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Modeling the beta distribution in short intervals

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Dedicated to Professors Antanas Laurinčikas and Eugenijus Manstavičius on the occasion of their 70th birthdays

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Abstract. We prove that any beta distribution can be simulated by means of a sequence of distributions defined via multiplicative functions in a short interval.

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1 Introduction and result

The letters c and C with or without subscripts denote constants. Both notations $f = O(g)$ or $f \ll g$ mean that $|f| \leqslant C|g|$ for some positive constant C, which may be absolute or depend upon various parameters. In such cases, we sometimes indicate this by a subscript. All asymptotic relations are meant as $x \to \infty$.

The beta distribution $B(a, b)$ with parameters $a, b > 0$ is concentrated on the interval $t \in [0,\ 1]$ and defined by

$$B(t; a, b) := \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^t \frac{\mathrm{d}v}{v^{1-a}(1-v)^{1-b}}.$$

When $a = b = 1/2$, this distribution is known as the arcsine law.

Let $f, g : \mathbb{N} \to [0; \infty)$ be multiplicative functions. Set

$$T_f(m, v) := \sum_{d|n,\, d\leqslant v} f(d), \quad T_f(m, m) =: T_f(m), \quad m \in \mathbb{N},\ v \in \mathbb{R},$$

and

$$G(x, y; g) := \sum_{x<n\leqslant x+y} g(n), \quad x, y \geqslant 0.$$

When $G(x,y;g) \neq 0$, we define

$$F(x,y,u;g,f) := \frac{1}{G(x,y;g)} \sum_{x<m\leqslant x+y} \frac{g(m)T_f(m,m^u)}{T_f(m)}. \tag{1.1}$$

In 1979, Deshouillers, Dress, and Tenenbaum [8] proved proved the following:

DDT theorem. *Uniformly in* $u \in [0;1]$,

$$F(0,x,u;1,1) = B\left(u, \frac{1}{2}, \frac{1}{2}\right) + O\left(\frac{1}{\sqrt{\ln x}}\right).$$

Manstavičius [10] noticed that not only the arcsine law but also some other beta distributions can occur as limits for the means of type (1.1). Later a few generalizations of the DDT theorem were obtained. To describe these results, we need some additional definitions.

Definition 1. Let $f : \mathbb{N} \to [0;\infty)$ be a multiplicative function such that $f(p^k) \leqslant C$ for some $C > 0$ and all $k \in \mathbb{N}$ and primes p. We say that f belongs to the class $\mathcal{G}(\varkappa, \delta)$, $\varkappa \geqslant 0$, $0 \leqslant \delta < 1$, if the function defined by the series

$$\sum_p \frac{f(p) - \varkappa}{p^s}, \quad s = \sigma + \mathrm{i}\tau,\ \sigma > 1,$$

for some $0 < c \leqslant 1/2$, has an analytic continuation $P(s)$ into the region

$$\sigma \geqslant \sigma(\tau) := 1 - \frac{c}{\ln(|\tau| + 3)},$$

where $P(s)$ is holomorphic, and $|P(s)| \leqslant \delta \log(|\tau| + 1) + c_0$ for some $c_0 \geqslant 0$.

In [4], it was proved that

$$F(0,x,u;1,f) = B(u,\, 1-\alpha,\, \alpha) + O\left(\frac{1}{\ln^{1-\alpha} x} + \frac{1}{\ln^{\alpha} x}\right)$$

when $f \in \mathcal{G}(\alpha, \delta)$, $\alpha \in (0;1)$. The particular cases $\alpha = 1$ or $\alpha = 0$ were considered in [1]. It was shown that only improper limit laws can occur in these cases.

A more general beta distribution can be modeled using the mean (1.1) with weight $g \not\equiv 1$. In [7], it was proved that the distributions of type (1.1) can approach any beta law $B(\varkappa - \alpha, \alpha)$ with $0 < \alpha < \varkappa < 1$. Later this result was extended.

Definition 2. We say that a pair of multiplicative functions (g, f) belongs to the class $\mathcal{M}(\varkappa, \alpha; \delta_1, \delta_2)$ if $g \in \mathcal{G}(\varkappa, \delta_1)$ and $g/T_f \in \mathcal{G}(\alpha, \delta_2)$ with some $\delta_1, \delta_2 > 0$, $\delta_1 + \delta_2 < 1$.

In [3], it was proved that any beta distribution $B(a,b)$ can be a limit law for the mean (1.1) when a pair of multiplicative functions $(g,f) \in \mathcal{M}(a+b,\, b, \delta_1, \delta_2)$, $a, b > 0$.

