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On the distribution of polynomial discriminants: Totally real case

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Dedicated to Professors Antanas Laurinčikas and Eugenijus Manstavičius
on the occasion of their 70th birthdays

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Abstract. We study the distribution of the discriminant $D(P)$ of polynomials P from the class $\mathcal{P}_n(Q)$ of all integer polynomials of degree n and height at most Q. We evaluate the asymptotic number of polynomials $P \in \mathcal{P}_n(Q)$ having all real roots and satisfying the inequality $|D(P)| \leqslant X$ as $Q \to \infty$ and $X/Q^{2n-2} \to 0$.

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1 Introduction and main results

1.1 Basic definitions

The number

$$D(P) = a_n^{2n-2} \prod_{1 \leqslant i < j \leqslant n} (\alpha_i - \alpha_j)^2 \tag{1.1}$$

is called the discriminant of the polynomial $P(x) = \sum_{k=0}^{n} a_k x^k = a_n \prod_{j=1}^{n} (x - \alpha_j)$

As a function of the polynomial coefficients, the discriminant can be written as a $(2n-1) \times (2n-1)$ determinant:

$$D(P) = (-1)^{n(n-1)/2} \begin{vmatrix} 1 & a_{n-1} & \cdots & a_2 & a_1 & a_0 & & & \\ & a_n & a_{n-1} & \cdots & a_2 & a_1 & a_0 & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & a_n & a_{n-1} & \cdots & a_2 & a_1 & a_0 \\ n & \cdots & 3a_3 & 2a_2 & a_1 & & & & \\ & na_n & \cdots & 3a_3 & 2a_2 & a_1 & & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & & na_n & \cdots & 3a_3 & 2a_2 & a_1 & \\ & & & & na_n & \cdots & 3a_3 & 2a_2 & a_1 \end{vmatrix},$$

where the empty entries are filled with zeros. The discriminant of P coincides with the resultant of P and P' up to the factor $(-1)^{n(n-1)/2}a_n^{-1}$.

Let $h(P)$ denote the height of the polynomial P. Our proof applies to the cases where $h(P)$ is defined to be the naive height $H(P) = \max_{0\leqslant k\leqslant n} |a_k|$, or the Mahler measure $M(P) = a_n \prod_{j=1}^{n} \max(1, |\alpha_j|)$, or the length $L(P) = \sum_{k=0}^{n} |a_k|$ of the polynomial P. In fact, in our proof, we use some properties shared by these height functions rather than an explicit form of $h(P)$. These essential properties of the height functions are summarized in Section 3.1. So, we will not further specify which particular height function h is involved in the definitions.

We define the *signature* of P as the number of pairs of complex (nonreal) roots of P (each root is counted with its multiplicity).

Everywhere in the paper the degree $n \geqslant 4$ of polynomials is fixed; the parameter Q bounding heights of polynomials from above grows to infinity. We identify real polynomials of degree n with their vectors of coefficients in $\mathbb{R}^{n+1}$ and treat sets of real polynomials as subsets of $\mathbb{R}^{n+1}$.

For $n \in \mathbb{N}$ and $Q > 1$, denote by $\mathcal{P}_n(Q)$ the set of all integer polynomials of degree n and height $h(P) \leqslant Q$.

For $X \geqslant 0$ and $0 \leqslant s \leqslant n/2$, define the counting functions

$$N_n(Q, X) := \#\{P \in \mathcal{P}_n(Q)\colon\ |D(P)| \leqslant X\}, \tag{1.2}$$

$$N_{n,s}(Q, X) := \#\{P \in \mathcal{P}_n(Q)\colon\ \operatorname{signature}(P) = s,\ |D(P)| \leqslant X\}. \tag{1.3}$$

Evidently,

$$N_n(Q, X) = \sum_{0\leqslant s\leqslant n/2} N_{n,s}(Q, X).$$

1.2 Notation

Here we explain the asymptotic notation we use. The expression "$f(x) \sim g(x)$ as $x \to x_0$" is equivalent to $\lim_{x\to x_0} f(x)/g(x) = 1$. The statement "$f(x) \ll_n g(x)$ as $x \to x_0$" means the existence of a positive constant c_n (depending on n only) such that $|f(x)| \leqslant c_n|g(x)|$ for all x in a neighborhood of x_0. Note that $f(x) \ll g(x)$ is tantamount to $f(x) = O(g(x))$. The notation $f(x) \asymp_n g(x)$ means the two-sided asymptotic inequality $g(x) \ll_n f(x) \ll_n g(x)$.

For a finite set S, we denote its cardinality by $\#S$. For a set $S \subset \mathbb{R}^m$, the $\operatorname{mes}_d S$ denotes its d-dimensional Lebesgue measure ($d \leqslant m$).

To simplify the notation of quantities used in the paper, for some of them, we do not indicate explicitly their dependence on n, since everywhere the degree n is fixed.

1.3 Background

The discriminant of a polynomial characterizes at large the distances between the roots of the polynomial [7, 10]. So the discriminant and its properties can be of big help in various problems, especially in the theory of Diophantine approximations. See [26] as an example of such an application to the cubic case of the Mahler conjecture [23, 25]; the proof in [26] is built upon Davenport's estimate [9] of the number of integer cubic polynomials having small discriminant. This state of things raises a great interest in the distribution of the values of the polynomial discriminant.

There are a number of effective results aimed at problems related to algorithmic solution of Diophantine equations. In this approach, usually the effective bounds on the extreme values of some quantities are of interest. In the 1970s, Győry established a series of such effective estimates [14, 15, 16, 17] concerning the distribution of the discriminants of integer polynomials; several of these results are improved in [19]. A lot of further references and details on this topic can be found in the survey [19] by Győry and the book [11] by Evertse and Győry.

