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On values of the Riemann zeta function at positive integers

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Abstract. We give new proofs of some known results on the values of the Riemann zeta function at positive integers and obtain some new theorems related to these values. Considering even zeta values as $\zeta(2n) = \eta_n \pi^{2n}$, we obtain the generating functions of the sequences η_n and $(-1)^n\eta_n$. Using the Riemann–Lebesgue lemma, we give recurrence relations for $\zeta(2n)$ and $\zeta(2n+1)$. Furthermore, we prove some series equations for $\sum_{k=1}^{\infty}(-1)^{k-1}\zeta(p+k)/k$.

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1 Introduction

The function $\zeta(s)$ defined by the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \mathrm{Re}(s) > 1,$$

is called the Riemann zeta function. It is an analytic function in the half-plane $\mathrm{Re}(s) > 1$, and it is well known that $\zeta(s)$ has an analytic continuation into $\mathbb{C} \setminus \{1\}$. The results in the region $\mathrm{Re}(s) < 1$ (for instance, $\zeta(0) = -1/2$ (see [3])) are stated for the analytic extension of $\zeta(s)$ to that region.

For even positive integers, we have the well-known relationship between zeta values and Bernoulli numbers:

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}. \tag{1.1}$$

Here $B_0 = 1, B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42, \ldots,$ and for $n \geqslant 1$, we have $B_{2n+1} = 0$ (this result was first published by Euler in 1740). The Bernoulli numbers have the generating function

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{\mathrm{e}^x - 1}, \quad |x| < 2\pi. \tag{1.2}$$

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By means of (1.1) the following relation can be derived (see [16]):

$$\zeta(2n)=\sum_{m=1}^{\infty}\frac{1}{m^{2n}}=\eta_n\pi^{2n},\quad n\geqslant 1, \tag{1.3}$$

where the numbers η_n ($n\in\mathbb{N}$) satisfy the recurrence relation

$$\eta_n=\sum_{m=1}^{n-1}(-1)^{m-1}\frac{\eta_{n-m}}{(2m+1)!}+(-1)^{n+1}\frac{n}{(2n+1)!},\quad n\geqslant 1. \tag{1.4}$$

Note that the empty sum $\sum_{m=1}^{0}$ is understood to be zero. From (1.4) we have $\eta_1=1/6$, $\eta_2=1/90$, $\eta_3=1/945$, and so on.

Furthermore, since $\zeta(0)=-1/2$, we define $\eta_0:=-1/2$, and hence we can rewrite (1.4) as

$$\eta_n=\sum_{m=1}^{n}(-1)^{m-1}\frac{\eta_{n-m}}{(2m+1)!}+(-1)^{n+1}\frac{1}{2(2n)!}$$

or, alternatively,

$$\sum_{m=0}^{n}(-1)^{m-1}\frac{\eta_{n-m}}{(2m+1)!}=(-1)^{n}\frac{1}{2(2n)!}. \tag{1.5}$$

Remark 1. Considering (1.1) and (1.3) together, we have

$$\eta_n=(-1)^{n+1}\frac{2^{2n-1}}{(2n)!}B_{2n}=(-1)^{n+1}\frac{2^{n-1}}{(2n-1)!!n!}B_{2n},\quad n\geqslant 1. \tag{1.6}$$

Here the double factorial $(2n-1)!!$ is the product of all odd integers from 1 to $2n-1$.

There is an enormous literature on the Riemann zeta function at positive integers. Therefore it is impossible to mention all of them, and we will try to give place to those that are directly related to our study. Now let us summarize our work. In Section 2, we obtain the generating functions of the sequences η_n and $(-1)^n\eta_n$. This gives us an opportunity to obtain some properties of η_n and $\zeta(2n)$. Most of these results are already known, but we relate them to η_n.

In Section 3, in the light of the Riemann–Lebesgue lemma and some integral formulas, we give recurrence relations to evaluate $\zeta(2n)$ and $\zeta(2n+1)$. As a consequence of these relations, we obtain the following integral representations for $\zeta(3)$ and $\zeta(5)$:

$$\zeta(3)=\frac{8}{7}\int_0^{\pi/2}x(2\ln 2-x\cot x)\,\mathrm{d}x$$

and

$$\zeta(5)=\int_0^{\pi/2}\left(\frac{32}{93}x^2-\frac{48}{217}\pi^2\right)x^2\cot x\,\mathrm{d}x+\frac{22}{651}\pi^4\ln 2.$$

Although there exists an evaluation formula for $\zeta(5)$ in [6, p. 204], our formula is much simpler.

