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The difference schemes for solving singularly perturbed three-point boundary value problem

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Abstract. In this paper, we propose and analyze numerical treatment for a singularly perturbed convection–diffusion boundary value problem with nonlocal condition. First, the boundary layer behavior of the exact solution and its first derivative have been estimated. Then we construct a finite difference scheme on a uniform mesh. We prove the uniform convergence of the proposed difference scheme and give an error estimate. We also present numerical examples, which demonstrate computational efficiency of the proposed method.

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1 Introduction

In this work, we treat the following singularly perturbed boundary value problem with nonlocal boundary condition:

$$Lu := \varepsilon u''(x) + a(x)u'(x) = f(x), \quad x \in \Omega, \tag{1.1}$$

$$u'(0) = \frac{A}{\varepsilon}, \tag{1.2}$$

$$u(0) + \gamma u(l_1) = Bu(l) + d, \quad l_1 \in \Omega, \tag{1.3}$$

where $0 < \varepsilon \ll 1$ is a small positive perturbation parameter, A, B, γ, and d are given constants, l_1 and l are given real numbers, and $\Omega = (0, l)$ and $\bar{\Omega} = [0, l]$. We assume that $a(x) \geqslant \alpha > 0$ and $f(x)$ are sufficiently smooth functions on $\bar{\Omega}$. Under these assumptions, singularly perturbed nonlocal problem (1.1)–(1.3) possesses a unique solution indicating a boundary layer of exponential type at $x = 0$.

Differential equations with small positive parameter that multiplies the highest order derivative are said to be singularly perturbed, and normally boundary layers occur in their solutions. These equations play an important role in today's advanced scientific computations. Many mathematical models starting from fluid dynamics to the problems in mathematical biology are modeled by singularly perturbed differential equations, such as quantum mechanics, astrophysics, chemical reactor theory, heat transport problem, meteorology, reaction–diffusion process, oceanography, Navier–Stokes flows with Reynolds numbers, and heat transfer problem with large Peclet numbers. More details about these problems can be found in [20, 25] and references therein.

Due to the presence of boundary layers, standard numerical methods for solving such problems may give rise to difficulties and do not give accurate results for small values of ε. Hence it is necessary to develop suitable numerical methods that uniformly converge with respect to ε. There are two types of such methods, fitted operator methods and fitted mesh methods. In the past few decades, various ε-uniform numerical methods are proposed in the literature for solving singularly perturbed problems [12, 13, 19, 21, 22, 23, 24, 27].

Differential equations with conditions connecting the values of the unknown solution at the boundary with values in the interior are said to be nonlocal boundary value problems. Boundary value problems with nonlocal conditions have been initiated by Il'in and Moiseev [16, 17], motivated by the work of Bitsadze and Samarskii [5] on nonlocal linear elliptic boundary value problems. This kind of problems arise in a variety of different areas of applied mathematics and physics. Typical examples include the vibrations of a guy wire of a uniform cross-section, mathematical models of a large number of phenomena in catalytic processes in chemistry and biology, problems of semiconductors, problems of hydromechanics, heat transfer problems, and some other physical phenomena [1, 15, 26]. In recent years, there has been increasing interest in studying boundary value problems with nonlocal or integral boundary conditions exhibiting boundary layers. The existence and uniqueness of solutions of nonlocal problems and also their numerical solution have been addressed by many authors [2, 3, 4, 6, 7, 8, 9, 10, 11, 14, 18, 28].

Motivated by the works mentioned, we give an ε-uniformly convergent numerical method for solving singularly perturbed three-point boundary value problems. This paper is organized as follows. In Section 2, we indicate the asymptotic behavior of the exact solution and its first derivative with respect to ε. In Section 3, we construct a finite difference discretization on a uniform mesh. An approximation for the nonlocal condition has been presented using simple deviation. In Section 4, we show the ε-uniform convergence of the numerical method and give the error estimate. In Section 5, we present some numerical experiments supporting the theoretical results. Finally, this paper ends with conclusion.

Notation. Throughout the paper, C denotes any generic positive constant independent of ε and the mesh parameter. Some specific fixed constants of this kind are indicated by subscripting C. For any continuous function $g(x)$ defined on the corresponding interval, we use the maximum norm $\|g\|_\infty = \max_{[0,l]} |g(x)|$ and $\|g\|_1 = \int_0^l |g(x)|\, \mathrm{d}x$.

2 Asymptotic estimates

In this section, we analyze asymptotic behavior of the exact solution of problem (1.1)–(1.3), which is needed in the analysis of numerical methods. In the following, we first prove bounds for the solution of the singularly perturbed nonlocal problem (1.1)–(1.3) and its derivative. Since the problem includes the convection term, a single boundary layer near $x = 0$ is present, and a nonlocality condition is assumed to be outside the boundary layer region.

