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Convergence analysis of the spectral collocation methods for two-dimensional nonlinear weakly singular Volterra integral equations*

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Abstract. We apply Jacobi spectral collocation approximation to a two-dimensional nonlinear weakly singular Volterra integral equation with smooth solutions. Under reasonable assumptions on the nonlinearity, we carry out complete convergence analysis of the numerical approximation in the L^∞-norm and weighted L^2-norm. The provided numerical examples show that the proposed spectral method enjoys spectral accuracy.

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1 Introduction

This paper is concerned with the numerical solutions of the following two-dimensional nonlinear Volterra integral equations with weakly singular kernels:

$$y(s,t) = \int_0^s \int_0^t (s-\sigma)^{-\mu}(t-\tau)^{-\nu} K\bigl(s,t,\sigma,\tau,y(\sigma,\tau)\bigr)\,\mathrm{d}\tau\,\mathrm{d}\sigma + f(s,t), \quad (s,t)\in \Lambda = [0,T]^2, \tag{1.1}$$

where $0<\mu,\nu<1$, $y(s,t)$ is an unknown function, $f(s,t)$ and $K(s,t,\sigma,\tau,y)$ are given continuous functions defined on Λ and $\Theta = D\times\mathbb{R}$ ($D =: \{(s,t,\sigma,\tau)\colon\ 0\leqslant\sigma\leqslant s\leqslant T,\ 0\leqslant\tau\leqslant t\leqslant T\}$), respectively. $K(s,t,\sigma,\tau,y)$ is nonlinear in the variable y with continuous $\partial^2 K/(\partial y\partial s)$ and $\partial^2 K/(\partial y\partial t)$ on Θ. We consider the case that Eq. (1.1) has a unique smooth solution.

Two-dimensional integro-differential equations are models of many problems arising in engineering and mechanics fields (see, e.g., [18, 40]); therefore their numerical simulations have received considerable attention, and many efficient numerical methods for such equations have been proposed, such as the differential

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transform method [12, 13, 16, 38], the Tau method [36, 37, 39], the Euler-type method [24], the Runge–Kutta method [3], the expansion and operational matrices method [35], and the collocation methods [14].

As we know, spectral methods provide highly accurate approximations for the solutions of smooth problems and have been used for the numerical computations of certain partial differential equations; see, for example, [5, 6, 7, 28, 29]. In the last decade, many one-dimensional Volterra integral or integro-differential equations were solved by spectral methods. In [4, 41], spectral Galerkin methods were developed to solve Volterra integral equations with smooth kernels, and in [50], spectral Galerkin methods were investigated for Volterra integral equations with weakly singular kernels. Xie and Tao [34, 47] considered some spectral and pseudo-spectral Galerkin approaches for smooth Volterra integral or integro-differential equations. Li, Tang, and Xu [22] extended the work of [47] to weakly singular Volterra integral equations. The authors of [2, 19, 33, 43, 45, 46, 48] proposed a Legendre spectral collocation method for Volterra integral or integro-differential equations with smooth kernels. Meanwhile, they established a vigorous error analysis in the L^∞- and L^2-norms and proved that the numerical errors decayed exponentially. In [9, 10, 32, 44], a Jacobi spectral collocation method was successfully introduced to approximate smooth solutions of weakly singular Volterra integral or integro-differential equations. Chen and Tang [11] proposed and analyzed a Jacobi spectral collocation approximation for weakly singular Volterra integral equations with nonsmooth solutions, where some function transformations and variable transformations were employed to transform the equation into a new Volterra integral equation which better regularity, so that the orthogonal polynomial theory can be directly applied. The analytical and numerical techniques used in [11] can be extended to weakly singular Volterra integral equations with pantograph delays [49]. In [21], the Jacobi spectral collocation method was also developed to solve weakly singular Volterra integral equations with nonsmooth solutions, but in the convergence analysis of this work, only variable transformations were used. Also, the Chebyshev spectral collocation method was applied directly to a weakly singular Volterra integral equation with nonsmooth solutions in [17]. All these achievements relate to one-step spectral methods. Recently, some authors developed multistep spectral methods for Volterra integral equations [15, 30, 31, 42, 51, 52]

Nevertheless, to the best of our knowledge, there have been no concerns regarding theoretical analysis of spectral methods for two-dimensional weakly singular integro-differential equations. Our main motivation in this paper is to construct and analyze a Jacobi spectral collocation approximation for Eq. (1.1).

The rest of this paper is organized as follows. In Section 2, we introduce the Jacobi-collocation method for two-dimensional nonlinear Volterra integral equations with weakly singular kernels and smooth solutions. Some preliminaries and useful lemmas for establishing the convergence analysis are provided in Section 3. The error estimations in the L^∞-norm and weighted L^2-norm are given in Section 4. In Section 5, we present numerical experiments to demonstrate the theoretical results obtained in Section 4. The final section is for the conclusion and future work.

2 Jacobi-collocation method

For an integer $N > 0$, we set $\mathcal{P}_N = \tilde{\mathcal{P}}_N \times \tilde{\mathcal{P}}_N$, where $\tilde{\mathcal{P}}_N$ is the space of the single-variable polynomials of degree up to N. For the weight function $\omega^{-\mu,-\mu}(x) = (1-x^2)^{-\mu}$, we denote the collocation points by $\{x_i\}_{i=0}^N$, which is the set of $(N+1)$ Jacobi–Gauss points, and by $\{w_i\}_{i=0}^N$ the corresponding weights. Similarly, for the weight function $\omega^{-\nu,-\nu}(y) = (1-y^2)^{-\nu}$, we denote the collocation points by $\{y_j\}_{j=0}^N$, which is the set of $(N+1)$ Jacobi–Gauss points, and by $\{\rho_j\}_{j=0}^N$ the corresponding weights. Let $\bar{\Omega} = [-1,1]^2$. For any function $v \in C(\bar{\Omega})$, we can define the Lagrange interpolating polynomial $\mathcal{I}_N^{-\mu,-\nu} v \in \mathcal{P}_N$ satisfying

$$\mathcal{I}_N^{-\mu,-\nu} v(x_i, y_j) = v(x_i, y_j), \quad 0 \leqslant i, j \leqslant N;$$

see, for example, [6, 28]. The Lagrange interpolation polynomial can be written in the form

$$\mathcal{I}_N^{-\mu,-\nu} v(x,y) = \sum_{i=0}^N \sum_{j=0}^N v(x_i, y_j) F_i(x) F_j(y),$$

where $F_i(x)$ and $F_j(y)$ are the Lagrange interpolation basis functions associated with $\{x_i\}_{i=0}^N$ and $\{y_j\}_{j=0}^N$, respectively.

To naturally apply the spectral collocation method, we make the change of variables

$$s=\frac{T}{2}(1+x),\qquad t=\frac{T}{2}(1+y),\qquad \sigma=\frac{T}{2}(1+\xi),\qquad \tau=\frac{T}{2}(1+\eta) \tag{2.1}$$

and let

$$u(x,y)=y\biggl(\frac{T}{2}(1+x),\,\frac{T}{2}(1+y)\biggr),\qquad g(x,y)=f\biggl(\frac{T}{2}(1+x),\frac{T}{2}(1+y)\biggr),$$

$$\tilde{K}\bigl(x,y,\xi,\eta,u(\xi,\eta)\bigr)=\biggl(\frac{T}{2}\biggr)^{2-\mu-\nu}K\biggl(\frac{T}{2}(1+x),\,\frac{T}{2}(1+y),\,\frac{T}{2}(1+\xi),\,\frac{T}{2}(1+\eta),\,u(\xi,\eta)\biggr),$$

so that (1.1) can be written as

$$u(x,y)=g(x,y)+\int_{-1}^{x}\int_{-1}^{y}(x-\xi)^{-\mu}(y-\eta)^{-\nu}\tilde{K}\bigl(x,y,\xi,\eta,u(\xi,\eta)\bigr)\,\mathrm{d}\eta\,\mathrm{d}\xi,\quad (x,y)\in\bar{\Omega}. \tag{2.2}$$

