DOI 10.1007/s10986-020-09473-x
Lithuanian Mathematical Journal, Vol. 60, No. 2, April, 2020, pp. 161–172

A note on the uniform asymptotic behavior of the finite-time ruin probability in a nonstandard renewal risk model*

Yuquan Cang, Yang Yang[1], and Xixi Shi

School of Statistics and Mathematics, Nanjing Audit University, Nanjing 211815, China
(e-mail: cangyuquan@163.com; yangyangmath@163.com; sxx18392065497@163.com)

Received July 5, 2018; revised May 25, 2019

Abstract. Consider a nonstandard renewal risk model in which claims and interarrival times form a sequence of independent and identically distributed random pairs, with each pair obeying arbitrary dependence or size-dependence structure. In the case of heavy-tailed claims, we obtain the asymptotic behavior of finite-time ruin probability with the uniformity in time in some infinite regions.

MSC: 62P05, 62E20, 91B30

Keywords: non-standard renewal risk model, finite-time ruin probability, asymptotics, size-dependence, heavy tail, uniformity

1 Introduction

Consider a continuous-time renewal risk model with no interest rate. In such a model, nonnegative claims Z_k, $k \in \mathbb{N}$, arrive at successive claim arrival times τ_k, $k \in \mathbb{N}$, with $\tau_0 = 0$. For each $k \in \mathbb{N}$, denote by $\theta_k = \tau_k - \tau_{k-1}$ the nonnegative interarrival time. Assume that random pairs (Z_k, θ_k), $k \in \mathbb{N}$, are independent and identically distributed (i.i.d.) copies of a generic random vector (Z, θ), allowing some dependence structure between Z and θ. Denote the claim distribution by $B = 1 - \overline{B}$ and the finite means of claim and interarrival time by $\mathbf{E}Z = b > 0$ and $\mathbf{E}\theta = 1/\lambda > 0$, respectively. Then the claim arrival times constitute a renewal counting process

$$N(t) = \sup\{k \in \mathbb{N}\colon\ \tau_k \leqslant t\}, \quad t \geqslant 0, \tag{1.1}$$

with mean function $\lambda(t) = \mathbf{E}N(t)$. In this way, the aggregate amount of claims is a random sum of the form

$$S(t) = \sum_{i=1}^{N(t)} Z_i, \quad t \geqslant 0, \tag{1.2}$$

[1] Corresponding author.

* The research was supported by the National Natural Science Foundation of China (No. 71471090), Natural Science Foundation of the Jiangsu Higher Education Institutions (No. 19KJA180003), PAPD, the High Level Talent Project of Six Talents Peak of Jiangsu Province (No. JY-039), the Project of Construction for Superior Subjects of Mathematics/Statistics of Jiangsu Higher Education Institutions.

where a summation over an empty index set is zero by convention. For any $t \geqslant 0$, the surplus process of an insurance company is quantified as

$$R(t) = x + ct - S(t),$$

where $x \geqslant 0$ is the initial reserve, and $c > 0$ is the constant premium rate. Then the finite-time ruin probability by time $t \geqslant 0$ is defined as

$$\psi(x,t) = \mathbf{P}\Bigl(\inf_{0\leqslant s\leqslant t} R(s) < 0 \Bigm| R(0) = x\Bigr) = \mathbf{P}\left(\max_{1\leqslant k\leqslant N(t)} \sum_{i=1}^{k} (Z_i - c\theta_i) > x\right). \tag{1.3}$$

Throughout the paper, we assume that the safety loading condition

$$\mu = \frac{c}{\lambda} - b > 0$$

is satisfied and θ is nondegenerate at zero. Under some dependence structure, we aim to find a precise asymptotic expansion for the finite-time ruin probability with the uniformity with respect to time t.

The asymptotic behavior for ruin probabilities with heavy-tailed claims has been studied by quite a few researchers. Some early works were considered in independent models, which require that the claims and the interarrival times are both i.i.d. random variables (r.v.s) and the two sequences are independent. Under the assumption that the integrated-tail distribution of B is subexponential, Veraverbeke [20] and Embrechts and Veraverbeke [6] obtained the asymptotic relation for the infinite-time ruin probability

$$\psi(x,\infty) = \mathbf{P}\Bigl(\inf_{s\geqslant 0} R(s) < 0 \Bigm| R(0) = x\Bigr) \sim \frac{1}{\mu}\int_x^\infty \overline{B}(u)\,\mathrm{d}u$$

as $x \to \infty$, where the symbol $\sim$ means that the quotient of both side tends to 1. In the presence of consistently-varying-tailed claims, Tang [15] established the uniform asymptotic formula

$$\psi(x,t) \sim \frac{1}{\mu}\int_x^{x+\mu\lambda(t)} \overline{B}(u)\,\mathrm{d}u \tag{1.4}$$

uniformly for all $t \in \varLambda = \{t \in \mathbb{R}_+ \colon \lambda(t) > 0\}$, that is,

$$\lim_{x\to\infty} \sup_{t\in\varLambda} \left| \frac{\psi(x,t)}{\mu^{-1}\int_x^{x+\mu\lambda(t)} \overline{B}(u)\,\mathrm{d}u} - 1\right| = 0.$$

Note that the uniform relation (1.4) contains the asymptotics for both finite-time and infinite-time ruin probabilities. Later, Leipus and Šiaulys [10] further studied an independent renewal risk model with strongly subexponential claims. Under some technical conditions on the hazard function $Q(x) = -\log \overline{B}(x)$ and the hazard rate $q(x) = Q'(x)$, they obtained that (1.4) holds uniformly for t in some narrower regions. Some related results can be found in [7, 12, 16, 18, 19, 23, 24] and references therein.