Lemma 1. *Let* $a, f \in C[0, l]$ *and* $1 + \gamma - B \neq 0$. *Then the solution* $u(x)$ *of problem* (1.1)–(1.3) *and its derivative satisfy the following bounds*:

$$\|u\|_\infty \leqslant C_0, \tag{2.1}$$

where

$$C_0 = c_0^{-1}\bigl[|d| + \alpha^{-1}\bigl(|B| + |\gamma|\bigr)\bigl(|A| + \|f\|_1\bigr)\bigr] + \alpha^{-1}\bigl(|A| + \|f\|_1\bigr),$$
$$c_0 = |1 + \gamma - B|,$$

and

$$\left|u'(x)\right| \leqslant C\left(1+\frac{1}{\varepsilon}\mathrm{e}^{-\alpha x/\varepsilon}\right), \quad x \in \bar{\Omega}. \tag{2.2}$$

Proof. We first prove (2.1). We can write Eq. (1.1) in the form

$$\begin{aligned} u'(x) &= u'(0)\mathrm{e}^{-(1/\varepsilon)\int_0^x a(\eta)\,\mathrm{d}\eta} + \frac{1}{\varepsilon}\int_0^x f(\xi)\mathrm{e}^{-(1/\varepsilon)\int_\xi^x a(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\xi \\ &= \frac{A}{\varepsilon}\mathrm{e}^{-(1/\varepsilon)\int_0^x a(\eta)\,\mathrm{d}\eta} + \frac{1}{\varepsilon}\int_0^x f(\xi)\mathrm{e}^{-(1/\varepsilon)\int_\xi^x a(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\xi. \end{aligned} \tag{2.3}$$

Integrating Eq. (2.3) from 0 to x, we get

$$\begin{aligned} u(x) &= u(0) + \frac{A}{\varepsilon}\int_0^x \mathrm{e}^{-(1/\varepsilon)\int_0^\tau a(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\tau + \frac{1}{\varepsilon}\int_0^x \mathrm{d}\tau\int_0^\tau f(\xi)\mathrm{e}^{-(1/\varepsilon)\int_\xi^\tau a(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\xi \\ &= u(0) + \frac{A}{\varepsilon}\int_0^x \mathrm{e}^{-(1/\varepsilon)\int_0^\tau a(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\tau + \frac{1}{\varepsilon}\int_0^x \mathrm{d}\xi f(\xi)\int_\xi^x \mathrm{e}^{-(1/\varepsilon)\int_\xi^\tau a(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\tau. \end{aligned} \tag{2.4}$$

Taking into account the boundary condition (1.3), we obtain

$$\begin{aligned} u(0) = \frac{1}{1+\gamma-B}\biggl\{ &d + \frac{AB}{\varepsilon}\int_0^l \mathrm{e}^{-(1/\varepsilon)\int_0^\tau a(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\tau + \frac{B}{\varepsilon}\int_0^l \mathrm{d}\xi f(\xi)\int_\xi^l \mathrm{e}^{-(1/\varepsilon)\int_\xi^\tau a(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\tau \\ &- \frac{A\gamma}{\varepsilon}\int_0^{l_1} \mathrm{e}^{-(1/\varepsilon)\int_0^\tau a(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\tau - \frac{\gamma}{\varepsilon}\int_0^{l_1} \mathrm{d}\xi f(\xi)\int_\xi^{l_1} \mathrm{e}^{-(1/\varepsilon)\int_\xi^\tau a(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\tau\biggr\}. \end{aligned} \tag{2.5}$$

From (2.5) it follows that

$$\begin{aligned} \left|u(0)\right| &\leqslant c_0^{-1}\biggl\{|d| + \frac{|A||B|}{\varepsilon}\int_0^l \mathrm{e}^{-\alpha\tau/\varepsilon}\,\mathrm{d}\tau + \frac{|B|}{\varepsilon}\int_0^l \mathrm{d}\xi\left|f(\xi)\right|\int_\xi^l \mathrm{e}^{-\alpha(\tau-\xi)/\varepsilon}\,\mathrm{d}\tau \\ &\quad + \frac{|A||\gamma|}{\varepsilon}\int_0^{l_1} \mathrm{e}^{-\alpha\tau/\varepsilon}\,\mathrm{d}\tau + \frac{|\gamma|}{\varepsilon}\int_0^{l_1} \mathrm{d}\xi\left|f(\xi)\right|\int_\xi^{l_1} \mathrm{e}^{-\alpha(\tau-\xi)/\varepsilon}\,\mathrm{d}\tau\biggr\} \\ &\leqslant c_0^{-1}\biggl\{|d| + \alpha^{-1}|A||B|\bigl(1-\mathrm{e}^{-\alpha l/\varepsilon}\bigr) + \alpha^{-1}|B|\int_0^l \left|f(\xi)\right|\bigl(1-\mathrm{e}^{-\alpha(l-\xi)/\varepsilon}\bigr)\,\mathrm{d}\xi \\ &\quad + \alpha^{-1}|A||\gamma|\bigl(1-\mathrm{e}^{-\alpha l_1/\varepsilon}\bigr) + \alpha^{-1}|\gamma|\int_0^{l_1} \left|f(\xi)\right|\bigl(1-\mathrm{e}^{-\alpha(l_1-\xi)/\varepsilon}\bigr)\,\mathrm{d}\xi\biggr\} \end{aligned}$$

$$\leqslant c_0^{-1}\left\{|d|+\alpha^{-1}|A||B|+\alpha^{-1}|B|\int_0^l\bigl|f(\xi)\bigr|\,\mathrm{d}\xi+\alpha^{-1}|A||\gamma|+\alpha^{-1}|\gamma|\int_0^l\bigl|f(\xi)\bigr|\,\mathrm{d}\xi\right\}$$
$$\leqslant c_0^{-1}\bigl\{|d|+\alpha^{-1}|A||B|+\alpha^{-1}|B|\|f\|_1+\alpha^{-1}|A||\gamma|+\alpha^{-1}|\gamma|\|f\|_1\bigr\}.$$