In [5,6,9], it was shown that similar problems can be considered in "short" intervals $[x; x+y]$. Namely, an analogue of the DDT theorem was proved [5,6]:

$$F(x,y,u;1,1) = B\left(u, \frac{1}{2}, \frac{1}{2}\right) + O\left(\frac{1}{\sqrt{\ln x}}\right)$$

uniformly in $0 \leqslant u \leqslant 1$ and $x^{19/24+\epsilon} \leqslant y \leqslant x$ for arbitrary $\epsilon > 0$.

In this paper, we generalize this result, showing that any beta distribution can be modeled by the means of type (1.1) if the pair of the multiplicative functions (g, f) satisfies some regularity conditions.

DEFINITION 3. Let $g : \mathbb{N} \to [0,\infty)$ be a multiplicative function such that $g(p^k) \leqslant C$ for some $C > 0$ and all $k \in \mathbb{N}$ and primes p. We say that g belongs to the class $\mathcal{K}(\varkappa,\beta)$, $\varkappa,\beta \geqslant 0$, if the functions defined by the series

$$P_1(s) := \sum_p \frac{g(p)-\varkappa}{p^s}, \qquad P_2(s) := \sum_p \frac{(g(p)-\varkappa)^2}{p^{2s}},$$
$$P_3(s) := \sum_p \frac{g(p^2)-\beta}{p^{2s}}, \quad s = \sigma + \mathrm{i}\tau,\ \sigma > 1,$$

can be analytically continued to the holomorphic functions into the region $\sigma \geqslant 1/2 - \delta$ for some $\delta > 0$, and there $|P_1(s)| + |P_2(s)| + |P_3(s)| \leqslant c_0$ for some $c_0 \geqslant 0$.

Remark. If $g \in \mathcal{K}(\varkappa,\beta_1)$, $g/T_f \in \mathcal{K}(\alpha,\beta_2)$, then $(g,f) \in \mathcal{M}(\varkappa,\alpha,0,0)$.

Theorem 1. *Let $f : \mathbb{N} \to [0;\infty)$ be a multiplicative function. If $g \in \mathcal{K}(\varkappa,\varkappa)$ is a strongly multiplicative function and $g/T_f \in \mathcal{K}(\alpha,\beta)$, $0 < \alpha < \varkappa$, then for any $\varepsilon > 0$,*

$$\bigl|F(x,y,u;g,f) - B(u;\varkappa-\alpha,\,\alpha)\bigr| \ll \frac{1}{\ln^{\varkappa-\alpha} x} + \frac{1}{\ln^{\alpha} x} + \frac{(\ln\ln x)^{\chi(\varkappa-\alpha)} + (\ln\ln x)^{\chi(\alpha)}}{\ln x}$$

uniformly in $0 \leqslant u \leqslant 1$ and $x^{19/24+\varepsilon} < y \leqslant x$. Here $\chi(1) = 1$ and $\chi(v) = 0$ for $v \neq 1$.

2 Preliminaries

Let $w, z \in \mathbb{C}$ and ϱ, γ, A, B, $M > 0$ are some constants. As usual, let $\zeta(s)$ be the Riemann zeta function. Suppose that f is an arithmetic function that satisfies the following conditions:

(i) for any $\delta > 0$, we have

$$\bigl|f(n)\bigr| \ll_\delta M n^\delta \quad (n \geqslant 1), \tag{2.1}$$

where the implied constant depends only on δ;

(ii) for $\sigma > 1$,

$$\sum_{n=1}^{\infty} \bigl|f(n)\bigr| n^{-\sigma} \leqslant M(\sigma-1)^{-\varrho}; \tag{2.2}$$

(iii) the Dirichlet series

$$\mathcal{F}(s,z,w) := \zeta(s)^{-z}\zeta(2s)^{-w}\sum_{n=1}^{\infty} f(n)n^{-s}$$

can be analytically continued to a holomorphic function in some open set containing $\sigma \geqslant 1/2$, and, in this region, $\mathcal{G}(s,z,w)$ satisfies the bound

$$\bigl|\mathcal{F}(s,z,w)\bigr| \leqslant M\bigl(|\tau|+1\bigr)^{\max\{\gamma(1-\sigma),\,0\}} \ln^A\bigl(|\tau|+1\bigr) \tag{2.3}$$

uniformly in $|z| \leqslant B$ and $|w| \leqslant C$.