A recent new trend of the study is introducing various dependence structures to model each pair of the claim and interarrival time. For instance, if the deductible amount applied to each loss is raised, then the claim size on each loss reduces, whereas the interarrival time increases, since small losses less than the deductible amount are retained by the insured. The present work is an attempt to extend the study of uniform asymptotics for finite-time ruin probability to the case allowing (both positive and negative) dependence between claims and their interarrival times.

Such a nonstandard renewal risk model was first initiated by Albrecher and Teugels [1]. In this framework, various dependence structures have been introduced to model the claim and its corresponding interarrival time. We refer to [2, 3, 4, 11, 22] among others. In particular, Li et al. [11] introduced a time-dependence structure, which is defined via the conditional tail probability of a claim size given the interarrival time prior to the claim, whereas Chen and Yuen [3] proposed a new dependence structure, namely the size-dependence structure, via the conditional distribution of the interarrival time, provided that the subsequent claim size is large. Precisely speaking, the dependence structure of (Z, θ) is described as follows.

ASSUMPTION 1. There exist a nonnegative r.v. θ^* and some large $x_0 > 0$ such that for all $x \geqslant x_0$ and $t \in [0, \infty)$,

$$\mathbf{P}(\theta > t \mid Z > x) \leqslant \mathbf{P}(\theta^* > t).$$

As pointed out in [3], Assumption 11 means that Z becoming large does not drag θ to infinity, and the size-dependence structure (Assumption 1) seems more natural and general than the time-dependence structure in [11] in view of the perception that the waiting time for a large claim depends on the claim size but not vice versa. In the model of [11], Z and θ are said to be time-dependent if $\mathbf{P}(Z > x \mid \theta = t) \sim \mathbf{P}(Z > x)h(t)$ uniformly for $t \geqslant 0$ as $x \to \infty$, where $h : [0, \infty) \mapsto (0, \infty)$ is a measurable function. As shown in [3], if Z and θ are time-dependent, then, as $x \to \infty$, uniformly in $t \geqslant 0$,

$$\mathbf{P}(\theta > t \mid Z > x) = \int_t^\infty \frac{\mathbf{P}(Z > x \mid \theta = s)}{\mathbf{P}(Z > x)} \, \mathbf{P}(\theta \in \mathrm{d}s) \leqslant 2 \int_t^\infty h(s) \, \mathbf{P}(\theta \in \mathrm{d}s).$$

Define a proper distribution G_0 on $[0, \infty)$ by $G_0(\mathrm{d}s) = h(s)\mathbf{P}(\theta \in \mathrm{d}s)$. Then the right-hand side is equal to $2\overline{G_0}(t)$. In this way, a nonnegative r.v. θ^* distributed by $G^* = \max\{1 - 2\overline{G_0}, 0\}$ can serve as the stochastic upper bound for θ conditional on $(Z > x)$ for all large x. Both of these two dependence structures imply asymptotic independence and contain many commonly used copulas.

Under Assumption 1 and in the presence of consistently-varying-tailed claims, Chen and Yuen [3] derived a precise large-deviation result for the aggregate amount of claims $S(t)$ in (1.2). Motivated by [10] and [3], in this paper, we employ arbitrary dependence or size-dependence structure to model the dependence of (Z, θ) and aim to establish the uniform upper and lower bounds for finite-time ruin probability.

The rest of this paper is organized as follows. In Section 2, we present the main result of this paper after recalling various preliminaries. In Section 3, we prepare a series of auxiliary lemmas, and in Section 4, we give the proof of the main result.

2 Preliminaries and main result

Throughout this paper, all limit relationships hold for x tending to ∞. For two positive functions f and g, we write $f(x) \sim g(x)$ if $\lim f(x)/g(x) = 1$, $f(x) \lesssim g(x)$ if $\limsup f(x)/g(x) \leqslant 1$, and $f(x) = o(g(x))$ if $\lim f(x)/g(x) = 0$. We often equip limit relationships with certain uniformity. For instance, for two positive bivariate functions $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$, we say that $f(x, t) \lesssim g(x, t)$ uniformly for $t \in A \neq \emptyset$ if

$$\lim_{x \to \infty} \sup_{t \in A} \frac{f(x, t)}{g(x, t)} \leqslant 1.$$

Additionally, for a real number x, we denote by $x^+ = \max\{x, 0\}$ its positive part and by $[x]$ the greatest integer smaller than or equal to x.

We are only interested in the case of heavy-tailed claims. A distribution V on $\mathbb{R}$ is said to be heavy-tailed, denoted by $V \in \mathcal{H}$, if $\int_{-\infty}^{+\infty} \mathrm{e}^{sx} \, V(\mathrm{d}x) = \infty$ for any $s > 0$; otherwise, it is said to be light-tailed, denoted by $V \in \mathcal{H}^c$. All heavy-tailed distributions can be further divided into lightly heavy-tailed and heavily heavy-tailed ones, which were proposed by Su et al. [14]. A distribution V on $\mathbb{R}$ is said to be lightly heavy-tailed, denoted by $V \in \mathcal{H}_1$, if $V \in \mathcal{H}$ and $\int_0^\infty x^s V(\mathrm{d}x) < \infty$ for any $s > 0$; otherwise, it is said to be heavily heavy-tailed (if it is still heavy-tailed), denoted by $V \in \mathcal{H}\backslash\mathcal{H}_1$. It is easy to verify that lognormal distributions and all heavy-tailed Weibull distributions are lightly heavy-tailed, whereas Pareto distributions are heavily heavy-tailed. By definition a distribution V on $\mathbb{R}$ is said to be long-tailed, denoted by $V \in \mathcal{L}$, if

$$\overline{V}(x+l) \sim \overline{V}(x)$$

for any fixed $l \in \mathbb{R}$. Another important subclass of heavy-tailed distributions is that of dominatedly-varying-tailed distributions, which are heavily heavy-tailed. By definition a distribution is said to be dominatedly-varying-tailed, denoted by $V \in \mathcal{D}$ on $\mathbb{R}$, if for any $0 < y < 1$,

$$\limsup_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} < \infty.$$

In particular, a distribution V on $\mathbb{R}$ is said to be consistently-varying-tailed, denoted by $V \in \mathcal{C}$, if

$$\lim_{y\uparrow 1} \limsup_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} = 1.$$

