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On the rate of convergence in the central limit theorem for random sums of strongly mixing random variables

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Abstract. We present upper bounds for $\sup_{x\in\mathbb{R}} |\mathbf{P}\{Z_N < x\} - \Phi(x)|$, where $\Phi(x)$ is the standard normal distribution function, for random sums $Z_N = S_N/\sqrt{\mathbf{V}S_N}$ with variances $\mathbf{V}S_N > 0$ ($S_N = X_1 + \dots + X_N$) of centered strongly mixing or uniformly strongly mixing random variables $X_1, X_2, \dots$. Here the number of summands N is a nonnegative integer-valued random variable independent of $X_1, X_2, \dots$.

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1 Introduction and main results

Let $X_1, X_2, \dots$ be a sequence of real centered random variables (r.v.s). For $a \leqslant b$, we denote by $\mathcal{F}_a^b$ the σ-algebra of events generated by r.v.s $X_a, X_{a+1}, \dots, X_b$. As usual, $\mathbb{R}$ is the real line, $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{N}_0 = \{0, 1, 2, \dots\}$, and $\mathbf{1}_A$ is the indicator of an event A.

We consider weak dependence conditions defined between the "past" and "future" in terms of the strong mixing coefficient $\alpha(\tau)$ introduced by Rosenblatt (1956) and by the uniformly strong mixing coefficient $\varphi(\tau)$ introduced by Ibragimov (1959). We say that a sequence of r.v.s $X_1, X_2, \dots$ satisfies the strong mixing (s.m.) condition (or $X_1, X_2, \dots$ are strongly mixing) with the s.m. coefficient $\alpha(\tau)$ if

$$\alpha(\tau) = \sup_{t\in\mathbb{N}} \sup_{A\in\mathcal{F}_1^t,\, B\in\mathcal{F}_{t+\tau}^\infty} \bigl|\mathbf{P}(AB) - \mathbf{P}(A)\mathbf{P}(B)\bigr| \underset{\tau\to\infty}{\longrightarrow} 0 \tag{1.1}$$

(see [12]). We say that a sequence of r.v.s $X_1, X_2, \dots$ satisfies the uniformly strong mixing (u.s.m.) condition (or $X_1, X_2, \dots$ are uniformly strongly mixing) with the u.s.m. coefficient $\varphi(\tau)$ if

$$\varphi(\tau) = \sup_{t\in\mathbb{N}} \sup_{\substack{A\in\mathcal{F}_1^t, B\in\mathcal{F}_{t+\tau}^\infty \\ \mathbf{P}(A)>0}} \frac{|\mathbf{P}(AB) - \mathbf{P}(A)\mathbf{P}(B)|}{\mathbf{P}(A)} \underset{\tau\to\infty}{\longrightarrow} 0 \tag{1.2}$$

(see [4]).

In what follows, $\Phi(x)$ is the standard normal distribution function. By $C(\cdot)$ with an index or without it we denote a positive finite factor depending only on the quantities indicated in the parentheses (not necessarily the same at different occurrences).

Recall the two following results for sums with a fixed number n of summands of s.m. and u.s.m. r.v.s.

Theorem A. (*See* [15, *Thm.* 10].) *Let a sequence of real r.v.s* $X_1, X_2, \ldots$ *with* $\mathbf{E}X_i = 0$ *and* $\mathbf{E}|X_i|^s < \infty$, *where* $2 < s \leqslant 3$, *for all* $i = 1, \ldots, n$, *satisfy the s.m. condition* (1.1) *with the s.m. coefficient* $\alpha(\tau) \leqslant K\mathrm{e}^{-\mu\tau}$, *where* $0 < K < \infty$ *and* $\mu > 0$ *are constants. Denote*

$$\Delta_n = \sup_{x\in\mathbb{R}} \bigl|\mathbf{P}\{Z_n < x\} - \Phi(x)\bigr|, \qquad Z_n = \frac{1}{B_n}\sum_{i=1}^{n} X_i, \qquad B_n^2 = \mathbf{E}\Biggl(\sum_{i=1}^{n} X_i\Biggr)^2,$$

where the variance $B_n^2 > 0$. *Then, for all* $n = 1, 2, \ldots,$

$$\Delta_n \leqslant C_0(K,\mu,s)\frac{n\max_{1\leqslant i\leqslant n}\mathbf{E}|X_i|^s}{B_n^s}\ln^{s-1}(1+n). \tag{1.3}$$

The following corollary immediately follows from Theorem A.

Corollary A. *Let the conditions of Theorem* A *be satisfied. Suppose, in addition, that the variances* B_n^2 *of the sums* $S_n = X_1 + \cdots + X_n$ *satisfy the condition*

$$B_n^2 \geqslant c_0 n, \tag{1.4}$$

where $0 < c_0 < \infty$ *is a constant. Then, for all* $n = 1, 2, \ldots,$

$$\Delta_n \leqslant C(K,\mu,c_0,s)\frac{\max_{1\leqslant i\leqslant n}\mathbf{E}|X_i|^s}{n^{(s-2)/2}}\ln^{s-1}(1+n). \tag{1.5}$$

Now, we present a particular result of Theorem 1 by Rio [11].

Theorem B. (*See* [11].) *Let a strictly stationary sequence of real r.v.s* $X, X_1, X_2, \ldots$ *with* $\mathbf{E}X = 0$ *and* $|X_i| \leqslant M < \infty$, $i = 1, 2, \ldots,$ *satisfy the u.s.m. condition* (1.2) *with the u.s.m. coefficient* $\varphi(\tau)$ *such that*

$$A = \sum_{\tau=1}^{\infty} \tau\varphi(\tau) < \infty. \tag{1.6}$$

Suppose that the variances $B_n^2 = \mathbf{V}S_n$ *of the sums* $S_n = X_1 + \cdots + X_n$ *satisfy the condition*

$$\lim_{n\to\infty} B_n^2 = \infty. \tag{1.7}$$

Denote

$$\Delta_n = \sup_{x\in\mathbb{R}} \bigl|\mathbf{P}\{Z_n < x\} - \Phi(x)\bigr|, \qquad Z_n = \frac{S_n}{B_n}, \qquad \sigma^2 = \lim_{n\to\infty}\frac{B_n^2}{n}.$$

Then, for all $n = 1, 2, \ldots,$

$$\Delta_n \leqslant C_0(A,M,\sigma)\frac{1}{\sqrt{n}}. \tag{1.8}$$

In this paper, we are interested in estimates of the quantity

$$\Delta = \sup_{x\in\mathbb{R}} \bigl|\mathbf{P}\{Z_N < x\} - \Phi(x)\bigr|,$$

where

$$Z_N = \frac{S_N}{\sqrt{\mathbf{V}S_N}}, \qquad S_N = \sum_{i=1}^{N} X_i, \qquad S_0 = 0,$$

assuming that the variance $\mathbf{V}S_N > 0$, the number of summands N is a nonnegative integer-valued r.v. independent of $X_1, X_2, \dots$, and the centered summands $X_1, X_2, \dots$ are s.m. or u.s.m. r.v.s.