Another approach is concerned about the asymptotic statistics of polynomials with small discriminants. In this paper, we are mostly interested in problems from this second area. Now, we shortly tell about results directly related to the subject of the paper.

A possible way to state the problem about the distribution of polynomial discriminants is to ask to find lower and upper bounds, as close as possible, of the form

$$Q^{f_*(v)} \ll_n N_n\big(Q, Q^{2n-2-2v}\big) \ll_n Q^{f^*(v)}, \tag{1.4}$$

where $f_*(v)$ and $f^*(v)$ are decreasing functions of v such that $0 \leqslant v \leqslant n-1$. The reason why the bound $Q^{2n-2-2v}$ on the discriminant is convenient is the following: if a polynomial $P \in \mathcal{P}_n(Q)$ has the leading coefficient $|a_n| \gg Q$ and the discriminant $|D(P)| < Q^{2n-2-2v}$, then according to (1.1) its roots α_j satisfy $\prod_{1\leqslant j<k\leqslant n} |\alpha_j - \alpha_k| \ll Q^{-v}$.

In 2008, Bernik, Götze, and Kukso [5] proved that, for $0 < v < 1/2$,

$$N_n\big(Q, Q^{2n-2-2v}\big) \gg_n Q^{n+1-2v}.$$

For degrees $n = 2$ and $n = 3$, the reader can find upper bounds in [12, 20, 21] and [3]. According to [12], the function $N_2(Q, X)$ can be estimated as

$$N_2(Q, X) = \kappa_2\, QX + O\big(X^{3/2} \ln Q + (Q \ln Q)^{3/2}\big).$$

In [20], the asymptotics of $N_3(Q, X)$ was obtained:

$$N_3(Q, X) = \kappa_3\, Q^{2/3} X^{5/6} + O\big(X \ln Q + Q^3\big).$$

Here $\kappa_2 = 4(\ln 2 + 1) = 6.77\ldots$ and $\kappa_3 = 26.95\ldots$ are explicit constants (for the exact value of κ_3, see [20, (1.6)]); the implicit big-O-constants are absolute. The asymptotic formulae for $N_2(Q, X)$ and $N_3(Q, X)$ imply that

$$N_2\big(Q, Q^{2-2v}\big) \sim \kappa_2\, Q^{3-2v} \qquad \text{for } 0 < v < \frac{3}{4},$$

$$N_3\big(Q, Q^{4-2v}\big) \sim \kappa_3\, Q^{4-(5/3)v} \quad \text{for } 0 < v < \frac{3}{5}.$$

Also, there are papers concerning the problem of establishing estimates like (1.4) in similar settings with p-adic norm (instead of the usual absolute value) used to bound discriminant values in (1.2) and (1.3) (see [4, 6, 18]). See also the survey [2].

For general n, the best up-to-date result regarding bounds on $N_n(Q, Q^{2n-2-2v})$ is that by Beresnevich, Bernik, Götze [1], who proved that

$$N_n\big(Q, Q^{2n-2-2v}\big) \gg_n Q^{n+1-((n+2)/n)v} \quad \text{for } 0 < v < n-1. \tag{1.5}$$

In addition, using probabilistic methods, Götze and Zaporozhets [13] proved the existence of a continuous function ϕ_n such that

$$\sup_{-\infty\leqslant a<b\leqslant\infty} \left| \mathbf{P}\left\{ a \leqslant \frac{D(P)}{Q^{2n-2}} \leqslant b \right\} - \int_a^b \phi_n(x)\, \mathrm{d}x \right| \ll_n \frac{1}{\log Q}, \tag{1.6}$$

where $\mathbf{P}\{A\}$ denotes the probability of an event A, and the polynomial P is picked at random uniformly from $\mathcal{P}_n(Q)$.

Here we prove that the difference in the left-hand side of (1.6) can be estimated by C_n/Q instead of $C_n/\log Q$. Moreover, we get such results for every particular signature s.

Our main result is the exact asymptotics of $\#\mathcal{P}_n^{(0)}(Q,v)$ as $Q\to\infty$. In the present paper, we extend the ideas from [22], where the lower bound (1.5) was proved for $0<v<n/(n+2)$. Another essential ingredient of our proof is the Selberg integral [24]. We are going to prove that

$$N_{n,0}\bigl(Q,Q^{2n-2-2v}\bigr)\asymp_n Q^{n+1-((n+2)/n)v}\quad\text{for } 0<v<\frac{n}{n+2}.$$

1.4 Main results

According to the following theorem, for every s and sufficiently large Q, the overall shape of the function $N_{n,s}(Q,X)$ can be asymptotically described by a continuous function $f_{n,s}$.

Theorem 1. *Let $n\geqslant 3$ be an integer. For every $s\geqslant 0$, there exists a positive continuous function $f_{n,s}:\mathbb{R}\to\mathbb{R}$ such that, for any real X,*

$$\left|N_{n,s}(Q,X)-Q^{n+1}f_{n,s}\left(\frac{X}{Q^{2n-2}}\right)\right|\ll_n Q^n,$$

where the implicit constant depend on n only.

The function $f_{n,s}(\delta)$ is increasing as δ grows, and $\lim_{\delta\to 0}f_{n,s}(\delta)=0$.

In Theorem 1, the parameter X may be a function of Q; the statement still holds.