The class $\mathcal{D}$ can be characterized by the Matuszewska index: For a distribution V on $\mathbb{R}$, its upper Matuszewska index is defined as

$$J_V^+ = -\lim_{y\to\infty} \frac{\log \overline{V}_*(y)}{\log y} \quad \text{with} \quad \overline{V}_*(y) := \liminf_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} \quad \text{for } y > 1.$$

Clearly, $V \in \mathcal{D}$ if and only if $J_V^+ < \infty$. It is well known that $\mathcal{C} \subset \mathcal{L} \cap \mathcal{D}$; see [8].

We are now in a position to state our main result.

Theorem 1. *Consider the nonstandard renewal risk model with $B \in \mathcal{C}$.*

(i) *We have*

$$\psi(x,t) \lesssim \frac{1}{\mu} \int_x^{x+\mu\lambda t} \overline{B}(u)\,\mathrm{d}u \tag{2.1}$$

uniformly for all $t \in [f_1(x), \infty)$, where $f_1(x)$ is an arbitrary function diverging to ∞.

(ii) *Under Assumption 1, if $\mathbf{E}Z^r < \infty$ and $\mathbf{E}\theta^r < \infty$ for some $r > 1$, then*

$$\psi(x,t) \gtrsim \frac{1}{\mu} \int_x^{x+\mu\lambda t} \overline{B}(u)\,\mathrm{d}u \tag{2.2}$$

uniformly for all $t \in [f_1(x), f_2(x)]$, where $f_1(x) \leqslant f_2(x)$ are two arbitrary functions diverging to ∞ and satisfying $f_2(x) = o(x)$.

Remark 1. When estimating the uniform upper bound, arbitrary dependence is allowed between Z and θ; whereas (Z, θ) is equipped with the size-dependence structure to investigate the uniform lower bound. Combining all conditions in Theorem 1, we can derive the precise equivalence (1.4) holding uniformly for all $t \in [f_1(x), f_2(x)]$. Leipus and Šiaulys [10] considered the independent continuous renewal risk model and obtained the same uniform asymptotic relation (1.4). However, they required some critical technical restrictions on the hazard function and hazard rate of B, so the absolutely continuity of B is necessarily needed; see Assumptions A and B in that paper. It is worth noting that in our main result, it is not necessary to assume the claim size Z to be continuously distributed. By the way, arbitrary dependence or the size-dependence is employed to model the dependence of (Z, θ).

3 Some auxiliary lemmas

We start this section by preparing some auxiliary lemmas. Recalling Assumption 1, introduce a nonnegative r.v. θ_1^* that is independent of all other sources of randomness and is identically distributed as θ conditional on $(Z > x)$. Correspondingly to $N(t)$ defined in (1.1), introduce the delayed renewal counting process

$$N^*(t) = \sup\{k \in \mathbb{N}\colon \tau_k^* \leqslant t\}, \quad t \geqslant 0,$$

where $\tau_1^* = \theta_1^*$ and $\tau_k^* = \theta_1^* + \sum_{i=2}^k \theta_i,\ k \geqslant 2$. Note that the distribution of θ_1^* depends on x due to $(Z > x)$, and so does that of $N^*(t)$. The first lemma establishes the law of large numbers for $N^*(t)$, $t \geqslant 0$, which slightly differs from Lemma 2.1 of [3].

Lemma 1. *Under Assumption* 1*, we have that, for any* $0 < \delta < \lambda$*,*

$$\lim_{x\to\infty} \sup_{t\in[f_1(x),\infty)} \mathbf{P}\left(\left|\frac{N^*(t)}{t} - \lambda\right| > \delta\right) = 0.$$

Proof. We follow the line of the proof of Lemma 2.1 in [3]. Note that, for any $0 < \delta < \lambda$,

$$\begin{aligned}\mathbf{P}\left(\left|\frac{N^*(t)}{t} - \lambda\right| > \delta\right) &\leqslant \mathbf{P}\big(N^*(t) < \big[(\lambda-\delta)t\big] + 1\big) + \mathbf{P}\big(N^*(t) > \big[(\lambda+\delta)t\big]\big) \\ &\leqslant \mathbf{P}\left(\theta_1^* + \sum_{i=2}^{[(\lambda-\delta)t]+1} \theta_i > t\right) + \mathbf{P}\left(\sum_{i=2}^{[(\lambda+\delta)t]} \theta_i \leqslant t\right). \end{aligned} \tag{3.1}$$