Assume that Eq. (2.2) holds at the collocation point-pairs (x_i,y_j) on $\bar{\Omega}$, that is,

$$u(x_i,y_j)=g(x_i,y_j)+\int_{-1}^{x_i}\int_{-1}^{y_j}(x_i-\xi)^{-\mu}(y_j-\eta)^{-\nu}\tilde{K}\bigl(x_i,y_j,\xi,\eta,u(\xi,\eta)\bigr)\,\mathrm{d}\eta\,\mathrm{d}\xi,\quad 0\leqslant i,j\leqslant N. \tag{2.3}$$

We will use the Jacobi–Gauss quadrature formulas to compute the integral term in (2.3). For this purpose, we make two linear transformations for (2.3):

$$\xi=\xi(x_i,\theta)=\frac{1+x_i}{2}\theta+\frac{x_i-1}{2},\quad \eta=\eta(y_j,\zeta)=\frac{1+y_j}{2}\zeta+\frac{y_j-1}{2},\quad -1\leqslant\theta,\zeta\leqslant 1. \tag{2.4}$$

Then the integral term in (2.3) becomes

$$\begin{aligned}&\int_{-1}^{x_i}\int_{-1}^{y_j}(x_i-\xi)^{-\mu}(y_j-\eta)^{-\nu}\tilde{K}\bigl(x_i,y_j,\xi,\eta,u(\xi,\eta)\bigr)\,\mathrm{d}\eta\,\mathrm{d}\xi\\ &\quad=\int_{-1}^{1}\int_{-1}^{1}(1-\theta)^{-\mu}(1-\zeta)^{-\nu}K_1\bigl(x_i,y_j,\xi(x_i,\theta),\eta(y_j,\zeta),u\bigl(\xi(x_i,\theta),\eta(y_j,\zeta)\bigr)\bigr)\,\mathrm{d}\zeta\,\mathrm{d}\theta,\end{aligned}$$

where

$$\begin{aligned}&K_1\bigl(x_i,y_j,\xi(x_i,\theta),\eta(y_j,\zeta),u\bigl(\xi(x_i,\theta),\eta(y_j,\zeta)\bigr)\bigr)\\ &\quad=\biggl(\frac{1+x_i}{2}\biggr)^{1-\mu}\biggl(\frac{1+y_j}{2}\biggr)^{1-\nu}\tilde{K}\bigl(x_i,y_j,\xi(x_i,\theta),\eta(y_j,\zeta),u\bigl(\xi(x_i,\theta),\eta(y_j,\zeta)\bigr)\bigr).\end{aligned}$$

Next, by the $(N+1)$-point Jacobi–Gauss quadrature formulas, we can estimate the new integral as follows:

$$\int_{-1}^{1}\int_{-1}^{1}(1-\theta)^{-\mu}(1-\zeta)^{-\nu}K_1\big(x_i,y_j,\xi(x_i,\theta),\eta(y_j,\zeta),u\big(\xi(x_i,\theta),\eta(y_j,\zeta)\big)\big)\,\mathrm{d}\zeta\,\mathrm{d}\theta$$
$$\approx\sum_{k=0}^{N}\sum_{l=0}^{N}K_1\big(x_i,y_j,\xi(x_i,\bar\theta_k),\eta(y_j,\bar\zeta_l),u\big(\xi(x_i,\bar\theta_k),\eta(y_j,\bar\zeta_l)\big)\big)\bar\rho_l\bar\varpi_k,$$

where $\{\bar\theta_k\}_{k=0}^N$ and $\{\bar\zeta_l\}_{l=0}^N$ are the sets of $(N+1)$ Jacobi–Gauss points corresponding to the weight functions $\omega^{-\mu,0}(\theta)=(1-\theta)^{-\mu}$ and $\omega^{-\nu,0}(\zeta)=(1-\zeta)^{-\nu}$, respectively.

We use u_{ij}, $0\leqslant i,j\leqslant N$, to approximate the function value $u(x_i,y_j)$, $0\leqslant i,j\leqslant N$, and use

$$u_N(x,y)=\sum_{m=0}^{N}\sum_{n=0}^{N}u_{mn}F_m(x)F_n(y) \tag{2.5}$$

to approximate the function $u(x,y)$, namely, $u(x_i,y_j)\approx u_{ij}$, $u(x,y)\approx u_N(x,y)$, and

$$u\big(\xi(x_i,\bar\theta_k),\eta(y_j,\bar\zeta_l)\big)\approx\sum_{m=0}^{N}\sum_{n=0}^{N}u_{mn}F_m\big(\xi(x_i,\bar\theta_k)\big)F_n\big(\eta(y_j,\bar\zeta_l)\big).$$

Then, the Jacobi collocation method is to seek $u_N(x,y)$ such that $\{u_{ij}\}_{i,j=0}^N$ satisfies the following collocation equations:

$$\begin{aligned}u_{ij}=&\sum_{k=0}^{N}\sum_{l=0}^{N}K_1\Bigg(x_i,\ y_j,\ \xi(x_i,\bar\theta_k),\ \eta(y_j,\bar\zeta_l),\ \sum_{m=0}^{N}\sum_{n=0}^{N}u_{mn}F_m\big(\xi(x_i,\bar\theta_k)\big)F_n(\eta(y_j,\bar\zeta_l))\Bigg)\bar\rho_l\bar\varpi_k\\&+g(x_i,y_j),\quad 0\leqslant i,j\leqslant N.\end{aligned} \tag{2.6}$$

3 Some preliminaries and useful lemmas

We first introduce some weighted Sobolev spaces on $\Omega=(-1,1)^2$. Set

$$\omega_{\alpha,\beta}(x,y)=(1-x)^{-\mu}(1-y)^{-\nu}(1+x)^{\alpha}(1+y)^{\beta},$$

where $-1<\alpha,\beta<1$. We define the weighted space

$$L^2_{\omega_{\alpha,\beta}}(\Omega)=\big\{v:\Omega\to\mathbb{R}\text{ measurable and }\|v\|_{\omega_{\alpha,\beta}}<\infty\big\}$$

equipped with the norm

$$\|v\|_{\omega_{\alpha,\beta}}=\iint_{\Omega}\big|v(x,y)\big|^2\omega_{\alpha,\beta}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

and the inner product

$$(u,v)_{\omega_{\alpha,\beta}}=\iint_{\Omega}u(x,y)v(x,y)\omega_{\alpha,\beta}(x,y)\,\mathrm{d}x\,\mathrm{d}y,\quad u,v\in L^2_{\omega_{\alpha,\beta}}(\Omega).$$

For any integer $m \geqslant 0$, the weighted Sobolev space is defined by

$$H^m_{\omega_{\alpha,\beta}}(\Omega) = \left\{ v\colon\ v \in L^2_{\omega_{\alpha,\beta}}(\Omega),\ \frac{\partial^{p+q} v}{\partial x^p \partial y^q} \in L^2_{\omega_{\alpha,\beta}}(\Omega),\ (p,q) \in \mathbb{N}^2,\ p+q \leqslant m \right\}$$

with the norm

$$\|v\|_{m,\,\omega_{\alpha,\beta}} = \left(\sum_{p+q=0}^{m} \left\| \frac{\partial^{p+q} v}{\partial x^p \partial y^q} \right\|^2_{\omega_{\alpha,\beta}} \right)^{1/2}.$$

Next, we introduce the discrete inner product:

$$(\phi,\psi)_N = \sum_{k=0}^{N} \sum_{l=0}^{N} \phi(\bar\theta_k,\bar\zeta_l)\psi(\bar\theta_k,\bar\zeta_l)\bar\rho_l \bar w_k, \quad \phi,\psi \in C(\bar\Omega).$$

From [5, 6, 8] we get the following lemma.

Lemma 1. *If $v \in H^m_{\omega_{0,0}}(\Omega)$ for some $m \geqslant 1$ and $\psi \in \mathcal{P}_N$, then there exists a constant C independent of N such that*

$$\left|(v,\psi)_{\omega_{0,0}} - (v,\psi)_N\right| \leqslant CN^{-m}\|v\|_{m,\omega_{0,0}}\|\psi\|_{\omega_{0,0}}.$$

From [23] we have the following result on the Lebesgue constant for Lagrange interpolation polynomials associated with the zeros of Jacobi polynomials.

