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Subharmonic solutions for a class of ordinary p-Laplacian systems*

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Abstract. In this paper, we study the existence of subharmonic solutions for ordinary p-Laplacian systems under a new growth condition. An existence theorem is obtained by using the generalized mountain pass theorem, which generalizes and improves some recent results in the literature.

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1 Introduction and main results

We consider the existence of subharmonic solutions for the following ordinary p-Laplacian system:

$$\left(\left|u'(t)\right|^{p-2}u'(t)\right)' + \nabla F\left(t, u(t)\right) = 0 \tag{1.1}$$

for a.e. $t \in \mathbb{R}$, where $p > 1$, $T > 0$, and $F : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is T-periodic in t for all $x \in \mathbb{R}^N$ and satisfies the following assumption:

(A) $F(t,x)$ is measurable in t for each $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$ and $b \in L^1(0,T;\mathbb{R}^+)$ such that

$$\left|F(t,x)\right| \leqslant a\left(|x|\right)b(t), \qquad \left|\nabla F(t,x)\right| \leqslant a\left(|x|\right)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, where $\nabla F(t,x)$ denotes the gradient of $F(t,x)$ in x.

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In 1978, Rabinowitz [22] published his pioneer paper on the existence of periodic solutions for the following second-order Hamiltonian system:

$$\begin{cases} u''(t) + \nabla F\big(t, u(t)\big) = 0, \\ u(0) - u(T) = u'(0) - u'(T) = 0 \end{cases} \tag{1.2}$$

under the Ambrosetti–Rabinowitz superquadratic condition: there exist $\mu > 2$ and $L^* > 0$ such that, for all $|x| \geqslant L^*$,

$$0 < \mu F(t, x) \leqslant \big(\nabla F(t, x), x\big) \tag{1.3}$$

for a.e. $t \in [0, T]$. From then on, various conditions have been applied to study the existence and multiplicity of periodic solutions for Hamiltonian systems by using the critical point theory; see [2, 5, 6, 7, 8, 9, 10, 14, 16, 20, 21, 23, 24, 25, 26, 27] and references therein.

Over the last few decades, many researchers try to replace the Ambrosetti–Rabinowitz superquadratic condition (1.3) by other superquadratic conditions. Some new superquadratic conditions are discovered. Especially, Fei [6] studied the existence of periodic solutions for problem (1.2) under a kind of new superquadratic condition. Afterward, under the more general superquadratic condition that there is $\mu_1 > 0$ such that

$$\liminf_{|x| \to \infty} \frac{(\nabla F(t, x), x) - 2F(t, x)}{|x|^{\mu_1}} > 0$$

uniformly for a.e. $t \in [0, T]$, Tao and Tang [27] investigated the existence of periodic and subharmonic solutions of problem (1.2). They generalized the corresponding results of [6].

Recently, the existence and multiplicity of solutions for p-Laplacian systems (1.1) have been studied by many authors; see [4, 11, 12, 13, 15, 17, 18, 19, 28, 29, 30] and references therein. Mawhin [19] generalized the Hartman–Knobloch results to perturbations of a vector p-Laplacian ordinary operator. Using the saddle point theorem, Xu and Tang [29] obtained some existence theorems for periodic solutions of problem (1.1). Zhang and Ma [30] investigated the existence of periodic and subharmonic solutions for systems (1.1), which extended the results of [10, 29]. With the perturbation technique and the dual least action principle, Lian et al. [15] proved some existence results for periodic and subharmonic solutions for systems (1.1). Li et al. [11] studied the existence of periodic solutions for systems (1.1) and proved the following result.