Since Z_1 is independent of θ_i, $i \geqslant 2$, by Assumption 1, we have that for any small $\varepsilon \in (0, \delta)$ and sufficiently large x, the first term in (3.1) is bounded by

$$\begin{aligned}&\mathbf{P}\left(\theta^* + \sum_{i=2}^{[(\lambda-\delta)t]+1} \theta_i > t\right) \\ &\quad\leqslant \mathbf{P}\left(\theta^* > \frac{(\delta-\varepsilon)t}{\lambda-\varepsilon}\right) + \mathbf{P}\left(\sum_{i=2}^{[(\lambda-\delta)t]+1} \theta_i > \frac{(\lambda-\delta)t}{\lambda-\varepsilon}\right) \\ &\quad\leqslant \mathbf{P}\left(\theta^* > \frac{\delta-\varepsilon}{\lambda-\varepsilon}\cdot f_1(x)\right) + \mathbf{P}\left(\frac{1}{t}\sum_{i=2}^{[(\lambda-\delta)t]+1}\left(\theta_i - \frac{1}{\lambda}\right) > \frac{\varepsilon(\lambda-\delta)}{\lambda(\lambda-\varepsilon)}\right), \end{aligned} \tag{3.2}$$

where θ^* is defined in Assumption 1. Both terms on the right-hand side of (3.2) vanish uniformly for all $t \in [f_1(x), \infty)$ due to the fact that θ^* is proper and the law of large numbers. Again by the law of large numbers, we can prove that the second term in (3.1) converges to 0 uniformly for all $t \in [f_1(x), \infty)$. This completes the proof of the lemma. □

The second lemma is Theorem 1 of [9].

Lemma 2. *Let interarrival times θ_k, $k \in \mathbb{N}$, be i.i.d. nonnegative r.v.s with finite mean $1/\lambda$. Then for any $\kappa > \lambda$, there exists a positive number ϵ such that*

$$\lim_{t\to\infty} \sum_{n>\kappa t} (1+\epsilon)^n \mathbf{P}(\tau_n \leqslant t) = 0.$$

Clearly, Lemma 2 implies that, for any $p > 0$,

$$\lim_{t\to\infty} \sum_{n>\kappa t} n^p \mathbf{P}(\tau_n \leqslant t) = 0. \tag{3.3}$$

The following lemma is a restatement of Lemma 2.3 of [3].

Lemma 3. *Let (ξ_k, θ_k), $k \in \mathbb{N}$, be i.i.d. copies of a generic random pair (ξ, θ), where ξ is distributed by $F \in \mathcal{D}$, and θ is nonnegative. Then for any $p > J_F^+$, there is a constant $C > 0$ such that*

$$\mathbf{P}\left(\sum_{k=1}^{n} \xi_k > x,\ \tau_n \leqslant t\right) \leqslant C n^{p+1} \overline{F}(x) \mathbf{P}(\tau_{n-1} \leqslant t)$$

uniformly for all $x \geqslant 0$, $t \geqslant 0$, and $n \in \mathbb{N}$.

Note that ξ and θ in Lemma 3 can be arbitrarily dependent. The next lemma plays a key role in the proof of the uniform asymptotic upper bound for the finite-time ruin probability. It is motivated by Corollary 5.2 of Tang [16], in which he established a uniformly equivalent version of (3.4) in the case of independent ξ and θ. However, arbitrary dependence between ξ and θ is allowed in our lemma.

Lemma 4. *Let (ξ_k, θ_k), $k \in \mathbb{N}$, be i.i.d. copies of a generic random pair (ξ, θ), where ξ is distributed by $F \in \mathcal{C}$, and θ is nonnegative with finite mean $1/\lambda$. Let $N(t)$ be a renewal counting process defined in* (1.1)*. If $\mathbf{E}\xi < 0$, then*

$$\mathbf{P}\left(\max_{1\leqslant k\leqslant N(t)} \sum_{i=1}^{k} \xi_i > x\right) \lesssim \frac{1}{|\mathbf{E}\xi|} \int_x^{x+|\mathbf{E}\xi|\lambda t} \overline{F}(y)\,\mathrm{d}y \tag{3.4}$$

uniformly for all $t \in [f_1(x), \infty)$.

Proof. For any decreasing and positive function $\epsilon(t)$ such that $\epsilon(t) \downarrow 0$, $t \to \infty$, according to whether $N(t)$ is greater than $(1+\epsilon(t))\lambda t$ or not, we split

$$\mathbf{P}\left(\max_{1\leqslant k\leqslant N(t)} \sum_{i=1}^{k} \xi_i > x\right) = I_1(x,t) + I_2(x,t).$$

According to Lemma 3.4(i) of [10] and Lemma 9 of [5], since $F \in \mathcal{C}$ and $\mathbf{E}\xi < 0$, we have

$$\begin{aligned} I_2(x,t) &\leqslant \mathbf{P}\left(\max_{1\leqslant k\leqslant [(1+\epsilon(t))\lambda t]} \sum_{i=1}^{k} \xi_i > x\right) \sim \frac{1}{|\mathbf{E}\xi|} \int_x^{x+|\mathbf{E}\xi|[(1+\epsilon(t))\lambda t]} \overline{F}(y)\,\mathrm{d}y \\ &= \frac{1}{|\mathbf{E}\xi|} \int_x^{x+|\mathbf{E}\xi|\lambda t} \overline{F}(y)\,\mathrm{d}y \left(1 + \frac{\int_{x+|\mathbf{E}\xi|\lambda t}^{x+|\mathbf{E}\xi|[(1+\epsilon(t))\lambda t]} \overline{F}(y)\,\mathrm{d}y}{\int_x^{x+|\mathbf{E}\xi|\lambda t} \overline{F}(y)\,\mathrm{d}y}\right) \leqslant \frac{1+\epsilon(t)}{|\mathbf{E}\xi|} \int_x^{x+|\mathbf{E}\xi|\lambda t} \overline{F}(y)\,\mathrm{d}y, \end{aligned} \tag{3.5}$$

which holds uniformly for all $t \in [1/\lambda, \infty)$. As for $I_1(x,t)$, by Lemma 3 and (3.3) we have