Obviously, if a polynomial P is picked at random uniformly from $\mathcal{P}_n(Q)$, then for $\delta\geqslant 0$,

$$\mathbf{P}\left\{\frac{|D(P)|}{Q^{2n-2}}\leqslant\delta\right\}=\frac{1}{\#\mathcal{P}_n(Q)}\sum_{0\leqslant s\leqslant n/2}N_{n,s}\bigl(Q,\delta Q^{2n-2}\bigr).$$

Moreover, since $\#\mathcal{P}_n(Q)=2Q(2Q+1)^n$, for the function ϕ_n in (1.6), we have

$$\int_{-\delta}^{\delta}\phi_n(x)\,\mathrm{d}x=2^{-n-1}\sum_{0\leqslant s\leqslant n/2}f_{n,s}(\delta).$$

Thus Theorem 1 shows that the difference in (1.6) can be estimated as C_nQ^{-1}, where the constant C_n depends on n only.

In the case $s=0$, that is, when all the roots are real, we prove the following theorem.

Theorem 2. *Let $n\geqslant 2$ be an integer. In the totally real case (that is, for $s=0$), we have*

$$f_{n,0}(\delta)\sim\lambda_{n,0}\,\delta^{(n+2)/(2n)}\quad\textit{as } \delta\to 0.$$

The constant $\lambda_{n,0}$ can be computed explicitly and depends only on n and the height function h.

Remark. The corresponding results (with naive height as $h(P)$) for $n=2$ and $n=3$ are obtained in [12] and [20].

Regarding the problem of obtaining estimates of the form (1.4), we get the following corollary.

Corollary 1. *Let $\epsilon\in(0,n/(2n+2))$ be a fixed small number. For all $v\in[\epsilon,(1-\epsilon)n/(n+2)]$, we have*

$$N_{n,0}\bigl(Q,Q^{2n-2-2v}\bigr)\sim\lambda_{n,0}\,Q^{n+1-((n+2)/n)v}\quad\textit{as } Q\to\infty,$$

where $\lambda_{n,0}$ is the same as in Theorem 2.

Proof. According to Theorem 1,

$$N_{n,0}\bigl(Q, Q^{2n-2-2v}\bigr) = Q^{n+1} f_{n,0}\bigl(Q^{-2v}\bigr) + O\bigl(Q^n\bigr),$$

where the big-O-constant depends only on n. Dividing both sides of the latter inequality by $Q^{n+1-((n+2)/n)v} = Q^{n+1}(Q^{-2v})^{(n+2)/(2n)}$, we obtain:

$$\frac{N_{n,0}(Q, Q^{2n-2-2v})}{Q^{n+1-((n+2)/n)v}} = \frac{f_{n,0}(Q^{-2v})}{(Q^{-2v})^{(n+2)/(2n)}} + O\bigl(Q^{((n+2)/n)v-1}\bigr).$$

The big-O-constant here remains the same.

Obviously, if v is bounded from below by arbitrary fixed $\epsilon_1 > 0$, then from Theorem 2 we have

$$\lim_{Q\to\infty} \frac{f_{n,0}(Q^{-2v})}{(Q^{-2v})^{(n+2)/(2n)}} = \lambda_{n,0},$$

where the rate of convergence depends on ϵ_1.

On the other hand, we need the remainder term $O(Q^{((n+2)/n)v-1})$ to vanish. This happens if for arbitrary but fixed $\epsilon_2 > 0$, $v \leqslant (1-\epsilon_2)n/(n+2)$ when we have $O(Q^{((n+2)/n)v-1}) = O(Q^{-\epsilon_2})$. Taking $\epsilon_1 = \epsilon_2 = \epsilon$ for simplicity, we get the statement of the corollary. In addition, we require $\epsilon < n/(2n+2)$ so that the range of v would be nonempty. Note that the rate of convergence depends on ϵ. □

Outline of the paper

Section 2 includes auxiliary propositions necessary to prove the main results; the reader can skip this section in the first reading. In Section 3, we express the distribution function of the discriminant via the volumes of specific regions. In Section 4, we study the asymptotic behavior of the corresponding volume for small values of the discriminant of polynomials having all the roots real.

2 Auxiliary statements

Lemma 1. (*See Davenport* [8].) *Let $\mathcal{D} \subset \mathbb{R}^d$ be a bounded region formed by points $(x_1, \dots, x_d)$ satisfying a finite collection of algebraic inequalities*

$$F_i(x_1, \dots, x_d) \geqslant 0, \quad 1 \leqslant i \leqslant k,$$

where F_i is a polynomial of degree $\deg F_i \leqslant m$ with real coefficients. Let

$$\Lambda(\mathcal{D}) = \mathcal{D} \cap \mathbb{Z}^d.$$

Then

$$\bigl|\#\Lambda(\mathcal{D}) - \operatorname{mes}_d \mathcal{D}\bigr| \leqslant C \max(\bar{V}, 1),$$

where the constant C depends only on d, k, m; the quantity $\bar{V}$ is the maximum of all r-dimensional measures of projections of $\mathcal{D}$ onto all the coordinate subspaces obtained by making $d-r$ coordinates of points in $\mathcal{D}$ equal to zero, r taking all values from 1 to $d-1$, that is,

$$\bar{V}(\mathcal{D}) := \max_{1\leqslant r<d}\bigl\{\bar{V}_r(\mathcal{D})\bigr\}, \quad \bar{V}_r(\mathcal{D}) := \max_{\substack{\mathcal{J}\subset\{1,\dots,d\}\\ \#\mathcal{J}=r}}\bigl\{\operatorname{mes}_r \operatorname{Proj}_{\mathcal{J}} \mathcal{D}\bigr\},$$

where $\operatorname{Proj}_{\mathcal{J}} \mathcal{D}$ is the orthogonal projection of $\mathcal{D}$ onto the coordinate subspace formed by coordinates with indices in $\mathcal{J}$.