So, we obtain

$$\bigl|u(0)\bigr|\leqslant c_0^{-1}\bigl\{|d|+\alpha^{-1}\bigl(|B|+|\gamma|\bigr)\bigl(|A|+\|f\|_1\bigr)\bigr\}. \tag{2.6}$$

From (2.4) we see that

$$\bigl|u(x)\bigr|\leqslant\bigl|u(0)\bigr|+\frac{|A|}{\varepsilon}\int_0^x\mathrm{e}^{-(1/\varepsilon)\int_0^\tau a(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\tau+\frac{1}{\varepsilon}\int_0^x\mathrm{d}\xi\bigl|f(\xi)\bigr|\int_\xi^x\mathrm{e}^{-(1/\varepsilon)\int_\xi^\tau a(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\tau$$
$$\leqslant\bigl|u(0)\bigr|+|A|\alpha^{-1}\bigl(1-\mathrm{e}^{-\alpha l/\varepsilon}\bigr)+\alpha^{-1}\int_0^l\bigl|f(\xi)\bigr|\bigl(1-\mathrm{e}^{-\alpha(l-\xi)/\varepsilon}\bigr)\,\mathrm{d}\xi$$
$$\leqslant\bigl|u(0)\bigr|+|A|\alpha^{-1}+\alpha^{-1}\int_0^l\bigl|f(\xi)\bigr|\,\mathrm{d}\xi,$$

which, together with (2.6), leads to (2.1).

Next, from (2.3) it follows that

$$\bigl|u'(x)\bigr|\leqslant\frac{|A|}{\varepsilon}\mathrm{e}^{-(1/\varepsilon)\int_0^x a(\eta)\,\mathrm{d}\eta}+\frac{1}{\varepsilon}\int_0^x\bigl|f(\xi)\bigr|\mathrm{e}^{-(1/\varepsilon)\int_\xi^x a(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\xi$$
$$\leqslant\frac{|A|}{\varepsilon}\mathrm{e}^{-\alpha x/\varepsilon}+\alpha^{-1}\max_{0\leqslant t\leqslant x}\bigl|f(t)\bigr|\bigl(1-\mathrm{e}^{-\alpha x/\varepsilon}\bigr)$$
$$\leqslant\frac{|A|}{\varepsilon}\mathrm{e}^{-\alpha x/\varepsilon}+\alpha^{-1}\|f\|_\infty,$$

which implies (2.2) and completes the proof of the lemma. □

3 Discrete problem

We further denote by ω_h the uniform mesh on Ω:

$$\omega_h=\left\{x_i=ih,\ i=1,2,\dots,N-1;\ h=\frac{l}{N}\right\},\qquad\bar{\omega}_h=\omega_h\cup\bigl\{x_0=0,\ x_N=l\bigr\}.$$

To simplify the notation, we set $g_i=g(x_i)$ for any function $g(x)$, whereas y_i denotes an approximation of $u(x)$ at x_i. For any mesh function $g(x_i)$ defined on $\bar{\omega}_h$, we use

$$g_{\bar{x},i}=\frac{g_i-g_{i-1}}{h},\qquad g_{x,i}=\frac{g_{i+1}-g_i}{h},\qquad g_{\mathring{x},i}=\frac{g_{x,i}+g_{\bar{x},i}}{2},\qquad g_{\bar{x}x,i}=\frac{g_{x,i}-g_{\bar{x},i}}{h}$$

and

$$\|g\|_\infty\equiv\|g\|_{\infty,\bar{\omega}_N}:=\max_{0\leqslant i\leqslant N}|g_i|,\qquad\|g\|_{1,\omega_h}=h\sum_{i=1}^{N-1}|g_i|.$$

To obtain a difference approximation for (1.1), we integrate (1.1) over (x_{i-1}, x_{i+1}):

$$h^{-1}\int_{x_{i-1}}^{x_{i+1}} Lu(x)\varphi_i(x)\,\mathrm{d}x = h^{-1}\int_{x_{i-1}}^{x_{i+1}} f(x)\varphi_i(x)\,\mathrm{d}x, \quad 1 \leqslant i \leqslant N-1, \tag{3.1}$$

with the basis functions $\{\varphi_i(x)\}_{i=1}^{N-1}$ of the form

$$\varphi_i(x) = \begin{cases} \varphi_i^{(1)}(x) := \frac{\mathrm{e}^{a_i(x-x_{i-1})/\varepsilon}-1}{\mathrm{e}^{a_i h/\varepsilon}-1}, & x_{i-1} < x < x_i, \\ \varphi_i^{(2)}(x) := \frac{1-\mathrm{e}^{-a_i(x_{i+1}-x)/\varepsilon}}{1-\mathrm{e}^{-a_i h/\varepsilon}}, & x_i < x < x_{i+1}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\varphi_i^{(1)}(x)$ and $\varphi_i^{(2)}(x)$, respectively, are the solutions of the following problems:

$$\begin{aligned} &\varepsilon\varphi_i'' - a_i\varphi_i' = 0, \quad x_{i-1} < x < x_i, \\ &\varphi_i(x_{i-1}) = 0, \qquad \varphi_i(x_i) = 1, \end{aligned}$$

and

$$\begin{aligned} &\varepsilon\varphi_i'' - a_i\varphi_i' = 0, \quad x_i < x < x_{i+1}, \\ &\varphi_i(x_i) = 1, \qquad \varphi_i(x_{i+1}) = 0. \end{aligned}$$