$$\begin{aligned} I_1(x,t) &\leqslant \sum_{n>(1+\epsilon(t))\lambda t} \mathbf{P}\Bigg(\max_{1\leqslant k\leqslant n} \sum_{i=1}^{k} \xi_i > x,\ N(t)=n\Bigg) \\ &\leqslant \sum_{n>(1+\epsilon(t))\lambda t} \mathbf{P}\Bigg(\sum_{i=1}^{n} \xi_k^{+} > x,\ \tau_n \leqslant t\Bigg) = o\big(\overline{F}(x)\big), \end{aligned}$$

which holds uniformly for all $t \in [f_1(x), \infty)$. Then since $F \in \mathcal{C} \subset \mathcal{L}$, we have

$$\begin{aligned} &\limsup_{x\to\infty} \sup_{t\in[f_1(x),\infty)} \frac{I_1(x,t)}{\int_x^{x+|\mathbf{E}\xi|\lambda t} \overline{F}(y)\,\mathrm{d}y} \\ &\quad\leqslant \limsup_{x\to\infty} \sup_{t\in[f_1(x),\infty)} \frac{I_1(x,t)}{\int_x^{x+1} \overline{F}(y)\,\mathrm{d}y} \leqslant \limsup_{x\to\infty} \sup_{t\in[f_1(x),\infty)} \frac{I_1(x,t)}{\overline{F}(x+1)} = 0. \end{aligned} \tag{3.6}$$

From (3.5) and (3.6) we obtain that relation (3.4) holds uniformly for all $t \in [f_1(x), \infty)$. □

4 Proof of main result

4.1 Proof of upper bound (2.1)

Before proving the upper bound, we cite two lemmas. The first lemma refers to Lemma 3.5 of [17].

Lemma 5. *For a distribution $V \in \mathcal{D}$ with $J_V^+ > 0$, we have that, for any $p > J_V^+$,*

$$x^{-p} = o\big(\overline{V}(x)\big). \tag{4.1}$$

The second lemma is due to Theorem 1 of [21], where some properties of the distributions whose tails can dominate all light-tailed and lightly heavy-tailed tail distributions are given. A distribution V on $[0,\infty)$ is said to belong to the class of distributions dominating all light-tailed and lightly heavy-tailed tail distributions, denoted by $V \in \mathcal{D}(\mathcal{H}_1 \cup \mathcal{H}^c)$, if for any distribution $W \in \mathcal{H}_1 \cup \mathcal{H}^c$, $\sup_{x>0}(\overline{W}(x)/\overline{V}(x)) < \infty$.

Lemma 6. *Let V and W be distributions on* $[0,\infty)$.

(i) $V \in \mathcal{D}(\mathcal{H}_1 \cup \mathcal{H}^c)$ *if and only if relation* (4.1) *holds for some* $p > 0$.
(ii) *If $W \in \mathcal{H}_1 \cup \mathcal{H}^c$ and $V \in \mathcal{D}(\mathcal{H}_1 \cup \mathcal{H}^c)$, then $\overline{W}(x) = o(\overline{V}(x))$.*

Now we begin to establish relation (2.1) uniformly for all $t \in [f_1(x), \infty)$. For sufficiently small $\Delta \in (0,1)$, by (1.3) we have

$$\begin{aligned} \psi(x,t) &\leqslant \mathbf{P}\Bigg(\max_{1\leqslant k\leqslant N(t)} \sum_{i=1}^{k}\bigg(Z_i - \frac{c(1-\Delta)}{\lambda}\bigg) > x - x^{1/2}\Bigg) \\ &\quad + \mathbf{P}\Bigg(\max_{1\leqslant k\leqslant N(t)} \sum_{i=1}^{k}\bigg(\frac{1-\Delta}{\lambda} - \theta_i\bigg) > \frac{x^{1/2}}{c}\Bigg) \\ &=: \psi_1(x,t) + \psi_2(x,t). \end{aligned} \tag{4.2}$$

We first deal with $\psi_2(x,t)$ in (4.2). According to Lemma 3.3 of [10], by $\mathbf{E}((1-\varDelta)/\lambda-\theta)<0$ we have

$$\psi_2(x,t)\leqslant \mathbf{P}\left(\max_{k\geqslant 1}\sum_{i=1}^{k}\left(\frac{1-\varDelta}{\lambda}-\theta_i\right)>\frac{x^{1/2}}{c}\right)\leqslant C_1\mathrm{e}^{-C_2x^{1/2}}$$

for some positive constants C_1 and C_2 depending on λ and $\varDelta$. Thus

$$\sup_{t\in[f_1(x),\infty)}\frac{\psi_2(x,t)}{\int_x^{x+\mu\lambda t}\overline{B}(u)\,\mathrm{d}u}\leqslant\frac{C_1\mathrm{e}^{-C_2x^{1/2}}}{\int_x^{x+1}\overline{B}(u)\,\mathrm{d}u}\leqslant\frac{C_1\mathrm{e}^{-C_2x^{1/2}}}{\overline{B}(x+1)}. \tag{4.3}$$

By Lemma 5 and Lemma 6 (1) we have $B\in\mathcal{D}(\mathcal{H}_1\cup\mathcal{H}^c)$. Note that the numerator on the right-hand side of (4.3) is a heavy-tailed Weibull tail distribution. Then Lemma 6 (2) gives

$$\limsup_{x\to\infty}\sup_{t\in[f_1(x),\infty)}\frac{\psi_2(x,t)}{\int_x^{x+\mu\lambda t}\overline{B}(u)\,\mathrm{d}u}=0. \tag{4.4}$$