Lemma 2. *Let $n \geqslant 2$ be an integer. Consider the change between the real variables*

$$(a_0, a_1, \ldots, a_n) \quad \textit{and} \quad (b, z_1, \ldots, z_n)$$

given by the relation

$$\sum_{k=1}^{n} a_k x^k = b \prod_{j=1}^{n} (x - z_j). \tag{2.1}$$

Then the Jacobian of this change equals

$$\left| \frac{\partial(a_0, a_1, \ldots, a_n)}{\partial(b; z_1, \ldots, z_n)} \right| = b^n \prod_{1 \leqslant i < j \leqslant n} |z_i - z_j|. \tag{2.2}$$

Proof. In this proof, the notation for the elementary symmetric polynomials differs from (4.3) used in Section 4. For a finite set $\mathcal{A} \subset \mathbb{N}$, let us denote

$$\sigma_k[\mathcal{A}] := \sum_{1 \leqslant j_1 < \cdots < j_k} z_{j_1} \ldots z_{j_k}, \quad \text{where } z_j := 0 \text{ if } j \notin \mathcal{A}.$$

Take by definition $\sigma_0[\mathcal{A}] := 1$ for any $\mathcal{A} \subset \mathbb{N}$ (including $\mathcal{A} = \emptyset$).

For any $j \in \mathbb{N}$ and $k \geqslant 1$, we have

$$\sigma_k[\mathcal{A}] = z_j \sigma_{k-1}\big[\mathcal{A} \setminus \{j\}\big] + \sigma_k\big[\mathcal{A} \setminus \{j\}\big]. \tag{2.3}$$

Equality (2.1) is equivalent to

$$a_k = (-1)^{n-k} b \sigma_{n-k}[\mathcal{A}] \quad \text{with } \mathcal{A} = \{1, 2, \ldots, n\}.$$

With the help of (2.3) we can express the determinant J_0 of the Jacobi matrix as follows:

$$J_0 := \begin{vmatrix} \frac{\partial a_n}{\partial b} & \frac{\partial a_n}{\partial z_1} & \ldots & \frac{\partial a_n}{\partial z_n} \\ \vdots & \vdots & \ldots & \vdots \\ \frac{\partial a_0}{\partial b} & \frac{\partial a_0}{\partial z_1} & \ldots & \frac{\partial a_0}{\partial z_n} \end{vmatrix} = (-1)^{n(n+1)/2}\, b^n J[\mathcal{A}],$$

where

$$J[\mathcal{A}] = \begin{vmatrix} \sigma_0[\mathcal{A} \setminus \{1\}] & \sigma_0[\mathcal{A} \setminus \{2\}] & \ldots & \sigma_0[\mathcal{A} \setminus \{n\}] \\ \sigma_1[\mathcal{A} \setminus \{1\}] & \sigma_1[\mathcal{A} \setminus \{2\}] & \ldots & \sigma_1[\mathcal{A} \setminus \{n\}] \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{n-1}[\mathcal{A} \setminus \{1\}] & \sigma_{n-1}[\mathcal{A} \setminus \{2\}] & \ldots & \sigma_{n-1}[\mathcal{A} \setminus \{n\}] \end{vmatrix}.$$

Subtracting the first column from the others and using (2.3), we can easily check that

$$J[\mathcal{A}] = J[\mathcal{A}_{-1}] \prod_{j=2}^{n} (z_1 - z_j), \quad \text{where } \mathcal{A}_{-1} := \mathcal{A} \setminus \{1\}.$$

Repeating this reduction procedure, we finally obtain the last nontrivial determinant for $\mathcal{A}_{-(n-2)} := \{n-1, n\}$:

$$J[\mathcal{A}_{-(n-2)}] = \begin{vmatrix} \sigma_0[\mathcal{A}_{-(n-2)} \setminus \{n-1\}] & \sigma_0[\mathcal{A}_{-(n-2)} \setminus \{n\}] \\ \sigma_1[\mathcal{A}_{-(n-2)} \setminus \{n-1\}] & \sigma_1[\mathcal{A}_{-(n-2)} \setminus \{n\}] \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ z_n & z_{n-1} \end{vmatrix}.$$

Hence we have

$$J_0 = (-1)^{n(n+1)/2}\, b^n \prod_{1\leqslant j<k\leqslant n} (z_j - z_k).$$

Taking the absolute value in the latter equation finishes the proof. □

3 Counting integer polynomials

In this section, we reduce counting integer polynomials to evaluating volumes of some regions.

To simplify the notation in this and the next sections, we do not indicate explicitly the dependence of some quantities on n, since everywhere the degree n is fixed.

3.1 Height functions

There are a number of height functions defined on real polynomials, for example, the naive height, Mahler measure, length, and so on. Our result for each of them can be obtained in the same way. So, we summarize their essential properties in a general notion of a height function and prove our theorem in a general form.

A continuous function $h : \mathbb{R}^{n+1} \to [0, +\infty)$ is called a *height function* if for all $\mathbf{v} = (v_n, \ldots, v_1, v_0) \in \mathbb{R}^{n+1}$, it satisfies the following properties:

(i) $h(t\mathbf{v}) = |t| h(\mathbf{v})$ for all real t;
(ii) $h(\mathbf{v}) = 0$ if and only if $\mathbf{v} = \mathbf{0}$;
(iii) $h(v_0, v_1, \ldots, v_n) = h(v_n, \ldots, v_1, v_0)$;
(iv) $h(v_0, -v_1, \ldots, (-1)^k v_k, \ldots, (-1)^n v_n) = h(v_0, v_1, \ldots, v_k, \ldots, v_n)$.

Note that, in terms of polynomials, property (iii) is equivalent to $h(x^n P(x^{-1})) = h(P(x))$. Property (iv) is equivalent to $h(P(-x)) = h(P(x))$.