There are not many results on the rate of convergence in the central limit theorem for random sums with weakly dependent summands. Strictly stationary and uniformly strongly mixing sequences, assuming that the number of summands and summands are dependent, were considered in [9]. Similar results for strictly stationary sequences of martingales have been obtained in [8]. A stationary sequence of m-dependent r.v.s, assuming that the number of summands and summands are independent, was investigated in the recent paper [10]. Without the rate of convergence, the asymptotic normality of random sums of stationary m-dependent random variables was investigated in [14], in the recent paper [6], and that of martingales in [13].

However, the author has not found any published results on the upper bounds of the quantity Δ for random sums with summands satisfying the s.m. condition.

To investigate the asymptotic normality and the rate of convergence for random sums of independent (and dependent as well) summands, we use, as usual, the additional r.v.s

$$A_N = \sum_{i=1}^{N} \mathbf{E}X_i, \qquad B_N^2 = \sum_{i=1}^{N} \mathbf{V}X_i, \qquad l_{r,N} = \sum_{i=1}^{N} \mathbf{E}|X_i|^r.$$

Now, seemingly for the first time, we introduce the additional r.v.s

$$\kappa_N^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} \operatorname{cov}(X_i, X_j) = B_N^2 + 2\sum_{1\leqslant i<j\leqslant N} \operatorname{cov}(X_i, X_j),$$

which are very useful to investigate asymptotic the normality and the rate of convergence for random sums of dependent (including weakly dependent) summands. Here $\operatorname{cov}(\xi,\eta) = \mathbf{E}\xi\eta - \mathbf{E}\xi\mathbf{E}\eta$ is the covariance of real r.v.s ξ and η. Moreover, we assume that $\sum_{i=1}^{0}(\cdot) = 0$.

The main results of this paper are Theorems 1–4.

The following statement is valid in the case where the summands satisfy the s.m. condition.

Theorem 1. *Let a sequence of real r.v.s* $X_1, X_2, \dots$ *with* $\mathbf{E}X_i = 0$ *and* $\mathbf{E}|X_i|^s \leqslant d_s < \infty$*, where* $2 < s \leqslant 3$*,* $i = 1, 2, \dots,$ *satisfy the s.m. condition* (1.1) *with the s.m. coefficient* $\alpha(\tau) \leqslant K\mathrm{e}^{-\mu\tau}$*, where* $0 < K < \infty$ *and* $\mu > 0$ *are constants. Let* N *be a nonnegative integer-valued r.v. independent of* $X_1, X_2, \dots$*. Then, for all* $\alpha \in (0,1)$*,*

$$\Delta \leqslant \frac{C_0 d_s}{\alpha^{s/2}} \frac{\mathbf{E}N \ln^{s-1}(1+N)}{(\mathbf{E}\kappa_N^2)^{s/2}} + \max\biggl\{\frac{1}{\sqrt{2\pi\mathrm{e}\alpha}(1+\sqrt{\alpha})}, \frac{1}{1-\alpha}\biggr\} \frac{\mathbf{E}|\kappa_N^2 - \mathbf{E}\kappa_N^2|}{\mathbf{E}\kappa_N^2}, \tag{1.9}$$

where the factor $C_0 = C_0(K, \mu, s)$ *is taken from* (1.3) *of Theorem* A.

Note that Theorem A follows from Theorem 1 in the particular case of the fixed number n ($N = n$) of summands.

In particular, if the summands are identically distributed with zero mixed moments, then we have the following result.

Corollary 1. *Let real identically distributed r.v.s $X, X_1, X_2, \ldots$ with $\mathbf{E}X = 0$, $0 < \sigma^2 = \mathbf{E}X^2$, $\mathbf{E}|X|^s \leqslant d_s < \infty$, where $2 < s \leqslant 3$, and $\mathbf{E}X_iX_j = 0$, $1 \leqslant i \neq j \leqslant \infty$, satisfy the s.m. condition* (1.1) *with the s.m. coefficient $\alpha(\tau) \leqslant K\mathrm{e}^{-\mu\tau}$, where $0 < K < \infty$ and $\mu > 0$ are constants. Let N with $0 < \mathbf{E}N < \infty$ be a nonnegative integer-valued r.v. independent of $X_1, X_2, \ldots$. Then $Z_N = S_N/(\sigma\sqrt{\mathbf{E}N})$, and, for all $\alpha \in (0, 1)$,*

$$\varDelta \leqslant \frac{C_0 d_s}{\alpha^{s/2}\sigma^s}\frac{\mathbf{E}N\ln^{s-1}(1+N)}{(\mathbf{E}N)^{s/2}} + \max\left\{\frac{1}{\sqrt{2\pi\mathrm{e}\alpha}(1+\sqrt{\alpha})}, \frac{1}{1-\alpha}\right\}\frac{\mathbf{E}|N-\mathbf{E}N|}{\mathbf{E}N}. \tag{1.10}$$

To present the results for three concrete random indices N, we use the definition of the τ-shifted $\mathcal{L}$ distribution (τ-shifted Poisson distribution, τ-shifted binomial distribution, τ-shifted negative binomial distribution, and so on), first introduced in the paper [16]. For completeness, we recall these definitions. We write $\xi \sim \mathcal{L}$ if the distribution of a r.v. ξ is $\mathcal{L}$.

DEFINITION 1. We say that a discrete r.v. N is distributed by the τ-shifted $\mathcal{L}$ distribution ($\tau \geqslant 0$) (for short, $N - \tau \sim \mathcal{L}$), or N is the τ-shifted r.v., if for any discrete r.v. $\xi \sim \mathcal{L}$ taking values x_k with probabilities p_k,

$$\mathbf{P}\{N = x_k + \tau\} = \mathbf{P}\{\xi = x_k\} = p_k. \tag{1.11}$$

In particular, the 0-shifted $\mathcal{L}$ distribution coincides with the $\mathcal{L}$ distribution.

DEFINITION 2. We say that a r.v. N is distributed by the τ-shifted Poisson distribution with parameters $\tau \in \mathbb{N}_0$ and $\lambda > 0$ (for short, $N - \tau \sim \mathcal{P}(\lambda)$) if

$$\mathbf{P}\{N = k + \tau\} = \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}, \quad k = 0, 1, 2, \ldots. \tag{1.12}$$

DEFINITION 3. We say that a r.v. N is distributed by the τ-shifted binomial distribution with parameters $\tau \in \mathbb{N}_0$, $n \in \mathbb{N}$, and $0 < p < 1$ (for short, $N - \tau \sim \mathcal{B}(n, p)$) if

$$\mathbf{P}\{N = k + \tau\} = \binom{n}{k}p^k(1-p)^{n-k}, \quad k = 0, 1, \ldots, n. \tag{1.13}$$

DEFINITION 4. We say that a r.v. N is distributed by the τ-shifted negative binomial distribution with parameters $\tau \in \mathbb{N}_0$, $r \in \mathbb{N}$, and $0 < p < 1$ (for short, $N - \tau \sim \mathcal{NB}(r, p)$ if

$$\mathbf{P}\{N = k + \tau\} = \binom{k-1}{r-1}p^r(1-p)^{k-r}, \quad k = r, r+1, \ldots. \tag{1.14}$$

Now, we present the following statement for three presented τ-shifted $\mathcal{L}$ distributions.