Theorem 1. (*See* [11, *Thm.* 1.4].) *Suppose that $F(t, x)$ satisfies the following conditions*:

(H0) $F(t, x) \geqslant 0$ *for all* $(t, x) \in [0, T] \times \mathbb{R}^N$;
(H1) $\lim_{|x| \to 0} F(t, x)/|x|^p = 0$ *uniformly for a.e.* $t \in [0, T]$;
(H2) *There exist constants* $\mu_2 > p$ *and* $L_0 > 0$ *and a function* $W \in L^1(0, T; \mathbb{R})$ *such that, for all* $x \in \mathbb{R}^N$ *with* $|x| \geqslant L_0$,

$$\mu_2 F(t, x) - \big(\nabla F(t, x), x\big) \leqslant W(t)|x|^p \quad \textit{for a.e. } t \in [0, T]$$

and

$$\limsup_{|x| \to \infty} \frac{\mu_2 F(t, x) - (\nabla F(t, x), x)}{|x|^p} \leqslant 0 \quad \textit{uniformly for a.e. } t \in [0, T];$$

(H3) *There exists* $\Omega \subset [0, T]$ *with* $\operatorname{meas} \Omega > 0$ *such that*

$$\liminf_{|x| \to \infty} \frac{F(t, x)}{|x|^p} > 0 \quad \textit{uniformly for a.e. } t \in \Omega.$$

Then system (1.1) *possesses a nonconstant T-periodic solution.*

By using the generalized mountain pass theorem, Ma and Zhang [17] extended the results of [27] to systems (1.1) and obtained the following theorem.

Theorem 2. (*See* [17, *Thm.* 1].) *Assume that* F *satisfies* (A), (H0), (H1)*, and the following conditions*:

(H4) $\liminf_{|x|\to\infty} F(t,x)/|x|^p > 0$ *uniformly for a.e.* $t \in [0,T]$;

(H5) $\limsup_{|x|\to\infty} F(t,x)/|x|^r \leqslant M < +\infty$ *uniformly for some* $M > 0$ *and a.e.* $t \in [0,T]$;

(H6) $\liminf_{|x|\to\infty}((\nabla F(t,x),x) - pF(t,x))/|x|^\lambda \geqslant \varrho > 0$ *uniformly for some* $\varrho > 0$ *and a.e.* $t \in [0,T]$,

where $r > p$ *and* $\lambda > r - p$*. Then problem* (1.1) *has a sequence of distinct periodic solutions with period* k_iT *satisfying* $k_i \in \mathbb{N}$ *and* $k_i \to \infty$ *as* $i \to \infty$.

In this paper, motivated by the works [17, 26, 27], we consider the existence of subharmonic solutions for problem (1.1) under a new growth condition. The main result is the following theorem.

Theorem 3. *Suppose that* $F(t,x)$ *satisfies* (H0), (H1), *and the following conditions*:

(H7) $F(t,x)/|x|^p \to +\infty$ *as* $|x| \to \infty$ *uniformly for a.e.* $t \in [0,T]$;

(H8) *There exist constants* a_0 *and* $L > 0$ *such that*

$$\bigl(\nabla F(t,x),x\bigr) - pF(t,x) \geqslant \frac{a_0}{|x|^p}F(t,x)$$

for all $x \in \mathbb{R}^N$ *with* $|x| \geqslant L$ *and a.e.* $t \in [0,T]$.

Then system (1.1) *has a sequence of distinct periodic solutions with period* k_iT *satisfying* $k_i \in \mathbb{N}$ *and* $k_i \to \infty$ *as* $i \to \infty$.

Remark 1. For second-order Hamiltonian system, the corresponding condition (H8) belongs to Tang and Wu [26]. Clearly, condition (H8) is weaker than (H6). There are functions F satisfying our assumptions of Theorem 3 and not satisfying the conditions of Theorem 2. For example, set

$$F(t,x) = |x|^p \ln\bigl(1+|x|^p\bigr) + \sin|x|^p - \ln\bigl(1+|x|^p\bigr)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. Then we have

$$\liminf_{|x|\to\infty} \frac{(\nabla F(t,x),x) - pF(t,x)}{|x|^\lambda} = 0$$

for any $\lambda > 0$. Hence, F does not satisfy the conditions of Theorem 2 but satisfies Theorem 3 with $a_0 = 1$.