We use the same notation for the height of a vector and of a polynomial.

Lemma 3. *For any $\mathbf{v} \in \mathbb{R}^{n+1} \setminus (-1,1)^{n+1}$, we have $h(\mathbf{v}) \geqslant h_0$, where h_0 is a positive constant depending only on the height function h and parameter n.*

Proof. The set $S_0 := \{\mathbf{v} \in \mathbb{R}^{n+1}\colon \|\mathbf{v}\|_\infty = 1\}$ is compact. From the extreme value theorem we have that there exists a value h_0 such that $h(\mathbf{v}) \geqslant h_0$ for all $\mathbf{v} \in S_0$.

Now, noticing that $h(\mathbf{v}) = \|\mathbf{v}\|_\infty h(\mathbf{v}_0)$, where $\mathbf{v}_0 := \|\mathbf{v}\|_\infty^{-1}\mathbf{v} \in S_0$ and $\|\mathbf{v}\|_\infty \geqslant 1$ for all $\mathbf{v} \in \mathbb{R}^{n+1} \setminus (-1,1)^{n+1}$, we have the lemma. □

3.2 Counting integer points via volume

We represent real polynomials of degree n by their vectors of coefficients in $\mathbb{R}^{n+1}$. Thus $N_{n,s}(Q,X)$ equals the number of integral points in the set

$$\mathcal{D}_s := \left\{ \mathbf{a} \in \mathbb{R}^{n+1}\colon \ \text{signature}\left(\sum_{i=0}^{n} a_j x^j \right) = s,\ a_n \neq 0,\ h(\mathbf{a}) \leqslant Q,\ \big|D(\mathbf{a})\big| \leqslant X \right\},$$

where $D(\mathbf{a})$ is the discriminant as a function of the coefficients $\mathbf{a}$ of the polynomial $\sum_{i=0}^{n} a_j x^j$. Note that $D(a_0, \ldots, a_n)$ is a homogeneous polynomial of degree $2n-2$ in variables $a_0, \ldots, a_n$.

From Lemma 1 we have

$$\left| \#\big(\mathcal{D}_s \cap \mathbb{Z}^{n+1}\big) - Q^{n+1} \operatorname{mes}_{n+1} \widetilde{\mathcal{D}}_s\left(\frac{X}{Q^{2n-2}} \right) \right| \leqslant c\, Q^n,$$

where c is a constant depending on n only, and

$$\widetilde{\mathcal{D}}_s(\delta) := \left\{\mathbf{a} \in \mathbb{R}^{n+1}\colon\ \text{signature}\left(\sum_{i=0}^{n} a_j x^j\right) = s,\ a_n \neq 0,\ h(\mathbf{a}) \leqslant 1,\ \left|D(\mathbf{a})\right| \leqslant \delta\right\}.$$

Obviously, the function $f_{n,s}(\delta)$ in Theorem 1 must equal $\text{mes}_{n+1}\,\widetilde{\mathcal{D}}_s(\delta)$:

$$f_{n,s}(\delta) := \text{mes}_{n+1}\,\widetilde{\mathcal{D}}_s(\delta).$$

Lemma 4. *For each* $0 \leqslant s \leqslant n/2$*, the function* $f_{n,s}(\delta)$ *is continuous and increasing* (*as* δ *grows*)*. In particular,*

$$\lim_{\delta\to 0} f_{n,s}(\delta) = 0.$$

Proof. The monotonicity of $f_{n,s}$ is obvious. Note that $f_{n,s}(\delta) = 0$ for negative δ because $\text{mes}_{n+1}\,\emptyset = 0$. Evidently, $f_{n,s}(\delta + 0) = \lim_{\epsilon\to+0} f_{n,s}(\delta+\epsilon) = f_{n,s}(\delta)$. Now note that, for all δ,

$$f_{n,s}(\delta) - f_{n,s}(\delta - 0) \leqslant \text{mes}_{n+1}\left\{\mathbf{a} \in \mathbb{R}^{n+1}\colon\ h(\mathbf{a}) \leqslant 1,\ D(\mathbf{a}) = \delta\right\} = 0$$

because $D(\mathbf{a})$ is a nonconstant polynomial in the variables $\mathbf{a}$. Thus $f_{n,s}(\delta)$ is a continuous function, and $f(0) = \lim_{\delta\to 0} f(\delta) = 0$. □

4 Proof of Theorem 2

In this section, we find the asymptotics of $f_{n,0}(\delta) = \text{mes}_{n+1}\,\widetilde{\mathcal{D}}_0(\delta)$ as $\delta \to +0$.

By definition

$$f_{n,0}(\delta) = \int\limits_{\widetilde{\mathcal{D}}_0(\delta)} \mathrm{d}a_0\,\mathrm{d}a_1 \cdots \mathrm{d}a_n.$$

Changing the variables $(a_0,\dots,a_n)$ in this integral to $(b;\alpha_1,\alpha_2,\dots,\alpha_n)$, where $b = a_n$, and α_i are the roots of $\sum_{j=0}^{n} a_j x^j$ (the corresponding Jacobian equals $b^n\sqrt{\Delta(\boldsymbol{\alpha})}$; see Lemma 2), we obtain

$$f_{n,0}(\delta) = \frac{1}{n!}\int\limits_{B_\delta} b^n\sqrt{\Delta(\boldsymbol{\alpha})}\,\mathrm{d}b\,\mathrm{d}\boldsymbol{\alpha}, \quad \Delta(\boldsymbol{\alpha}) = \prod_{1\leqslant i<j\leqslant n} (\alpha_i - \alpha_j)^2,$$

where $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_n) \in \mathbb{R}^n$, and the region $B_\delta \subset \mathbb{R}^{n+1}$ is defined by

$$B_\delta := \left\{(b;\boldsymbol{\alpha}) \in \mathbb{R}^{n+1}\colon\ |b|h(p_{\boldsymbol{\alpha}}) \leqslant 1,\ b^{2n-2}\Delta(\boldsymbol{\alpha}) \leqslant \delta\right\}, \quad p_{\boldsymbol{\alpha}}(x) = \prod_{i=1}^{n}(x - \alpha_i). \tag{4.1}$$

The factor $(n!)^{-1}$ arises because of the symmetry of the roots $\alpha_1,\dots,\alpha_n$.