Lemma 2. *Let $\{F_m(x)\}_{m=0}^N$ and $\{F_n(y)\}_{n=0}^N$ be the Nth Lagrange interpolation polynomials associated with the Jacobi–Gauss points $\{x_i\}_{i=0}^N$ and $\{y_j\}_{j=0}^N$, respectively. Then*

$$\left\|\mathcal{I}_N^{-\mu,-\nu}\right\|_\infty := \max_{(x,y)\in\bar\Omega} \sum_{m=0}^{N}\sum_{n=0}^{N} \left|F_m(x)F_n(y)\right| = \begin{cases} \mathcal{O}(\log^2 N), & \frac12 \leqslant \mu,\ \nu < 1, \\ \mathcal{O}(N^{1-\mu-\nu}), & 0 < \mu,\ \nu < \frac12, \\ \mathcal{O}(N^{1/2-\nu}\log N), & \frac12 \leqslant \mu < 1,\ 0 < \nu < \frac12, \\ \mathcal{O}(N^{1/2-\mu}\log N), & 0 < \mu < \frac12,\ \frac12 \leqslant \nu < 1. \end{cases}$$

For the interpolation error, we have the following result.

Lemma 3. *For any function $v \in H^m_{\omega_{-\mu,\,-\nu}}(\Omega)$ $(m \geqslant 1)$, we have*

$$\left\|v - \mathcal{I}_N^{-\mu,-\nu} v\right\|_{\omega_{-\mu,-\nu}} \leqslant CN^{-m}\|v\|_{m,\,\omega_{-\mu,-\nu}}, \tag{3.1}$$

$$\left\|v - \mathcal{I}_N^{-\mu,-\nu} v\right\|_{\infty} \leqslant CN^{4-m}\|v\|_{m,\,\omega_{-\mu,-\nu}}. \tag{3.2}$$

Proof. For (3.1), we can see [5, 6, 8]. We now prove (3.2). For any integer r, m such that $0 \leqslant r \leqslant m$, $m > 1$, we have the following interpolation error estimate (see [5, 6, 8]:

$$\left\|v - \mathcal{I}_N^{-\mu,-\nu} v\right\|_{r,\,\omega_{-\mu,-\nu}} \leqslant CN^{2r-m}\|v\|_{m,\,\omega_{-\mu,-\nu}}, \quad v \in H^m_{\omega_{-\mu,-\nu}}(\Omega). \tag{3.3}$$

From [1] we know that $H^2_{\omega_{-\mu,-\nu}}(\Omega)$ is embedded in $C(\bar\Omega)$, namely,

$$\|v\|_\infty \leqslant C\|v\|_{2,\,\omega_{-\mu,-\nu}}, \quad v \in H^2_{\omega_{-\mu,-\nu}}(\Omega). \tag{3.4}$$

Then, (3.3) and (3.4) yield inequality (3.2). □

In our analysis, we will apply the two-dimensional Gronwall lemma. The following result can be found in [25].

Lemma 4. *Suppose that*

$$w(s,t) \leqslant a(s,t) + b(s,t) \int_0^s \int_0^t c(\sigma,\tau) w(\sigma,\tau)\, \mathrm{d}\tau\, \mathrm{d}\sigma, \quad (s,t) \in \Lambda,$$

where $w(s,t)$, $a(s,t)$, $b(s,t)$, and $c(s,t)$ are nonnegative continuous functions defined on Λ. Then, on Λ, we have

$$w(s,t) \leqslant a(s,t) + b(s,t) \int_0^s \int_0^t \exp\biggl(\int_\sigma^s \int_\tau^t b(r,w) c(r,w)\, \mathrm{d}w\, \mathrm{d}r \biggr) a(\sigma,\tau) c(\sigma,\tau)\, \mathrm{d}\tau\, \mathrm{d}\sigma.$$

By Lemma 4 we can obtain the following result.

Lemma 5. *Assume that $v(x,y)$ and $h(x,y)$ are nonnegative continuous function defined on $\bar{\Omega}$ and satisfying*

$$v(x,y) \leqslant h(x,y) + M \int_{-1}^x \int_{-1}^y (x-\xi)^{-\mu} (y-\eta)^{-\nu} v(\xi,\eta)\, \mathrm{d}\eta\, \mathrm{d}\xi, \tag{3.5}$$

where M is a positive constant. Then, there is a constant C such that

$$v(x,y) \leqslant h(x,y) + C \int_{-1}^x \int_{-1}^y (x-\xi)^{-\mu} (y-\eta)^{-\nu} h(\xi,\eta)\, \mathrm{d}\eta\, \mathrm{d}\xi, \quad (x,y) \in \bar{\Omega}. \tag{3.6}$$

Proof. By the changes of variables (2.1), we can rewrite (3.5) as follows:

$$w(s,t) \leqslant a(s,t) + M \biggl(\frac{2}{T} \biggr)^{2-\mu-\nu} \int_0^s \int_0^t (s-\sigma)^{-\mu} (t-\tau)^{-\nu} w(\sigma,\tau)\, \mathrm{d}\tau\, \mathrm{d}\sigma,$$

where $(s,t) \in \Lambda$, and

$$w(s,t) = v\biggl(\frac{2}{T}s - 1, \frac{2}{T}t - 1 \biggr), \qquad a(s,t) = h\biggl(\frac{2}{T}s - 1, \frac{2}{T}t - 1 \biggr).$$

Set

$$b(s,t) = M \biggl(\frac{2}{T} \biggr)^{2-\mu-\nu}, \qquad c(\sigma,\tau) = (s-\sigma)^{-\mu} (t-\tau)^{-\nu}.$$

It is obvious that $a(s,t)$, $b(s,t)$, $c(s,t)$, and $w(s,t)$ satisfy the conditions in Lemma 4. Then we have

$$w(s,t) \leqslant a(s,t) + b(s,t) \int_0^s \int_0^t \exp\biggl(\int_\sigma^s \int_\tau^t b(r,w) c(r,w)\, \mathrm{d}w\, \mathrm{d}r \biggr) a(\sigma,\tau) c(\sigma,\tau)\, \mathrm{d}\tau\, \mathrm{d}\sigma.$$

Clearly,

$$\left|\int_\sigma^s\int_\tau^t b(r,w)c(r,w)\,\mathrm{d}w\,\mathrm{d}r\right| = M\left(\frac{2}{T}\right)^{2-\mu-\nu}\left|\int_\sigma^s\int_\tau^t (s-r)^{-\mu}(t-w)^{-\nu}\,\mathrm{d}w\,\mathrm{d}r\right|$$
$$= \left(\frac{2}{T}\right)^{2-\mu-\nu}\frac{M}{(1-\mu)(1-\nu)}(s-\sigma)^{1-\mu}(t-\tau)^{1-\nu} \leqslant C.$$

Consequently,

$$w(s,t) \leqslant a(s,t) + C\int_0^s\int_0^t a(\sigma,\tau)c(\sigma,\tau)\,\mathrm{d}\tau\,\mathrm{d}\sigma$$
$$= a(s,t) + C\int_0^s\int_0^t (s-\sigma)^{-\mu}(t-\tau)^{-\nu}a(\sigma,\tau)\,\mathrm{d}\tau\,\mathrm{d}\sigma. \tag{3.7}$$

Thus, (3.6) is obtained by applying the changes of variables (2.1) to (3.7). □

From now on, for $m \geqslant 0$ and $\kappa \in [0,1]$, $C^{m,\kappa}(\bar{\Omega})$ is the space of functions whose mth derivatives are Hölder continuous with exponent κ, endowed with the norm

$$\|v\|_{m,\kappa} = \max_{0\leqslant p+q\leqslant m}\max_{(x,y)\in\bar{\Omega}}\left|\frac{\partial^{p+q}v(x,y)}{\partial x^p\partial y^q}\right|$$
$$+ \max_{0\leqslant p+q\leqslant m}\sup_{(x',y')\neq(x'',y'')\in\bar{\Omega}}\frac{|\partial^{p+q}v(x',y')/\partial x^p\partial y^q - \partial^{p+q}v(x'',y'')/\partial x^p\partial y^q|}{((x'-x'')^2+(y'-y'')^2)^{\kappa/2}}.$$

We will need a result of Ragozin [26,27], which states that, for integer $m \geqslant 0$ and $\kappa \in (0,1)$, there exists a constant $C_{m,\kappa} > 0$ such that, for any function $v \in C^{m,\kappa}(\bar{\Omega})$, there exists a polynomial function $\mathcal{G}_N v \in \mathcal{P}_N$ such that

$$\|v - \mathcal{G}_N v\|_\infty \leqslant C_{m,\kappa}N^{-(m+\kappa)}\|v\|_{m,\kappa}. \tag{3.8}$$

In fact, as stated in [26] and [27], $\mathcal{G}_N$ is a linear operator from $C^{m,\kappa}(\bar{\Omega})$ into $\mathcal{P}_N$.