We next turn to $\psi_1(x,t)$. For sufficiently small $\varDelta\in(0,\lambda\mu/c)$, write $\xi=Z-c(1-\varDelta)/\lambda$ with distribution F and mean $\mathbf{E}\xi=-\mu+c\varDelta/\lambda<0$. By Lemma 4 we obtain that, uniformly for all $t\in[f_1(x),\infty)$,

$$\begin{aligned}\psi_1(x,t)&\lesssim\frac{1}{|\mathbf{E}\xi|}\int\limits_{x-x^{1/2}}^{x-x^{1/2}+|\mathbf{E}\xi|\lambda t}\overline{F}(u)\,\mathrm{d}u\leqslant\frac{1}{|\mathbf{E}\xi|}\int\limits_{x-x^{1/2}}^{x-x^{1/2}+|\mathbf{E}\xi|\lambda t}\overline{B}(u)\,\mathrm{d}u\\&=\frac{1-x^{-1/2}}{|\mathbf{E}\xi|}\int\limits_{x}^{x+|\mathbf{E}\xi|\lambda t/(1-x^{-1/2})}\overline{B}\big(v\big(1-x^{-1/2}\big)\big)\,\mathrm{d}v\\&\leqslant\frac{1}{|\mathbf{E}\xi|}\sup_{v\geqslant x}\frac{\overline{B}(v(1-x^{-1/2}))}{\overline{B}(v)}\int\limits_{x}^{x+|\mathbf{E}\xi|\lambda t/(1-x^{-1/2})}\overline{B}(v)\,\mathrm{d}v\\&\lesssim\frac{1}{|\mathbf{E}\xi|}\int\limits_{x}^{x+\mu\lambda t}\overline{B}(v)\,\mathrm{d}v,\end{aligned} \tag{4.5}$$

where the last step is due to $|\mathbf{E}\xi|=\mu-c\varDelta/\lambda<\mu$ and $B\in\mathcal{C}$. Therefore the desired relation (2.1) follows from (4.4) and (4.5) by letting ϵ and $\varDelta$ tend to 0.

4.2 Proof of lower bound (2.2)

To prove the lower bound, we first give a lemma, which presents the law of large numbers for independent but not necessarily identically distributed r.v.s with finite moments of order greater than 1.

Lemma 7. *Let ξ_k, $k\in\mathbb{N}$, be a sequence of independent r.v.s with $\mathbf{E}\xi_k=0$ and $\mathbf{E}|\xi_k|^r\leqslant C<\infty$ for some $r>1$ and $C>0$. Then for any $\varepsilon>0$,*

$$\lim_{n\to\infty}\mathbf{P}\left(\frac{1}{n}\left|\sum_{k=1}^{n}\xi_k\right|>\varepsilon\right)=0.$$

Proof. Without loss of generality, we set $1 < r \leqslant 2$. By Markov's inequality we have

$$\mathbf{P}\left(\frac{1}{n}\left|\sum_{k=1}^{n}\xi_k\right| > \varepsilon\right) \leqslant \frac{1}{n^r\varepsilon^r}\mathbf{E}\left|\sum_{k=1}^{n}\xi_k\right|^r \leqslant \frac{1}{n^r\varepsilon^r}\left(2-\frac{1}{n}\right)\sum_{k=1}^{n}\mathbf{E}|\xi_k|^r$$
$$\leqslant \frac{C(2n-1)}{n^r\varepsilon^r} \to 0$$

as $n \to \infty$, where the second step is due to 2.6.20 on p. 82 of [13]. □

In what follows, we deal with the uniform lower bound (2.2). For any $0 < \delta < 1$ and $v > 0$, by (1.3) we have

$$\psi(x,t) \geqslant \sum_{|n-\lambda t|\leqslant\delta\lambda t}\mathbf{P}\left(\max_{1\leqslant k\leqslant n}\sum_{i=1}^{k}(Z_i - c\theta_i) > x,\ N(t)=n\right)$$
$$\geqslant \sum_{|n-\lambda t|\leqslant\delta\lambda t}\mathbf{P}\left(\max_{1\leqslant k\leqslant n}\sum_{i=1}^{k}(Z_i - c\theta_i) > x,\ N(t)=n,\ \bigcup_{j=1}^{n}\{Z_j > (1+v)x\}\right).$$

By Bonferroni's inequality we have

$$\psi(x,t) \geqslant \tilde{\psi}_1(x,t) - \tilde{\psi}_2(x,t) = \sum_{|n-\lambda t|\leqslant\delta\lambda t}\bigl(J_1(x,t) - J_2(x,t)\bigr), \tag{4.6}$$

where

$$J_1(x,t) = \sum_{j=1}^{n}\mathbf{P}\left(\max_{1\leqslant k\leqslant n}\sum_{i=1}^{k}(Z_i - c\theta_i) > x,\ N(t)=n,\ Z_j > (1+v)x\right)$$

and

$$J_2(x,t) = \sum_{1\leqslant i<j\leqslant n}\mathbf{P}\bigl(N(t)=n,\ Z_i > (1+v)x,\ Z_j > (1+v)x\bigr).$$

For $J_1(x,t)$, we have

$$J_1(x,t) \geqslant \sum_{j=1}^{n}\mathbf{P}\left(\max_{1\leqslant k\leqslant n}\sum_{i=1}^{k}(Z_i - c\theta_i) > x,\ N(t)=n,\ Z_j > (1+v)x,\ \theta_j \leqslant \frac{vx}{2c}\right)$$
$$\geqslant \sum_{j=1}^{n}\mathbf{P}\left(\sum_{i=1,\, i\neq j}^{n}(Z_i - c\theta_i) > -\frac{vx}{2},\ N(t)=n,\ Z_j > (1+v)x,\ \theta_j \leqslant \frac{vx}{2c}\right)$$
$$= \overline{B}\bigl((1+v)x\bigr)\sum_{j=1}^{n}\mathbf{P}\left(\sum_{i=1,\, i\neq j}^{n}(Z_i - c\theta_i) > -\frac{vx}{2},\ N(t)=n,\ \theta_j \leqslant \frac{vx}{2c}\ \Big|\ Z_j > (1+v)x\right).$$