Denote

$$K(\boldsymbol{\alpha}) := \frac{1}{h(p_{\boldsymbol{\alpha}})} = \frac{1}{h(\sigma_0(\boldsymbol{\alpha}), -\sigma_1(\boldsymbol{\alpha}),\dots,(-1)^n\sigma_n(\boldsymbol{\alpha}))}, \tag{4.2}$$

where $\sigma_k(\boldsymbol{\alpha})$ are the elementary symmetric polynomials of the variables $\alpha_1,\dots,\alpha_n$:

$$\sigma_k(\boldsymbol{\alpha}) := \begin{cases} 1, & k = 0, \\ \sum_{1\leqslant j_1<j_2<\cdots<j_k\leqslant n} \alpha_{j_1}\alpha_{j_2}\cdots\alpha_{j_k}, & 1 \leqslant k \leqslant n. \end{cases} \tag{4.3}$$

The two restrictions on b in (4.1) can be joined in the single inequality $|b| \leqslant \Psi(\boldsymbol{\alpha})$, where

$$\Psi(\boldsymbol{\alpha}) = \min\biggl\{ \biggl(\frac{\delta}{\Delta(\boldsymbol{\alpha})}\biggr)^{1/(2n-2)}, K(\boldsymbol{\alpha}) \biggr\}.$$

Integrating over b from $-\Psi(\boldsymbol{\alpha})$ to $\Psi(\boldsymbol{\alpha})$, we get

$$f_{n,0}(\delta) = \frac{2}{(n+1)!} \int\limits_{\mathbb{R}^n} \sqrt{\Delta(\boldsymbol{\alpha})}\, \Psi(\boldsymbol{\alpha})^{n+1} \,\mathrm{d}\boldsymbol{\alpha}.$$

Note that both these integrals converge since $\operatorname{mes}_{n+1} \widetilde{\mathcal{D}}_0(\delta) \leqslant \operatorname{mes}_{n+1}\{\mathbf{a} \in \mathbb{R}^{n+1}\colon\ h(\mathbf{a}) \leqslant 1\}$.

To ensure the correctness of the transformations below, we restrict the range of one of the roots α_i.

Lemma 5.

$$f_{n,0}(\delta) = \frac{4}{(n+1)!} \int\limits_{\mathbb{R}^{n-1}\times[-1,1]} \sqrt{\Delta(\boldsymbol{\alpha})}\, \Psi(\boldsymbol{\alpha})^{n+1} \,\mathrm{d}\boldsymbol{\alpha}.$$

Proof. Let us make the change of variables $\alpha_i = \beta_i^{-1}$ in the integral. For this change, the Jacobian equals

$$\left| \det\biggl(\frac{\partial \alpha_i}{\partial \beta_j}\biggr)_{i,j=1}^{n} \right| = \biggl(\prod_{k=1}^{n} \beta_k\biggr)^{-2}.$$

We have

$$\Delta(\boldsymbol{\alpha}) = \prod_{1\leqslant i<j\leqslant n} \biggl(\frac{1}{\beta_i} - \frac{1}{\beta_j}\biggr)^2 = \biggl(\prod_{k=1}^{n} \beta_k\biggr)^{-(2n-2)} \Delta(\boldsymbol{\beta}),$$

$$\sigma_k(\boldsymbol{\alpha}) = \biggl(\prod_{k=1}^{n} \beta_k\biggr)^{-1} \sigma_{n-k}(\boldsymbol{\beta}).$$

So, from the properties of the height function we have $K(\boldsymbol{\alpha}) = K(\boldsymbol{\beta})|\prod_{k=1}^{n}\beta_k|$, and thus

$$\Psi(\boldsymbol{\alpha}) = \Psi(\boldsymbol{\beta}) \left| \prod_{k=1}^{n} \beta_k \right|.$$

Now we have

$$\int\limits_{\mathbb{R}^{n-1}\times[-1,1]} \sqrt{\Delta(\boldsymbol{\alpha})}\, \Psi(\boldsymbol{\alpha})^{n+1} \,\mathrm{d}\alpha_1 \cdots \mathrm{d}\alpha_n = \int\limits_{\mathbb{R}^{n-1}\times(\mathbb{R}\setminus[-1,1])} \sqrt{\Delta(\boldsymbol{\beta})}\, \Psi(\boldsymbol{\beta})^{n+1} \,\mathrm{d}\beta_1 \cdots \mathrm{d}\beta_n.$$

The lemma is proved. □

Now, let us make one more change of variables with parameter ρ:

$$\alpha_j = \tau + \rho\,\theta_j, \quad 1 \leqslant j \leqslant n-1, \qquad \alpha_n = \tau.$$

The Jacobian of this change equals ρ^{n-1}. The parameter ρ will be specified later. To shorten the notation, denote

$$\boldsymbol{\theta} = (\theta_1, \dots, \theta_{n-1}, 0) \in \mathbb{R}^n, \qquad \boldsymbol{\tau} = (\tau, \dots, \tau, \tau) \in \mathbb{R}^n.$$