In the last section, we give some ideas for possible evaluation of the series

$$\sigma_p = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\zeta(p+k)}{k}, \quad p \geqslant 1,$$

in a *closed form*. Meanwhile, we obtain an interesting equation between σ_p and a series consisting of the Riemann zeta function and the *skew-harmonic numbers*

$$H_n^- = 1 - \frac{1}{2} + \cdots + (-1)^{n-1}\frac{1}{n}.$$

2 Constants related to $\zeta(s)$ at even integers

It is common to use the Taylor series expansion of $z \cot z$ in obtaining the values of the Riemann zeta function at special points (see, e.g., [9, 13, 14, 19]). In this section, we use (1.5) to obtain the generating function of the sequences η_n and $(-1)^n\eta_n$ in terms of $z \cot z$ and $z \coth z$.

Proposition 1. *Let η_n be defined as in* (1.3). *Then*

$$\sum_{n=0}^{\infty} \eta_n z^n = -\frac{1}{2}\sqrt{z}\cot\sqrt{z} \tag{2.1}$$

and

$$\sum_{n=0}^{\infty} (-1)^n \eta_n z^n = -\frac{1}{2}\sqrt{z}\coth\sqrt{z}$$

for $|z| < \pi^2$.

Proof. From (1.5) we have

$$\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}(-1)^{m-1}\frac{\eta_{n-m}}{(2m+1)!}\right)z^{2n} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}.$$

We can write the left-hand side as a product of two series:

$$-\frac{1}{z}\left(\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!}\right)\left(\sum_{n=0}^{\infty}\eta_n z^{2n}\right) = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{z^{2n}}{(2n)!}.$$

Considering the well-known series expansions of sin and cos functions, we have

$$-\frac{1}{z}\sin z\left(\sum_{n=0}^{\infty}\eta_n z^{2n}\right) = \frac{1}{2}\cos z,$$

and hence

$$\sum_{n=0}^{\infty}\eta_n z^{2n} = -\frac{z}{2}\cot z \tag{2.2}$$

for $|z| < \pi$. This completes the proof of the first equality. For the second one, considering (1.1) and (1.3) together, we have

$$2(-1)^{n+1}\eta_n = \frac{2^{2n}}{(2n)!}B_{2n}.$$

On the other hand, from (1.2) we have

$$1 - \frac{1}{2}z + \sum_{n=1}^{\infty} B_{2n}\frac{z^{2n}}{(2n)!} = \frac{z}{\mathrm{e}^z - 1}, \quad |z| < 2\pi.$$

Hence

$$1 - z + \sum_{n=1}^{\infty} 2(-1)^{n+1}\eta_n z^{2n} = \frac{2z}{\mathrm{e}^{2z} - 1}, \quad |z| < \pi,$$

and recalling that $\eta_0 = -1/2$, we have

$$\sum_{n=0}^{\infty}(-1)^{n+1}\eta_n z^{2n} = \frac{z}{2}\frac{\mathrm{e}^{2z}+1}{\mathrm{e}^{2z}-1} = \frac{1}{2}z\frac{\mathrm{e}^z + \mathrm{e}^{-z}}{\mathrm{e}^z - \mathrm{e}^{-z}} = \frac{1}{2}z\coth z$$

for $|z| < \pi$, as claimed. □

Remark 2. Since $\cos z$ is an even function and $\sin z$ is an odd function, $\cos\sqrt{z}$ and $\sqrt{z}/\sin\sqrt{z}$ ($|z| < \pi^2$) are analytic functions. Hence their product $\sqrt{z}\cot\sqrt{z}$ ($|z| < \pi^2$) is also analytic (provided that everywhere the same branch of the function $\sqrt{z}$ is used). The same is also valid for $\sqrt{z}\coth\sqrt{z}$.

Corollary 1. *In view of* (2.1)*, we have*

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{2^{2n}} = 0$$

or, equivalently,

$$\sum_{n=1}^{\infty}\frac{\zeta(2n)}{2^{2n}} = \frac{1}{2}$$

(*since* $\zeta(0) = -1/2$) *and*

$$\sum_{n=0}^{\infty}\frac{\zeta(2n)}{4^{2n}} = -\frac{\pi}{8}.$$

Also, we have

$$\sum_{n=0}^{\infty}\eta_n = \sum_{n=0}^{\infty}\frac{\zeta(2n)}{\pi^{2n}} = -\frac{1}{2}\cot 1 \approx -0.321\,05.$$

Remark 3. Putting $z = \pi x$ ($0 \leqslant x < 1$) in (2.2), we get

$$\sum_{n=0}^{\infty}\eta_n\pi^{2n}x^{2n} = -\frac{\pi}{2}x\cot(\pi x), \quad 0 \leqslant x < 1.$$

Since $\eta_n \pi^{2n} = \zeta(2n)$, we have the following well-known equation [14, p. 36]:

$$2\sum_{n=0}^{\infty} \zeta(2n)x^{2n} = -\pi x \cot(\pi x), \quad 0 \leqslant x < 1.$$

The next corollary gives a representation for η_n in terms of the Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$. The Stirling number $\left\{ {n \atop k} \right\}$ stands for the number of ways to partition a set of n things into k nonempty subsets (see [4, 10]). For the second equation in the corollary, see Rzadkowski [15].