Lemma 1. (*See* [5, *Cor.* 1.2].) *Let* $w, z \in \mathbb{C}$, $\varrho > 0$, $\gamma \geqslant 0$, $A \geqslant 0$, $B > 0$, $C > 0$, *and* $M > 0$ *some constants. Suppose that an arithmetic function* f *satisfies conditions* (2.1)–(2.3). *Then for any* $\epsilon > 0$, *we have*

$$\sum_{x<n\leqslant x+y} f(n) = y \ln^{z-1} x \left(\lambda_0(z,w) + O\left(\frac{M}{\ln x} \right) \right)$$

uniformly in $x^{1-\theta+\epsilon} \leqslant y \leqslant x$, $0 < |z| \leqslant B$, *and* $|w| \leqslant C$, *where*

$$\lambda_0(\kappa, w) := \frac{\mathcal{F}(1,z,w)\zeta^w(2)}{\Gamma(z)},$$

and $\theta = 5/(12+5\gamma)$. *The implied constant in the O term depends only on* A, B, C, ρ, γ, *and* ϵ.

For $\varkappa > 0$ and any multiplicative function θ, set

$$A(\varkappa, \theta) := \frac{1}{\Gamma(\varkappa)} \prod_p \left(1 - \frac{1}{p}\right)^{\varkappa} \sum_{k=0}^{\infty} \frac{\theta(p^k)}{p^k}.$$

Lemma 2. *Let* φ *and* ψ *be nonnegative multiplicative functions such that*

$$\varphi(p^k) \leqslant C_1, \quad \psi(p^k) \leqslant C_1, \quad \textit{and} \quad \psi(p^{k+1}) \leqslant \psi(p^k) \tag{2.4}$$

for $k \in \mathbb{N}$. *Assume furthermore that* $\varphi \cdot \psi \in \mathcal{K}(\varkappa, \beta)$, $\varkappa, \beta > 0$. *Then for any* $\varepsilon > 0$,

$$\sum_{x<n\leqslant x+y} \varphi(n)\psi(nd) = \frac{y}{\ln^{1-\varkappa}(\mathrm{e}x)} \left(A(\varkappa, \varphi \cdot \psi) \cdot \tilde{h}(d; \varphi, \psi) + O\left(\frac{\hat{h}(d; \varphi, \psi)}{\ln(\mathrm{e}x)} \right) \right)$$

uniformly jn $x \geqslant 1$ *and* $x^{7/12+\varepsilon} \leqslant y \leqslant x$. *Here the multiplicative functions* $\tilde{h}$ *and* $\hat{h}$ *are defined by*

$$\tilde{h}(p^k; \varphi, \psi) := \left(\sum_{j=0}^{\infty} \frac{\varphi(p^j)\psi(p^j)}{p^j} \right)^{-1} \sum_{j=0}^{\infty} \frac{\varphi(p^j)\psi(p^{k+j})}{p^j},$$
$$\hat{h}(p^k; \varphi, \psi) := \left(1 + \frac{c_1}{p^{\sigma_0}} \right) \sum_{j=0}^{\infty} \frac{\varphi(p^j)\psi(p^{k+j})}{p^{j\sigma_0}}.$$

Here $\sigma_0 = 1/2 - \delta$, $0 < \delta < 1/2$, *and* $c_1 \geqslant 0$ *is a constant depending on* $\varkappa$, β, δ, *and* C_1.

Remark. If (2.4) holds, then

$$\tilde{h}(p^k; \varphi, \psi) = \psi(p^k) + O(p^{-1}),$$
$$\hat{h}(p^k; \varphi, \psi) = \psi(p^k) + O(p^{-\sigma_0}) \tag{2.5}$$

for $k \in \mathbb{N}$. Hence $\tilde{h} \in \mathcal{K}(\varkappa, \beta)$ and $\hat{h} \in \mathcal{G}(\varkappa, 0)$, provided that $\psi \in \mathcal{K}(\varkappa, \beta)$. We will further frequently use this property.