Then the functions under the integral take the form

$$\Delta(\boldsymbol{\tau} + \rho\,\boldsymbol{\theta}) = \rho^{n(n-1)} \widetilde{\Delta}(\boldsymbol{\theta}), \qquad \widetilde{\Delta}(\boldsymbol{\theta}) := \prod_{k=1}^{n-1} \theta_k^2 \prod_{1\leqslant i<j\leqslant n-1} (\theta_i - \theta_j)^2,$$

$$\Psi(\boldsymbol{\tau} + \rho\,\boldsymbol{\theta}) = \min\left\{ \left(\frac{\delta}{\rho^{n(n-1)}} \right)^{1/(2n-2)} \widetilde{\Delta}(\boldsymbol{\theta})^{-1/(2n-2)},\ K(\boldsymbol{\tau} + \rho\,\boldsymbol{\theta}) \right\}.$$

The latter equality suggests the choice of ρ according to

$$\rho^{n(n-1)} = \delta.$$

To avoid complicated exponents in expressions, we keep using ρ, but from now it is a function of δ. Thus, after transformations, we obtain

$$f_{n,0}(\delta) = \frac{4}{(n+1)!}\, \rho^{(n-1)(n+2)/2} \int_{-1}^{1} \widetilde{J}(\rho, \tau)\, \mathrm{d}\tau, \tag{4.4}$$

where

$$\widetilde{J}(\rho, \tau) := \int_{\mathbb{R}^{n-1}} \sqrt{\widetilde{\Delta}(\boldsymbol{\theta})}\, \widetilde{\Psi}(\rho; \tau, \boldsymbol{\theta})^{n+1}\, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{n-1} \tag{4.5}$$

with

$$\widetilde{\Psi}(\rho; \tau, \boldsymbol{\theta}) := \min\bigl\{ \widetilde{\Delta}(\boldsymbol{\theta})^{-1/(2n-2)},\ K(\boldsymbol{\tau} + \rho\,\boldsymbol{\theta}) \bigr\}.$$

Now the problem is to find out whether $\lim_{\rho\to 0} \widetilde{J}(\rho, \tau)$ exists, and if it does, then which value it has. As we will see, this limit is a close relative to the function

$$G(\chi) := \int_{\mathbb{R}^{n-1}} \sqrt{\widetilde{\Delta}(\boldsymbol{\theta})}\, \min\bigl\{ \widetilde{\Delta}(\boldsymbol{\theta})^{-1/(2n-2)}, \chi \bigr\}^{n+1}\, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{n-1}. \tag{4.6}$$

Lemma 6. *The integral $G(\chi)$ converges and equals*

$$G(\chi) = \frac{2n(n+1)}{(n+2)}\, \chi^{2/n} \int_{[-1,1]^{n-2}} \left(\prod_{i=1}^{n-2} |\xi_i|\, (1 - \xi_i) \prod_{1\leqslant i<j\leqslant n-2} |\xi_i - \xi_j| \right)^{-2/n} \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{n-2}.$$

Proof. First, note that $\widetilde{\Delta}(\boldsymbol{\theta})$ is a symmetric function of $\theta_1, \dots, \theta_{n-1}$ and that $\widetilde{\Delta}(-\boldsymbol{\theta}) = \widetilde{\Delta}(\boldsymbol{\theta})$. Hence we can write

$$G(\chi) = 2(n-1) \int_{|\theta_j| \leqslant \theta_{n-1}} \sqrt{\widetilde{\Delta}(\boldsymbol{\theta})}\, \min\bigl\{ \chi, \widetilde{\Delta}(\boldsymbol{\theta})^{-1/(2n-2)} \bigr\}^{n+1}\, \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_{n-1}.$$

Next, we use the fact that $\widetilde{\Delta}(\boldsymbol{\theta})$ is a homogeneous polynomial of total degree $\deg \widetilde{\Delta}(\boldsymbol{\theta}) = n(n-1)$. In the latter integral, we change variables

$$\theta_{n-1} = r, \qquad \theta_j = r\xi_j, \quad 1 \leqslant j \leqslant n-2,$$

and obtain

$$\frac{G(\chi)}{2(n-1)} = \int\limits_{[-1,1]^{n-2}} \left[\int\limits_0^\infty r^{n(n-1)/2+n-2} \sqrt{\widetilde{\Delta}(\boldsymbol{\xi},1)} \min\bigl\{\chi, r^{-n/2}\widetilde{\Delta}(\boldsymbol{\xi},1)^{-1/(2n-2)}\bigr\}^{n+1}\,\mathrm{d}r \right] \mathrm{d}\boldsymbol{\xi}, \tag{4.7}$$

where $\boldsymbol{\xi} = (\xi_1, \dots, \xi_{n-2}) \in \mathbb{R}^{n-2}$ and $\mathrm{d}\boldsymbol{\xi} = \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{n-2}$.