Theorem 2. *Let real identically distributed r.v.s $X, X_1, X_2, \ldots$ with $\mathbf{E}X = 0$, $0 < \sigma^2 = \mathbf{E}X^2$, $\mathbf{E}|X|^s \leqslant d_s < \infty$, where $2 < s \leqslant 3$, and $\mathbf{E}X_iX_j = 0$, $1 \leqslant i \neq j \leqslant \infty$, satisfy the s.m. condition* (1.1) *with the s.m. coefficient $\alpha(\tau) \leqslant K\mathrm{e}^{-\mu\tau}$, where $0 < K < \infty$ and $\mu > 0$ are constants. Let N be a nonnegative integer-valued r.v. independent of $X_1, X_2, \ldots$. Then $Z_N = S_N/(\sigma\sqrt{\mathbf{E}N})$, and*:

(i) *If $N - \tau \sim \mathcal{P}(\lambda)$ with $\tau \in \mathbb{N}_0$ and $\lambda > 0$, then*

$$\varDelta \leqslant C_1\frac{\ln^{s-1}(1+\tau+\lambda)}{(\tau+\lambda)^{(s-2)/2}}. \tag{1.15}$$

(ii) *If* $N-\tau \sim \mathcal{B}(n,p)$ *with* $\tau \in \mathbb{N}_0$, $n \in \mathbb{N}$, *and* $0<p<1$, *then*

$$\Delta \leqslant C_2 \frac{\ln^{s-1}(1+\tau+np)}{(\tau+np)^{(s-2)/2}}. \tag{1.16}$$

(iii) *If* $N-\tau \sim \mathcal{NB}(r,p)$ *with* $\tau \in \mathbb{N}_0$, $r \in \mathbb{N}$, *and* $0<p<1$, *then*

$$\Delta \leqslant C_3 \frac{\ln^{s-1}(1+\tau+r/p)}{(\tau p+r)^{(s-2)/2}}. \tag{1.17}$$

Here $C_i = C_i(K,\mu,\sigma,d_s,s)$, $i=1,2,3$.

Since the 0-shifted $\mathcal{L}$ distribution coincides with the $\mathcal{L}$ distribution, substituting $\tau=0$ into Theorem 2, we obtain the corresponding estimates of Δ for a Poisson random sum, a binomial random sum, and a negative binomial random sum.

The estimates in Theorem 1, Corollary 1, and Theorem 2 contain logarithmic factors. The corresponding estimates contain no logarithmic factors in the case where the summands satisfy the u.s.m. condition. Namely, the following statement is valid.

Theorem 3. *Let a strictly stationary sequence of real r.v.s* $X, X_1, X_2, \dots$ *with* $\mathbf{E}X=0$ *and* $|X_i| \leqslant M < \infty$, $i=1,2,\dots$, *satisfy the u.s.m. condition* (1.2) *and* (1.6), *and let the variance* $\mathbf{V}(X_1+\dots+X_n)$ *of the sum* $X_1+\dots+X_n$ *satisfy condition* (1.7). *Let* N *be a nonnegative integer-valued r.v. independent of* $X_1, X_2, \dots$. *Then*

$$\Delta \leqslant C_0 \mathbf{E}\frac{1}{\sqrt{N}}\mathbf{1}_{\{N\geqslant 1\}} + 1.04\frac{\mathbf{E}|\kappa_N^2 - \mathbf{E}\kappa_N^2|}{\mathbf{E}\kappa_N^2}, \tag{1.18}$$

where the factor $C_0 = C_0(A,M,\sigma)$ *is from* (1.8) *of Theorem* B.

Note that Theorem B follows from Theorem 3 in the particular case of the fixed number n ($N=n$) of summands.

In particular, if the summands have zero mixed moments, we have the following result.

Corollary 2. *Let a strictly stationary sequence of real r.v.s* $X, X_1, X_2, \dots$ *with* $\mathbf{E}X=0$, $|X_i| \leqslant M < \infty$, $i=1,2,\dots$, $0<\sigma^2=\mathbf{E}X^2$, *and* $\mathbf{E}X_iX_j = 0$ *for* $1 \leqslant i \neq j \leqslant \infty$, *satisfy the u.s.m. condition* (1.2) *and* (1.6). *Let* N *with* $0<\mathbf{E}N<\infty$ *be a nonnegative integer-valued r.v. independent of* $X_1, X_2, \dots$. *Then* $Z_N = S_N/(\sigma\sqrt{\mathbf{E}N})$, *and*

$$\Delta \leqslant C_0 \mathbf{E}\frac{1}{\sqrt{N}}\mathbf{1}_{\{N\geqslant 1\}} + 1.04\frac{\mathbf{E}|N-\mathbf{E}N|}{\mathbf{E}N}. \tag{1.19}$$

As in Corollary 1 (where summands satisfy the s.m. condition (1.1)), the corresponding statement is valid for three presented τ-shifted $\mathcal{L}$ distributions when the summands satisfy the u.s.m. condition (1.2) and (1.6). Namely, we have the following statement.

Theorem 4. *Let a strictly stationary sequence of real r.v.s* $X, X_1, X_2, \dots$ *with* $\mathbf{E}X=0$, $|X_i| \leqslant M < \infty$, $i=1,2,\dots$, $0<\sigma^2=\mathbf{E}X^2$, *and* $\mathbf{E}X_iX_j=0$ *for* $1 \leqslant i \neq j \leqslant \infty$, *satisfy the u.s.m. condition* (1.2) *and* (1.6). *Let* N *be a nonnegative integer-valued r.v. independent of* $X_1, X_2, \dots$. *Then* $Z_N = S_N/(\sigma\sqrt{\mathbf{E}N})$, *and*:

(i) *If* $N-\tau \sim \mathcal{P}(\lambda)$ *with* $\tau \in \mathbb{N}_0$ *and* $\lambda>0$, *then*

$$\Delta \leqslant (2C_0+1.04)\frac{1}{\sqrt{\tau+\lambda}}. \tag{1.20}$$

(ii) *If* $N-\tau \sim \mathcal{B}(n,p)$ *with* $\tau \in \mathbb{N}_0$, $n \in \mathbb{N}$, *and* $0<p<1$, *then*

$$\Delta \leqslant (2C_0+1.04)\frac{1}{\sqrt{\tau+np}}. \tag{1.21}$$

(iii) *If* $N-\tau \sim \mathcal{NB}(r,p)$ *with* $\tau \in \mathbb{N}_0$, $r \in \mathbb{N}$, *and* $0<p<1$, *then*

$$\Delta \leqslant (C_0+1.04)\frac{1}{\sqrt{\tau p+r}}. \tag{1.22}$$

Here $C_0 = C_0(A,M,\sigma)$ *is taken from* (1.8) *of Theorem* B.

It remains to repeat the idea for $\tau = 0$: taking $\tau = 0$ in Theorem 4, we obtain the corresponding estimates of Δ for Poisson, binomial, and negative binomial random sums.

2 Auxiliary results

The relationships between the first moments of the random sum S_N and the corresponding moment characteristics of the r.v.s A_N and κ_N^2 are given in the following lemma.