Rearranging (3.1) gives

$$-\varepsilon h^{-1}\int_{x_{i-1}}^{x_{i+1}} \varphi_i'(x)u'(x)\,\mathrm{d}x + a_i h^{-1}\int_{x_{i-1}}^{x_{i+1}} \varphi_i(x)u'(x)\,\mathrm{d}x = f_i - R_i, \quad 1 \leqslant i \leqslant N-1, \tag{3.2}$$

with

$$R_i = h^{-1}\int_{x_{i-1}}^{x_{i+1}} \bigl[a(x) - a(x_i)\bigr]\varphi_i(x)u'(x)\,\mathrm{d}x + h^{-1}\int_{x_{i-1}}^{x_{i+1}} \bigl[f(x_i) - f(x)\bigr]\varphi_i(x)\,\mathrm{d}x. \tag{3.3}$$

Using the interpolating quadrature rules (2.1) and (2.2) from [3] with weight functions $\varphi_i(x)$ on subintervals (x_{i-1}, x_i) and (x_i, x_{i+1}) from (3.2), we obtain the following precise relation:

$$\begin{aligned} &-\varepsilon h^{-1}\int_{x_{i-1}}^{x_{i+1}} \varphi_i'(x)u'(x)\,\mathrm{d}x + a_i h^{-1}\int_{x_{i-1}}^{x_{i+1}} \varphi_i(x)u'(x)\,\mathrm{d}x \\ &\quad = -\varepsilon h^{-1}u_{\bar{x},i} + a_i h^{-1}u_{\bar{x},i}\int_{x_{i-1}}^{x_i} \varphi_i^{(1)}(x)\,\mathrm{d}x + a_i h^{-1}u_{x,i}\int_{x_i}^{x_{i+1}} \varphi_i^{(2)}(x)\,\mathrm{d}x + \varepsilon h^{-1}u_{x,i} \\ &\quad = \varepsilon u_{\bar{x}x,i} + a_i\bigl(\chi_i^{(1)}u_{\bar{x},i} + \chi_i^{(2)}u_{x,i}\bigr), \end{aligned} \tag{3.4}$$

where

$$\chi_i^{(1)} = h^{-1}\int_{x_{i-1}}^{x_i} \varphi_i^{(1)}(x)\,\mathrm{d}x = \frac{\varepsilon}{ha_i} - \frac{1}{\mathrm{e}^{a_i h/\varepsilon} - 1},$$

$$\chi_i^{(2)} = h^{-1} \int_{x_i}^{x_{i+1}} \varphi_i^{(2)}(x)\,\mathrm{d}x = \frac{1}{1-\mathrm{e}^{-a_i h/\varepsilon}} - \frac{\varepsilon}{h a_i}.$$

Substituting

$$u_{\bar{x},i} = u_{\mathring{x},i} - \frac{h}{2} u_{\bar{x}x,i}, \qquad u_{x,i} = u_{\mathring{x},i} + \frac{h}{2} u_{\bar{x}x,i}$$

into (3.4), we get

$$\varepsilon u_{\bar{x}x,i} + a_i\big(\chi_i^{(1)} u_{\bar{x},i} + \chi_i^{(2)} u_{x,i}\big) = \varepsilon\theta_i u_{\bar{x}x,i} + a_i u_{\mathring{x},i}, \tag{3.5}$$

where

$$\theta_i = 1 + \frac{a_i h}{2}\varepsilon\big(\chi_i^{(2)} - \chi_i^{(1)}\big) = \gamma_i \coth\gamma_i, \quad \gamma_i = \frac{a_i h}{2\varepsilon}, \tag{3.6}$$

$$\chi_i^{(1)} + \chi_i^{(2)} = h^{-1} \int_{x_{i-1}}^{x_i} \varphi_i^{(1)}(x)\,\mathrm{d}x + h^{-1} \int_{x_i}^{x_{i+1}} \varphi_i^{(2)}(x)\,\mathrm{d}x = 1.$$

As a consequence, for Eq. (1.1), from (3.2) and (3.5) we obtain the approximate relation

$$\ell u_i := \varepsilon\theta_i u_{\bar{x}x,i} + a_i u_{\mathring{x},i} = f_i - R_i, \quad 1 \leqslant i \leqslant N-1. \tag{3.7}$$

It is now necessary to define an approximation for boundary condition (1.2). Similarly to the process in (3.7), we start with the identity

$$\int_{x_0}^{x_1} Lu(x)\varphi_0(x)\,\mathrm{d}x = \int_{x_0}^{x_1} f(x)\varphi_0(x)\,\mathrm{d}x, \tag{3.8}$$

where

$$\varphi_0(x) = \begin{cases} \frac{1-\mathrm{e}^{-a_0(x_1-x)/\varepsilon}}{1-\mathrm{e}^{-a_0 h/\varepsilon}}, & x_0 < x < x_1, \\ 0, & x \notin (x_0, x_1). \end{cases}$$