Remark 2. Set F as in Remark 1; it does not satisfy the conditions of Theorem 1. Especially, F does not satisfy condition (H2). So, Theorem 3 is a new result on the existence of periodic solutions for system (1.1).

If $\Omega = [0,T]$ in (H3), then the conditions of Theorem 3 are much weaker than the conditions of Theorem 1, and if $\operatorname{meas}([0,T]\setminus\Omega) > 0$, the conditions of Theorem 3 and the conditions of Theorem 1 cannot imply each other. There are functions F satisfying the assumptions of Theorem 1 and not satisfying the conditions of Theorem 3. For example, let

$$F(t,x) = \psi(t)|x|^{p+1}$$

for all $(t,x) \in [0,T]\times\mathbb{R}^N$, where

$$\psi(t) = \begin{cases} \sin(2\pi t/T), & t \in [0,T/2], \\ 0, & t \in [T/2,T]. \end{cases}$$

Take $\Omega = [T/6,T/4]$ and $\mu_2 = p+1$. A straightforward computation implies that F satisfies the conditions of Theorem 1 but does not satisfy condition (H7).

Moreover, if $\operatorname{meas}([0,T]\setminus\Omega) > 0$, then we do not know whether system (1.1) has subharmonic solutions under the conditions of Theorem 1.

2 Proof of the main results

Let us consider the functional φ_k on $W^{1,p}_{kT}$ given by

$$\varphi_k(u) = \frac{1}{p}\int_0^{kT} |u'|^p \,\mathrm{d}t - \int_0^{kT} F(t,u)\,\mathrm{d}t \tag{2.1}$$

for each $u \in W^{1,p}_{kT}$, where

$$W^{1,p}_{kT} = \bigl\{u\colon [0,kT] \to \mathbb{R}^N \bigm| u \text{ is absolutely continuous, } u(0) = u(kT), \text{and } u' \in L^p\bigl(0,kT;\mathbb{R}^N\bigr)\bigr\}$$

is a reflexive Banach space with the norm

$$\|u\| = \left(\int_0^{kT} \bigl|u(t)\bigr|^p \,\mathrm{d}t + \int_0^{kT} \bigl|u'(t)\bigr|^p \,\mathrm{d}t\right)^{1/p} \quad \text{for } u \in W^{1,p}_{kT}.$$

For $u \in W^{1,p}_{kT}$, let

$$\overline{u} = \frac{1}{kT}\int_0^{kT} u(t)\,\mathrm{d}t, \qquad \widetilde{u} = u(t) - \overline{u},$$

and

$$\widetilde{W}^{1,p}_{kT} = \bigl\{u \in W^{1,p}_{kT} \bigm| \overline{u} = 0\bigr\}.$$

Then we have

$$W^{1,p}_{kT} = \widetilde{W}^{1,p}_{kT} \oplus \mathbb{R}^N$$

and

$$\|u\|_{L^p} \leqslant C_k \|u'\|_{L^p} \quad \text{(Wirtinger's inequality)},$$
$$\|u\|_{\infty} \leqslant C_k \|u'\|_{L^p} \quad \text{(Sobolev inequality)}$$

for all $u \in \widetilde{W}^{1,p}_{kT}$, where C_k is a positive constant.

It follows from assumption (A) that the functional φ_k is continuously differentiable on $W^{1,p}_{kT}$. Moreover, we have

$$\bigl\langle \varphi_k'(u), v\bigr\rangle = \int_0^{kT} |u'|^{p-2}(u', v')\,\mathrm{d}t - \int_0^{kT} \bigl(\nabla F(t,u), v\bigr)\,\mathrm{d}t$$

for all $u, v \in W^{1,p}_{kT}$. It is well known that the problem of finding kT-periodic solutions of problem (1.1) is equal to that of seeking the critical points of φ_k.

