x \leqslant \begin{cases} \frac{1}{l+1}\big(\mathbf{E}|\xi|^{l+1} + \frac{2^{(l+1)/2}\Gamma((l+2)/2)}{\sqrt{\pi}}\big) & \text{if } l \geqslant 0, \\ \frac{1}{l+1}\big(\mathbf{E}|\xi|^{l+1} + 1\big) & \text{if } 0 \leqslant l \leqslant 1, \\ \frac{1}{l+1}\big(\mathbf{E}|\xi|^{l+1} + \frac{2\sqrt{2}}{\sqrt{\pi}}\big) & \text{if } 1 < l \leqslant 2, \end{cases} \tag{2.3}$$

where Γ is the gamma function. Moreover, if $\mathbf{E}\xi^2 = 1$, then, for all $0 \leqslant l \leqslant 1$,

$$\int_{-\infty}^{\infty} |x|^l \big|\mathbf{P}\{\xi < x\} - \Phi(x)\big| \,\mathrm{d}x \leqslant \frac{2}{l+1}. \tag{2.4}$$

Furthermore, to estimate $\mathcal{I}_{l,n}$ for $1 < l \leqslant 1 + \delta$, we use an estimate, which follows from estimate (58) in [15] under the condition of exponentially decreasing u.s.m. coefficient $\varphi(\tau)$ and truncation level $t = 1$.

Lemma 4. *Let $X_1, X_2, \ldots$ be a sequence of r.v.s with $\mathbf{E}X_i = 0$ and $\mathbf{E}|X_i|^\kappa < \infty$ for $i = 1, \ldots, n$, where $\kappa \geqslant 2$, satisfying the u.s.m. condition* (1.1) *with $\varphi(\tau) \leqslant K\mathrm{e}^{-\mu\tau}$, where $0 < K < \infty$ and $\mu > 0$ are constants. Let $L_{2,n} = \sum_{i=1}^n \mathbf{E}X_i^2/\mathbf{E}S_n^2 \leqslant C_*$. Denote*

$$\mathcal{I}_{l,n} = \int_{-\infty}^{\infty} |x|^l \big|\mathbf{P}\{Z_n < x\} - \Phi(x)\big| \,\mathrm{d}x, \qquad Z_n = \frac{S_n}{\sqrt{\mathbf{E}S_n^2}}, \qquad S_n = \sum_{i=1}^n X_i,$$

where $\mathbf{E}S_n^2 > 0$. Then, for all $1 < l \leqslant \kappa - 1$,

$$\mathcal{I}_{l,n} \leqslant C_{\mathbf{0}}\Bigg(1 + \frac{1}{(\mathbf{E}S_n^2)^{(l+1)/2}} \sum_{i=1}^n \mathbf{E}|X_i|^{l+1} \mathbf{1}_{\{|X_i| > \sqrt{\mathbf{E}S_n^2}\}}\Bigg), \tag{2.5}$$

where $C_{\mathbf{0}} = C(K, \mu, l, C_*)$.

Lemma 5. (*See* [16].) *For all $a > 0$ and $l \geqslant 0$, we have*

$$\begin{aligned} \int_{-\infty}^{\infty} |x|^l \big|\Phi(xa) - \Phi(x)\big| \,\mathrm{d}x &= \frac{2^{(l+2)/2}\Gamma((l+2)/2)}{\sqrt{2\pi}} |a - 1| \int_0^1 \frac{\mathrm{d}t}{(\gamma(t))^{(l+2)/2}} & (2.6) \\ &\leqslant \frac{2^{(l+2)/2}\Gamma((l+2)/2)}{\sqrt{2\pi}(l+1)} \cdot \left|1 - \frac{1}{a^{l+1}}\right|, & (2.7) \end{aligned}$$

where $\gamma(t) = [1 + t(a-1)]^2 > 0$.