Note that the function $\varphi_0(x)$ is the solution of the following problem:

$$\begin{aligned} &\varepsilon\varphi_0'' - a_0\varphi_0' = 0, \quad x_0 < x < x_1, \\ &\varphi_0(x_0) = 1, \qquad \varphi_0(x_1) = 0. \end{aligned}$$

Rearranging (3.8), we take

$$-\varepsilon \int_{x_0}^{x_1} \varphi_0'(x) u'(x)\,\mathrm{d}x + a_0 \int_{x_0}^{x_1} \varphi_0(x) u'(x)\,\mathrm{d}x = A + f_0 \int_{x_0}^{x_1} \varphi_0(x)\,\mathrm{d}x + r^{(0)},$$

where

$$r^{(0)} = \ell u_0 - A = \int_{x_0}^{x_1} \big[a_0 - a(x)\big]\varphi_0(x) u'(x)\,\mathrm{d}x + \int_{x_0}^{x_1} \big[f(x) - f_0\big]\varphi_0(x)\,\mathrm{d}x. \tag{3.9}$$

By arguments similar to the process (3.7) we have

$$-\varepsilon\int_{x_0}^{x_1}\varphi_0'(x)u'(x)\,\mathrm{d}x + a_0\int_{x_0}^{x_1}\varphi_0(x)u'(x)\,\mathrm{d}x = \varepsilon u_{x,0} + a_0 u_{x,0}\int_{x_0}^{x_1}\varphi_0(x)\,\mathrm{d}x = \varepsilon\theta_0 u_{x,0},$$

where

$$\theta_0 = 1 + \frac{a_0}{\varepsilon}\int_{x_0}^{x_1}\varphi_0(x)\,\mathrm{d}x = \frac{a_0 h}{\varepsilon(1-\mathrm{e}^{-a_0 h/\varepsilon})}. \tag{3.10}$$

For boundary condition (1.2), we can write an approximate relation in the form

$$\ell_0 u := \varepsilon\theta_0 u_{x,0} = A + \kappa_0 f_0 + r^{(0)}, \tag{3.11}$$

where

$$\kappa_0 = \frac{h}{1-\mathrm{e}^{-a_0 h/\varepsilon}} - \frac{\varepsilon}{a_0}. \tag{3.12}$$

Next, we introduce an approach for the boundary condition (1.3). Let x_{N_0} be the mesh point nearest to l_1. By Taylor's formula with respect to x_{N_0} we can write

$$u(x) = u(x_{N_0}) + (x - x_{N_0})u'(\xi), \quad \xi\in(x_{N_0}, l_1). \tag{3.13}$$

Substituting $x = l_1$ into (3.13), for the boundary condition (1.3), we obtain

$$u_0 + \gamma u_{N_0} + r^{(N)} = Bu_N + d, \tag{3.14}$$

where

$$r^{(N)} = \gamma(l_1 - x_{N_0})u'(\xi), \quad \xi\in(x_{N_0}, l_1). \tag{3.15}$$

As a consequence of (3.7), (3.11), and (3.14), we propose the following difference scheme for approximating problem (1.1)–(1.3):

$$\ell y_i := \varepsilon\theta_i y_{\bar{x}x,i} + a_i y_{\mathring{x},i} = f_i, \quad 1\leqslant i\leqslant N-1, \tag{3.16}$$

$$\ell_0 y := \varepsilon\theta_0 y_{x,0} = A + \kappa_0 f_0, \tag{3.17}$$

$$y_0 + \gamma y_{N_0} = By_N + d, \tag{3.18}$$

where θ_i, θ_0, and κ_0 are given by (3.6), (3.10), and (3.12), respectively.

4 Analysis of the method

Let $z_i = y_i - u_i$, $0\leqslant i\leqslant N$. Then the error function in the numerical solution satisfies

$$\varepsilon\theta_i z_{\bar{x}x,i} + a_i z_{\mathring{x},i} = R_i, \quad 1\leqslant i\leqslant N-1, \tag{4.1}$$

$$\varepsilon\theta_0 z_{x,0} = -r^{(0)}, \tag{4.2}$$

$$z_0 + \gamma z_{N_0} = Bz_N + r^{(N)}, \tag{4.3}$$

where the errors R_i, $r^{(0)}$, and $r^{(N)}$ are defined by (3.3), (3.9), and (3.15), respectively.

Lemma 2. *If $a, f \in C^1[0,l]$, then the errors R_i, $r^{(0)}$, and $r^{(N)}$ satisfy the following inequalities*:

$$\|R\|_{1,\omega_h} \leqslant Ch, \tag{4.4}$$

$$\left|r^{(0)}\right| \leqslant Ch, \tag{4.5}$$

$$\left|r^{(N)}\right| \leqslant Ch. \tag{4.6}$$

Proof. We can express the remainder term R_i as follows:

$$R_i = \ell u_i - f_i = R_i^{(1)} + R_i^{(2)}, \tag{4.7}$$

where

$$R_i^{(1)} = h^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} \big[a(x) - a(x_i)\big] \varphi_i(x) u'(x)\,\mathrm{d}x, \tag{4.8}$$

$$R_i^{(2)} = h^{-1} \int\limits_{x_{i-1}}^{x_{i+1}} \big[f(x_i) - f(x)\big] \varphi_i(x)\,\mathrm{d}x. \tag{4.9}$$