As shown in [1], a deformation lemma can be proved with the weaker condition (C) of Cerami [3] replacing the usual Palais–Smale condition, and it turns out that the generalized mountain pass theorem [24, Thm. 5.3] holds under condition (C).

Theorem 4. (*See* [24, *Generalized Mountain Pass Theorem*].) *Let E be a real Banach space with $E = X \oplus V$, where V is finite dimensional. Suppose that $I \in C^1(E,\mathbb{R})$ satisfies condition* (C) *and*

(i) *there are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geqslant \alpha$, and*
(ii) *there are $e \in \partial B_1 \cap X$ and $r > \rho$ such that if $Q = (\bar{B}_r \cap V) \oplus \{se \mid 0 < s < r\}$, then $I|_{\partial Q} \leqslant 0$.*

Then I possesses a critical value $c \geqslant \alpha$, which can be characterized as

$$c = \inf_{h\in\Gamma} \max_{u\in Q} I\big(h(u)\big),$$

where $\Gamma = \{h \in C(\bar{Q}, E) \mid h = id \text{ on } \partial Q\}$.

Now, we can prove our result.

Proof of Theorem 3. First of all, we will prove that φ_k satisfies condition (C), that is, for every sequence $\{u_n\} \subset W^{1,p}_{kT}$, $\{u_n\}$ has a convergent subsequence if $\{\varphi_k(u_n)\}$ is bounded and $(1 + \|u_n\|)\|\varphi_k'(u_n)\| \to 0$ as $n \to \infty$.

Then there exists a positive constant M_0 such that

$$\big|\varphi_k(u_n)\big| \leqslant M_0, \quad \big(1 + \|u_n\|\big)\big\|\varphi_k'(u_n)\big\| \leqslant M_0 \tag{2.2}$$

for all $n \in \mathbb{N}$. By a standard argument we only need to prove that $\{u_n\}$ is a bounded sequence in $W^{1,p}_{kT}$. Otherwise, we can assume that $\|u_n\| \to +\infty$ as $n \to \infty$. Let $v_n = u_n/\|u_n\|$, and then $\|v_n\| = 1$. Taking if necessary a subsequence, we can suppose that, as $n \to \infty$,

$$\begin{aligned} &v_n \rightharpoonup v \quad \text{weakly in } W^{1,p}_{kT}, \\ &v_n \to v \quad \text{strongly in } C\big(0, kT; \mathbb{R}^N\big). \end{aligned} \tag{2.3}$$

From (2.1), (2.2), and (2.3) we obtain

$$\left| \int_0^{kT} \frac{F(t, u_n)}{\|u_n\|^p}\,\mathrm{d}t - \frac{1}{p} \right| \leqslant \frac{|\varphi_k(u_n)|}{\|u_n\|^p} + \frac{1}{p}\int_0^{kT} |v_n|^p\,\mathrm{d}t \leqslant \frac{M_0}{\|u_n\|^p} + \frac{kT}{p}\|v_n\|_\infty^p. \tag{2.4}$$

If $v \not\equiv 0$, then letting $\Omega_1^k = \{t \in [0, kT]\colon |v(t)| > 0\}$, we have $|\Omega_1^k| > 0$. Since $\|u_n\| \to +\infty$, we get $|u_n| \to +\infty$ as $n \to \infty$ for a.e. $t \in \Omega_1^k$. By (H7) we have

$$\lim_{n\to+\infty} \frac{F(t, u_n)}{|u_n|^p} = +\infty$$

a.e. on Ω_1^k. Then we deduce from the Fatou lemma that

$$\liminf_{n\to\infty} \int_0^{kT} \frac{F(t, u_n)}{\|u_n\|^p}\,\mathrm{d}t \geqslant \liminf_{n\to\infty} \int_{\Omega_1^k} \frac{F(t, u_n)}{|u_n|^p}|v_n|^p\,\mathrm{d}t = +\infty,$$

which contradicts to (2.4). So, $\|u_n\|$ is bounded.