In Lemmas 6, 7, and 8, we present some useful estimates of $(\mathbf{E}|N-\mathbf{E}N|)/\mathbf{E}N$, where the random number N is a τ-shifted Poisson r.v., a τ-shifted binomial r.v., and a τ-shifted negative binomial r.v., respectively.

Lemma 6. (*See* [17].) *If* $N-\tau \sim \mathcal{P}(\lambda)$ *with* $\tau \in \mathbb{N}_0$ *and* $\lambda > 0$*, then*

$$\frac{\mathbf{E}|N-\mathbf{E}N|}{\mathbf{E}N} \leqslant \frac{1}{\sqrt{\tau+\lambda}}. \tag{2.8}$$

Lemma 7. (*See* [17].) *If* $N-\tau \sim \mathcal{B}(n,p)$ *with* $\tau \in \mathbb{N}_0$, $n \in \mathbb{N}$, *and* $0<p<1$*, then*

$$\frac{\mathbf{E}|N-\mathbf{E}N|}{\mathbf{E}N} < \frac{1}{\sqrt{\tau+np}}. \tag{2.9}$$

Lemma 8. (*See* [17].) *If* $N-\tau \sim \mathcal{NB}(r,p)$ *with* $\tau \in \mathbb{N}_0$, $r \in \mathbb{N}$, *and* $0<p<1$*, then*

$$\frac{\mathbf{E}|N-\mathbf{E}N|}{\mathbf{E}N} < \frac{1}{\sqrt{\tau p+r}}. \tag{2.10}$$

3 Basic inequality for $\mathcal{I}_l$ with $\mathrm{E}X_i = 0, i = 1, 2, \dots$

Denote

$$\mathcal{I}_l = \int_{-\infty}^{\infty} |x|^l \big|\Delta(x)\big| \,\mathrm{d}x, \qquad \Delta(x) = \mathbf{P}\big\{S_N < x\sqrt{\mathbf{E}S_N^2}\big\} - \Phi(x), \qquad S_N = \sum_{i=1}^{N} X_i,$$

$$\xi_k = \frac{S_k}{\sqrt{\mathbf{E}S_k^2}}, \quad a_k = \frac{\sqrt{\mathbf{E}S_N^2}}{\sqrt{\mathbf{E}S_k^2}}, \quad S_k = \sum_{i=1}^{k} X_i, \quad k = 1, 2, \dots,$$

where $X_1, X_2, \dots$ are (arbitrarily dependent) r.v.s, $\mathbf{E}S_N^2 > 0$, and $S_0 = 0$. It is clear that if N is a nonnegative integer-valued r.v. with $p_k = \mathbf{P}\{N = k\}$, $k = 0, 1, 2, \dots$, independent of $X_1, X_2, \dots$, then, for all $x \in \mathbb{R}$,

$$\Delta(x) = \sum_{k=0}^{\infty} \big[\mathbf{P}\big\{S_k < x\sqrt{\mathbf{E}S_N^2}\big\} - \Phi(x)\big] p_k.$$

Let

$$K(\alpha) = \big\{k \in \mathbb{N}\colon \big|\mathbf{E}S_k^2 - \mathbf{E}S_N^2\big| \leqslant (1-\alpha)\mathbf{E}S_N^2\big\}$$

and

$$\overline{K}(\alpha) = \big\{k \in \mathbb{N}\colon \big|\mathbf{E}S_k^2 - \mathbf{E}S_N^2\big| > (1-\alpha)\mathbf{E}S_N^2\big\}$$

for $\alpha \in (0, 1)$.