Let us first prove (4.8). Using the mean value theorem for the functions in (4.8), we get

$$\big|R_i^{(1)}\big| \leqslant C \int\limits_{x_{i-1}}^{x_{i+1}} \big|u'(x)\big|\big|\varphi_i(x)\big|\,\mathrm{d}x. \tag{4.10}$$

Substituting (2.2) into (4.10), since $0 < \varphi_i(x) \leqslant 1$, we obtain

$$\begin{aligned} \big\|R^{(1)}\big\|_{1,\omega_h} &\leqslant Ch \sum_{i=1}^{N-1} \int\limits_{x_{i-1}}^{x_{i+1}} \big|u'(x)\big|\,\mathrm{d}x \leqslant Ch \int\limits_0^l \big|u'(x)\big|\,\mathrm{d}x \leqslant Ch \int\limits_0^l \left(1 + \frac{1}{\varepsilon}\mathrm{e}^{-\alpha x/\varepsilon}\right)\mathrm{d}x \\ &\leqslant Ch\big(l + \alpha^{-1}\big(1 - \mathrm{e}^{-\alpha l/\varepsilon}\big)\big). \end{aligned} \tag{4.11}$$

We now prove (4.9). Also, using the mean value theorem for the functions in (4.9), we have $|R_i^{(2)}| \leqslant C\int_{x_{i-1}}^{x_{i+1}} |\varphi_i(x)|\,\mathrm{d}x$. Consequently,

$$\big\|R^{(2)}\big\|_{1,\omega_h} \leqslant Ch. \tag{4.12}$$

Substituting (4.11) and (4.12) into (4.7), we obtain (4.4).

Next, we estimate the remainder term $r^{(0)}$. Using the mean value theorem for the functions in (3.9), since $0 < \varphi_0(x) \leqslant 1$, we obtain

$$\begin{aligned} \big|r^{(0)}\big| &\leqslant Ch \int\limits_{x_0}^{x_1} \big|u'(x)\big|\varphi_0(x)\,\mathrm{d}x + Ch \int\limits_{x_0}^{x_1} \varphi_0(x)\,\mathrm{d}x \leqslant Ch\left(\int\limits_{x_0}^{x_1} \left(1 + \frac{1}{\varepsilon}\mathrm{e}^{-\alpha x/\varepsilon}\right)\mathrm{d}x + h\right) \\ &\leqslant Ch\big[(2h + \alpha^{-1}\big(1 - \mathrm{e}^{-\alpha h/\varepsilon}\big)\big], \end{aligned}$$

which leads to (4.5).

It remains to estimate the remainder term $r^{(N)}$. Since we accept l_1 except for the boundary layer domain, $u'(x)$ is bounded. From (3.15) we obtain

$$\left|r^{(N)}\right| \leqslant \left|\gamma(l_1 - x_{N_0})\right|\left|u'(\xi)\right| \leqslant Ch. \qquad \square$$

Lemma 3. *Let the error function z_i be the solution of problem* (4.1)–(4.3), *and let $1+\gamma-B \neq 0$. Then*

$$\|z\|_{\infty,\bar{\omega}_N} \leqslant C\left\{\left|r^{(0)}\right| + \left|r^{(N)}\right| + \|R\|_{1,\omega_h}\right\}. \tag{4.13}$$

Proof. If $v_i = z_{x,i}$, then we can write Eq. (4.1) as follows:

$$\varepsilon\theta_i v_{\bar{x},i} + \frac{a_i}{2}(v_i + v_{i-1}) = R_i, \quad 1 \leqslant i \leqslant N-1. \tag{4.14}$$

From (4.14) we get

$$v_i = \frac{\varepsilon\theta_i - 0.5ha_i}{\varepsilon\theta_i + 0.5ha_i} v_{i-1} + \frac{hR_i}{\varepsilon\theta_i + 0.5ha_i}.$$

Solving this first-order difference equation with respect to v_i and setting the boundary condition

$$v_0 = -\frac{r^{(0)}}{\varepsilon\theta_0},$$

we have

$$v_i = -\frac{r^{(0)}}{\varepsilon\theta_0} Q_i + \sum_{k=1}^{i} \varphi_k Q_{ik}, \tag{4.15}$$

where

$$Q_{ik} = \begin{cases} 1, & k = i, \\ \prod_{j=k+1}^{i} \frac{\varepsilon\theta_j - 0.5ha_j}{\varepsilon\theta_j + 0.5ha_j}, & 1 \leqslant k \leqslant i-1, \end{cases} \qquad \varphi_k = \frac{hR_k}{\varepsilon\theta_k + 0.5ha_k}.$$

From (4.15) we take

$$z_{i+1} = z_i - \frac{hr^{(0)}}{\varepsilon\theta_0} Q_i + h\sum_{k=1}^{i} \varphi_k Q_{ik}. \tag{4.16}$$

Solving the first-order difference equation (4.16), we obtain

$$\begin{aligned} z_i &= z_1 - h\sum_{k=1}^{i-1} \frac{r^{(0)}}{\varepsilon\theta_0} Q_k + h\sum_{k=1}^{i-1}\sum_{j=1}^{k} \varphi_j Q_{kj} \\ &= z_0 - \frac{hr^{(0)}}{\varepsilon\theta_0} - h\sum_{k=1}^{i-1} \frac{r^{(0)}}{\varepsilon\theta_0} Q_k + h\sum_{k=1}^{i-1}\sum_{j=1}^{k} \varphi_j Q_{kj}, \quad i \geqslant 2. \end{aligned} \tag{4.17}$$