If $v \equiv 0$, then by (2.4) we have

$$\lim_{n\to\infty} \int_0^{kT} \frac{F(t, u_n)}{\|u_n\|^p}\,\mathrm{d}t = \frac{1}{p}. \tag{2.5}$$

Assumption (A) yields

$$\big|F(t, x)\big| \leqslant a_1 b(t), \qquad \big|\nabla F(t, x)\big| \leqslant a_1 b(t) \tag{2.6}$$

for all $x \in \mathbb{R}^N$ with $|x| \leqslant L$ and a.e. $t \in [0, T]$, where $a_1 = \max_{0\leqslant s\leqslant L} a(s)$.

By (H1) there exists a constant $L_1 < L$ such that

$$\left|F(t,x)\right| \leqslant |x|^p \tag{2.7}$$

for all $x \in \mathbb{R}^N$ with $|x| \leqslant L_1$ and a.e. $t \in [0,T]$. It follows from (H8), (2.6), and (2.7) that

$$\frac{a_0}{|x|^p} F(t,x) \leqslant \left(\nabla F(t,x), x\right) - pF(t,x) + a_0 + L_2 a_1 b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$, where $L_2 = a_0 L_1^{-p} + L + p$. So, we get

$$\begin{aligned}\int_0^{kT} \frac{|F(t,u_n)|}{|u_n|^p}\,\mathrm{d}t &\leqslant a_0^{-1} \int_0^{kT} \left(\left(\nabla F(t,u_n), u_n\right) - pF(t,u_n)\right) \mathrm{d}t + kT + a_0^{-1} L_2 a_1 \|b\|_{L^1_{kT}} \\ &= a_0^{-1}\left(p\varphi_k(u_n) - \left\langle \varphi_k'(u_n), u_n\right\rangle\right) + kT + a_0^{-1} L_2 a_1 \|b\|_{L^1_{kT}} \\ &\leqslant (p+1) a_0^{-1} M_0 + kT + a_0^{-1} L_2 a_1 \|b\|_{L^1_{kT}}.\end{aligned}$$

Then we obtain

$$\begin{aligned}\left|\int_0^{kT} \frac{F(t,u_n)}{\|u_n\|^p}\,\mathrm{d}t\right| &= \left|\int_0^{kT} \frac{F(t,u_n)}{|u_n|^p} |v_n|^p\,\mathrm{d}t\right| \leqslant \|v_n\|_\infty^p \int_0^{kT} \frac{|F(t,u_n)|}{|u_n|^p}\,\mathrm{d}t \\ &\leqslant \|v_n\|_\infty^p \left((p+1) a_0^{-1} M_0 + kT + a_0^{-1} L_2 a_1 \|b\|_{L^1_{kT}}\right) \to 0\end{aligned}$$

as $n \to \infty$, which is a contradiction to (2.5). So, $\|u_n\|$ is bounded.

Now, by the generalized mountain pass theorem with condition (C) we only need to show that

(G1) $\inf_{u \in S_k} \varphi_k(u) > 0$;
(G2) $\sup_{u \in Q_k} \varphi_k(u) < +\infty$, $\sup_{u \in \partial Q_k} \varphi_k(u) \leqslant 0$,

where $S_k = \widetilde{W}_{kT}^{1,p} \cap \partial B_\rho$, $Q_k = \{x + se \mid x \in \mathbb{R}^N \cap B_{r_2},\ s \in [0, r_1]\}$, $r_2 > 0$, $\rho < r_1$, $e \in \widetilde{W}_{kT}^{1,p}$, and $B_r = \{u \in W_{kT}^{1,p}\colon \|u\| \leqslant r\}$.

It follows from (H1) that there exist two positive constants ε and δ with $\varepsilon < 1/(pC_k)$ and $\delta < C_k$ such that

$$F(t,x) \leqslant \varepsilon |x|^p \tag{2.8}$$

for all $|x| \leqslant \delta$ and a.e. $t \in [0,kT]$.