First, we observe that $\mathbf{E}S_k^2 \geqslant \alpha \mathbf{E}S_N^2 > 0$ for $k \in K(\alpha)$. Since

$$\big[\mathbf{P}\big\{S_0 < x\sqrt{\mathbf{E}S_N^2}\big\} - \Phi(x)\big] p_0 = \big[\mathbf{1}_{\{x>0\}} - \Phi(x)\big] p_0$$

and

$$\sum_{k=1}^{\infty}\bigl[\mathbf{P}\bigl\{S_k < x\sqrt{\mathbf{E}S_N^2}\bigr\} - \Phi(x)\bigr]p_k$$
$$= \sum_{k\in K(\alpha)}\bigl[\mathbf{P}\{\xi_k < xa_k\} - \Phi(xa_k)\bigr]p_k + \sum_{k\in K(\alpha)}\bigl[\Phi(xa_k) - \Phi(x)\bigr]p_k$$
$$+ \sum_{k\in \overline{K}(\alpha)}\bigl[\mathbf{P}\{\xi_k < xa_k\} - \Phi(x)\bigr]p_k,$$

we can state the following:

Basic inequality for $\mathcal{I}_l$. Let $X_1, X_2, \ldots$ be (arbitrarily dependent, not necessarily identically distributed) r.v.s with $\mathbf{E}X_i = 0$ for all $i = 1, 2, \ldots$, and let N be a nonnegative integer-valued r.v. with $p_k = \mathbf{P}\{N = k\}$, $k = 0, 1, 2, \ldots$, independent of $X_1, X_2, \ldots$. Then, for all $l \geqslant 0$,

$$\mathcal{I}_l \leqslant \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4, \tag{3.1}$$

where

$$\sum\nolimits_1 = \sum_{k\in K(\alpha)}\int_{-\infty}^{\infty}|x|^l\bigl|\mathbf{P}\{\xi_k < xa_k\} - \Phi(xa_k)\bigr|\,\mathrm{d}x p_k,$$

$$\sum\nolimits_2 = \sum_{k\in K(\alpha)}\int_{-\infty}^{\infty}|x|^l\bigl|\Phi(xa_k) - \Phi(x)\bigr|\,\mathrm{d}x p_k,$$

$$\sum\nolimits_3 = \int_{-\infty}^{\infty}|x|^l\bigl|\mathbf{1}_{\{x>0\}} - \Phi(x)\bigr|\,\mathrm{d}x p_0,$$

$$\sum\nolimits_4 = \sum_{k\in \overline{K}(\alpha)}\int_{-\infty}^{\infty}|x|^l\bigl|\mathbf{P}\bigl\{S_k < x\sqrt{\mathbf{E}S_N^2}\bigr\} - \Phi(x)\bigr|\,\mathrm{d}x p_k,$$

where $K(\alpha) = \{k \in \mathbb{N}\colon |\mathbf{E}S_k^2 - \mathbf{E}S_N^2| \leqslant (1-\alpha)\mathbf{E}S_N^2\}$ and $\overline{K}(\alpha) = \{k \in \mathbb{N}\colon |\mathbf{E}S_k^2 - \mathbf{E}S_N^2| > (1-\alpha)\mathbf{E}S_N^2\}$ for $\alpha \in (0, 1)$.

4 Proofs of Theorems 1–3 and Corollaries 1, 2

In what follows, the sequence of r.v.s $X_1, X_2, \ldots$ satisfies the u.s.m. condition (1.1).

First of all, note that under the condition $\mathbf{E}X_i = 0$ for all $i = 1, 2, \ldots$,

$$\kappa_N^2 = \sum_{i=1}^{N}\sum_{j=1}^{N}\mathbf{E}X_iX_j = b_N^2 + 2\sum_{1\leqslant i<j\leqslant N}\mathbf{E}X_iX_j,$$

where $b_N^2 = \sum_{i=1}^{N}\mathbf{E}X_i^2$.