Corollary 2. *For $n \geqslant 1$, we have*

$$\eta_n = (-1)^n \frac{2^{2n-1} n}{(2^{2n}-1)(2n)!} \sum_{k=1}^{2n-1} (-1)^k \frac{k!}{2^k} \left\{ {2n-1 \atop k} \right\}$$

and

$$B_{2n} = -\frac{n}{2^{2n}-1} \sum_{k=1}^{2n-1} (-1)^k \frac{k!}{2^k} \left\{ {2n-1 \atop k} \right\}.$$

Proof. Knopf [12] gave the following Maclaurin expansion of $z \cot z$:

$$z \cot z = 1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n} n z^{2n}}{(2^{2n}-1)(2n)!} \sum_{k=1}^{2n-1} \frac{(-1)^k k!}{2^k} \left\{ {2n-1 \atop k} \right\}, \quad |z| < \pi.$$

By taking into account these equations together with Proposition 1, we get

$$\sum_{n=0}^{\infty} \eta_n z^{2n} = -\frac{z}{2} \cot z = \frac{-1}{2} + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n-1} n z^{2n}}{(2^{2n}-1)(2n)!} \sum_{k=1}^{2n-1} \frac{(-1)^k k!}{2^k} \left\{ {2n-1 \atop k} \right\}.$$

Comparison of coefficients gives $\eta_0 = -1/2$ and

$$\eta_n = (-1)^n \frac{2^{2n-1} n}{(2^{2n}-1)(2n)!} \sum_{k=1}^{2n-1} \frac{(-1)^k k!}{2^k} \left\{ {2n-1 \atop k} \right\}, \quad n \geqslant 1.$$

The second equation can be easily verified considering the first equation together with (1.6). □

Remark 4. The following interesting partial fraction expression of the cotangent function belongs to Euler [7]:

$$\pi \cot(\pi x) = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x+n} + \frac{1}{x-n} \right)$$

for $x \in \mathbb{R} \setminus \mathbb{Z}$. Using this and Remark 3, we have

$$\sum_{n=1}^{\infty} \eta_n (\pi x)^{2n} = -\frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{x}{x+n} + \frac{x}{x-n} \right) = \sum_{n=1}^{\infty} \frac{x^2}{n^2 - x^2}, \quad x \notin \mathbb{Z}.$$

3 Values of the Riemann zeta function at positive integers

3.1 A recurrence formula for the Riemann zeta function at even integers

There is a vast literature on the particular values of the Riemann zeta function. Here by the help of the Riemann–Lebesgue lemma we have some recurrence formulas, including particular values of $\zeta(s)$.

Theorem 1.

$$\zeta(2m+2) = \sum_{j=0}^{m-1} (-1)^{m-j-1} \frac{2^{2j+1}-1}{2^{2m+2}-1} \frac{(2\pi)^{2m-2j}\zeta(2j+2)}{(2m-2j)!} + \frac{(-1)^m (2\pi)^{2m+2}}{(2^{2m+2}-1)8(m+1)(2m+1)!}$$

and

$$\sum_{j=0}^{m-1} (-1)^j \left(\frac{1}{2^{2j+1}} - 1 \right) \frac{\zeta(2j+2)}{(2m-2j-1)!\pi^{2j}} = -\frac{\pi^2}{2(2m+1)(2m)!}.$$

Proof. Consider the well-known formula

$$\frac{1}{2} + \sum_{n=1}^{k} \cos nx = \frac{\sin \frac{2k+1}{2} x}{2 \sin \frac{x}{2}}. \tag{3.1}$$

Hence

$$\frac{1}{2} x^{2m+1} + \sum_{n=1}^{k} x^{2m+1} \cos nx = x^{2m+1} \frac{\sin \frac{2k+1}{2} x}{2 \sin \frac{x}{2}}.$$

Integrating both sides on the interval $[0, \pi]$, we get

$$\frac{\pi^{2m+2}}{2(2m+2)} + \sum_{n=1}^{k} \int_0^\pi x^{2m+1} \cos(nx)\, \mathrm{d}x = \int_0^\pi x^{2m+1} \frac{\sin \frac{2k+1}{2} x}{2 \sin \frac{x}{2}}\, \mathrm{d}x. \tag{3.2}$$