Lemma 1. *Let* $X_1, X_2, \dots$ *be arbitrarily dependent, not necessarily identically distributed, r.v.s with* $\mathbf{E}X_i^2 < \infty$ *for all* $i = 1,2,\dots$, *and let* N *be a nonnegative integer-valued r.v. independent of* $X_1, X_2, \dots$. *Denote*

$$S_N = \sum_{i=1}^{N} X_i, \qquad A_N = \sum_{i=1}^{N} \mathbf{E}X_i,$$
$$B_N^2 = \sum_{i=1}^{N} \mathbf{V}X_i, \qquad \kappa_N^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} \operatorname{cov}(X_i, X_j).$$

Then

$$\mathbf{E}S_N = \mathbf{E}A_N, \tag{2.1}$$
$$\mathbf{E}S_N^2 = \mathbf{E}\kappa_N^2 + \mathbf{E}A_N^2, \tag{2.2}$$
$$\mathbf{V}S_N = \mathbf{E}\kappa_N^2 + \mathbf{V}A_N. \tag{2.3}$$

If r.v.s $X_1, X_2, \dots$ *are independent, then*

$$\mathbf{E}S_N^2 = \mathbf{E}B_N^2 + \mathbf{E}A_N^2, \tag{2.4}$$
$$\mathbf{V}S_N = \mathbf{E}B_N^2 + \mathbf{V}A_N. \tag{2.5}$$

Proof. Denote $p_k = \mathbf{P}\{N=k\}$, $k = 0,1,2,\dots$. Since N is independent of $X_1, X_2, \dots$, $A_k = \mathbf{E}S_k$ for $k = 1,2,\dots$, and $A_0 = \mathbf{E}S_0 = 0$, we have

$$\mathbf{E}S_N = \sum_{k=0}^{\infty} \mathbf{E}S_k p_k = \sum_{k=1}^{\infty} \mathbf{E}S_k p_k = \sum_{k=1}^{\infty} A_k p_k = \sum_{k=0}^{\infty} A_k p_k = \mathbf{E}A_N,$$

and (2.1) is proved.

Since $\kappa_k^2 = \mathbf{V}S_k$ for $k = 1, 2, \dots$, $\kappa_0^2 = 0$, and $A_0^2 = 0$, (2.2) follows from the relations

$$\begin{aligned}\mathbf{E}S_N^2 &= \sum_{k=0}^{\infty} \mathbf{E}S_k^2 p_k = \sum_{k=1}^{\infty} \mathbf{E}S_k^2 p_k = \sum_{k=1}^{\infty}\sum_{i=1}^{k}\sum_{j=1}^{k} \mathbf{E}X_iX_j p_k \\ &= \sum_{k=1}^{\infty} \kappa_k^2 p_k + \sum_{k=1}^{\infty} A_k^2 p_k = \sum_{k=0}^{\infty} \kappa_k^2 p_k + \sum_{k=0}^{\infty} A_k^2 p_k \\ &= \mathbf{E}\kappa_N^2 + \mathbf{E}A_N^2.\end{aligned}$$

Relation (2.3) follows from (2.2) and (2.1):

$$\mathbf{V}S_N = \mathbf{E}S_N^2 - (\mathbf{E}S_N)^2 = \mathbf{E}\kappa_N^2 + \mathbf{E}A_N^2 - (\mathbf{E}S_N)^2 = \mathbf{E}\kappa_N^2 + \mathbf{V}A_N.$$

Observing that $\kappa_N^2 = B_N^2$ for independent summands, we have that (2.2) and (2.3) reduce to (2.4) and (2.5). Lemma 1 is proved. □

To obtain upper estimates of the second moment $\mathbf{E}S_N^2$ and of the variance $\mathbf{V}S_N$ of the random sum $S_N = X_1 + \cdots + X_N$ for weakly dependent summands (satisfying the s.m. or the u.s.m. condition), we need the corresponding estimates of the sums $S_n = X_1 + \cdots + X_n$ with a fixed number n of summands. Such inequalities are well known, and therefore, we give them without proof.

The following estimates are valid for the sum S_n with a fixed number n of summands satisfying the s.m. condition.

Lemma 2. (*See, e.g.,* [15, *formula* (14)].) *Let a sequence of real r.v.s* $X_1, X_2, \dots$ *satisfy the s.m. condition* (1.1). *Denote*

$$S_n = \sum_{i=1}^{n} X_i, \qquad l_{s,n}^* = \sum_{i=1}^{n} \mathbf{E}^{2/s}|X_i|^s,$$

$$C_n\big(\alpha^{(s-2)/s}\big) = 1 + 16\sum_{\tau=1}^{n-1}\big(\alpha(\tau)\big)^{(s-2)/s}, \qquad C_n(\alpha) = 1 + 8\sum_{\tau=1}^{n-1}\alpha(\tau).$$

Then:

(i) *If* $\mathbf{E}|X_i|^s < \infty$, *where* $2 < s \leqslant 3$, *for all* $i = 1, \dots, n$, *then*

$$\mathbf{V}S_n \leqslant C_n\big(\alpha^{(s-2)/s}\big) l_{s,n}^*. \tag{2.6}$$

(ii) *If* $\mathbf{P}\{|X_i| \leqslant M\} = 1$ *for all* $i = 1, \dots, n$ *with a nonrandom constant* $M > 0$, *then*

$$\mathbf{V}S_n \leqslant C_n(\alpha) n M^2. \tag{2.7}$$

Note that to obtain inequality (2.6), it suffices to use the inequality in [3, p. 278, Cor. A.2]; to prove (2.7), we use inequality (1.4) in [5, p. 364, Lemma 1.2] or the inequality in [3, p. 277, Thm. A.5].

The corresponding estimates are valid for the sum S_n with a fixed number n of summands satisfying the u.s.m. condition.

Lemma 3. *Let a sequence of real r.v.s* $X_1, X_2, \dots$ *satisfy the u.s.m. condition* (1.2). *Denote*

$$S_n = \sum_{i=1}^{n} X_i, \qquad C_n\big(\varphi^{1/2}\big) = 1 + 4\sum_{\tau=1}^{n-1}\varphi^{1/2}(\tau), \qquad C_n(\varphi) = 1 + 4\sum_{\tau=1}^{n-1}\varphi(\tau).$$

Then:

(i) *If* $\mathbf{E}X_i^2 < \infty$ *for all* $i = 1, \dots, n$, *then*

$$\mathbf{V}S_n \leqslant C_n\big(\varphi^{1/2}\big) \sum_{i=1}^{n} \mathbf{V}X_i. \tag{2.8}$$

(ii) *If* $\mathbf{P}\{|X_i| \leqslant M\} = 1$ *for all* $i = 1, \dots, n$ *with a nonrandom constant* $M > 0$, *then*

$$\mathbf{V}S_n \leqslant C_n(\varphi) n M^2. \tag{2.9}$$

To obtain inequality (2.8), it suffices to use inequality (1.3) in [5, p. 363, Lemma 1.1] or the inequality in [3, p. 278, Thm. A.6]; to prove (2.9), we use inequality (20.29) in [2, p. 171, Lemma 2].

Now, in Lemmas 4 and 5, we present upper estimates of the second moment $\mathbf{E}S_N^2$ and of the variance $\mathbf{V}S_N$ of the random sum S_N for summands satisfying the s.m. or the u.s.m condition, respectively.