Proof of Theorem 1. For the sum ξ_k with fixed number k of summands, we use the notation

$$\mathcal{I}_{l,k} = \int\limits_{-\infty}^{\infty} |x|^l \big|\mathbf{P}\{\xi_k < x\} - \Phi(x)\big| \, \mathrm{d}x, \quad k = 1, 2, \dots .$$

Estimation of $\sum_1$. First, we observe that

$$\sum\nolimits_1 = \sum_{k \in K(\alpha)} \frac{1}{a_k^{l+1}} \mathcal{I}_{l,k} p_k. \tag{4.1}$$

Since $a_k \geqslant 1/\sqrt{2-\alpha}$ and $\mathbf{E}S_k^2 \geqslant \alpha \mathbf{E}S_N^2$ for $k \in K(\alpha)$, using (1.2) of Theorem A, from (4.1) we obtain that, for all $0 \leqslant l \leqslant 1+\delta$,

$$\begin{aligned} \sum\nolimits_1 &\leqslant (2-\alpha)^{(l+1)/2} C_0 \sum_{k \in K(\alpha)} \frac{1}{(\mathbf{E}S_k^2)^{(2+\delta)/2}} \sum_{i=1}^{k} \mathbf{E}|X_i|^{2+\delta} \ln^{1+\delta}(1+k) p_k \\ &\leqslant \frac{(2-\alpha)^{(l+1)/2} C_0}{\alpha^{(2+\delta)/2}} \frac{\mathbf{E} l_{2+\delta,N} \ln^{1+\delta}(1+N) \mathbf{1}_{\{N \in K(\alpha)\}}}{(\mathbf{E}S_N^2)^{(2+\delta)/2}}, \end{aligned} \tag{4.2}$$

where $l_{2+\delta,k} = \sum_{i=1}^{k} \mathbf{E}|X_i|^{2+\delta}$ and C_0 is taken from (1.2) of Theorem A.

Estimation of $\sum_2$. To estimate $\sum_2$, using (2.6) of Lemma 5, we get that, for all $l \geqslant 0$,

$$\int\limits_{-\infty}^{\infty} |x|^l \big|\Phi(xa_k) - \Phi(x)\big| \, \mathrm{d}x = \frac{2^{(l+2)/2}\Gamma((l+2)/2)}{\sqrt{2\pi}} |a_k - 1| \int\limits_0^1 \frac{\mathrm{d}t}{(\gamma_k(t))^{(l+2)/2}}, \tag{4.3}$$

where $\gamma_k(t) = [1 + t(a_k - 1)]^2$.

Since

$$\frac{1}{\sqrt{2-\alpha}} \leqslant a_k \leqslant \frac{1}{\sqrt{\alpha}}$$

for $k \in K(\alpha)$, we easily obtain that

$$\gamma_k(t) \geqslant \frac{1}{2-\alpha} \tag{4.4}$$

for $0 \leqslant t \leqslant 1$, $\alpha \in (0,1)$, and $k \in K(\alpha)$ (see [16, p. 255]).

The upper bound of $|a_k - 1|$ for $k \in K(\alpha)$ easily follows:

$$|a_k - 1| = \frac{|\mathbf{E}S_k^2 - \mathbf{E}S_N^2|}{\sqrt{\mathbf{E}S_k^2}(\sqrt{\mathbf{E}S_k^2} + \sqrt{\mathbf{E}S_N^2})} \leqslant \frac{1}{\alpha + \sqrt{\alpha}} \frac{|\mathbf{E}S_k^2 - \mathbf{E}S_N^2|}{\mathbf{E}S_N^2}. \tag{4.5}$$

Substituting (4.5) and (4.4) into (4.3), we obtain that, for all $l \geqslant 0$ and $k \in K(\alpha)$,

$$\int\limits_{-\infty}^{\infty} |x|^l \big|\Phi(xa_k) - \Phi(x)\big| \, \mathrm{d}x \leqslant \frac{[2(2-\alpha)]^{(l+2)/2}\Gamma((l+2)/2)}{\sqrt{2\pi}(\alpha + \sqrt{\alpha})} \frac{|\kappa_k^2 - \mathbf{E}S_N^2|}{\mathbf{E}S_N^2}. \tag{4.6}$$