Recalling the formula (see [9, p. 436])

$$\int_0^\pi x^{2m+1} \cos(nx)\, \mathrm{d}x = \frac{(-1)^n}{n^{2m+2}} \sum_{j=0}^{m} (-1)^j \frac{(2m+1)!}{(2m-2j)!} (n\pi)^{2m-2j} + (-1)^{m+1} \frac{1}{n^{2m+2}} (2m+1)!$$

and employing this in (3.2), we get

$$\int_0^\pi \frac{x^{2m+1}}{2 \sin \frac{x}{2}} \sin \frac{2k+1}{2} x\, \mathrm{d}x = \frac{\pi^{2(m+1)}}{4(m+1)} + \sum_{j=0}^{m} \frac{(-1)^j (2m+1)! \pi^{2m-2j}}{(2m-2j)!} \sum_{n=1}^{k} \frac{(-1)^n n^{2m-2j}}{n^{2m+2}} + (-1)^{m+1} (2m+1)! \sum_{n=1}^{k} \frac{1}{n^{2m+2}}.$$

Now if we take the limit as $k \to \infty$, then from the Riemann–Lebesgue lemma it follows that

$$\lim_{k\to\infty} \int_0^\pi \frac{x^{2m+1}}{2\sin\frac{x}{2}} \cos\frac{2k+1}{2} x \, \mathrm{d}x = 0,$$

and hence we obtain

$$\frac{\pi^{2(m+1)}}{4(m+1)} + \sum_{j=0}^{m} (-1)^j \frac{(2m+1)!}{(2m-2j)!} \pi^{2m-2j} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2j+2}} + (-1)^{m+1}(2m+1)! \sum_{n=1}^{\infty} \frac{1}{n^{2m+2}} = 0.$$

Here recalling that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2j+2}} = \left(\frac{1}{2^{2j+1}} - 1\right)\zeta(2j+2),$$

we have

$$\sum_{j=0}^{m} (-1)^j (\frac{1}{2^{2j+1}} - 1) \frac{\zeta(2j+2)}{(2m-2j)!\pi^{2j}} = \frac{(-1)^m \zeta(2m+2)}{\pi^{2m}} - \frac{\pi^2}{4(m+1)(2m+1)!},$$

which completes the proof of the first equality. For the second one, multiplying both sides of (3.1) by x^{2m}, we have

$$\frac{1}{2}x^{2m} + \sum_{n=1}^{k} x^{2m} \cos nx = x^{2m} \frac{\sin\frac{2k+1}{2}x}{2\sin\frac{x}{2}}.$$

Integrating both sides on the interval $[0, \pi]$, we get

$$\frac{\pi^{2m+1}}{2(2m+1)} + \sum_{n=1}^{k} \int_0^\pi x^{2m} \cos(nx) \, \mathrm{d}x = \int_0^\pi x^{2m} \frac{\sin\frac{2k+1}{2}x}{2\sin\frac{x}{2}} \, \mathrm{d}x. \tag{3.3}$$

Recalling the formula (see [9, p. 436])

$$\int_0^\pi x^{2m} \cos(nx) \, \mathrm{d}x = \frac{(-1)^n}{n^{2m+1}} \sum_{j=0}^{m-1} (-1)^j \frac{(2m)!}{(2m-2j-1)!} (n\pi)^{2m-2j-1}$$

and employing this in (3.3), we get

$$\int_0^\pi \frac{x^{2m}}{2\sin\frac{x}{2}} \sin\frac{2k+1}{2} x \, \mathrm{d}x = \frac{\pi^{2m+1}}{2(2m+1)} + \sum_{j=0}^{m-1} (-1)^j \frac{(2m)!}{(2m-2j-1)!} \pi^{2m-2j-1} \sum_{n=1}^{k} \frac{(-1)^n}{n^{2j+2}}.$$

Now if we take the limit as $k \to \infty$, then it follows from the Riemann–Lebesgue lemma that

$$\sum_{j=0}^{m-1} (-1)^j (\frac{1}{2^{2j+1}} - 1) \frac{\zeta(2j+2)}{(2m-2j-1)!\pi^{2j+1}} = -\frac{\pi}{2(2m+1)(2m)!}. \qquad \square$$

3.2 An integral representation of the Apéry constant $\zeta(3)$ and a recurrence formula for $\zeta(2n+1)$

In 1979, Apéry proved that $\zeta(3) = 1.2020569031\ldots$ is irrational (see [2, 20]). Even though there exist elementary formulas for $\zeta(2), \zeta(4), \zeta(6), \ldots$, there exist no elementary formulas for $\zeta(2k+1)$, $k \geqslant 1$.

Here we give an integral representation for $\zeta(3)$ by using the Riemann–Lebesgue lemma.