As in Lemma 1, we construct a conditional r.v. $\theta_1^* = (\theta \mid Z > (1+v)x)$ and the corresponding delayed renewal counting process $N^*(t)$, $t \geqslant 0$. It is easy to see that

$$J_1(x,t) \geqslant n\overline{B}\bigl((1+v)x\bigr)\mathbf{P}\left(\sum_{i=2}^{n}(Z_i - c\theta_i) > -\frac{vx}{2},\ N^*(t)=n,\ \theta_1^* \leqslant \frac{vx}{2c}\right),$$

implying

$$\begin{aligned}\tilde{\psi}_1(x,t) &\geqslant (1-\delta)\lambda t\overline{B}\big((1+v)x\big)\\&\quad\times\mathbf{P}\left(\sum_{i=2}^{[(1-\delta)\lambda t]}Z_i - c\sum_{i=2}^{[(1+\delta)\lambda t]}\theta_i > -\frac{vx}{2},\ \left|\frac{N^*(t)}{\lambda t}-1\right|\leqslant\delta,\ \theta_1^*\leqslant\frac{vx}{2c}\right)\\&\geqslant(1-\delta)\lambda t\overline{B}\big((1+v)x\big)\big(J_{11}(x,t)-J_{12}(x,t)-J_{13}(x)\big),\end{aligned}\tag{4.7}$$

where

$$J_{11}(x,t)=\mathbf{P}\left(\sum_{i=2}^{[(1-\delta)\lambda t]}Z_i - c\sum_{i=2}^{[(1+\delta)\lambda t]}\theta_i > -\frac{vx}{2}\right),$$

$$J_{12}(x,t)=\mathbf{P}\left(\left|\frac{N^*(t)}{\lambda t}-1\right|>\delta\right),\qquad J_{13}(x)=\mathbf{P}\left(\theta_1^*>\frac{vx}{2c}\right).$$

By Lemma 1 we have

$$\lim_{x\to\infty}\sup_{t\in[f_1(x),\infty)}J_{12}(x,t)=0.\tag{4.8}$$

By Assumption 1 we have

$$\lim_{x\to\infty}J_{13}(x)=\lim_{x\to\infty}\mathbf{P}\left(\theta>\frac{vx}{2c}\;\Big|\;Z>(1+v)x\right)\leqslant\lim_{x\to\infty}\mathbf{P}\left(\theta^*>\frac{vx}{2c}\right)=0.\tag{4.9}$$

As for $J_{11}(x,t)$, for all $t\in[f_1(x),f_2(x)]$,

$$\begin{aligned}J_{11}(x,t)&=\mathbf{P}\left(\sum_{i=2}^{[(1-\delta)\lambda t]}(Z_i-c\theta_i)-c\sum_{i=[(1-\delta)\lambda t]+1}^{[(1+\delta)\lambda t]}\theta_i>-\frac{vx}{2}\right)\\&\geqslant\mathbf{P}\left(\sum_{i=2}^{[(1-\delta)\lambda t]}\left((Z_i-c\theta_i)-\left(b-\frac{c}{\lambda}\right)\right)-c\sum_{i=[(1-\delta)\lambda t]+1}^{[(1+\delta)\lambda t]}\left(\theta_i-\frac{1}{\lambda}\right)>-\frac{vx}{2}+c(1+\delta)t\right)\\&\geqslant\mathbf{P}\left(\frac{1}{t}\left(\sum_{i=2}^{[(1-\delta)\lambda t]}\left((Z_i-c\theta_i)-\left(b-\frac{c}{\lambda}\right)\right)-c\sum_{i=[(1-\delta)\lambda t]+1}^{[(1+\delta)\lambda t]}\left(\theta_i-\frac{1}{\lambda}\right)\right)\right.\\&\qquad\left.>-\frac{vx}{2f_2(x)}+c(1+\delta)\right).\end{aligned}$$

Since $\mathbf{E}Z^r<\infty$ and $\mathbf{E}\theta^r<\infty$, we have $\mathbf{E}|Z-c\theta|^r<\infty$ by C_r-inequality. Then from Lemma 7 we obtain that

$$\liminf_{x\to\infty}\inf_{t\in[f_1(x),f_2(x)]}J_{11}(x,t)=1.\tag{4.10}$$

Since $B\in\mathcal{C}$, plugging (4.8)–(4.10) into (4.7) yields

$$\liminf_{\delta\downarrow 0}\liminf_{v\downarrow 0}\liminf_{x\to\infty}\inf_{t\in[f_1(x),f_2(x)]}\frac{\tilde{\psi}_1(x,t)}{\lambda t\overline{B}(x)}\geqslant 1.\tag{4.11}$$

We finally estimate $\tilde{\psi}_2(x,t)$. Interchanging the order of summations, since $B \in \mathcal{D}$, we obtain that, uniformly for all $t \in [f_1(x), f_2(x)]$,

$$\begin{aligned}
\tilde{\psi}_2(x,t) &= \big(\overline{B}\big((1+v)x\big)\big)^2 \sum_{|n-\lambda t| \leqslant \delta\lambda t} \sum_{1 \leqslant i<j \leqslant n} \mathbf{P}\big(N(t)=n \mid Z_i > (1+v)x,\ Z_j > (1+v)x\big) \\
&\leqslant \big(\overline{B}\big((1+v)x\big)\big)^2 \sum_{1 \leqslant i<j \leqslant (1+\delta)\lambda t} \mathbf{P}\bigg(\bigg|\frac{N(t)}{\lambda t} - 1\bigg| \leqslant \delta \biggm| Z_i > (1+v)x,\ Z_j > (1+v)x\bigg) \\
&\leqslant \big(\overline{B}\big((1+v)x\big)\big)^2 \big((1+\delta)\lambda t\big)^2 = o\big(\lambda t \overline{B}(x)\big),
\end{aligned} \tag{4.12}$$