Lemma 4. *Let a sequence of real r.v.s* $X_1, X_2, \dots$ *satisfy the s.m. condition* (1.1), *and let* N *be a nonnegative integer-valued r.v. independent of* $X_1, X_2, \dots$. *Denote*

$$S_N = \sum_{i=1}^{N} X_i, \qquad A_N = \sum_{i=1}^{N} \mathbf{E}X_i, \qquad l_{s,N}^* = \sum_{i=1}^{N} \mathbf{E}^{2/s}|X_i|^s.$$

Then:

(i) *If* $\mathbf{E}|X_i|^s < \infty$, *where* $2 < s \leqslant 3$, *for all* $i = 1, 2, \dots$, *then*

$$\mathbf{E}S_N^2 \leqslant C_\infty\big(\alpha^{(s-2)/s}\big)\mathbf{E}l_{s,N}^* + \mathbf{E}A_N^2, \tag{2.10}$$

$$\mathbf{V}S_N \leqslant C_\infty\big(\alpha^{(s-2)/s}\big)\mathbf{E}l_{s,N}^* + \mathbf{V}A_N. \tag{2.11}$$

(ii) *If* $\mathbf{P}\{|X_i| \leqslant M\} = 1$ *for all* $i = 1, 2, \dots$ *with a nonrandom constant* $M > 0$, *then*

$$\mathbf{E}S_N^2 \leqslant C_\infty(\alpha) M^2 \mathbf{E}N + \mathbf{E}A_N^2, \tag{2.12}$$

$$\mathbf{V}S_N \leqslant C_\infty(\alpha) M^2 \mathbf{E}N + \mathbf{V}A_N. \tag{2.13}$$

Here the factors $C_\infty(\alpha^{(s-2)/s})$ *and* $C_\infty(\alpha)$ *are taken from Lemma* 2.

Proof. Since $A_k = \mathbf{E}S_k$ for all $k = 1, 2, \dots$ and $A_0 = 0$, we rewrite $\mathbf{E}S_N^2$ as follows:

$$\mathbf{E}S_N^2 = \sum_{k=0}^{\infty} \mathbf{E}S_k^2 p_k = \sum_{k=0}^{\infty} \big(\mathbf{E}S_k^2 - (\mathbf{E}S_k)^2\big)p_k + \sum_{k=0}^{\infty} (\mathbf{E}S_k)^2 p_k = \sum_{k=1}^{\infty} \mathbf{V}S_k p_k + \mathbf{E}A_N^2.$$

In this equality, estimating $\mathbf{V}S_k$ according to (2.6) and (2.7), we obtain inequalities (2.10) and (2.12):

$$\mathbf{E}S_N^2 \leqslant C_\infty\big(\alpha^{(s-2)/s}\big) \sum_{k=1}^{\infty} l_{s,k}^* p_k + \mathbf{E}A_N^2 = C_\infty\big(\alpha^{(s-2)/s}\big)\mathbf{E}l_{s,N}^* + \mathbf{E}A_N^2,$$

$$\mathbf{E}S_N^2 \leqslant C_\infty(\alpha) M^2 \sum_{k=1}^{\infty} k p_k + \mathbf{E}A_N^2 = C_\infty(\alpha) M^2 \mathbf{E}N + \mathbf{E}A_N^2,$$

and then, since $\mathbf{E}S_N = \mathbf{E}A_N$ by (2.1), from (2.10) and (2.12) we obtain (2.11) and (2.13).

Lemma 4 is proved. □

Lemma 5. *Let a sequence of real r.v.s $X_1, X_2, \dots$ satisfy the u.s.m. condition* (1.2)*, and let N be a nonnegative integer-valued r.v. independent of $X_1, X_2, \dots$. Denote*

$$S_N = \sum_{i=1}^{N} X_i, \qquad A_N = \sum_{i=1}^{N} \mathbf{E}X_i, \qquad B_N^2 = \sum_{i=1}^{N} \mathbf{V}X_i.$$

Then:

(i) *If $\mathbf{E}X_i^2 < \infty$ for all $i = 1, 2, \dots$, then*

$$\mathbf{E}S_N^2 \leqslant C_\infty(\varphi^{1/2})\mathbf{E}B_N^2 + \mathbf{E}A_N^2, \tag{2.14}$$

$$\mathbf{V}S_N \leqslant C_\infty(\varphi^{1/2})\mathbf{E}B_N^2 + \mathbf{V}A_N. \tag{2.15}$$

(ii) *If $\mathbf{P}\{|X_i| \leqslant M\} = 1$ for all $i = 1, 2, \dots$ with a nonrandom constant $M > 0$, then*

$$\mathbf{E}S_N^2 \leqslant C_\infty(\varphi)M^2\mathbf{E}N + \mathbf{E}A_N^2, \tag{2.16}$$

$$\mathbf{V}S_N \leqslant C_\infty(\varphi)M^2\mathbf{E}N + \mathbf{V}A_N. \tag{2.17}$$

Here the factors $C_\infty(\varphi^{1/2})$ and $C_\infty(\varphi)$ are taken from Lemma 3.

The proof of Lemma 5 is similar to that of Lemma 4, so we omit the details and indicate only one difference: while estimating the variance $\mathbf{V}S_k$, instead of inequalities (2.6) and (2.7) in Lemma 4, we use inequalities (2.8) and (2.9), respectively.

Now, in Lemmas 6, 7, and 8, we present some useful estimates of $\mathbf{E}(1/\sqrt{1+N})$ (or $\mathbf{E}(1/\sqrt{N})$) and $\mathbf{E}|N - \mathbf{E}N|/\mathbf{E}N$ when the random number N is a τ-shifted Poisson r.v., a τ-shifted binomial r.v., and a τ-shifted negative binomial r.v.

Lemma 6. *If $N - \tau \sim \mathcal{P}(\lambda)$ with $\tau \in \mathbb{N}_0$ and $\lambda > 0$, then*

$$\frac{1}{\sqrt{1+\tau+\lambda}} \leqslant \mathbf{E}\frac{1}{\sqrt{1+N}} \leqslant \frac{\sqrt{2}}{\sqrt{1+\tau+\lambda}}, \tag{2.18}$$

$$\frac{\mathbf{E}|N - \mathbf{E}N|}{\mathbf{E}N} \leqslant \frac{1}{\sqrt{\tau+\lambda}}. \tag{2.19}$$

Proof. Let $p_k = \mathbf{P}\{N = k + \tau\} = (\lambda^k/k!)\mathrm{e}^{-\lambda}$, $k = 0, 1, 2, \dots$. Since

$$\mathbf{E}\frac{1}{1+N} = \sum_{k=0}^{\infty} \frac{1}{1+\tau+k} p_k \leqslant \frac{1}{1+\tau},$$

$$\mathbf{E}\frac{1}{1+N} = \mathrm{e}^{-\lambda}\sum_{k=0}^{\infty} \frac{1}{1+\tau+k}\frac{\lambda^k}{k!} \leqslant \mathrm{e}^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{(k+1)!} = \frac{1}{\lambda}\left(1 - \mathrm{e}^{-\lambda}\right) < \frac{1}{\lambda},$$

and

$$\max\{1+\tau,\, \lambda\} \geqslant \frac{1+\tau+\lambda}{2} \quad \text{for} \quad \tau \geqslant 0 \text{ and } \lambda > 0,$$

we have

$$\mathbf{E}\frac{1}{1+N} \leqslant \min\left\{\frac{1}{1+\tau}, \frac{1}{\lambda}\right\} = \frac{1}{\max\{1+\tau,\, \lambda\}} \leqslant \frac{2}{1+\tau+\lambda}. \tag{2.20}$$

The function $f(x) = 1/\sqrt{1+x}$ is convex for $x \in (-1, \infty)$. Therefore, by Jensen's inequality, Lyapunov's inequality, and (2.20) we get that

$$\frac{1}{\sqrt{1+\tau+\lambda}} = \frac{1}{\sqrt{1+\mathbf{E}N}} = f(\mathbf{E}N) \leqslant \mathbf{E}f(N) = \mathbf{E}\frac{1}{\sqrt{1+N}}$$
$$\leqslant \left(\mathbf{E}\frac{1}{1+N}\right)^{1/2} \leqslant \frac{\sqrt{2}}{\sqrt{1+\tau+\lambda}},$$

and (2.18) is proved.