Theorem 2.

$$\zeta(3) = \frac{8}{7}\int_0^{\pi/2} x(2\ln 2 - x\cot x)\,\mathrm{d}x.$$

Proof. The well-known formula

$$\cos\alpha - \cos\beta = 2\sin\frac{\beta-\alpha}{2}\cos\frac{\beta+\alpha}{2}$$

with $\alpha = ((2k-1)/2)x$ and $\beta = ((2k+1)/2)x$ ($k \in \mathbb{N}$) gives

$$\cos\frac{2k-1}{2}x - \cos\frac{2k+1}{2}x = 2\sin\frac{x}{2}\sin kx.$$

By summing these equalities from $k = 1$ to $k = n$ we get famous formula

$$\sin x + \sin 2x + \cdots + \sin nx = \frac{1}{2}\cot\frac{x}{2} - \frac{1}{2\sin\frac{x}{2}}\cos\left(\frac{2n+1}{2}x\right). \tag{3.4}$$

Multiplying both sides of (3.4) by x^2 and integrating over $[0,\pi]$, we have

$$\sum_{k=1}^{n}\int_0^{\pi} x^2\sin kx\,\mathrm{d}x = \frac{1}{2}\int_0^{\pi} x^2\cot\frac{x}{2}\,\mathrm{d}x - \int_0^{\pi}\frac{x^2}{2\sin\frac{x}{2}}\cos\left(\frac{2n+1}{2}x\right)\mathrm{d}x. \tag{3.5}$$

Elementary calculation shows that

$$\int_0^{\pi} x^2\sin kx\,\mathrm{d}x = -\frac{\pi^2}{k}\cos k\pi + \frac{2}{k^3}(\cos k\pi - 1) \tag{3.6}$$

and

$$\frac{1}{2}\int_0^{\pi} x^2\cot\frac{x}{2}\,\mathrm{d}x = 4\int_0^{\pi/2} x^2\cot x\,\mathrm{d}x. \tag{3.7}$$

Since the function

$$\varphi(x) = \begin{cases} \frac{x^2}{2\sin\frac{x}{2}}, & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is integrable on $[0,\pi]$, it follows from the Riemann–Lebesgue lemma that

$$\lim_{n\to\infty}\int_0^{\pi}\frac{x^2}{2\sin\frac{x}{2}}\cos\left(\frac{2n+1}{2}x\right)\mathrm{d}x = 0.$$

Using this in (3.5), we have

$$\sum_{k=1}^{\infty}\int\limits_0^{\pi} x^2\sin kx\,\mathrm{d}x = 4\int\limits_0^{\pi/2} x^2\cot x\,\mathrm{d}x.$$

We may use (3.6) and (3.7) to conclude that

$$-\pi^2\sum_{k=1}^{\infty}\frac{1}{k}\cos k\pi + 2\sum_{k=1}^{\infty}\frac{\cos k\pi - 1}{k^3} = 4\int\limits_0^{\pi/2} x^2\cot x\,\mathrm{d}x. \tag{3.8}$$

Since

$$\sum_{k=1}^{\infty}\frac{1}{k}\cos k\pi = -\frac{1}{1}+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\cdots = -\ln 2$$

and

$$\sum_{k=1}^{\infty}\frac{\cos k\pi - 1}{k^3} = -2\left(\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\cdots\right) = -\frac{7}{4}\zeta(3),$$

from (3.8) we get that

$$\pi^2\ln 2 - \frac{7}{2}\zeta(3) = 4\int\limits_0^{\pi/2} x^2\cot x\,\mathrm{d}x.$$

As a result, we get

$$\zeta(3) = \frac{2}{7}\left[\pi^2\ln 2 - 4\int\limits_0^{\pi/2} x^2\cot x\,\mathrm{d}x\right] = \frac{8}{7}\left\{\int\limits_0^{\pi/2}\left[(2\ln 2)x - x^2\cot x\right]\mathrm{d}x\right\}$$

$$= \frac{8}{7}\int\limits_0^{\pi/2} x(2\ln 2 - x\cot x)\,\mathrm{d}x. \qquad \square$$

Remark 5. Considering the equation

$$\zeta(3) = \frac{2}{7}\left[\pi^2\ln 2 - 4\int\limits_0^{\pi/2} x^2\cot x\,\mathrm{d}x\right]$$

with the formula (see in [9, p. 436])

$$\int\limits_0^{\pi/2} x^2\cot x\,\mathrm{d}x = \left(\frac{\pi}{2}\right)^2\left(\frac{1}{2} - \sum_{k=1}^{\infty}\frac{1}{4^k(1+k)}\zeta(2k)\right),$$

we have

$$\zeta(3)=\frac{\pi^2}{7}\left[(\ln 4-1)+\sum_{k=1}^{\infty}\frac{1}{4^k(1+k)}\zeta(2k)\right].$$

which was proven, among others, independently in [5,6,8,19]; their proofs are based on the integration of the cotangent function.

Now we give a recurrence formula for $\zeta(2m+1)$.