Proof of Lemma 2. We modify the proof of Lemma 3.1 in [2]. Suppose that $x \geqslant 3$. As usual, let $\zeta(s)$ be the Riemann zeta function. Introduce the Dirichlet series

$$F_d(s) := \sum_{n=1}^{\infty} \frac{\varphi(n)\psi(nd)}{n^s}, \qquad G_d(s,\varkappa,w) := \zeta^{-\varkappa}(s)\zeta^{-w}(2s)F_d(s).$$

For $\operatorname{Re} s > 1$, we have

$$G_d(s,\varkappa,w) = \prod_p \left(1-\frac{1}{p^s}\right)^{\varkappa}\left(1-\frac{1}{p^{2s}}\right)^{w} \sum_{k=0}^{\infty} \frac{\varphi(p^k)\psi(p^{k+\alpha_p(d)})}{p^{ks}}. \tag{2.6}$$

Here $\alpha_p(d)$ is defined by $p^{\alpha_p(d)} \,\|\, d$. Setting

$$\gamma(p,s,\varkappa,w) := \begin{cases} (1-\frac{1}{p^s})^{\varkappa}(1-\frac{1}{p^{2s}})^{w} & \text{if } p \leqslant p_0,\\ (\sum_{k=0}^{\infty} \frac{\varphi(p^k)\psi(p^k)}{p^{sk}})^{-1} & \text{if } p > p_0, \end{cases}$$

we choose $p_0 = p_0(\delta, C_1)$ and $c_1 = c_1(\delta, C_1, \varkappa, w)$ such that, for $\sigma \geqslant \sigma_0$,

$$\left|\gamma(p,s,\varkappa,w)\right| \leqslant 1 + \frac{c_1}{p^{\sigma_0}} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{C_1^2}{p_0^{k\sigma}} < 1. \tag{2.7}$$

Let $\mathbb{P}'$ be the subset of primes p such that $p \nmid d$ or $p > p_0$. Then the Euler product (2.6) can be written in the form

$$G_d(s,\varkappa,w) = L(s,\varkappa,w) \cdot g(d;s,\varkappa,w)$$

with

$$L(s,\varkappa,w) := \prod_{p\in\mathbb{P}'} \left(1-\frac{1}{p^s}\right)^{\varkappa}\left(1-\frac{1}{p^{2s}}\right)^{w} \sum_{k=0}^{\infty} \frac{\varphi(p^k)\psi(p^k)}{p^{ks}}$$

and the multiplicative function $g(\cdot;s,\varkappa,w)$ defined by

$$g\big(p^k;s,\varkappa,w\big) := \gamma(p,s,\varkappa,w) \sum_{j=0}^{\infty} \frac{\varphi(p^j)\psi(p^{j+k})}{p^{js}}.$$

We have that, for $\sigma \geqslant \sigma_0$, the multiplicative functions $g(d;s,\varkappa,w)$ and $\hat{h}(d;\varphi,\psi)$ are related by the inequality

$$\left|g(d;s,\varkappa,w)\right| \leqslant \hat{h}(d;\varphi,g). \tag{2.8}$$

Taking the exponent and logarithm, which is allowed by (2.7), we can write

$$L(s,\varkappa,w) = H(s) \cdot \mathrm{e}^{E(s,\varkappa,w)},$$

where

$$E(s,\varkappa,w) := \sum_p \frac{\varphi(p)\psi(p)-\varkappa}{p^s} + \sum_p \frac{1}{p^{2s}}\left(\varphi\psi\big(p^2\big) - \frac{(\varphi\psi(p))^2}{2} + \frac{\varkappa}{2} - w\right).$$

and $H(s)$ is analytic and bounded for $\sigma \geqslant \sigma_0$. Moreover, we can take $w = \beta - \varkappa(\varkappa+1)/2$. Then the assumptions of lemma allow us to assert that the function $E(s,\varkappa,w)$ has an analytic continuation into the region $\sigma \geqslant \sigma_0$, and in this domain,

$$\left|L(s,\varkappa,w)\right| \ll 1.$$

This, together with (2.8), allows an analytic continuation of $G_d(s,\varkappa,w)$ into the region $\sigma \geqslant \sigma_0$ and yields there the estimate

$$\left|G_d(s,\varkappa,w)\right| \ll \hat{h}(d;\varphi,\psi).$$

Note that

$$G_d(1,\varkappa,w) = \Gamma(\varkappa)\cdot A(\varkappa,\varphi\cdot\psi)\cdot\zeta^{-w}(2)\cdot\tilde{h}(d;\varphi,\psi) \ll \hat{h}(d;\varphi,\psi)$$

and for $\sigma > 1$,

$$F_d(\sigma) = G_d(\sigma,\varkappa,w)\cdot\zeta^{\varkappa}(\sigma)\cdot\zeta^{w}(2\sigma) \ll \hat{h}(d;\varphi,\psi).$$