Since $\mathbf{E}N = \tau + \lambda$ and $\mathbf{V}N = \lambda$ for a r.v. $N - \tau \sim \mathcal{P}(\lambda)$, we obtain

$$\frac{\mathbf{E}|N - \mathbf{E}N|}{\mathbf{E}N} \leqslant \frac{\sqrt{\mathbf{V}N}}{\mathbf{E}N} \leqslant \frac{1}{\sqrt{\tau+\lambda}},$$

so that (2.19) and thus Lemma 6 are proved. □

Lemma 7. *If* $N - \tau \sim \mathcal{B}(n, p)$ *with* $\tau \in \mathbb{N}_0$, $n \in \mathbb{N}$, *and* $0 < p < 1$, *then*

$$\frac{1}{\sqrt{1+\tau+np}} \leqslant \mathbf{E}\frac{1}{\sqrt{1+N}} \leqslant \frac{\sqrt{2}}{\sqrt{1+\tau+np}}, \tag{2.21}$$

$$\frac{\mathbf{E}|N - \mathbf{E}N|}{\mathbf{E}N} < \frac{1}{\sqrt{\tau+np}}. \tag{2.22}$$

Proof. Let $p_k = \mathbf{P}\{N = k + \tau\} = \binom{n}{k} p^k q^{n-k}$, $k = 0, 1, \ldots, n$, $0 < p < 1$, $q = 1 - p$. Then, as in the proof of Lemma 6 (with $f(x) = 1/\sqrt{1+x}$), we make sure that (2.21) is valid:

$$\mathbf{E}\frac{1}{1+N} = \sum_{k=0}^{n} \frac{1}{1+\tau+k} p_k \leqslant \frac{1}{1+\tau},$$

$$\mathbf{E}\frac{1}{1+N} = \sum_{k=0}^{n} \frac{1}{1+\tau+k} p_k \leqslant \sum_{k=0}^{n} \frac{n!}{(k+1)!(n-k)!} p^k q^{n-k}$$
$$= \frac{1}{(n+1)p} \sum_{l=1}^{n+1} \frac{(n+1)!}{l!((n+1)-l)!} p^l q^{(n+1)-l}$$
$$= \frac{1-q^{n+1}}{(n+1)p} < \frac{1}{np},$$

$$\mathbf{E}\frac{1}{1+N} \leqslant \min\left\{\frac{1}{1+\tau}, \frac{1}{np}\right\} = \frac{1}{\max\{1+\tau, np\}} \leqslant \frac{2}{1+\tau+np},$$

$$\frac{1}{\sqrt{1+\tau+np}} = \frac{1}{\sqrt{1+\mathbf{E}N}} = f(\mathbf{E}N) \leqslant \mathbf{E}f(N) = \mathbf{E}\frac{1}{\sqrt{1+N}}$$
$$\leqslant \left(\mathbf{E}\frac{1}{1+N}\right)^{1/2} \leqslant \frac{\sqrt{2}}{\sqrt{1+\tau+np}}.$$

Here we used the fact that $\mathbf{E}N = \tau + np$ for a r.v. $N - \tau \sim \mathcal{B}(n,p)$.

Moreover, $\mathbf{V}N = npq$ for a r.v. $N - \tau \sim \mathcal{B}(n,p)$. Therefore

$$\frac{\mathbf{E}|N - \mathbf{E}N|}{\mathbf{E}N} \leqslant \frac{\sqrt{\mathbf{V}N}}{\mathbf{E}N} < \frac{1}{\sqrt{\tau+np}}.$$

Lemma 7 is proved. □

Lemma 8. *Let* $N - \tau \sim \mathcal{NB}(r,p)$ *with* $\tau \in \mathbb{N}_0$, $r \in \mathbb{N}$, *and* $0 < p < 1$. *Then*

$$\frac{1}{\sqrt{\tau + r/p}} \leqslant \mathbf{E}\frac{1}{\sqrt{N}} \leqslant \frac{1}{\sqrt{\tau+r}}, \tag{2.23}$$
$$\frac{\mathbf{E}|N - \mathbf{E}N|}{\mathbf{E}N} < \frac{1}{\sqrt{\tau p + r}}. \tag{2.24}$$

Proof. Let $p_k = \mathbf{P}\{N = k + \tau\} = \binom{k-1}{r-1}p^r q^{k-r}$, $k = r, r+1, r+2, \ldots$, $0 < p < 1$, $q = 1 - p$. Then

$$\mathbf{E}\frac{1}{N} = \sum_{k=r}^{\infty} \frac{1}{\tau + k} p_k \leqslant \frac{1}{\tau + r}, \tag{2.25}$$

because $\sum_{k=r}^{\infty} p_k = 1$.

The function $f(x) = 1/\sqrt{x}$ is convex for $x \in (0, \infty)$. Therefore, by Jensen's inequality, Lyapunov's inequality, (2.25), and the equality $\mathbf{E}N = \tau + r/p$ for a r.v. $N - \tau \sim \mathcal{NB}(r,p)$, we get that

$$\frac{1}{\sqrt{\tau + r/p}} = \frac{1}{\sqrt{\mathbf{E}N}} = f(\mathbf{E}N) \leqslant \mathbf{E}f(N) = \mathbf{E}\frac{1}{\sqrt{N}}$$
$$\leqslant \left(\mathbf{E}\frac{1}{N}\right)^{1/2} \leqslant \frac{1}{\sqrt{\tau+r}},$$

and (2.23) is proved.

Since $\mathbf{E}N = \tau + r/p$ and $\mathbf{V}N = r(1-p)/p^2$ for a r.v. $N - \tau \sim \mathcal{NB}(r,p)$, we obtain

$$\frac{\mathbf{E}|N - \mathbf{E}N|}{\mathbf{E}N} \leqslant \frac{\sqrt{\mathbf{V}N}}{\mathbf{E}N} < \frac{1}{\sqrt{\tau p + r}},$$

so that (2.24) and thus Lemma 8 are proved. □

3 Decomposition of $\Delta(x)$ with $\mathbf{E}X_i = 0, i = 1, 2, \dots$

In addition, denote

$$\Delta(x) = \mathbf{P}\{S_N < x\sqrt{\mathbf{V}S_N}\} - \Phi(x), \qquad S_N = \sum_{i=1}^{N} X_i,$$

$$\xi_k = \frac{S_k}{\sqrt{\mathbf{V}S_k}}, \quad a_k = \frac{\sqrt{\mathbf{V}S_N}}{\sqrt{\mathbf{V}S_k}}, \quad S_k = \sum_{i=1}^{k} X_i, \quad k = 1, 2, \dots,$$

where $X_1, X_2, \dots$ are arbitrarily dependent r.v.s, $\mathbf{V}S_N > 0$, and $S_0 = 0$. It is clear that if N is a nonnegative integer-valued r.v. with $p_k = \mathbf{P}\{N = k\}$, $k = 0, 1, 2, \dots$, independent of $X_1, X_2, \dots$, then, for all $x \in \mathbb{R}$,

$$\Delta(x) = \sum_{k=0}^{\infty} \big[\mathbf{P}\{S_k < x\sqrt{\mathbf{V}S_N}\} - \Phi(x)\big] p_k.$$

Let $K(\alpha) = \{k \in \mathbb{N}\colon |\mathbf{V}S_k - \mathbf{V}S_N| \leqslant (1-\alpha)\mathbf{V}S_N\}$ and $\overline{K}(\alpha) = \{k \in \mathbb{N}\colon |\mathbf{V}S_k - \mathbf{V}S_N| > (1-\alpha)\mathbf{V}S_N\}$ for $\alpha \in (0, 1)$.