We will further need the following lemma.

Lemma 6. *Let $\bar{R}(x,\xi), \partial\bar{R}(x,\xi)/\partial x \in C(\bar{\Omega})$, and $0 < \kappa_\mu < 1-\mu$. Then for any function $z(x) \in C([-1,1])$, there exists a positive constant C such that*

$$\frac{|\int_{-1}^{x'}(x'-\xi)^{-\mu}\bar{R}(x',\xi)z(\xi)\,\mathrm{d}\xi - \int_{-1}^{x''}(x''-\xi)^{-\mu}\bar{R}(x'',\xi)z(\xi)\,\mathrm{d}\xi|}{|x'-x''|^{\kappa_\mu}} \leqslant C\|z\|_\infty \tag{3.9}$$

for any $x', x'' \in [-1,1]$ such that $x' \neq x''$.

Similarly, let $\tilde{R}(y,\eta)$, $\partial\tilde{R}(y,\eta)/\partial y \in C(\bar{\Omega})$, and $0 < \kappa_\nu < 1-\nu$. Then for any function $h(y) \in C([-1,1])$, there exists a positive constant C such that

$$\frac{|\int_{-1}^{y'}(y'-\eta)^{-\nu}\tilde{R}(y',\eta)h(\eta)\,\mathrm{d}\eta - \int_{-1}^{y''}(y''-\eta)^{-\nu}\tilde{R}(y'',\eta)h(\eta)\,\mathrm{d}\eta|}{|y'-y''|^{\kappa_\nu}} \leqslant C\|h\|_\infty \tag{3.10}$$

for any $y', y'' \in [-1,1]$ such that $y' \neq y''$.

Proof. We now prove (3.9). Without loss of generality, we assume that $x' < x''$. Observe that

$$\left|\int_{-1}^{x'}(x'-\xi)^{-\mu}\bar{R}(x',\xi)z(\xi)\,\mathrm{d}\xi-\int_{-1}^{x''}(x''-\xi)^{-\mu}\bar{R}(x'',\xi)z(\xi)\,\mathrm{d}\xi\right|\leqslant I_1+I_2+I_3, \tag{3.11}$$

where

$$I_1=\left|\int_{-1}^{x'}(x'-\xi)^{-\mu}\big(\bar{R}(x',\xi)-\bar{R}(x'',\xi)\big)z(\xi)\,\mathrm{d}\xi\right|,$$

$$I_2=\left|\int_{-1}^{x'}\big((x'-\xi)^{-\mu}-(x''-\xi)^{-\mu}\big)\bar{R}(x'',\xi)z(\xi)\,\mathrm{d}\xi\right|,$$

$$I_3=\left|\int_{x'}^{x''}(x''-\xi)^{-\mu}\bar{R}(x'',\xi)z(\xi)\,\mathrm{d}\xi\right|.$$

Applying the Lagrange midvalue differential theorem to I_1 gives

$$I_1\leqslant\left\|\frac{\partial\bar{R}}{\partial x}\right\|_\infty(x''-x')\int_{-1}^{x'}(x'-\xi)^{-\mu}\big|z(\xi)\big|\,\mathrm{d}\xi\leqslant C(x''-x')\|z\|_\infty. \tag{3.12}$$

We now estimate I_2. It is clear that

$$\begin{aligned} I_2&\leqslant\|\bar{R}\|_\infty\|z\|_\infty\int_{-1}^{x'}\big|(x'-\xi)^{-\mu}-(x''-\xi)^{-\mu}\big|\,\mathrm{d}\xi\\ &\leqslant C\big((x''-x')^{1-\mu}+(x'+1)^{1-\mu}-(x''+1)^{1-\mu}\big)\|z\|_\infty\\ &\leqslant C(x''-x')^{1-\mu}\|z\|_\infty. \end{aligned} \tag{3.13}$$

Finally, we have

$$I_3\leqslant\|\bar{R}\|_\infty\|z\|_\infty\int_{x'}^{x''}(x''-\xi)^{-\mu}\,\mathrm{d}\xi\leqslant C(x''-x')^{1-\mu}\|z\|_\infty. \tag{3.14}$$

Consequently, from (3.11)–(3.14) we have

$$\begin{aligned} &\frac{|\int_{-1}^{x'}(x'-\xi)^{-\mu}\bar{R}(x',\xi)z(\xi)\,\mathrm{d}\xi-\int_{-1}^{x''}(x''-\xi)^{-\mu}\bar{R}(x'',\xi)z(\xi)\,\mathrm{d}\xi|}{|x'-x''|^{\kappa_\mu}}\\ &\quad\leqslant C((x''-x')^{1-\kappa_\mu}+(x''-x')^{1-\mu-\kappa_\mu})\|z\|_\infty\leqslant C\|z\|_\infty \end{aligned}$$

for $0<\kappa_\mu<1-\mu$. Thus, we have proven (3.9). The proof of (3.10) is similar to that of (3.9). □

Let $R(x, y, \xi, \eta)$, $\partial R(x, y, \xi, \eta)/\partial x$, $\partial R(x, y, \xi, \eta)/\partial y \in C(\bar{\Omega} \times \bar{\Omega})$. We further define a linear weakly singular integral operator $\mathcal{M}$:

$$\mathcal{M}v = \int_{-1}^{x} \int_{-1}^{y} (x - \xi)^{-\mu} (y - \eta)^{-\nu} R(x, y, \xi, \eta) v(\xi, \eta) \, \mathrm{d}\eta \, \mathrm{d}\xi. \tag{3.15}$$

From Lemma 6 we can show that $\mathcal{M}$ is compact as an operator from $C(\bar{\Omega})$ to $C^{0,\kappa}(\bar{\Omega})$ for any $0 < \kappa < \min\{\kappa_\mu, \kappa_\nu\}$; namely, we have the following lemma.