Moreover, (2.4) implies

$$\varphi(n)\psi(nd) \leqslant \psi(d)C_1^{\omega(n)} \prod_{\substack{p^k\|n\\(p,d)=1}} \psi\bigl(p^k\bigr) \leqslant \hat{h}(d;\varphi,\psi)C_1^{2\omega(n)}.$$

Since $\omega(n) = o(\ln n)$ and

$$\frac{y}{\ln^{1-\varkappa} x} = \frac{y}{\ln^{1-\varkappa}(\mathrm{e}x)}\left(1+O\left(\frac{1}{\ln x}\right)\right),$$

the proof of the lemma for $x \geqslant 3$ now follows from Lemma 1. If $x < 3$, then

$$\sum_{n\leqslant x}\varphi(n)\psi(nd) \leqslant \sum_{n\leqslant 2}\varphi(n)\psi(nd) \ll \hat{h}(d;\varphi,\psi).$$

In this case, the lemma immediately follows, since $\tilde{h}(d;\varphi,\psi) \leqslant \hat{h}(d;\varphi,\psi)$. $\square$

For $0 \leqslant u \leqslant 1,\ x \geqslant 1$, and $b \in \mathbb{R}$, we set

$$\Theta(x,u,b) := \sum_{m\leqslant x^u} \frac{a_m}{m\ln^b(\frac{\mathrm{e}x}{m})}, \quad a_m \geqslant 0.$$

This sum may be evaluated in terms of the integral

$$I(u;a,b,\eta) := \int_0^u \frac{\mathrm{d}v}{(\eta+v)^a(\eta+1-v)^b}, \quad \eta \geqslant 0,$$

provided that some information about the behavior of the sum

$$M(v) := \sum_{m\leqslant v} a_m, \quad v \geqslant 1,$$

is given.

The next lemma is a straightforward consequence of Lemma 4 in [3].

Lemma 3. *Assume that $x \geqslant \mathrm{e}$ and*

$$\left|M(v) - \frac{Av}{\ln^a(\mathrm{e}v)}\right| \leqslant \frac{Bv}{\ln^{a+1}(\mathrm{e}v)}$$

for some $a, A \in \mathbb{R}$, and $B \geqslant 0$ and all $1 \leqslant v \leqslant x$. Then

$$\left|\Theta(x,u,b) - \frac{A}{\ln^{a+b-1} x} I(u; a, b, \eta_x)\right| \\ \ll \frac{1}{\ln^{a+1} x} + \frac{1}{\ln^b x} + \frac{(\ln\ln x)^{\chi(a+1)} + (\ln\ln x)^{\chi(b)}}{\ln^{a+b} x}.$$

Here and in what follows, $\chi(1) = 1$ and $\chi(v) = 0$ for $v \neq 1$. The implicit constant in symbol $\ll$ depends at most on a, b, A, and B.

Lemma 4. *Assume that $a < 1$, $b < 1$, and $h \in \mathcal{G}(1-a, \delta)$, $u \in [0;1]$. Then for $x \geqslant 3$ and $0 \leqslant y \leqslant x$, we have*

$$\begin{aligned} S_x(y,u,b;h) &:= \sum_{m \leqslant (x+y)^u} \frac{h(m)}{m \ln^b(\frac{\mathrm{e}x}{m})} \\ &= A(1-a,\, h) \frac{1}{\ln^{a+b-1} x} \int_0^u \frac{\mathrm{d}s}{s^a(1-s)^b} \\ &\quad + O\left(\frac{1}{\ln^a x} + \frac{1}{\ln^b x} + \frac{(\ln\ln x)^{\chi(a+1)} + (\ln\ln x)^{\chi(b)}}{\ln^{a+b} x}\right). \end{aligned}$$

Moreover,

$$S_x(y,u,b;h) \ll \ln^{1-a-b} x. \tag{2.9}$$

Proof. We have

$$\sum_{m \leqslant (x+y)^u} \frac{h(m)}{m \ln^b(\frac{\mathrm{e}x}{m})} = \sum_{m \leqslant x^u} \frac{h(m)}{m \ln^b(\frac{\mathrm{e}x}{m})} + \sum_{x^u < m \leqslant (x+y)^u} \frac{h(m)}{m \ln^b(\frac{\mathrm{e}x}{m})}.$$

Suppose that $u \leqslant \eta_x := \ln^{-1} x$. Then

$$\sum_{m \leqslant (x+y)^u} \frac{h(m)}{m \ln^b(\frac{\mathrm{e}x}{m})} \ll \frac{1}{\ln^b x}$$

since $h(p^k) \ll 1$.