Theorem 3. *For $m \geqslant 1$, we have*

$$\begin{aligned}(-1)^m\left(2-\frac{1}{2^{2m}}\right)&\zeta(2m+1)\\&=\frac{2^{2m}}{(2m)!}\int_0^{\pi/2}x^{2m}\cot x\,\mathrm{d}x+\pi^{2m}\sum_{i=1}^{m-1}\frac{(-1)^{i+1}}{\pi^{2i}(2m-2i)!}\left(1-\frac{1}{2^{2i}}\right)\zeta(2i+1)-\frac{\pi^{2m}}{(2m)!}\ln 2.\end{aligned}$$

Proof. Multiplying both sides of (3.4) by x^{2m} and integrating over $[0,\pi]$, we get

$$\sum_{k=1}^{n}\int_0^{\pi}x^{2m}\sin kx\,\mathrm{d}x=\frac{1}{2}\int_0^{\pi}x^{2m}\cot\frac{x}{2}\,\mathrm{d}x-\int_0^{\pi}\frac{x^{2m}}{2\sin\frac{x}{2}}\cos\left(\frac{2n+1}{2}x\right)\mathrm{d}x. \tag{3.9}$$

Using the formula (see [9, p. 436])

$$\int_0^{\pi}x^{2m}\sin kx\,\mathrm{d}x=\frac{(-1)^{k+1}(2m)!}{k}\pi^{2m}\sum_{i=0}^{m}(-1)^i\frac{1}{(2m-2i)!k^{2i}\pi^{2i}}+(-1)^m\frac{(2m)!}{k^{2m+1}},$$

we have

$$\begin{aligned}\sum_{k=1}^{n}\int_0^{\pi}x^{2m}\sin kx\,\mathrm{d}x&=(2m)!\pi^{2m}\sum_{i=0}^{m}\frac{(-1)^i}{\pi^{2i}(2m-2i)!}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k^{2i+1}}+(-1)^m(2m)!\sum_{k=1}^{n}\frac{1}{k^{2m+1}}\\&=(2m)!\pi^{2m}\left[\frac{1}{(2m)!}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k}+\sum_{i=1}^{m}\frac{(-1)^i}{\pi^{2i}(2m-2i)!}\sum_{k=1}^{n}\frac{(-1)^{k+1}}{k^{2i+1}}\right]\\&\quad+(-1)^m(2m)!\sum_{k=1}^{n}\frac{1}{k^{2m+1}}.\end{aligned}$$

Taking the limit as $n\to\infty$, we get

$$\begin{aligned}\sum_{k=1}^{\infty}\int_0^{\pi}x^{2m}\sin kx\,\mathrm{d}x&=(2m)!\pi^{2m}\left[\frac{1}{(2m)!}\ln 2+\sum_{i=1}^{m}\frac{(-1)^i\zeta(2i+1)}{\pi^{2i}(2m-2i)!}\left(1-\frac{1}{2^{2i}}\right)\right]\\&\quad+(-1)^m(2m)!\zeta(2m+1).\end{aligned} \tag{3.10}$$

On the other hand,

$$\frac{1}{2}\int_0^{\pi} x^{2m}\cot\frac{x}{2}\,\mathrm{d}x = 2^{2m}\int_0^{\pi/2} x^{2m}\cot x\,\mathrm{d}x. \tag{3.11}$$

Since by the Riemann–Lebesgue lemma

$$\lim_{n\to\infty}\int_0^{\pi}\frac{x^{2m}}{2\sin\frac{x}{2}}\cos\left(\frac{2n+1}{2}x\right)\mathrm{d}x = 0,$$

using (3.9), (3.10), and (3.11), we conclude that

$$\pi^{2m}\ln 2 + (2m)!\pi^{2m}\sum_{i=1}^{m}\frac{(-1)^i\zeta(2i+1)}{\pi^{2i}(2m-2i)!}\left(1-\frac{1}{2^{2i}}\right) + (-1)^m(2m)!\zeta(2m+1)$$
$$= 2^{2m}\int_0^{\pi/2} x^{2m}\cot x\,\mathrm{d}x,$$

and therefore

$$\pi^{2m}\ln 2 + (2m)!\pi^{2m}\sum_{i=1}^{m-1}\frac{(-1)^i\zeta(2i+1)}{\pi^{2i}(2m-2i)!}\left(1-\frac{1}{2^{2i}}\right) + (-1)^m(2m)!\zeta(2m+1)\left(2-\frac{1}{2^{2m}}\right)$$
$$= 2^{2m}\int_0^{\pi/2} x^{2m}\cot x\,\mathrm{d}x.$$

Thus we arrive at

$$(-1)^m(2m)!(2-\frac{1}{2^{2m}})\zeta(2m+1)$$
$$= 2^{2m}\int_0^{\pi/2} x^{2m}\cot x\,\mathrm{d}x + (2m)!\pi^{2m}\sum_{i=1}^{m-1}\frac{(-1)^{i+1}}{\pi^{2i}(2m-2i)!}\left(1-\frac{1}{2^{2i}}\right)\zeta(2i+1) - \pi^{2m}\ln 2. \qquad \square$$