Here we used the fact that $\mathbf{E}S_k^2 = \kappa_k^2$ for $k = 1, 2, \ldots$ if $\mathbf{E}X_i = 0$ for $i = 1, 2, \ldots$. It only remains to substitute (4.6) into the expression of $\sum_2$. We obtain that, for all $l \geqslant 0$,

$$\sum\nolimits_2 \leqslant \frac{[2(2-\alpha)]^{(l+2)/2}\Gamma((l+2)/2)}{\sqrt{2\pi}(\alpha+\sqrt{\alpha})} \frac{\mathbf{E}|\kappa_N^2 - \mathbf{E}S_N^2|\mathbf{1}_{\{N \in K(\alpha)\}}}{\mathbf{E}S_N^2}. \tag{4.7}$$

Estimation of $\sum_3 + \sum_4$. Taking into account that $S_0 = 0$, from (2.3) of Lemma 3 the estimate of $\sum_3$ for $0 \leqslant l \leqslant 2$ follows:

$$\sum\nolimits_3 \leqslant \frac{p_0}{l+1} \cdot \begin{cases} 1 & \text{if } 0 \leqslant l \leqslant 1, \\ \frac{2\sqrt{2}}{\sqrt{\pi}} & \text{if } 1 < l \leqslant 2. \end{cases} \tag{4.8}$$

We easily see that

$$\sum\nolimits_4 \leqslant \sum\nolimits_{41} + \sum\nolimits_{42} + \sum\nolimits_{43}, \tag{4.9}$$

where

$$\sum\nolimits_{41} = \sum_{k \in \overline{K}^-(\alpha)} \frac{1}{a_k^{l+1}} \mathcal{I}_{l,k} p_k, \qquad \sum\nolimits_{42} = \sum_{k \in \overline{K}^+(\alpha)} \frac{1}{a_k^{l+1}} \mathcal{I}_{l,k} p_k,$$

$$\sum\nolimits_{43} = \sum_{k \in \overline{K}(\alpha)} \int_{-\infty}^{\infty} |x|^l \big|\Phi(xa_k) - \Phi(x)\big| \, \mathrm{d}x p_k,$$

and $\overline{K}(\alpha) = \overline{K}^-(\alpha) \cup \overline{K}^+(\alpha)$ is rewritten as the union of $\overline{K}^-(\alpha) = \{k \in \mathbb{N}\colon \mathbf{E}S_k^2 < \alpha \mathbf{E}S_N^2\}$ and $\overline{K}^+(\alpha) = \{k \in \mathbb{N}\colon \mathbf{E}S_k^2 > (2-\alpha)\mathbf{E}S_N^2\}$.

Since $1/a_k \leqslant \sqrt{\alpha}$ for $k \in \overline{K}^-(\alpha)$ and $\mathcal{I}_{l,k} \leqslant 2/(l+1)$ for $0 \leqslant l \leqslant 1$ by (2.4) of Lemma 3, using (4.8), we obtain that, for $0 \leqslant l \leqslant 1$,

$$\begin{aligned}
\sum\nolimits_3 + \sum\nolimits_{41} &\leqslant \frac{1}{l+1} p_0 + \frac{2\alpha^{(l+1)/2}}{l+1} \sum_{k \in \overline{K}^-(\alpha)} p_k \\
&\leqslant \max\big\{1, 2\alpha^{(l+1)/2}\big\} \frac{1}{l+1} \sum_{k \geqslant 0: |\kappa_k^2 - \mathbf{E}S_N^2| > (1-\alpha)\mathbf{E}S_N^2} p_k \\
&\leqslant \max\big\{1, 2\alpha^{(l+1)/2}\big\} \frac{1}{(l+1)(1-\alpha)} \frac{\mathbf{E}|\kappa_N^2 - \mathbf{E}S_N^2|\mathbf{1}_{\{N \in \overline{K}(\alpha) \cup \{0\}\}}}{\mathbf{E}S_N^2}.
\end{aligned} \tag{4.10}$$