Let us assume that $\eta_x < u \leqslant 1$. Then, applying Lemma 1 in [1], we have

$$\sum_{x^u < m \leqslant (x+y)^u} \frac{h(m)}{m \ln^b(\frac{\mathrm{e}x}{m})} \ll \frac{1}{x^u(1+(1-u)\ln x)^b} \sum_{m \leqslant (2x)^u} h(m) \ll \frac{(1+u\ln x)^{-a}}{(1+((1-u)\ln x)^b)}.$$

Thus, uniformly in $0 \leqslant u \leqslant 1$, we have

$$\sum_{m \leqslant (x+y)^u} \frac{h(m)}{m \ln^b(\frac{\mathrm{e}x}{m})} = \sum_{m \leqslant x^u} \frac{h(m)}{m \ln^b(\frac{\mathrm{e}x}{m})} + \frac{1}{\ln^a x} + \frac{1}{\ln^b x}. \tag{2.10}$$

In [3], it was shown that

$$I(u;a,b,\eta_x) = I(u;a,b,0) + O\bigl(\eta_x + \eta_x^{1-a} + \eta_x^{1-b}\bigr).$$

Moreover, $A(1-a,h) \ll 1$ when $h \in \mathcal{G}(1-a,\delta)$. Thus the proof of Lemma 4 follows from (2.10), Lemma 1 in [3], and Lemma 3. $\square$

3 Proof of Theorem 1

The distributions (1.1) can be written as follows:

$$F(x,y,u;g,f) = S(x) - R(x), \tag{3.1}$$

where

$$S(x) := \frac{1}{G(x,y;g)} \sum_{x<n\leqslant x+y} \frac{g(n)}{T_f(n)} \sum_{\substack{d\mid n\\ d\leqslant (x+y)^u}} f(d),$$

$$R(x) := \frac{1}{G(x,y;g)} \sum_{x<n\leqslant x+y} \frac{g(n)}{T_f(n)} \sum_{\substack{d\mid n\\ n^u<d\leqslant (x+y)^u}} f(d).$$

Since $g \in \mathcal{K}(\varkappa,\varkappa)$, Lemma 2 with $\varphi \equiv 1$, $\psi = g$, and $d = 1$ yields

$$\frac{1}{G(x,y;g)} = \frac{\ln^{1-\varkappa}(\mathrm{e}x)}{yA(\varkappa,g)}\biggl(1 + O\biggl(\frac{1}{\ln(\mathrm{e}x)}\biggr)\biggr). \tag{3.2}$$

Consider two cases. First, assume that $0 \leqslant u \leqslant 1/2$. Changing the order of summation, we have

$$R(x) \ll \frac{1}{G(x,y;g)} \sum_{x^u<d\leqslant (x+y)^u} f(d) \sum_{x/d<m\leqslant (x+y)/d} \frac{g(md)}{T_f(md)}. \tag{3.3}$$

If $u \in [0;1/2]$, then

$$\biggl(\frac{x}{d}\biggr)^{7/12+\varepsilon_1} \leqslant \frac{y}{d} \leqslant \frac{x}{d} \tag{3.4}$$

for some $\varepsilon_1 > 0$. Therefore, applying Lemma 2 for the inner sum in (3.3) and using (3.2), we get

$$R(x) \ll \ln^{1-\varkappa} x \sum_{x^u<d\leqslant (2x)^u} \frac{f(d)}{d\ln^{1-\alpha}(\frac{\mathrm{e}x}{d})} \hat{h}\biggl(d;1,\frac{g}{T_f}\biggr).$$

Note that (2.5) implies $f\hat{h} \in \mathcal{G}(\varkappa-\alpha,0)$. Therefore, taking $h := f\hat{h}$, $b = 1-\alpha$, and $a = 1-\varkappa+\alpha$ in Lemma 4, we have

$$R(x) \ll \frac{1}{\ln^{\alpha} x} + \frac{1}{\ln^{\varkappa-\alpha} x} + \frac{(\ln\ln x)^{\chi(\varkappa-\alpha)}}{\ln x}.$$

The main term in (3.1) is

$$S(x) = \frac{1}{G(x,y;g)} \sum_{d\leqslant (x+y)^u} f(d) \sum_{x/d<m\leqslant (x+y)/d} \frac{g(dm)}{T_f(dm)}.$$