For $u \in \widetilde{W}_{kT}^{1,p}$ with $\|u\| \leqslant \delta/C_k$, we have $\|u\|_\infty \leqslant \delta$. We obtain from (2.8) and Wirtinger's inequality that

$$\begin{aligned}\varphi_k(u) &= \frac{1}{p} \int_0^{kT} |u'|^p\,\mathrm{d}t - \int_0^{kT} F(t,u)\,\mathrm{d}t \geqslant \frac{1}{p} \int_0^{kT} |u'|^p\,\mathrm{d}t - \varepsilon \int_0^{kT} |u|^p\,\mathrm{d}t \\ &\geqslant \left(\frac{1}{p} - \varepsilon C_k\right) \|u'\|_{L^p}^p \geqslant \left(\frac{1}{p} - \varepsilon C_k\right) (1 + C_k)^{-1} \|u\|^p.\end{aligned}$$

Choose $\rho_k \in (0, \delta/C_k)$ to obtain

$$\inf_{u \in S_k} \varphi_k(u) > 0,$$

where $S_k = \widetilde{W}_{kT}^{1,p} \cap \partial B_{\rho_k}$. So, condition (G1) holds.

Let

$$\overline{W}_{kT}^{1,p} = \mathbb{R}^N \oplus \operatorname{span}\{e_k\},$$

where $e_k = (\sin(k^{-1}\omega t), 0, \dots, 0) \in \widetilde{W}_{kT}^{1,p}$ and $\omega = 2\pi/T$. Since $\dim(\overline{W}_T^{1,p}) < \infty$, all the norms are equivalent. For any $u \in \overline{W}_T^{1,p}$, there exists a positive constant M_1 such that

$$\|u\|_{L_T^p} \geqslant M_1 \|u\|_{L_T^2}. \tag{2.9}$$

By (H7), for $M_2 = (2\omega^p T/(pM_1^p))(2/T)^{p/2}$, there exists a positive constant $M_3 > M_1(T/2)^{1/2}T^{-1/p}$ such that

$$F(t,x) \geqslant M_2|x|^p$$

for all $|x| \geqslant M_3$ and a.e. $t \in [0,T]$. So, we have

$$F(t,x) \geqslant M_2|x|^p - M_2M_3^p \tag{2.10}$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0,T]$. Now, we get from (2.9) and (2.10) that

$$\begin{aligned}
\varphi_k(x+se_k) &= \frac{1}{p}\int_0^{kT} |se_k'|^p \,\mathrm{d}t - \int_0^{kT} F(t, x+se_k)\,\mathrm{d}t \\
&\leqslant \frac{1}{p}\left(\frac{\omega}{k}\right)^p |s|^p \int_0^{kT} \left|\cos\left(k^{-1}\omega t\right)\right|^p \mathrm{d}t - M_2 \int_0^{kT} |x+se_k|^p \,\mathrm{d}t + M_2M_3^p kT \\
&\leqslant \frac{1}{p}\left(\frac{\omega}{k}\right)^p |s|^p kT - kM_2 \int_0^{T} |x+se_1|^p \,\mathrm{d}t + M_2M_3^p kT \\
&\leqslant \frac{1}{p}\left(\frac{\omega}{k}\right)^p |s|^p kT - kM_2M_1^p \left(\int_0^{T} |x+se_1|^2 \,\mathrm{d}t\right)^{p/2} + M_2M_3^p kT \\
&= \frac{1}{p}\left(\frac{\omega}{k}\right)^p |s|^p kT - kM_2M_1^p \left(\int_0^{T} \left(|x|^2 + |se_1|^2\right) \mathrm{d}t\right)^{p/2} + M_2M_3^p kT \\
&= \frac{1}{p}\left(\frac{\omega}{k}\right)^p |s|^p kT - kM_2M_1^p \left(T|x|^2 + \frac{1}{2}Ts^2\right)^{p/2} + M_2M_3^p kT.
\end{aligned}$$