From condition (4.3) we get

$$|z_0| \leqslant \frac{1}{|1+\gamma-B|}\Bigg\{\left|r^{(N)}\right| + \frac{h|\gamma-B||r^{(0)}|}{\varepsilon\theta_0} + h|\gamma|\sum_{k=1}^{N_0-1}\frac{|r^{(0)}|}{\varepsilon\theta_0}Q_k + h|\gamma|\sum_{k=1}^{N_0-1}\sum_{j=1}^{k}\frac{h|R_j|}{\varepsilon\theta_j+0.5ha_j}Q_{kj} + h|B|\sum_{k=1}^{N-1}\frac{|r^{(0)}|}{\varepsilon\theta_0}Q_k + h|B|\sum_{k=1}^{N-1}\sum_{j=1}^{k}\frac{h|R_j|}{\varepsilon\theta_j+0.5ha_j}Q_{kj}\Bigg\}. \tag{4.18}$$

Next, since $\varepsilon\theta_i + 0.5ha_i > 0$ and $0 < (\varepsilon\theta_i - 0.5ha_i)/(\varepsilon\theta_i + 0.5ha_i) < 1$ $(1 \leqslant i \leqslant N)$, from (4.18) and (4.17) we can easily obtain (4.13). □

We now can state the convergence result of this paper.

Theorem 1. *Let* $a, f \in C^1[0,l]$. *Let* u *be the solution of* (1.1)–(1.3)*, and let* y *be the solution of* (3.16)–(3.18). *Then we have the following* ε*-uniform estimate*:

$$\|y-u\|_{\infty,\bar{\omega}_N} \leqslant Ch.$$

5 Algorithm and numerical results

In this section, we propose a technique for solving problem (3.16)–(3.18). In addition, we demonstrate the effectiveness of our method by applying it to two examples of problem (1.1)–(1.3).

First, reformulating (3.16), we can write

$$\ell y_i := \varepsilon\theta_i\frac{y_{x,i}-y_{\bar{x},i}}{h} + a_i\frac{y_{x,i}+y_{\bar{x},i}}{2} = f_i, \quad 1 \leqslant i \leqslant N-1,$$

and denoting $y_{x,i} = w_i$, we have

$$\varepsilon\theta_i\frac{w_i-w_{i-1}}{h} + a_i\frac{w_i+w_{i-1}}{2} = f_i.$$

From this we get

$$w_i = A_i w_{i-1} + F_i, \tag{5.1}$$

where

$$A_i = \frac{2\varepsilon\theta_i - a_i h}{2\varepsilon\theta_i + a_i h}, \qquad F_i = \frac{2hf_i}{2\varepsilon\theta_i + a_i h}.$$

From (3.17) and $y_{x,0} = w_0$, together with (5.1), we have

$$w_i = A_i w_{i-1} + F_i, \quad 1 \leqslant i \leqslant N-1,$$
$$w_0 = A_0,$$

where $A_0 = A/(\varepsilon\theta_0)$. Solving this first-order difference problem, we obtain

$$w_i = A_0\prod_{k=1}^{i}A_k + \sum_{k=1}^{i}\left(\prod_{j=k+1}^{i}A_j\right)F_k, \quad 1 \leqslant i \leqslant N-1.$$

Second, if we reconsider $y_{x,i} = w_i$ and $y_{x,0} = w_0$, we can write another first-order difference problem:

$$y_{i+1} = y_i + A_0 h \prod_{k=1}^{i} A_k + h \sum_{k=1}^{i} \left(\prod_{j=k+1}^{i} A_j \right) F_k, \quad 2 \leqslant i \leqslant N-1, \qquad y_1 = y_0 + A_0 h.$$

Solving this problem, we obtain

$$y_i = y_0 + A_0 h + h \sum_{k=1}^{i-1} \left[A_0 \prod_{m=1}^{k} A_m + \sum_{m=1}^{k} \left(\prod_{j=m+1}^{k} A_j \right) F_m \right], \quad 1 \leqslant i \leqslant N-1. \tag{5.2}$$

Finally, from (3.17) we have

$$\begin{aligned} y_0 = (1+\gamma-B)^{-1} \Bigg\{ & d + (B-\gamma) A_0 h - \gamma h \sum_{k=1}^{N_0-1} \left[A_0 \prod_{m=1}^{k} A_m + \sum_{m=1}^{k} \left(\prod_{j=m+1}^{k} A_j \right) F_m \right] \\ & + B h \sum_{k=1}^{N-1} \left[A_0 \prod_{m=1}^{k} A_m + \sum_{m=1}^{k} \left(\prod_{j=m+1}^{k} A_j \right) F_m \right] \Bigg\}. \end{aligned}$$

Thus, if we consider this value in (5.2), then we can solve problem (3.16)–(3.18) too.