Now let $1 < l \leqslant 1 + \delta$, $0 < \delta \leqslant 1$. In this case, instead of (2.4) of Lemma 3, we use (2.5) of Lemma 4, whereby for any fixed $k = 1, 2, \ldots$,

$$\mathcal{I}_{l,k} \leqslant C_{\mathbf{0}} \left(1 + \frac{1}{(\mathbf{E}S_k^2)^{(l+1)/2}} \sum_{i=1}^{k} \mathbf{E}|X_i|^{l+1} \mathbf{1}_{\{|X_i| > \sqrt{\mathbf{E}S_k^2}\}}\right),$$

and therefore

$$\frac{1}{a_k^{l+1}} \mathcal{I}_{l,k} \leqslant C_{\mathbf{0}} \frac{1}{a_k^{l+1}} + C_{\mathbf{0}} \frac{1}{(\mathbf{E}S_N^2)^{(l+1)/2}} \sum_{i=1}^{k} \mathbf{E}|X_i|^{l+1} \mathbf{1}_{\{|X_i| > \sqrt{\mathbf{E}S_k^2}\}}. \tag{4.11}$$

Thus, using (4.8) and (4.11) and recalling that $1/a_k \leqslant \sqrt{\alpha}$ for $k \in \overline{K}^-(\alpha)$, we obtain that, for $1 < l \leqslant 1+\delta$, $0 < \delta \leqslant 1$,

$$\begin{aligned}
\sum\nolimits_3 + \sum\nolimits_{41} &\leqslant \frac{2\sqrt{2}p_0}{\sqrt{\pi}(l+1)} + C_{\mathbf{0}}\alpha^{(l+1)/2} \sum_{k \in \overline{K}^-(\alpha)} p_k \\
&\quad + C_{\mathbf{0}} \frac{1}{(\mathbf{E}S_N^2)^{(l+1)/2}} \sum_{k \in \overline{K}^-(\alpha)} \sum_{i=1}^{k} \mathbf{E}|X_i|^{l+1} \mathbf{1}_{\{|X_i| > \sqrt{\mathbf{E}S_k^2}\}} p_k \\
&\leqslant \max\left\{ \frac{2\sqrt{2}}{\sqrt{\pi}(l+1)}, C_{\mathbf{0}}\alpha^{(l+1)/2} \right\} \sum_{k \in \overline{K}^-(\alpha) \cup \{0\}} p_k + C_{\mathbf{0}} \frac{\mathbf{E} l_{l+1,N} \mathbf{1}_{\{N \in \overline{K}^-(\alpha)\}}}{(\mathbf{E}S_N^2)^{(l+1)/2}} \\
&\leqslant \max\left\{ \frac{2\sqrt{2}}{\sqrt{\pi}(l+1)}, C_{\mathbf{0}}\alpha^{(l+1)/2} \right\} \frac{1}{1-\alpha} \frac{\mathbf{E}|\kappa_N^2 - \mathbf{E}S_N^2| \mathbf{1}_{\{N \in \overline{K}(\alpha) \cup \{0\}\}}}{\mathbf{E}S_N^2} \\
&\quad + C_{\mathbf{0}} \frac{\mathbf{E} l_{l+1,N} \mathbf{1}_{\{N \in \overline{K}^-(\alpha)\}}}{(\mathbf{E}S_N^2)^{(l+1)/2}}. \qquad (4.12)
\end{aligned}$$

To estimate $\sum_{42}$, we use (1.2) of Theorem A. Since $\mathbf{E}S_k^2 > (2-\alpha)\mathbf{E}S_N^2$ for $k \in \overline{K}^+(\alpha)$, we obtain that, for all $0 \leqslant l \leqslant 1+\delta$, $0 < \delta \leqslant 1$,