Remark 6. By means of Theorem 3 we can evaluate $\zeta(5)$ in terms of $\zeta(3)$ as

$$\zeta(5) = \frac{2}{93}\left[16\int_0^{\pi/2} x^4\cot x\,\mathrm{d}x + 9\pi^2\zeta(3) - \pi^4\ln 2\right].$$

In fact, as a consequence of Theorem 2, we can state $\zeta(5)$ nonrecursively:

$$\zeta(5) = \int_0^{\pi/2}\left(\frac{32}{93}x^2 - \frac{48}{217}\pi^2\right)x^2\cot x\,\mathrm{d}x + \frac{22}{651}\pi^4\ln 2.$$

Although there exists an evaluation formula for $\zeta(5)$ in [6, p. 204], our formula is much simpler.

4 Evaluation of a series with zeta values in a closed form

In this section, we would like to evaluate the series

$$\sigma_p = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \zeta(p+k)$$

for any real $p \geqslant 1$ in a "closed form". At least, it is desirable to evaluate the series for $p = 1$:

$$\sigma_1 = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \zeta(k+1).$$

It is interesting that the "similar" series

$$\sum_{k=2}^{\infty} \frac{(-1)^{k-1}}{k+1} \zeta(k)$$

is known and can be evaluated as (see [17, Eq. (5.1)] or [18, Eq. (457)])

$$\sum_{k=2}^{\infty} \frac{(-1)^{k}}{k+1} \zeta(k) = 1 + \frac{\gamma}{2} - \frac{\ln(2\pi)}{2}$$

from the representation (see [1])

$$\sum_{k=2}^{\infty} (-1)^k \zeta(k) x^k = x\big(\psi(x+1) + \gamma\big)$$

by integration over $[0, 1]$. This representation also gives

$$\sigma_1 = \int_0^1 \frac{1}{x} \big(\psi(x+1) + \gamma\big)\, \mathrm{d}x.$$

Note that because of $\psi(1) = -\gamma$, this integral has no problem at $x = 0$.

For our purpose, we use the polylogarithm

$$\mathrm{Li}_m(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^m}$$

with $\mathrm{Li}_m(1) = \zeta(m)$. The following proposition shows that the series σ_p can be expressed through a series with logarithms.

Theorem 4. *For every $p > 1$, we have*

$$\sigma_p = \sum_{n=1}^{\infty} \frac{1}{n^p} \ln\left(1 + \frac{1}{n}\right)$$

and

$$\sigma_p - \zeta'(p) = \sum_{n=1}^{\infty} \frac{1}{n^p} \ln(1+n). \tag{4.1}$$

Proof. First, we have

$$\sum_{n=1}^{\infty} \frac{\ln(1+n)}{n^p} = \sum_{n=1}^{\infty} \frac{1}{n^p} \ln\left[n\left(1+\frac{1}{n}\right)\right] = \sum_{n=1}^{\infty} \frac{\ln n}{n^p} + \sum_{n=1}^{\infty} \frac{1}{n^p} \ln\left(1+\frac{1}{n}\right)$$
$$= -\zeta'(p) + \sum_{n=1}^{\infty} \frac{1}{n^p} \ln\left(1+\frac{1}{n}\right).$$

Now using the power series expansion of the logarithm function, we get that

$$\sum_{n=1}^{\infty} \frac{\ln(1+n)}{n^p} = -\zeta'(p) + \sum_{n=1}^{\infty} \frac{1}{n^p}\left\{\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k}\left(\frac{1}{n}\right)^k\right\} = -\zeta'(p) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=1}^{\infty} \frac{1}{n^{k+p}}$$
$$= -\zeta'(p) + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \zeta(p+k). \qquad \square$$

Remark 7. The first equation in the proposition is also true for $p = 1$ as

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \zeta(k+1) = \sum_{n=1}^{\infty} \frac{1}{n} \ln\left(1+\frac{1}{n}\right)$$

because of $\ln(1+1/n) \sim 1/n$ $(n \to \infty)$.

Now we give an integral representation of σ_p in terms of the gamma function.

Corollary 3.

$$\sigma_p = \zeta'(p) + p \int_1^{\infty} \frac{\ln\Gamma([x+2])}{x^{p+1}} \,\mathrm{d}x,$$

where $[x]$ *denotes the integer part of* x*.*

Proof. It is known that the function

$$F(p) = \sum_{n=1}^{\infty} \frac{\ln(n+1)}{n^p}$$

has the integral representation (see [11, p. 64])

$$F(p) = p \int_1^{\infty} \frac{S(x)}{x^{p+1}} \,\mathrm{d}x, \tag{4.2}$$

where

$$S(x) = \sum_{n \leqslant x} \ln(n+1), \quad 1 \leqslant x < \infty.$$

Since $S(x) = \ln\Gamma([x+2])$, from (4.2) we have that

$$F(p) = p\int_1^\infty \frac{\ln\Gamma([x+2])}{x^{p+1}}\,\mathrm{d}x,$$

and therefore, by (4.1),

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\zeta(p+k) = \zeta'(p) + p\int_1^\infty \frac{\ln\Gamma([x+2])}{x^{p+1}}\,\mathrm{d}x. \qquad \square$$

In the following theorem, we establish some integral and series representations between zeta values and the polylogarithm function.