We start with the observation that $g/T_f \in \mathcal{K}(\alpha, \beta)$. In Lemma 2, taking $\varphi \equiv 1$ and $\psi = g/T_f$ and having in mind (3.4), we obtain

$$S(x) = S_1(x) + O\big(R_1(x)\big),$$

where

$$S_1(x) := \frac{yA(\alpha, \frac{g}{T_f})}{G(x, y; g)} \sum_{d \leqslant (x+y)^u} \frac{f(d)}{d \ln^{1-\alpha}(\frac{ex}{d})} \tilde{h}\left(d; 1, \frac{g}{T_f}\right),$$

$$R_1(x) := \frac{y}{G(x, y; g)} \sum_{d \leqslant 2x} \frac{f(d)\hat{h}(d; 1, \frac{g}{T_f})}{d \ln^{2-\alpha}(\frac{ex}{d})}.$$

By (2.5) , $f\hat{h} \in \mathcal{G}(\varkappa - \alpha, 0)$. Then, taking $a = 1 - \varkappa + \alpha$ and $b = 2 - \alpha$ in (2.9) and using (3.2), we deduce

$$R_1(x) \ll \frac{1}{\ln x}.$$

In view of (2.5), we can check that $f\tilde{h} \in \mathcal{K}(\varkappa - \alpha, \varkappa - \varkappa\beta/\alpha) \subset \mathcal{G}(\varkappa - \alpha, 0)$. Hence (3.2) and Lemma 4 yield

$$S_1(x) = \frac{A(\alpha, \frac{g}{T_f})A(\varkappa - \alpha,\, f\tilde{h})}{A(\varkappa, g)} \int_0^u \frac{\mathrm{d}s}{s^{1-\varkappa+\alpha}(1-s)^{1-\alpha}} + O\left(\frac{1}{\ln^\alpha x} + \frac{1}{\ln^{\varkappa-\alpha} x} + \frac{(\ln\ln x)^{\chi(\varkappa-\alpha)}}{\ln x}\right).$$

In [3], it was shown that

$$\frac{A(\alpha, \frac{g}{T_f})A(\varkappa - \alpha, f\tilde{h})}{A(\varkappa, g)} = \frac{\Gamma(\varkappa)}{\Gamma(\varkappa - \alpha)\Gamma(\alpha)}.$$

Now collecting all needed estimates in (3.1), we prove the theorem for $u \in [0; 1/2]$.

Now suppose that $1/2 < u \leqslant 1$. Changing the order of summation in $R(x)$, we obtain

$$R(x) = \frac{1}{G(x, y; g)}\left(\sum_{x/(x+y)^u < m \leqslant x^{1-u}} \; \sum_{x/m < d \leqslant (x+y)/m} \frac{f(d)g(md)}{T_f(md)} + \sum_{x^{1-u} < m \leqslant (x+y)^{1-u}} \; \sum_{m^{u/(1-u)} < d \leqslant (x+y)/m} \frac{f(d)g(md)}{T_f(md)}\right).$$

If $x^{1-u} < m \leqslant (x+y)^{1-u}$, then $m^{u/(1-u)} > x/m$ and $(x+y)^u \leqslant (x+y)/m$. Hence

$$R(x) \ll R_2(x) := \frac{1}{G(x, y; g)} \sum_{x/(x+y)^u < m \leqslant x^{1-u}} \; \sum_{x/m < d \leqslant (x+y)/m} \frac{f(d)g(md)}{T_f(md)}. \tag{3.5}$$

If $u \in (1/2; 1]$, then $m \leqslant (x+y)^{1-u} \leqslant (2x)^{1/2}$. By the assumptions of the theorem this implies

$$\frac{x}{m} \geqslant \frac{y}{m} \geqslant \left(\frac{x}{m}\right)^{7/12+\varepsilon_2} \tag{3.6}$$

for some $\varepsilon_2 > 0$. Since $fg/T_f \in \mathcal{K}(\varkappa - \alpha,\ \varkappa - \varkappa\beta/\alpha)$, we can apply Lemma 2 with $\varphi := f$ and $\psi := g/T_f$ to estimate the inner sum in (3.5). This, together with (3.2), yields

$$R_2(x) \ll \ln^{1-\varkappa} x \sum_{x/(x+y)^u < m \leqslant (x+y)^{1-u}} \frac{\hat{h}(m; f, \frac{g}{T_f})}{m \ln^{1-\varkappa+\alpha}(\frac{\mathrm{e}x}{m})}$$
$$\ll \ln^{1-\varkappa} x \biggl(S_x(y,\, 1-u,\, 1-\varkappa+\alpha;\, \hat{h}) - S_x\biggl(0,\, 1-u-\frac{\ln 2}{\ln x},\, 1-\varkappa+\alpha;\, \hat{h}\biggr)\biggr).$$