$$\begin{aligned}
\sum\nolimits_{42} &\leqslant C_0 \sum_{k \in \overline{K}^+(\alpha)} \frac{1}{a_k^{l+1}} \frac{1}{(\mathbf{E}S_k^2)^{(2+\delta)/2}} \sum_{i=1}^{k} \mathbf{E}|X_i|^{2+\delta} \ln^{1+\delta}(1+k) p_k \\
&= C_0 \sum_{k \in \overline{K}^+(\alpha)} \frac{1}{(\mathbf{E}S_N^2)^{(l+1)/2}} \frac{1}{(\mathbf{E}S_k^2)^{(1+\delta-l)/2}} \sum_{i=1}^{k} \mathbf{E}|X_i|^{2+\delta} \ln^{1+\delta}(1+k) p_k \\
&\leqslant C_0 \sum_{k \in \overline{K}^+(\alpha)} \frac{1}{(\mathbf{E}S_N^2)^{(l+1)/2}} \frac{\sum_{i=1}^{k} \mathbf{E}|X_i|^{2+\delta} \ln^{1+\delta}(1+k) p_k}{((2-\alpha)\mathbf{E}S_N^2)^{(1+\delta-l)/2}} \\
&= \frac{C_0}{(2-\alpha)^{(1+\delta-l)/2}} \frac{\mathbf{E} l_{2+\delta,N} \ln^{1+\delta}(1+N) \mathbf{1}_{\{N \in \overline{K}^+(\alpha)\}}}{(\mathbf{E}S_N^2)^{(2+\delta)/2}}. \qquad (4.13)
\end{aligned}$$

To estimate $\sum_{43}$, we use (2.7) of Lemma 5 and obtain that, for all $l \geqslant 0$,

$$\begin{aligned}
\sum\nolimits_{43} &\leqslant \frac{2^{(l+2)/2}\Gamma((l+2)/2)}{\sqrt{2\pi}(l+1)} \sum_{k \in \overline{K}(\alpha)} \left| 1 - \frac{1}{a_k^{l+1}} \right| p_k \\
&\leqslant \frac{2^{(l+2)/2}\Gamma((l+2)/2)}{\sqrt{2\pi}(l+1)} \frac{\mathbf{E}|\kappa_N^{l+1} - (\mathbf{E}S_N^2)^{(l+1)/2}| \mathbf{1}_{\{N \in \overline{K}(\alpha)\}}}{(\mathbf{E}S_N^2)^{(l+1)/2}}. \qquad (4.14)
\end{aligned}$$

Substituting (4.10) in the case $0 \leqslant l \leqslant 1$ ((4.12) in the case $1 < l \leqslant 1+\delta$), (4.13), and (4.14) into (4.9) and observing that the function $f(l) = |1 - 1/a^{l+1}|$, where $0 < a < \infty$, is nondecreasing for $l \in [-1, \infty)$, we obtain that

$$\sum\nolimits_3 + \sum\nolimits_4 \leqslant \biggl(\max\bigl\{1, 2\alpha^{(l+1)/2}\bigr\}\frac{1}{1-\alpha} + \frac{2^{(l+2)/2}\Gamma((l+2)/2)}{\sqrt{2\pi}}\biggr) \times \frac{1}{l+1}\frac{\mathbf{E}|\kappa_N^2 - \mathbf{E}S_N^2|\mathbf{1}_{\{N\in\overline{K}(\alpha)\cup\{0\}\}}}{\mathbf{E}S_N^2} + \frac{C_0}{(2-\alpha)^{(1+\delta-l)/2}}\frac{\mathbf{E}l_{2+\delta,N}\ln^{1+\delta}(1+N)\mathbf{1}_{\{N\in\overline{K}^+(\alpha)\}}}{(\mathbf{E}S_N^2)^{(2+\delta)/2}} \tag{4.15}$$