Lemma 7. *Let* $0 < \kappa < \min\{\kappa_\mu, \kappa_\nu\}$, *and let* $\mathcal{M}$ *be defined by* (3.15). *Then, for any function* $v(x, y) \in C(\bar{\Omega})$, *there exists a positive constant* C *such that*

$$\|\mathcal{M}v\|_{0,\kappa} \leqslant C\|v\|_\infty. \tag{3.16}$$

Proof. We only need to prove that $\mathcal{M}$ is Hölder continuous, that is,

$$\frac{|\mathcal{M}v(x', y') - \mathcal{M}v(x'', y'')|}{((x' - x'')^2 + (y' - y'')^2)^{\kappa/2}} \leqslant C\|v\|_\infty, \quad (x', y'), (x'', y'') \in \bar{\Omega},$$

for $0 < \sqrt{(x' - x'')^2 + (y' - y'')^2} < 1$ and $0 < \kappa < \min\{\kappa_\mu, \kappa_\nu\}$. From (3.15) we have

$$\begin{aligned} &\bigl|\mathcal{M}v(x', y') - \mathcal{M}v(x'', y'')\bigr| \\ &\quad = \Biggl| \int_{-1}^{x'} \int_{-1}^{y'} (x' - \xi)^{-\mu} (y' - \eta)^{-\nu} R(x', y', \xi, \eta) v(\xi, \eta) \, \mathrm{d}\eta \, \mathrm{d}\xi \\ &\qquad - \int_{-1}^{x''} \int_{-1}^{y''} (x'' - \xi)^{-\mu} (y'' - \eta)^{-\nu} R(x'', y'', \xi, \eta) v(\xi, \eta) \, \mathrm{d}\eta \, \mathrm{d}\xi \Biggr| \\ &\quad \leqslant E_1 + E_2, \end{aligned} \tag{3.17}$$

where

$$\begin{aligned} E_1 = \Biggl| &\int_{-1}^{x'} \int_{-1}^{y'} (x' - \xi)^{-\mu} (y' - \eta)^{-\nu} R(x', y', \xi, \eta) v(\xi, \eta) \, \mathrm{d}\eta \, \mathrm{d}\xi \\ &- \int_{-1}^{x'} \int_{-1}^{y''} (x' - \xi)^{-\mu} (y'' - \eta)^{-\nu} R(x', y'', \xi, \eta) v(\xi, \eta) \, \mathrm{d}\eta \, \mathrm{d}\xi \Biggr|, \end{aligned}$$

$$\begin{aligned} E_2 = \Biggl| &\int_{-1}^{x'} \int_{-1}^{y''} (x' - \xi)^{-\mu} (y'' - \eta)^{-\nu} R(x', y'', \xi, \eta) v(\xi, \eta) \, \mathrm{d}\eta \, \mathrm{d}\xi \\ &- \int_{-1}^{x''} \int_{-1}^{y''} (x'' - \xi)^{-\mu} (y'' - \eta)^{-\nu} R(x'', y'', \xi, \eta) v(\xi, \eta) \, \mathrm{d}\eta \, \mathrm{d}\xi \Biggr|. \end{aligned}$$

We now estimate the two terms. Observe that

$$E_1 \leqslant \int_{-1}^{x'} (x'-\xi)^{-\mu} \left| \int_{-1}^{y'} (y'-\eta)^{-\nu} R(x',y',\xi,\eta) v(\xi,\eta)\,\mathrm{d}\eta \right.$$
$$\left. - \int_{-1}^{y''} (y''-\eta)^{-\nu} R(x',y'',\xi,\eta) v(\xi,\eta)\,\mathrm{d}\eta \right| \mathrm{d}\xi,$$

which, together with (3.10), gives

$$\begin{aligned} E_1 &\leqslant C|y'-y''|^{\kappa_\nu} \int_{-1}^{x'} (x'-\xi)^{-\mu} \max_{y\in[-1,1]} \big|v(\xi,y)\big|\,\mathrm{d}\xi \\ &\leqslant C|y'-y''|^{\kappa_\nu} \|v\|_\infty \leqslant C\Big(\big((x'-x'')^2+(y'-y'')^2\big)^{\kappa_\nu/2}\Big)\|v\|_\infty \\ &\leqslant C\big((x'-x'')^2+(y'-y'')^2\big)^{\min\{\kappa_\mu,\kappa_\nu\}/2}\|v\|_\infty, \end{aligned} \tag{3.18}$$

where we used the condition $0 < \sqrt{(x'-x'')^2+(y'-y'')^2} < 1$. Similarly, we have

$$E_2 \leqslant C((x'-x'')^2+(y'-y'')^2)^{\min\{\kappa_\mu,\kappa_\nu\}/2}\|v\|_\infty. \tag{3.19}$$

Consequently, it follow from (3.17), (3.18), and (3.19) that

$$\begin{aligned} &\frac{|\mathcal{M}v(x',y')-\mathcal{M}v(x'',y'')|}{((x'-x'')^2+(y'-y'')^2)^{\kappa/2}} \\ &\quad = \big((x'-x'')^2+(y'-y'')^2\big)^{-\kappa/2}\big|\mathcal{M}v(x',y')-\mathcal{M}v(x'',y'')\big| \\ &\quad \leqslant C\big((x'-x'')^2+(y'-y'')^2\big)^{\min\{\kappa_\mu,\kappa_\nu\}-\kappa/2}\|v\|_\infty \leqslant C\|v\|_\infty \end{aligned}$$

for $0<\kappa<\min\{\kappa_\mu,\kappa_\nu\}$. Thus, we have proven (3.16). □

To prove the error estimate in the weighted L^2-norm, we need the two-dimensional Hardy inequality (see [20]).

Lemma 8. *For any measurable function $f \geqslant 0$, the two-dimensional Hardy inequality*:

$$\left(\int_a^b\int_c^d \big|(\mathcal{T}f)(x,y)\big|^q \breve{\omega}(x)\bar{\omega}(y)\,\mathrm{d}y\,\mathrm{d}x\right)^{1/q} \leqslant C\left(\int_a^b\int_c^d \big|f(x,y)\big|^P \hat{\omega}(x)\tilde{\omega}(y)\,\mathrm{d}y\,\mathrm{d}x\right)^{1/P}$$

hold if and only if

$$\sup_{(x,y)\in(a,b)\times(c,d)} \left(\int_x^b\int_y^d \breve{\omega}(s)\bar{\omega}(t)\,\mathrm{d}t\,\mathrm{d}s\right)^{1/q}\left(\int_a^x\int_c^y \big(\hat{\omega}(s)\tilde{\omega}(t)\big)^{1-p'}\,\mathrm{d}t\,\mathrm{d}s\right)^{1/p'} < \infty,$$

where $p' = p/(p-1)$, $1 < p \leqslant q < \infty$. Here $\mathcal{T}$ is an operator of the form

$$(\mathcal{T}f)(x,y) = \int_a^x \int_c^y R(x,y,s,t) f(s,t)\,\mathrm{d}t\,\mathrm{d}s$$

with a given kernel $R(x,y,s,t)$, $\breve{\omega}(x)\bar{\omega}(y)$ and $\hat{\omega}(x)\tilde{\omega}(y)$ are weight functions, and $-\infty \leqslant a < b \leqslant \infty$, $-\infty \leqslant c < d \leqslant \infty$.

We have the following estimate for the Lagrange interpolation based on the Jacobi collocation point-pairs.

Lemma 9. *For every bounded function $v(x,y)$, there exists a constant C independent of $v(x,y)$ such that*

$$\sup_N \|\mathcal{I}_N^{-\mu,-\nu} v\|_{\omega_{-\mu,-\nu}} \leqslant C\|v\|_\infty. \tag{3.20}$$

Proof. As the $(N+1)$-point Jacobi–Gauss quadrature formulas are accurate for the polynomials with degree not exceeding $2N+1$, that is,

$$\int_{-1}^1 p_1(x)(1-x^2)^{-\mu}\,\mathrm{d}x = \sum_{i=0}^N p_1(x_i)w_i, \quad p_1 \in \tilde{\mathcal{P}}_{2N+1}, \tag{3.21}$$

$$\int_{-1}^1 p_2(y)(1-y^2)^{-\nu}\,\mathrm{d}x = \sum_{j=0}^N p_2(y_j)\rho_j \ , \quad p_2 \in \tilde{\mathcal{P}}_{2N+1}. \tag{3.22}$$

Then (3.21) and (3.22) yield

$$\int_{-1}^1 \int_{-1}^1 p(x,y)\big(1-x^2\big)^{-\mu}\big(1-y^2\big)^{-\nu}\,\mathrm{d}y\,\mathrm{d}x = \sum_{i=0}^N \sum_{j=0}^N p(x_i,\rho_j)\rho_j w_i, \quad p \in \mathcal{P}_{2N+1}. \tag{3.23}$$