First, we observe that $\mathbf{V}S_k \geqslant \alpha\mathbf{V}S_N > 0$ if $k \in K(\alpha)$, because $\alpha \in (0, 1)$ and $\mathbf{V}S_N > 0$.

Since

$$\big[\mathbf{P}\{S_0 < x\sqrt{\mathbf{V}S_N}\} - \Phi(x)\big] p_0 = \big[\mathbf{1}_{\{x>0\}} - \Phi(x)\big] p_0$$

and

$$\begin{aligned}
&\sum_{k=1}^{\infty} \big[\mathbf{P}\{S_k < x\sqrt{\mathbf{V}S_N}\} - \Phi(x)\big] p_k \\
&\quad = \sum_{k \in K(\alpha)} \big[\mathbf{P}\{\xi_k < x a_k\} - \Phi(x a_k)\big] p_k + \sum_{k \in K(\alpha)} \big[\Phi(x a_k) - \Phi(x)\big] p_k \\
&\qquad + \sum_{k \in \overline{K}(\alpha)} \big[\mathbf{P}\{S_k < x\sqrt{\mathbf{V}S_N}\} - \Phi(x)\big] p_k,
\end{aligned}$$

we can state the following:

Proposition 1. *Let $X_1, X_2, \dots$ be arbitrarily dependent, not necessarily identically distributed, r.v.s with $\mathbf{E}X_i = 0$ for all $i = 1, 2, \dots$, and let N be a nonnegative integer-valued r.v. with $p_k = \mathbf{P}\{N = k\}$, $k = 0, 1, 2, \dots$, independent of $X_1, X_2, \dots$. Denote*

$$\Delta(x) = \mathbf{P}\{S_N < x\sqrt{\mathbf{V}S_N}\} - \Phi(x), \qquad S_N = \sum_{i=1}^{N} X_i,$$

$$\xi_k = \frac{S_k}{\sqrt{\mathbf{V}S_k}}, \quad a_k = \frac{\sqrt{\mathbf{V}S_N}}{\sqrt{\mathbf{V}S_k}}, \quad S_k = \sum_{i=1}^{k} X_i, \quad k = 1, 2, \dots,$$

where $S_0 = 0$ and $\mathbf{V}S_N > 0$. Then, for all $x \in \mathbb{R}$,

$$\Delta(x) = \Sigma_1(x) + \Sigma_2(x) + \Sigma_3(x) + \Sigma_4(x), \tag{3.1}$$

where

$$\Sigma_1(x) = \sum_{k\in K(\alpha)} \big[\mathbf{P}\{\xi_k < xa_k\} - \Phi(xa_k)\big]p_k,$$

$$\Sigma_2(x) = \sum_{k\in K(\alpha)} \big[\Phi(xa_k) - \Phi(x)\big]p_k,$$

$$\Sigma_3(x) = \big[\mathbf{1}_{\{x>0\}} - \Phi(x)\big]p_0,$$

$$\Sigma_4(x) = \sum_{k\in \overline{K}(\alpha)} \big[\mathbf{P}\{S_k < x\sqrt{\mathbf{V}S_N}\} - \Phi(x)\big]p_k,$$

where $K(\alpha) = \{k \in \mathbb{N}\colon |\mathbf{V}S_k - \mathbf{V}S_N| \leqslant (1-\alpha)\mathbf{V}S_N\}$ *and* $\overline{K}(\alpha) = \{k \in \mathbb{N}\colon |\mathbf{V}S_k - \mathbf{V}S_N| > (1-\alpha)\mathbf{V}S_N\}$ *for* $\alpha \in (0, 1)$.

4 Proofs of the results for random sums of strongly mixing random variables

Proof of Theorem 1. We estimate the quantities on the right-hand side of Eq. (3.1) under the condition that a sequence of real r.v.s $X_1, X_2, \dots$ with $\mathbf{E}X_i = 0$ and $\mathbf{E}|X_i|^s \leqslant d_s < \infty$, where $2 < s \leqslant 3$, for all $i = 1, 2, \dots$, satisfies the s.m. condition (1.1).

Estimation of $|\Sigma_1(x)|$. It is clear that

$$\big|\Sigma_1(x)\big| \leqslant \sum_{k\in K(\alpha)} \sup_{y\in\mathbb{R}} \big|\mathbf{P}\{S_k < y\sqrt{\mathbf{V}S_k}\} - \Phi(y)\big|p_k.$$

Therefore, to estimate the right-hand side of this inequality, we can use estimate (1.3) of Theorem A for the sum S_k with a fixed number $k = 1, 2, \dots$ of summands. Since $\mathbf{V}S_k \geqslant \alpha\mathbf{V}S_N$ for $k \in K(\alpha)$, we obtain that

$$\begin{aligned}\big|\Sigma_1(x)\big| &\leqslant C_0 d_s \sum_{k\in K(\alpha)} \frac{k}{(\mathbf{V}S_k)^{s/2}} \ln^{s-1}(1+k)p_k \\ &\leqslant \frac{C_0 d_s}{\alpha^{s/2}} \frac{1}{(\mathbf{V}S_N)^{s/2}} \sum_{k\in K(\alpha)} k\ln^{s-1}(1+k)p_k \\ &\leqslant \frac{C_0 d_s}{\alpha^{s/2}} \frac{\mathbf{E}N\ln^{s-1}(1+N)\mathbf{1}_{\{N\in K(\alpha)\}}}{(\mathbf{V}S_N)^{s/2}},\end{aligned} \tag{4.1}$$

where $C_0 = C_0(K, \mu, s)$ is taken from (1.3) of Theorem A.