for $0 \leqslant l \leqslant 1$ and

$$\sum\nolimits_3 + \sum\nolimits_4 \leqslant \biggl(\max\biggl\{\frac{2\sqrt{2}}{\sqrt{\pi}}, C_{\mathbf{0}}\alpha^{(l+1)/2}\biggr\}\frac{1}{1-\alpha} + \frac{2^{(l+2)/2}\Gamma((l+2)/2)}{\sqrt{2\pi}}\biggr) \times \frac{1}{l+1}\frac{\mathbf{E}|\kappa_N^{l+1} - (\mathbf{E}S_N^2)^{(l+1)/2}|\mathbf{1}_{\{N\in\overline{K}(\alpha)\cup\{0\}\}}}{(\mathbf{E}S_N^2)^{(l+1)/2}} + C_{\mathbf{0}}\frac{\mathbf{E}l_{l+1,N}\mathbf{1}_{\{N\in\overline{K}^-(\alpha)\}}}{(\mathbf{E}S_N^2)^{(l+1)/2}} + \frac{C_0}{(2-\alpha)^{(1+\delta-l)/2}}\frac{\mathbf{E}l_{2+\delta,N}\ln^{1+\delta}(1+N)\mathbf{1}_{\{N\in\overline{K}^+(\alpha)\}}}{(\mathbf{E}S_N^2)^{(2+\delta)/2}} \tag{4.16}$$

for $1 < l \leqslant 1+\delta$, $0 < \delta \leqslant 1$.

Substituting (4.2), (4.7), and (4.15) for $0 \leqslant l \leqslant 1$ ((4.16) for $1 < l \leqslant 1+\delta$) into (3.1), taking into account that $\mathbf{E}S_N^2 = \mathbf{E}\kappa_N^2$ in the case of $\mathbf{E}X_i = 0$ for $i = 1, 2, \dots$ and $\sum_{i=1}^{0}(\cdot) = 0$ (see (2.1) of Lemma 1), and taking a concrete $\alpha \in (0,1)$, for example, $\alpha = 1/2$, we obtain estimates (1.3) and (1.4) of Theorem 1.

Theorem 1 is proved. □

Proof of Corollary 1. The proof immediately follows from Theorem 1. □

Proof of Theorem 2. The proof immediately follows from Theorem 1 since for all $l \geqslant 1$,

$$\lambda_l \leqslant l\mathcal{I}_{l-1}. \qquad \square \tag{4.17}$$

Proof of Corollary 2. The proof immediately follows from Theorem 2. □

Proof of Theorem 3. Since $\mathcal{I}_l \leqslant 2/(l+1)$ for $0 \leqslant l \leqslant 1$ and $\lambda_l \leqslant 2$ for $1 \leqslant l \leqslant 2$ (see (2.4) and (4.17)), we assume, without loss of generality, that $\mathbf{E}N$ is sufficiently large. To estimate the first terms in (1.5) of Corollary 1 and in (1.6) of Corollary 2, we use the estimate

$$\mathbf{E}N\ln^{1+\delta}(1+N) \leqslant \sqrt{\mathbf{V}N}\mathbf{E}^{1/2}\ln^{2(1+\delta)}(1+N) + \mathbf{E}N\mathbf{E}\ln^{1+\delta}(1+N). \tag{4.18}$$

Now observing that the functions $f_1(x) = \ln^{2(1+\delta)}(\mathrm{e}^{1+2\delta} + 1 + x)$ and $f_2(x) = \ln^{1+\delta}(\mathrm{e}^{\delta} + 1 + x)$, where $0 < \delta \leqslant 1$, are strictly concave for all $x \in (-1, \infty)$, we obtain by Jensen's inequality that

$$\mathbf{E}\ln^{2(1+\delta)}(1+N) < \ln^{2(1+\delta)}\bigl(\mathrm{e}^{1+2\delta} + 1 + \mathbf{E}N\bigr), \tag{4.19}$$

$$\mathbf{E}\ln^{1+\delta}(1+N) < \ln^{1+\delta}\bigl(\mathrm{e}^{\delta} + 1 + \mathbf{E}N\bigr). \tag{4.20}$$

Now substituting (4.19) and (4.20) into (4.18), substituting the obtained inequality into (1.5) and (1.6) of Corollaries 1 and 2, respectively, and estimating, in the corresponding cases of the number N of summands, the second terms in (1.5) and (1.6) by (2.8) of Lemma 6, by (2.9) of Lemma 7, and by (2.10) of Lemma 8, we obtain (1.7)–(1.12) of Theorem 3.

Theorem 3 is proved. □

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