We have

$$\min\bigl\{\chi, r^{-n/2}\widetilde{\Delta}(\boldsymbol{\xi},1)^{-1/(2n-2)}\bigr\} = \begin{cases} \chi, & r \leqslant R, \\ r^{-n/2}\widetilde{\Delta}(\boldsymbol{\xi},1)^{-1/(2n-2)}, & r > R, \end{cases}$$

where

$$R = R(\chi, \boldsymbol{\xi}) := \chi^{-2/n}\,\widetilde{\Delta}(\boldsymbol{\xi},1)^{-1/(n(n-1))}.$$

Hence we split the inner integral (over the variable r) into two integrals:

$$\frac{G(\chi)}{2(n-1)} = \int\limits_{[-1,1]^{n-2}} \left[\int\limits_0^R + \int\limits_R^\infty \right] \mathrm{d}\boldsymbol{\xi}.$$

For the first inner integral, we get

$$\begin{aligned} \int\limits_0^R &= \sqrt{\widetilde{\Delta}(\boldsymbol{\xi},1)}\chi^{n+1} \int\limits_0^R r^{(n^2+n)/2-2}\,\mathrm{d}r = \frac{2}{n^2+n-2} R^{(n+2)(n-1)/2} \sqrt{\widetilde{\Delta}(\boldsymbol{\xi},1)}\chi^{n+1} \\ &= \frac{2}{n^2+n-2}\chi^{2/n}\widetilde{\Delta}(\boldsymbol{\xi},1)^{-1/n}. \end{aligned}$$

The second one amounts to

$$\int\limits_R^\infty = \widetilde{\Delta}(\boldsymbol{\xi},1)^{1/2-(n+1)/(2(n-1))} \int\limits_R^\infty r^{-2}\,\mathrm{d}r = \widetilde{\Delta}(\boldsymbol{\xi},1)^{-1/(n-1)} R^{-1} = \chi^{2/n}\widetilde{\Delta}(\boldsymbol{\xi},1)^{-1/n}.$$

Thus we obtain

$$G(\chi) = \frac{2n(n+1)}{n+2}\chi^{2/n} \int\limits_{[-1,1]^{n-2}} \widetilde{\Delta}(\boldsymbol{\xi},1)^{-1/n}\,\mathrm{d}\boldsymbol{\xi}. \tag{4.8}$$

All we need now is to prove that the integral in (4.8) converges. In its expanded form, this integral (denote it by $\widetilde{G}$) can be written as

$$\widetilde{G} = \int\limits_{[-1,1]^{n-2}} \prod_{i=1}^{n-2} |\xi_i|^{-2/n}(1-\xi_i)^{-2/n} \prod_{1 \leqslant i < j \leqslant n-2} |\xi_i - \xi_j|^{-2/n}\,\mathrm{d}\xi_1 \cdots \mathrm{d}\xi_{n-2}. \tag{4.9}$$

Here the Selberg integral [24] will help us. This special integral is defined as

$$S_m(\alpha,\beta,\gamma) := \int_0^1 \cdots \int_0^1 \prod_{i=1}^{m} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\leqslant i<j\leqslant m} |t_i-t_j|^{2\gamma}\,\mathrm{d}t_1\cdots\mathrm{d}t_m.$$

For any integer $m \geqslant 1$, the Selberg integral converges for

$$\mathrm{Re}(\alpha)>0,\quad \mathrm{Re}(\beta)>0,\quad \mathrm{Re}(\gamma) > -\min\left\{\frac{1}{m},\,\frac{\mathrm{Re}(\alpha)}{m-1},\,\frac{\mathrm{Re}(\beta)}{m-1}\right\}. \tag{4.10}$$

Taking a look at the integral (4.9), we see that

$$\widetilde{G} \leqslant 2^{n-2} S_{n-2}\left(\frac{n-2}{n},\,\frac{n-2}{n},\,-\frac{1}{n}\right).$$

Obviously, the convergence conditions (4.10) are satisfied:

$$-\frac{1}{n} > -\min\left\{\frac{1}{n-2},\,\frac{n-2}{n(n-3)}\right\}.$$

Hence $S_{n-2}((n-2)/n,\,(n-2)/n,\,-1/n)$ converges, and so does $\widetilde{G}$. Thus $G(\chi)$ converges too. □

Recall that $K(\boldsymbol{\alpha})$ is the reciprocal of the height of the monic polynomial $p_{\boldsymbol{\alpha}}$ (cf. (4.2)). Thus, for any ρ, τ and $\boldsymbol{\theta}$, we have $K(\boldsymbol{\tau}+\rho\,\boldsymbol{\theta}) \leqslant \chi_0$, where $\chi_0 := \sup_{\boldsymbol{\alpha}\in\mathbb{R}^n} K(\boldsymbol{\alpha})$ is finite (see Lemma 3). Hence the integrand in $\widetilde{J}(\rho,\tau)$ (see (4.5)) does not exceed the integrand in $G(\chi_0)$ (cf. (4.6)), and we can apply the Weierstrass M-test (for the uniform convergence of improper integrals), which tells us that, for all $\tau\in[-1,1]$, the following convergence is uniform:

$$\lim_{\rho\to 0} \widetilde{J}(\rho,\tau) = \widetilde{J}(0,\tau) = G\big(\widetilde{K}(\tau)\big),$$

where $\widetilde{K}(\tau) := K(\boldsymbol{\tau})$. Therefore from (4.4) we get

$$\begin{aligned}\lambda_{n,0} &= \lim_{\delta\to 0}\big(\delta^{-(n+2)(2n)} f_{n,0}(\delta)\big) = \frac{4}{(n+1)!}\int_{-1}^{1} G\big(\widetilde{K}(\tau)\big)\,\mathrm{d}\tau \\ &= \frac{8n(n+1)}{(n+2)!}\int_{-1}^{1}\widetilde{K}(\tau)^{2/n}\,\mathrm{d}\tau \int_{[-1,1]^{n-2}} \widetilde{\Delta}(\xi_1,\dots,\xi_{n-2},1)^{-1/n}\,\mathrm{d}\xi_1\cdots\mathrm{d}\xi_{n-2}.\end{aligned}$$

Theorem 2 is proved.

Note that the dependence of $\lambda_{n,0}$ on the height function h is absorbed solely in the factor $\int_{-1}^{1}\widetilde{K}(\tau)^{2/n}\,\mathrm{d}\tau$.

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