Let $\{J_m^{-\mu,-\mu}(x)\}_{m=0}^N$ and $\{J_m^{-\nu,-\nu}(y)\}_{m=0}^N$ be the sets of Jacobi polynomials with respect to the weight functions $\omega^{-\mu,-\mu}(x) = (1-x^2)^{-\mu}$, $\omega^{-\nu,-\nu}(y) = (1-y^2)^{-\nu}$, respectively. By (3.23) we have

$$\begin{aligned}
\|\mathcal{I}_N^{-\mu,-\nu} v\|^2_{\omega_{-\mu,-\nu}} &= \int_{-1}^1 \int_{-1}^1 \big(\mathcal{I}_N^{-\mu,-\nu} v\big)^2 \big(1-x^2\big)^{-\mu}\big(1-y^2\big)^{-\nu}\,\mathrm{d}y\,\mathrm{d}x \\
&= \sum_{i=0}^N \sum_{j=0}^N v^2(x_i,y_j)\rho_j w_i \leqslant \|v\|_\infty^2 \sum_{i=0}^N w_i \sum_{j=0}^N \rho_j \\
&= \bar{\gamma}\tilde{\gamma}\|v\|_\infty^2,
\end{aligned} \tag{3.24}$$

where $\bar{\gamma} = \int_{-1}^1 J_0^{-\mu,-\mu}(x)(1-x^2)^{-\mu}\,\mathrm{d}x$ and $\tilde{\gamma} = \int_{-1}^1 J_0^{-\nu,-\nu}(y)(1-y^2)^{-\nu}\,\mathrm{d}y$. From (3.24) we obtain the desired result (3.20). □

4 Convergence analysis

In this section, we show that the approximated solution obtained by the Jacobi collocation method (2.6) converges exponentially. Firstly, we will derive the error estimate in L^∞-norm.

Theorem 1. *Let $u(x,y)$ be the exact solution of Eq.* (2.2)*, and let the approximated solution $u_N(x,y)$ of the form* (2.5) *be obtained by the collocation equations* (2.6)*. If $u(x,y) \in H^m_{\omega_{-\mu,-\nu}}(\Omega)$ $(m \geqslant 5)$ and $\tilde{K}$ satisfies m times Lipschitz continuous conditions with its fifth argument, that is,*

$$\left|\frac{\partial^{p+q}\tilde{K}(x,y,\xi,\eta,u_1(\xi,\eta))}{\partial\xi^p\partial\eta^q} - \frac{\partial^{p+q}\tilde{K}(x,y,\xi,\eta,u_2(\xi,\eta))}{\partial\xi^p\partial\eta^q}\right| \leqslant L_{p+q}\left|u_1(\xi,\eta) - u_2(\xi,\eta)\right|, \quad p+q = 0,1,2,\dots,m, \tag{4.1}$$

then

$$\|u - u_N\|_\infty \leqslant \begin{cases} CN^{-m}(N^4\|u\|_{m,\omega_{-\mu,-\nu}} + \log^2 NK^*), & \frac{1}{2} \leqslant \mu,\nu < 1, \\ CN^{-m}(N^4\|u\|_{m,\omega_{-\mu,-\nu}} + N^{1-\mu-\nu}K^*), & 0 < \mu,\nu < \frac{1}{2}, \\ CN^{-m}(N^4\|u\|_{m,\omega_{-\mu,-\nu}} + N^{1/2-\nu}\log NK^*), & \frac{1}{2} \leqslant \mu < 1,\ 0 < \nu < \frac{1}{2}, \\ CN^{-m}(N^4\|u\|_{m,\omega_{-\mu,-\nu}} + N^{1/2-\mu}\log NK^*), & 0 < \mu < \frac{1}{2},\ \frac{1}{2} \leqslant \nu < 1, \end{cases} \tag{4.2}$$

provided that N is sufficiently large, where C is a constant independent of N, and

$$K^* = \max_{(x,y)\in\bar{\Omega}} \left\|\tilde{K}\left(x,y,\xi,\eta,u(\xi,\eta)\right)\right\|_{m,\omega_{0,0}}. \tag{4.3}$$

Proof. By (2.4) we can change the numerical scheme (2.6) to

$$u_{ij} = \int_{-1}^{x_i}\int_{-1}^{y_i} (x_i-\xi)^{-\mu}(y_j-\eta)^{-\nu}\tilde{K}\left(x_i,y_j,\xi,\eta,u_N(\xi,\eta)\right)\mathrm{d}\eta\,\mathrm{d}\xi + g(x_i,y_j) - I(x_i,y_j), \tag{4.4}$$

where u_N is defined by (2.5), and

$$\begin{aligned} I(x,y) = {} & \int_{-1}^{1}\int_{-1}^{1}(1-\theta)^{-\mu}(1-\zeta)^{-\nu}K_1\left(x,y,\xi(x,\theta),\eta(y,\zeta),u_N\left(\xi(x,\theta),\eta(y,\zeta)\right)\right)\mathrm{d}\zeta\,\mathrm{d}\theta \\ & - \sum_{k=0}^{N}\sum_{l=0}^{N} K_1\left(x,y,\xi(x,\bar{\theta}_k),\eta(y,\bar{\zeta}_l),u_N\left(\xi(x,\bar{\theta}_k),\eta(y,\bar{\zeta}_l)\right)\right)\bar{\rho}_l\bar{w}_k. \end{aligned}$$

Using the definition of $\|\cdot\|_{m,\omega_{0,0}}$ and Lemma 1 gives

$$\begin{aligned} \left|I(x,y)\right| & \leqslant CN^{-m}\left\|\tilde{K}\left(x,y,\xi(x,\cdot),\eta(y,\cdot),u_N\left(\xi(x,\cdot),\eta(y,\cdot)\right)\right)\right\|_{m,\omega_{0,0}} \\ & = \left(\sum_{p+q=0}^{m}\left\|\frac{\partial^{p+q}\tilde{K}(x,y,\xi(x,\theta),\eta(y,\zeta),u_N(\xi(x,\theta),\eta(y,\zeta)))}{\partial\theta^p\partial\zeta^q}\right\|^2_{\omega_{0,0}}\right)^{1/2}. \end{aligned}$$

Notng that

$$\begin{aligned}
&\left|\frac{\partial^{p+q}\tilde{K}(x,y,\xi(x,\theta),\eta(y,\zeta),u_N(\xi(x,\theta),\eta(y,\zeta)))}{\partial\theta^p\partial\zeta^q}\right| \\
&\quad= \left|\left(\frac{\partial\xi}{\partial\theta}\right)^p\left(\frac{\partial\eta}{\partial\zeta}\right)^q\frac{\partial^{p+q}\tilde{K}(x,y,\xi,\eta,u_N(\xi,\eta))}{\partial\xi^p\partial\eta^q}\right| \leqslant \left|\frac{\partial^{p+q}\tilde{K}(x,y,\xi,\eta,u_N(\xi,\eta))}{\partial\xi^p\partial\eta^q}\right| \\
&\quad\leqslant \left|\frac{\partial^{p+q}\tilde{K}(x,y,\xi,\eta,u(\xi,\eta))}{\partial\xi^p\partial\eta^q}\right| + \left|\frac{\partial^{p+q}\tilde{K}(x,y,\xi,\eta,u_N(\xi,\eta))}{\partial\xi^p\partial\eta^q} - \frac{\partial^{p+q}\tilde{K}(x,y,\xi,\eta,u(\xi,\eta))}{\partial\xi^p\partial\eta^q}\right|,
\end{aligned}$$

we get

$$\begin{aligned}
\left|I(x,y)\right| &\leqslant CN^{-m}\big(\big\|\tilde{K}\big(x,y,\xi,\eta,u(\xi,\eta)\big)\big\|_{m,\omega_{0,0}} \\
&\quad + \big\|\tilde{K}\big(x,y,\xi,\eta,u_N(\xi,\eta)\big) - \tilde{K}\big(x,y,\xi,\eta,u(\xi,\eta)\big)\big\|_{m,\omega_{0,0}}\big).
\end{aligned} \tag{4.5}$$