Since $M_2 = (2\omega^p T/(pM_1^p))(2/T)^{p/2}$, we have

$$\begin{aligned}
\varphi_k(x+se_k) &\leqslant \frac{1}{p}\left(\frac{\omega}{k}\right)^p |s|^p kT - kM_2M_1^p \left(\frac{1}{2}Ts^2\right)^{p/2} + M_2M_3^p kT \\
&\leqslant -\frac{1}{p}\omega^p |s|^p kT + M_2M_3^p kT
\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\varphi_k(x+se_k) &\leqslant \frac{1}{p}\left(\frac{\omega}{k}\right)^p |s|^p kT - kM_2M_1^p\big(T|x|^2\big)^{p/2} + M_2M_3^p kT \\ &\leqslant \frac{kT\omega^p}{p}\big(|s|^p - 2^{1+p/2}|x|^p\big) + M_2M_3^p kT \\ &\leqslant \frac{kT\omega^p}{p}\big(|s|^p - 2|x|^p\big) + M_2M_3^p kT. \end{aligned} \tag{2.12}$$

Let

$$r_1 = r_2 = r = 2^{1/p}\frac{M_3}{\omega}(pM_2)^{1/p},$$

so that $r \geqslant 1$. For $x + re_k \in \partial Q_k$, we get from (2.11) that

$$\varphi_k(x+re_k) \leqslant -\frac{1}{p}\omega^p r^p kT + M_2M_3^p kT < 0, \tag{2.13}$$

and, for $x + se_k \in \partial Q_k$ with $|x| = r$, we obtain from (2.12) that

$$\varphi_k(x+se_k) \leqslant \frac{kT\omega^p}{p}\big(|s|^p - 2|x|^p\big) + M_2M_3^p kT < 0. \tag{2.14}$$

If $s = 0$, by (H0) we get

$$\varphi_k(x) = -\int_0^{kT} F(t,x)\,\mathrm{d}t \leqslant 0 \tag{2.15}$$

for all $x \in \mathbb{R}^N$. By (2.13), (2.14), and (2.15) condition (G2) holds.

Moreover, we have

$$\begin{aligned}\varphi_k(x+se_k) &= \frac{1}{p}\int_0^{kT} |se_k'|^p\,\mathrm{d}t - \int_0^{kT} F(t, x+se_k)\,\mathrm{d}t \\ &\leqslant \frac{1}{p}\left(\frac{\omega}{k}\right)^p |s|^p \int_0^{kT} \big|\cos\big(k^{-1}\omega t\big)\big|^p\,\mathrm{d}t \\ &\leqslant \frac{1}{p}\left(\frac{\omega}{k}\right)^p |s|^p kT \leqslant 2M_3^p M_2 T\end{aligned}$$

for all $x + se_k \in Q_k$. Hence, φ_k has a critical point u_k for every $k \in \mathbb{N}$, and

$$\varphi_k(u_k) \leqslant 2M_3^p M_2 T.$$

Here it is easy to see that there is $k_1 \in \mathbb{N}$ such that $u_k \neq u_1$ for all $k \geqslant k_1$. Otherwise, we see that

$$\varphi_k(u_k) = k\varphi(u_1) \to \infty$$

as $k \to \infty$, which contradicts to the boundedness of $\varphi_k(u_k)$.

Reapplying what we have just shown, there is $k_2 > k_1$ such that $u_{k_1 k} \neq u_{k_1}$ for all $k_1 k \geqslant k_2$. Otherwise, we obtain

$$\varphi_{k_1 k}(u_{k_1 k}) = k\varphi_{k_1}(u_{k_1}) \to \infty$$

as $k \to \infty$, which contradicts to the boundedness of $\varphi_{k_1 k}(u_{k_1 k})$.

Now, it follows by the preceding that we have a sequence $\{u_{k_j}\}$ of distinct nonzero solutions of system (1.1), and the proof is complete. □

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