From (2.5) it follows that $\hat{h} \in \mathcal{G}(\alpha, 0)$. Applying Lemma 4 with $a = 1-\alpha$ and $b = 1-\varkappa+\alpha$, we derive

$$R_2(x) \ll \frac{1}{\ln^{\alpha} x} + \frac{1}{\ln^{\varkappa-\alpha} x} + \frac{(\ln\ln x)^{\chi(\varkappa-\alpha)} + (\ln\ln x)^{\chi(\alpha)}}{\ln x}. \tag{3.7}$$

Consider $S(x)$ in (3.1). We have

$$S(x) = 1 - S_2(x) - R_2(x), \tag{3.8}$$

where

$$S_2(x) := \frac{1}{G(x, y; g)} \sum_{m \leqslant (x+y)^{1-u}} \sum_{x/m < d \leqslant (x+y)/m} \frac{f(d)g(md)}{T_f(md)}.$$

We have that $fg/T_f \in \mathcal{K}(\varkappa - \alpha,\ \varkappa - \varkappa\beta/\alpha)$. In view of (3.6), Lemma 2 with $\varphi = f$ and $\psi = g/T_f$ yields

$$S_2(x) = S_3(x) + O\bigl(R_3(x)\bigr), \tag{3.9}$$

where

$$S_3(x) := \frac{yA(\varkappa-\alpha, \frac{fg}{T_f})}{G_x(x, y; g)} \sum_{d \leqslant (x+y)^{1-u}} \frac{\tilde{h}(d; f, \frac{g}{T_f})}{d \ln^{1-\varkappa+\alpha}(\frac{\mathrm{e}x}{d})},$$
$$R_3(x) := \frac{y}{G(x, y; g)} \sum_{d \leqslant 2x} \frac{\hat{h}(d; f, \frac{g}{T_f})}{d \ln^{2-\varkappa+\alpha}(\frac{\mathrm{e}x}{d})}.$$

Taking $a = 1-\alpha$ and $b = 2-\varkappa+\alpha$ in (2.9) and using (3.2), we get

$$R_3(x) \ll \frac{1}{\ln x}.$$

Note that (2.5) implies $\tilde{h} \in \mathcal{K}(\alpha, \beta) \subset \mathcal{G}(\alpha, 0)$. Then, choosing $h := \tilde{h}$, $b = 1-\varkappa+\alpha$, and $a = 1-\alpha$ in Lemma 4 and taking into account (3.2), we obtain

$$S_3(x) = \frac{A(\alpha, \tilde{h})A(\varkappa-\alpha,\ \frac{fg}{T_f})}{A(\varkappa, g)} \int_0^{1-u} \frac{\mathrm{d}s}{s^{1-\alpha}(1-s)^{1-\varkappa+\alpha}}$$
$$+ O\biggl(\frac{1}{\ln^{\alpha} x} + \frac{1}{\ln^{\varkappa-\alpha} x} + \frac{(\ln\ln x)^{\chi(\varkappa-\alpha)}}{\ln x}\biggr).$$

Routine calculations show that

$$\frac{A(\varkappa-\alpha,\,\frac{fg}{T_f})A(\alpha,\tilde{h})}{A(\varkappa,g)}=\frac{\Gamma(\varkappa)}{\Gamma(\varkappa-\alpha)\Gamma(\alpha)}.$$

Substituting these estimates of $S_3(x)$ and $R_3(x)$ into (3.9), from (3.8), (3.7), (3.5), and (3.1) we deduce

$$F(x,y,u;g,f)=1-\frac{\Gamma(\varkappa)}{\Gamma(\varkappa-\alpha)\Gamma(\alpha)}\int\limits_u^1\frac{\mathrm{d}s}{s^{1-\varkappa+\alpha}(1-s)^{1-\alpha}}$$
$$+O\biggl(\frac{1}{\ln^{\alpha}x}+\frac{1}{\ln^{\varkappa-\alpha}x}+\frac{(\ln\ln x)^{\chi(\varkappa-\alpha)}+(\ln\ln x)^{\chi(\alpha)}}{\ln x}\biggr)$$

uniformly for $1/2 < u \leqslant 1$.

This completes the proof of Theorem 1.

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