Using m times the Lipschitz continuous conditions (4.1), we have

$$\begin{aligned}
&\big\|\tilde{K}\big(x,y,\xi,\eta,u_N(\xi,\eta)\big) - \tilde{K}\big(x,y,\xi,\eta,u(\xi,\eta)\big)\big\|_{m,\omega_{0,0}} \\
&\quad= \left(\sum_{p+q=0}^{m}\left\|\frac{\partial^{p+q}\tilde{K}(x,y,\xi,\eta,u_N(\xi,\eta))}{\partial\xi^p\partial\eta^q} - \frac{\partial^{p+q}\tilde{K}(x,y,\xi,\eta,u(\xi,\eta))}{\partial\xi^p\partial\eta^q}\right\|^2_{\omega_{0,0}}\right)^{1/2} \\
&\quad\leqslant \sum_{p+q=0}^{m}\left\|\frac{\partial^{p+q}\tilde{K}(x,y,\xi,\eta,u_N(\xi,\eta))}{\partial\xi^p\partial\eta^q} - \frac{\partial^{p+q}\tilde{K}(x,y,\xi,\eta,u(\xi,\eta))}{\partial\xi^p\partial\eta^q}\right\|_{\omega_{0,0}} \\
&\quad\leqslant \sum_{p+q=0}^{m}L_{p+q}\big\|u_N(\xi,\eta) - u(\xi,\eta)\big\|_{\omega_{0,0}} \leqslant C\big\|u_N(\xi,\eta) - u(\xi,\eta)\big\|_{\omega_{0,0}}.
\end{aligned}$$

Consequently, (4.5) becomes

$$\left|I(x,y)\right| \leqslant CN^{-m}\big(\big\|\tilde{K}\big(x,y,\xi,\eta,u(\xi,\eta)\big)\big\|_{m,\omega_{0,0}} + \big\|u_N(\xi,\eta) - u(\xi,\eta)\big\|_{\omega_{0,0}}\big). \tag{4.6}$$

Subtracting (4.4) from (2.3) yields

$$\begin{aligned}
&u(x_i,y_j) - u_{ij} \\
&\quad= \int_{-1}^{x_i}\int_{-1}^{y_j}(x_i-\xi)^{-\mu}(y_j-\eta)^{-\nu}\big(\tilde{K}\big(x_i,y_j,\xi,\eta,u(\xi,\eta)\big) - \tilde{K}\big(x_i,y_j,\xi,\eta,u_N(\xi,\eta)\big)\big)\,\mathrm{d}\eta\,\mathrm{d}\xi \\
&\qquad + I(x_i,y_j).
\end{aligned} \tag{4.7}$$

Multiplying by $F_i(x)F_j(y)$ both sides of (4.7) and summing up from $i=0$ to N and from $j=0$ to N give

$$\begin{aligned}
&\mathcal{I}_N^{-\mu,-\nu}u(x,y) - u_N(x,y) \\
&\quad= \mathcal{I}_N^{-\mu,-\nu}\left(\int_{-1}^{x}\int_{-1}^{y}(x-\xi)^{-\mu}(y-\eta)^{-\nu}\big(\tilde{K}\big(x,y,\xi,\eta,u(\xi,\eta)\big) - \tilde{K}\big(x,y,\xi,\eta,u_N(\xi,\eta)\big)\big)\,\mathrm{d}\eta\,\mathrm{d}\xi\right) \\
&\qquad + \mathcal{I}_N^{-\mu,-\nu}I(x,y).
\end{aligned} \tag{4.8}$$

Let $e(x,y)=u(x,y)-u_N(x,y)$ denote the error functions. From (4.8) we obtain that

$$e(x,y)=\mathcal{I}_N^{-\mu,-\nu}\left(\int_{-1}^{x}\int_{-1}^{y}(x-\xi)^{-\mu}(y-\eta)^{-\nu}\big(\tilde{K}\big(x,y,\xi,\eta,u(\xi,\eta)\big)-\tilde{K}\big(x,y,\xi,\eta,u_N(\xi,\eta)\big)\big)\,\mathrm{d}\eta\,\mathrm{d}\xi\right)$$
$$+J_1(x,y)+J_2(x,y), \tag{4.9}$$

where

$$J_1(x,y)=\mathcal{I}_N^{-\mu,-\nu}I(x,y), \qquad J_2(x,y)=u(x,y)-\mathcal{I}_N^{-\mu,-\nu}u(x,y).$$

Applying the Lagrange midvalue differential theorem to (4.9), we get that there exists a function $\phi(\xi,\eta)=u_N(\xi,\eta)+\lambda e(\xi,\eta)$ $(0<\lambda<1)$ such that

$$e(x,y)=\mathcal{I}_N^{-\mu,-\nu}\left(\int_{-1}^{x}\int_{-1}^{y}(x-\xi)^{-\mu}(y-\eta)^{-\nu}\frac{\partial\tilde{K}(x,y,\xi,\eta,\phi(\xi,\eta))}{\partial u}e(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi\right)$$
$$+J_1(x,y)+J_2(x,y), \tag{4.10}$$

where $\partial\tilde{K}/\partial u$ denotes the partial derivative of $\tilde{K}$ with respect to its fifth argument. For simplicity of notation, we set

$$\mathcal{M}e=\int_{-1}^{x}\int_{-1}^{y}(x-\xi)^{-\mu}(y-\eta)^{-\nu}\frac{\partial\tilde{K}(x,y,\xi,\eta,\phi(\xi,\eta))}{\partial u}e(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi.$$

Then we can write (4.10) as

$$e(x,y)=\int_{-1}^{x}\int_{-1}^{y}(x-\xi)^{-\mu}(y-\eta)^{-\nu}\frac{\partial\tilde{K}(x,y,\xi,\eta,\phi(\xi,\eta))}{\partial u}e(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi$$
$$+J_1(x,y)+J_2(x,y)+J_3(x,y), \tag{4.11}$$

where

$$J_3(x,y)=\mathcal{I}_N^{-\mu,-\nu}(\mathcal{M}e)-\mathcal{M}e.$$

Denoting $\Delta:=\{(x,y,\xi,\eta)\colon\ -1\leqslant\xi\leqslant x,\ -1\leqslant\eta\leqslant y,\ -1\leqslant x,y\leqslant 1\}$, from (4.11) we have

$$\big|e(x,y)\big|\leqslant M\int_{-1}^{x}\int_{-1}^{y}(x-\xi)^{-\mu}(y-\eta)^{-\nu}\big|e(\xi,\eta)\big|\,\mathrm{d}\eta\,\mathrm{d}\xi+\big|J_1(x,y)+J_2(x,y)+J_3(x,y)\big|, \tag{4.12}$$

where

$$M=\max_{(x,y,\xi,\eta)\in\Delta}\frac{\partial\tilde{K}(x,y,\xi,\eta,\phi(\xi,\eta))}{\partial u}.$$

Applying Lemma 5, we deduce from (4.12) that

$$\big|e(x,y)\big|\leqslant C\int_{-1}^{x}\int_{-1}^{y}(x-\xi)^{-\mu}(y-\eta)^{-\nu}\big|J_1(\xi,\eta)+J_2(\xi,\eta)+J_3(\xi,\eta)\big|\,\mathrm{d}\eta\,\mathrm{d}\xi$$
$$+\big|J_1(x,y)+J_2(x,y)+J_3(x,y)\big|. \tag{4.13}$$

Then we have

$$\|e\|_\infty \leqslant C\big(\|J_1\|_\infty + \|J_2\|_\infty + \|J_3\|_\infty\big). \tag{4.14}$$

Firstly, it follows from Lemma 2 and from (4.6) that

$$\begin{aligned}
\|J_1\|_\infty &= \big\|\mathcal{I}_N^{-\mu,-\nu} I(x,y)\big\|_\infty \leqslant C \max_{(x,y)\in\bar{\Omega}} \big|I(x,y)\big| \max_{(x,y)\in\bar{\Omega}} \sum_{m=0}^{N}\sum_{n=0}^{N} \big|F_m(x)F_n(y)\big| \\
&\leqslant \begin{cases} CN^{-m}\log^2 N(K^* + \|e\|_\infty), & \frac{1}{2} \leqslant \mu,\ \nu < 1, \\ N^{1-m-\mu-\nu}(K^* + \|e\|_\infty), & 0 < \mu, \nu < \frac{1}{2}, \\ N^{1/2-m-\nu}\log N(K^* + \|e\|_\infty), & \frac{1}{2} \leqslant \mu < 1,\ 0 < \nu < \frac{1}{2}, \\ N^{1/2-m-\mu}\log N(K^* + \|e\|_\infty), & 0 < \mu < \frac{1}{2},\ \frac{1}{2} \leqslant \nu < 1. \end{cases}
\end{aligned} \tag{4.15}$$