Example 1. We consider the first test problem

$$\begin{aligned} &\varepsilon u''(x) + 2u'(x) = (\varepsilon - 2)\mathrm{e}^{-x}, \quad 0 < x < 1, \\ &u'(0) = \frac{1}{\varepsilon}, \qquad u(0) + \frac{1}{3} u\left(\frac{1}{4}\right) + u(1) = 1. \end{aligned}$$

Its exact solution is

$$u(x) = d_1 + d_2 \mathrm{e}^{-2x/\varepsilon} + \mathrm{e}^{-x},$$

where

$$d_1 = -\frac{3}{7}\left[\mathrm{e}^{-1} + \frac{1}{3}\mathrm{e}^{-1/4} + \left(1 + \mathrm{e}^{-2/\varepsilon} + \frac{1}{3}\mathrm{e}^{-1/(2\varepsilon)}\right) d_2\right], \qquad d_2 = -\frac{1+\varepsilon}{2}.$$

Example 2. We consider the second test problem

$$\begin{aligned} &\varepsilon u''(x) + 2u'(x) = (\varepsilon - 2)\mathrm{e}^{-x}, \quad 0 < x < 1, \\ &u'(0) = \frac{1}{\varepsilon}, \qquad u(0) + \frac{2}{3} u\left(\frac{3}{4}\right) + u(1) = 1. \end{aligned}$$

Its exact solution is

$$u(x) = d_1 + d_2 \mathrm{e}^{-2x/\varepsilon} + \mathrm{e}^{-x},$$

where

$$d_1 = -\frac{3}{8}\left[\mathrm{e}^{-1} + \frac{2}{3}\mathrm{e}^{-3/4} + \left(1 + \mathrm{e}^{-2/\varepsilon} + \frac{2}{3}\mathrm{e}^{-3/(2\varepsilon)}\right) d_2\right], \qquad d_2 = -\frac{1+\varepsilon}{2}.$$

Table 1. Exact errors, computed ε-uniform errors, and convergence rates for Example 1

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
2^{-4}	0.0032535	0.0008514	0.0002154	0.0000540	0.0000135	0.0000034
	1.93	1.98	1.99	2.00	1.99	
2^{-8}	0.0097452	0.0045248	0.0019363	0.0006924	0.0002017	0.0000528
	1.11	1.22	1.48	1.78	1.93	
2^{-12}	0.0103662	0.0051362	0.0025413	0.0012488	0.0006039	0.0002817
	1.01	1.02	1.03	1.05	1.10	
2^{-16}	0.0104051	0.0051745	0.0025793	0.0012867	0.0006417	0.0003195
	1.01	1.00	1.00	1.00	1.01	
2^{-20}	0.0104075	0.0051769	0.0025816	0.0012891	0.0006440	0.0003218
	1.01	1.00	1.00	1.00	1.00	
2^{-24}	0.0104076	0.0051770	0.0025818	0.0012892	0.0006442	0.0003220
	1.01	1.00	1.00	1.00	1.00	
e^N	0.0104076	0.0051770	0.0025818	0.0012892	0.0006442	0.0003220
p^N	1.01	1.00	1.00	1.00	1.00	

Table 2. Exact errors, computed ε-uniform errors, and convergence rates for Example 2

ε	$N=16$	$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
2^{-4}	0.0036800	0.0009640	0.0002440	0.0000612	0.0000153	0.0000038
	1.93	1.98	2.00	2.00	2.01	
2^{-8}	0.0109089	0.0050654	0.0021680	0.0007754	0.0002259	0.0000592
	1.11	1.22	1.48	1.78	1.93	
2^{-12}	0.0116034	0.0057492	0.0028446	0.0013979	0.0006760	0.0003154
	1.01	1.02	1.02	1.05	1.10	
2^{-16}	0.0116470	0.0057921	0.0028871	0.0014403	0.0007183	0.0003576
	1.01	1.00	1.00	1.00	1.01	
2^{-20}	0.0116497	0.0057948	0.0028898	0.0014429	0.0007209	0.0003602
	1.01	1.00	1.00	1.00	1.00	
2^{-24}	0.0116499	0.0057949	0.0028900	0.0014431	0.0007211	0.0003604
	1.01	1.00	1.00	1.00	1.00	
e^N	0.0116499	0.0057949	0.0028900	0.0014431	0.0007211	0.0003604
p^N	1.01	1.00	1.00	1.00	1.00	

We define the exact error e_ε^N and the computed parameter-uniform maximum pointwise error e^N as follows:

$$e_\varepsilon^N = \|y-u\|_\infty, \quad \bar{\omega}, \quad e^N = \max_\varepsilon e_\varepsilon^N.$$

where y is the numerical approximation to u for various values of N and ε. We also define the computed parameter-uniform convergence rate

$$p^N = \log_2 \frac{e^N}{e^{2N}}.$$

The values of ε for which we solve the test problems are $\varepsilon = 2^{-4i}$, $i = 1, 2, \ldots, 6$. The resulting errors and convergence rates are listed in Tables 1–2.

6 Conclusion

We have described a finite difference method on the uniform mesh for the solution of singularly perturbed three-point boundary value problems. The method was constructed by the integral identities with use of appropriate quadrature rules with the remainder terms in integral form. We have analyzed the ε-uniform convergence.

For two examples, we have computed the maximum absolute errors and convergence rates as predicted by the theory. In Tables 1 and 2, we give the results for different values of ε and N. The obtained theoretical results are confirmed by numerical experiments. The ideas presented here can be easily applied to solving more complicated boundary value problems for singularly perturbed equations with nonlocal boundary conditions.

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