Theorem 5.

$$\zeta(s+p) = \frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}\operatorname{Li}_p(\mathrm{e}^{-x})\,\mathrm{d}x$$

and

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\zeta(p+k)}{k} = \int_0^\infty \frac{1-\mathrm{e}^{-t}}{t}\operatorname{Li}_p(\mathrm{e}^{-t})\,\mathrm{d}x.$$

Proof. We start from the representation

$$\mathrm{e}^{-x} = \frac{1}{2\pi\mathrm{i}}\int_{(a)} x^{-s}\Gamma(s)\,\mathrm{d}s,$$

where the integration is on vertical line with abscissa $0 < a < 1$. Replacing x by xn, we have

$$\mathrm{e}^{-xn} = \frac{1}{2\pi\mathrm{i}}\int_{(a)} x^{-s}\frac{1}{n^s}\Gamma(s)\,\mathrm{d}s.$$

Dividing both sides by n^p and summing for $n = 1, 2, 3, \ldots$, we find

$$\sum_{n=1}^{\infty}\frac{\mathrm{e}^{-xn}}{n^p} = \frac{1}{2\pi\mathrm{i}}\int_{(a)} x^{-s}\sum_{n=1}^{\infty}\frac{1}{n^{s+p}}\Gamma(s)\,\mathrm{d}s,$$

that is,

$$\operatorname{Li}_p(\mathrm{e}^{-x}) = \frac{1}{2\pi\mathrm{i}}\int_{(a)} x^{-s}\zeta(s+p)\Gamma(s)\,\mathrm{d}s.$$

From this by suitable properties of the Mellin transform we have

$$\zeta(s+p) = \frac{1}{\Gamma(s)}\int_0^\infty x^{s-1}\operatorname{Li}_p\left(\mathrm{e}^{-x}\right)\mathrm{d}x.$$

Note that this formula is valid not only for s with $0 < \mathrm{Re}(s) < 1$, but also for all s with $\mathrm{Re}(s) > 0$. This fact is a consequence of the elementary asymptotic formula $\mathrm{Li}_p(\mathrm{e}^{-x}) \sim \mathrm{e}^{-x}$ as $x \to \infty$.

Further, putting $s = k$ and summing for k give

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\zeta(p+k)}{k} &= \int_0^{\infty} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^{k-1}}{k!} \mathrm{Li}_p\left(\mathrm{e}^{-x}\right) \mathrm{d}x \\ &= \int_0^{\infty} \frac{1-\mathrm{e}^{-x}}{x} \mathrm{Li}_p\left(\mathrm{e}^{-x}\right) \mathrm{d}x,\end{aligned}$$

which completes the proof. □

At the end of this section, we give a relation between σ_p and a series consisting of the Riemann zeta function and the skew-harmonic numbers H_n^- (see [4]) defined by

$$H_n^- = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n}.$$

Theorem 6.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}\zeta(p+k)}{k} = \ln 2 + \sum_{n=1}^{\infty} H_n^- \bigl(\zeta(n+p) - \zeta(n+p+1)\bigr).$$

Proof. The equation can be obtained from Abel's lemma, which says that if $\{a_n\}$ and $\{b_n\}$, $n \geqslant 1$, are two sequences and $A_n = a_1 + a_2 + \cdots + a_n$, then, for every $n \geqslant 1$,

$$\sum_{k=1}^{n} a_k b_k = b_n A_n + \sum_{k=1}^{n-1} A_k (b_k - b_{k+1}).$$

Now for $a_k = (-1)^{k-1}/k$ and $b_k = \zeta(p+k) - 1$, we have $A_k = H_k^-$ and thus find

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \bigl(\zeta(p+k) - 1\bigr) = H_n^- \bigl(\zeta(n+p) - 1\bigr) + \sum_{k=1}^{n-1} H_k^- \bigl(\zeta(k+p) - \zeta k + p + 1\bigr).$$

Since $H_n^- \to \ln 2$ and $\zeta(n+p) - 1 \to 0$ as $n \to \infty$, we find

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \bigl(\zeta(p+k) - 1\bigr) = \sum_{k=1}^{\infty} H_k^- \bigl(\zeta(k+p) - \zeta(k+p+1)\bigr).$$

At the same time,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \bigl(\zeta(p+k) - 1\bigr) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}\zeta(p+k)}{k} - \ln 2,$$

and the proposition follows. □

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