Estimation of $|\Sigma_2(x)|$. To estimate $|\Sigma_2(x)|$, we use the following estimate for the standard normal distribution function $\Phi(x)$ (see [7, p. 114]):

$$\sup_{x\in\mathbb{R}} \big|\Phi(xa) - \Phi(x)\big| \leqslant \frac{1}{\sqrt{2\pi e}} \cdot \begin{cases} a^{-1} - 1 & \text{if } 0 < a < 1, \\ a - 1 & \text{if } a \geqslant 1. \end{cases} \tag{4.2}$$

Since $\mathbf{V}S_k \geqslant \alpha \mathbf{V}S_N$ for $k \in K(\alpha)$, we observe that, for all $k \in K(\alpha)$,

$$|a_k - 1| = \frac{|\mathbf{V}S_k - \mathbf{V}S_N|}{\sqrt{\mathbf{V}S_k}(\sqrt{\mathbf{V}S_k} + \sqrt{\mathbf{V}S_N})} \leqslant \frac{1}{\sqrt{\alpha}(1+\sqrt{\alpha})} \frac{|\mathbf{V}S_k - \mathbf{V}S_N|}{\mathbf{V}S_N},$$
$$\left|a_k^{-1} - 1\right| = \frac{|\mathbf{V}S_k - \mathbf{V}S_N|}{\sqrt{\mathbf{V}S_N}(\sqrt{\mathbf{V}S_k} + \sqrt{\mathbf{V}S_N})} \leqslant \frac{1}{1+\sqrt{\alpha}} \frac{|\mathbf{V}S_k - \mathbf{V}S_N|}{\mathbf{V}S_N}.$$

Therefore, taking $a = a_k = \sqrt{\mathbf{V}S_N}/\sqrt{\mathbf{V}S_k}$ in (4.2) and using two last inequalities, we obtain that, for all $k \in K(\alpha)$,

$$\sup_{x\in\mathbb{R}} \left|\Phi(xa_k) - \Phi(x)\right| \leqslant \frac{1}{\sqrt{2\pi \mathrm{e}\alpha}(1+\sqrt{\alpha})} \frac{|\mathbf{V}S_k - \mathbf{V}S_N|}{\mathbf{V}S_N}. \tag{4.3}$$

Substituting (4.3) into the expression of $\Sigma_2(x)$ and observing that $\mathbf{V}S_k = \kappa_k^2$ for any fixed $k = 1, 2, \dots$, we obtain that

$$\left|\Sigma_2(x)\right| \leqslant \frac{1}{\sqrt{2\pi \mathrm{e}\alpha}(1+\sqrt{\alpha})} \frac{1}{\mathbf{V}S_N} \sum_{k\in K(\alpha)} \left|\kappa_k^2 - \mathbf{V}S_N\right| p_k \tag{4.4}$$
$$= \frac{1}{\sqrt{2\pi \mathrm{e}\alpha}(1+\sqrt{\alpha})} \frac{\mathbf{E}|\kappa_N^2 - \mathbf{V}S_N| \mathbf{1}_{\{N\in K(\alpha)\}}}{\mathbf{V}S_N}. \tag{4.5}$$

Estimation of $|\Sigma_3(x)| + |\Sigma_4(x)|$. We observe that $|\mathbf{V}S_k - \mathbf{V}S_N| > (1-\alpha)\mathbf{V}S_N$ for $k = 0$ with $\mathbf{V}S_N > 0$ and $S_0 = 0$. Therefore,

$$\left|\Sigma_3(x)\right| + \left|\Sigma_4(x)\right| \leqslant p_0 + \sum_{k\in \overline{K}(\alpha)} p_k = \sum_{k\geqslant 0:\ |\kappa_k^2 - \mathbf{V}S_N| > (1-\alpha)\mathbf{V}S_N} p_k$$
$$\leqslant \frac{\mathbf{E}\left|\kappa_N^2 - \mathbf{V}S_N\right| \mathbf{1}_{\{N\in \overline{K}(\alpha)\cup\{0\}\}}}{(1-\alpha)\mathbf{V}S_N}. \tag{4.6}$$

Substituting (4.1), (4.5), and (4.6) into (3.1) and taking into account that $\mathbf{V}S_N = \mathbf{E}\kappa_N^2$ in the case of $\mathbf{E}X_i = 0$ for all $i = 1, 2, \dots$ (see (2.3) of Lemma 1), we obtain estimate (1.9) of Theorem 1.

Theorem 1 is proved. □

Proof of Corollary 1. The proof immediately follows from Theorem 1. □

Proof of Theorem 2. Since $\Delta \leqslant 0.5416$ [1, p. 103], we assume, without loss of generality, that $\mathbf{E}N$ is sufficiently large. To estimate the first term in (1.10) of Corollary 1, we use the estimate

$$\mathbf{E}N \ln^{s-1}(1+N) \leqslant \sqrt{\mathbf{V}N}\, \mathbf{E}^{1/2} \ln^{2(s-1)}(1+N) + \mathbf{E}N \mathbf{E} \ln^{s-1}(1+N). \tag{4.7}$$

Now, observing that the functions $f_1(x) = \ln^{2(s-1)}(\mathrm{e}^{2s-3} + 1 + x)$ and $f_2(x) = \ln^{s-1}(\mathrm{e}^{s-2} + 1 + x)$, where $2 < s \leqslant 3$, are strictly concave for $x \in (-1, \infty)$, by Jensen's inequality we obtain that

$$\mathbf{E} \ln^{2(s-1)}(1+N) < \ln^{2(s-1)}\bigl(\mathrm{e}^{2s-3} + 1 + \mathbf{E}N\bigr), \tag{4.8}$$
$$\mathbf{E} \ln^{s-1}(1+N) < \ln^{s-1}\bigl(\mathrm{e}^{s-2} + 1 + \mathbf{E}N\bigr). \tag{4.9}$$

Now, substitute (4.8) and (4.9) into (4.7) and the obtained inequality into the first term in (1.10) of Corollary 1. Then, in the corresponding cases of the number N of summands, we can estimate the second term in (1.10) by (2.19) of Lemma 6, by (2.22) of Lemma 7, and by (2.24) of Lemma 8, obtaining (1.15), (1.16), and (1.17), respectively.

Theorem 2 is proved. □

5 Proofs of the results for random sums of uniformly strongly mixing random variables

Proof of Theorem 3. The proof of Theorem 3 is similar to that of Theorem 1. To estimate Δ, we also use decomposition (3.1). The difference is only that estimating $|\Sigma_1(x)|$, we use (1.8) of Theorem B instead of (1.3) of Theorem A in the proof of Theorem 1. So, we obtain that, under the conditions of Theorem 3,

$$\left|\Sigma_1(x)\right| \leqslant C_0 \mathbf{E} \frac{1}{\sqrt{N}} \mathbf{1}_{\{N \in K(\alpha)\}}, \tag{5.1}$$

where $C_0 = C_0(A, M, \sigma)$ is taken from (1.8) of Theorem B.

We observe that the estimates of $|\Sigma_2(x)|$ and of $|\Sigma_3(x)| + |\Sigma_4(x)|$ in (4.5) and (4.6) are valid for any dependence condition. Therefore, substituting (5.1), (4.5), and (4.6) into (3.1), we obtain (1.18) of Theorem 3.

Theorem 3 is proved. □

Proof of Corollary 2. The proof immediately follows from Theorem 3. □

Proof of Theorem 4. Substituting (2.18) and (2.19) of Lemma 6, (2.21) and (2.22) of Lemma 7, and (2.23) and (2.24) of Lemma 8 into (1.19) of Corollary 2, we obtain the estimates (1.20)–(1.22) of Theorem 4. □

Theorem 4 is proved. □

Final remark. More general statements for random sums with strongly mixed and uniformly strongly mixed summands are considered in Theorems 1 and 3, respectively. In Corollaries 1 and 2 and in Theorems 2 and 4, the estimates of the quantity Δ are given in the simplest case, where, in addition, the summands are uncorrelated. For correlated summands, the estimation of Δ in the statements analogous to Corollaries 1 and 2 and to Theorems 2 and 4 require a more detailed analysis of the quantity $\mathbf{E}|\kappa_N^2 - \mathbf{E}\kappa_N^2|/\mathbf{E}\kappa_N^2$, which is left to the future.

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