where the last step follows from the fact that $t\overline{B}((1+v)x) \leqslant x\overline{B}(x) \to 0$ by $\mathbf{E}Z = b < \infty$. Combining (4.11), (4.12), and (4.6), we obtain

$$\liminf_{x\to\infty} \inf_{t\in[f_1(x),f_2(x)]} \frac{\psi(x,t)}{\mu^{-1}\int_x^{x+\mu\lambda t} \overline{B}(u)\,\mathrm{d}u} \geqslant \liminf_{x\to\infty} \inf_{t\in[f_1(x),f_2(x)]} \frac{\psi(x,t)}{\lambda t \overline{B}(x)} \geqslant 1.$$

This completes the proof of the lower bound (2.2).

Acknowledgment. The authors are most grateful to the two referees for their very thorough reading of the paper and valuable suggestions.

References

1. H. Albrecher and J.L. Teugels, Exponential behavior in the presence of dependence in risk theory, *J. Appl. Probab.*, **43**(1):257–273, 2006.
2. A.L. Badescu, E.C.K. Cheung, and D. Landriault, Dependent risk models with bivariate phase-type distributions, *J. Appl. Probab.*, **46**(1):113–131, 2009.
3. Y. Chen and K.C. Yuen, Precise large deviations of aggregate claims in a size-dependent renewal risk model, *Insur. Math. Econ.*, **51**(2):457–461, 2012.
4. H. Cossette, E. Marceau, and F. Marri, On the compound Poisson risk model with dependence based on a generalized Farlie–Gumbel–Morgenstern copula, *Insur. Math. Econ.*, **43**(3):444–455, 2008.
5. D. Denisov, S. Foss, and D. Korshunov, Tail asymptotics for the supremum of a random walk when the mean is not finite, *Queueing Syst.*, **46**(1–2):15–33, 2004.
6. P. Embrechts and N. Veraverbeke, Estimates for the probability of ruin with special emphasis on the possibility of large claims, *Insur. Math. Econ.*, **1**(1):55ÂÍC–72, 1982.
7. R. Kaas and Q. Tang, Note on the tail behavior of random walk maxima with heavy tails and negative drift, *North Am. Actuar. J.*, **7**(3):57–61, 2003.
8. C. Klüppelberg, Subexponential distributions and integrated tails, *J. Appl. Probab.*, **25**(1):132–141, 1988.
9. J. Kočetova, R. Leipus, and J. Šiaulys, A property of the renewal counting process with application to the finite-time ruin probability, *Lith. Math. J.*, **49**(1):55–61, 2009.
10. R. Leipus and J. Šiaulys, Asymptotic behaviour of the finite-time ruin probability under subexponential claim sizes, *Insur. Math. Econ.*, **40**(3):498–508, 2007.
11. J. Li, Q. Tang, and R. Wu, Subexponential tails of discounted aggregate claims in a time-dependent renewal risk model, *Adv. Appl. Probab.*, **42**(4):1126–1146, 2010.

12. J. Peng and D. Wang, Uniform asymptotics for ruin probabilities in a dependent renewal risk model with stochastic return on investments, *Stochastics*, **90**(3):432–471, 2018.

13. V.V. Petrov, *Limit Theorems of Probability Theory: Sequences of Independent Random Variables*, Clarendon Press, Oxford, 1995.

14. C. Su, Z. Hu, and Q. Tang, Characterizations on heaviness of distribution tails on nonnegative variables, *Adv. Math.*, **32**:606–614, 2003 (in Chinese).

15. Q. Tang, Asymptotics for the finite time ruin probability in the renewal model with consistent variation, *Stoch. Models*, **20**(3):281–297, 2004.

16. Q. Tang, Uniform estimates for the tail probability of maxima over finite horizons with subexponential tails, *Probab. Eng. Inf. Sci.*, **18**(1):71–86, 2004.

17. Q. Tang and G. Tsitsiashvili, Precise estimates for the ruin probability in finite horizon in a discrete-time model with heavy-tailed insurance and financial risks, *Stochastic Processes Appl.*, **108**(2):299–325, 2003.

18. Q. Tang, G. Wang, and K.C. Yuen, Uniform tail asymptotics for the stochastic present value of aggregate claims in the renewal risk model, *Insur. Math. Econ.*, **46**(2):362–370, 2010.

19. Q. Tang and Y. Yang, Interplay of insurance and financial risks in a stochastic environment, *Scand. Actuar. J.*, **2019**(5):432–451, 2019.

20. N. Veraverbeke, Asymptotic behaviour of Wiener–Hopf factors of a random walk, *Stochastic Processes Appl.*, **5**(1): 27–37, 1977.

21. Y. Wang, F. Cheng, and Y. Yang, The dominant relations and their applications on some subclasses of heavy-tailed distributions, *Chin. J. Appl. Probab. Stat.*, **21**(1):21–30, 2005 (in Chinese).

22. Y. Yang, R. Leipus, J. Šiaulys, and Y. Cang, Uniform estimates for the finite-time ruin probability in the dependent renewal risk model, *J. Math. Anal. Appl.*, **383**(1):215–225, 2011.

23. Y. Yang, W. Su, and Z. Zhang, Estimating the discounted density of the deficit at ruin by Fourier cosine series expansion, *Stat. Probab. Lett.*, **146**:147–155, 2019.

24. Y. Yang, K. Wang, J. Liu, and Z. Zhang, Asymptotics for a bidimensional risk model with two geometric Lévy price processes, *J. Ind. Manag. Optim.*, **15**(2):481–505, 2019.