Next, by Lemma 3 we obtain that

$$\|J_2\|_\infty = \big\|u(x,y) - \mathcal{I}_N^{-\mu,-\nu} u(x,y)\big\|_\infty \leqslant CN^{4-m}\|u\|_{m,\omega_{-\mu,\,-\nu}}. \tag{4.16}$$

We now estimate the third term J_3. Note that

$$\mathcal{I}_N^{-\mu,-\nu} p(x,y) = p(x,y), \quad \text{i.e.,} \quad \big(\mathcal{I}_N^{-\mu,-\nu} - \mathcal{I}\big)p(x,y) = 0, \quad p \in \mathcal{P}_N, \tag{4.17}$$

where $\mathcal{I}$ denotes the identity operator. Using (3.8), Lemma 2, Lemma 7, and (4.17), we have

$$\begin{aligned}
\|J_3\|_\infty &= \big\|(\mathcal{I}_N^{-\mu,-\nu} - \mathcal{I})\mathcal{M}e\big\|_\infty = \big\|(\mathcal{I}_N^{-\mu,-\nu} - \mathcal{I})(\mathcal{M}e - \mathcal{G}_N\mathcal{M}e)\big\|_\infty \\
&\leqslant \big(1 + \big\|\mathcal{I}_N^{-\mu,-\nu}\big\|_\infty\big)\|\mathcal{M}e - \mathcal{G}_N\mathcal{M}e\|_\infty \leqslant CN^{-k}\big(1 + \big\|\mathcal{I}_N^{-\mu,-\nu}\big\|_\infty\big)\|\mathcal{M}e\|_{0,\kappa} \\
&\leqslant \begin{cases} CN^{-\kappa}\log^2 N\|e\|_\infty, & \frac{1}{2} \leqslant \mu, \nu < 1, \\ CN^{1-\kappa-\mu-\nu}\|e\|_\infty, & 0 < \mu, \nu < \frac{1}{2}, \\ CN^{1/2-\kappa-\nu}\log N\|e\|_\infty, & \frac{1}{2} \leqslant \mu < 1,\ 0 < \nu < \frac{1}{2}, \\ CN^{1/2-\kappa-\mu}\log N\|e\|_\infty, & 0 < \mu < \frac{1}{2},\ \frac{1}{2} \leqslant \nu < 1, \end{cases}
\end{aligned} \tag{4.18}$$

where in the last step we have used Lemma 7 under the following assumption:

$$\begin{cases} 0 < \kappa < \min\{\kappa_\mu, \kappa_\nu\}, & \frac{1}{2} \leqslant \mu, \nu < 1, \\ 1 - \mu - \nu < \kappa < \min\{\kappa_\mu, \kappa_\nu\}, & 0 < \mu, \nu < \frac{1}{2}, \\ \frac{1}{2} - \nu < \kappa < \min\{\kappa_\mu, \kappa_\nu\}, & \frac{1}{2} \leqslant \mu < 1,\ 0 < \nu < \frac{1}{2}, \\ \frac{1}{2} - \mu < \kappa < \min\{\kappa_\mu, \kappa_\nu\}, & 0 < \mu < \frac{1}{2},\ \frac{1}{2} \leqslant \nu < 1. \end{cases} \tag{4.19}$$

Thus, we obtain the desired estimate (4.2) by combining (4.14)–(4.16) and (4.18). □

Next, we provide the error estimate in weighted L^2-norm.

Theorem 2. *If the hypotheses given in Theorem* 1 *hold and* κ *satisfies condition* (4.19)*, then*

$$\|u-u_N\|_{\omega_{-\mu,-\nu}} \leqslant \begin{cases} CN^{-m}\big(N^{4-\kappa}\|u\|_{m,\omega_{-\mu,-\nu}}+K^*\big), & \frac{1}{2}\leqslant \mu,\nu<1, \\ CN^{-m}(N^{4-\kappa}\|u\|_{m,\omega_{-\mu,-\nu}}+K^*), & 0<\mu,\nu<\frac{1}{2}, \\ CN^{-m}(N^{4-\kappa}\|u\|_{m,\omega_{-\mu,-\nu}}+K^*), & \frac{1}{2}\leqslant \mu<1,\ 0<\nu<\frac{1}{2}, \\ CN^{-m}(N^{4-\kappa}\|u\|_{m,\omega_{-\mu,-\nu}}+K^*), & 0<\mu<\frac{1}{2},\ \frac{1}{2}\leqslant \nu<1, \end{cases} \tag{4.20}$$

provided that N *is sufficiently large, where* C *is a constant independent of* N*, and* K^* *is defined by* (4.3)*.*

Proof. It follows from (4.13), Lemma 5, and Lemma 8 that

$$\|e\|_{\omega_{-\mu,-\nu}} \leqslant C\big(\|J_1\|_{\omega_{-\mu,-\nu}}+\|J_2\|_{\omega_{-\mu,-\nu}}+\|J_3\|_{\omega_{-\mu,-\nu}}\big). \tag{4.21}$$

By Lemma 9 and (4.6) we have

$$\|J_1\|_{\omega_{-\mu,-\nu}} = \big\|\mathcal{I}_N^{-\mu,-\nu}I(x,y)\big\|_{\omega_{-\mu,-\nu}} \leqslant C\max_{(x,y)\in\bar{\Omega}}\big|I(x,y)\big| \leqslant CN^{-m}\big(K^*+\|e\|_\infty\big). \tag{4.22}$$

Applying Lemma 3 yields

$$\|J_2\|_{\omega_{-\mu,-\nu}} = \big\|u(x,y)-\mathcal{I}_N^{-\mu,-\nu}u(x,y)\big\|_{\omega_{-\mu,-\nu}} \leqslant CN^{-m}\|u\|_{m,\omega_{-\mu,-\nu}}. \tag{4.23}$$

From (3.8), Lemma 7, and Lemma 9 we obtain that

$$\begin{aligned} \|J_3\|_{\omega_{-\mu,-\nu}} &= \big\|\big(\mathcal{I}_N^{-\mu,-\nu}-\mathcal{I}\big)\mathcal{M}e\big\|_{\omega_{-\mu,-\nu}} = \big\|\big(\mathcal{I}_N^{-\mu,-\nu}-\mathcal{I}\big)(\mathcal{M}e-\mathcal{G}_N\mathcal{M}e)\big\|_{\omega_{-\mu,-\nu}} \\ &\leqslant \big\|\mathcal{I}_N^{-\mu,-\nu}(\mathcal{M}e-\mathcal{G}_N\mathcal{M}e)\big\|_{\omega_{-\mu,-\nu}} + \|\mathcal{M}e-\mathcal{G}_N\mathcal{M}e\|_{\omega_{-\mu,-\nu}} \\ &\leqslant C\|\mathcal{M}e-\mathcal{G}_N\mathcal{M}e\|_\infty \leqslant CN^{-k}\|\mathcal{M}e\|_{0,\kappa} \leqslant CN^{-k}\|e\|_\infty. \end{aligned}$$

Then using the convergence result in Theorem 1, we have

$$\|J_3\|_{\omega_{-\mu,-\nu}} \leqslant \begin{cases} CN^{-m-\kappa}(N^4\|u\|_{m,\omega_{-\mu,-\nu}}+\log^2 NK^*), & \frac{1}{2}\leqslant \mu,\nu<1, \\ CN^{-m-\kappa}(N^4\|u\|_{m,\omega_{-\mu,-\nu}}+N^{1-\mu-\nu}K^*), & 0<\mu,\nu<\frac{1}{2}, \\ CN^{-m-\kappa}(N^4\|u\|_{m,\omega_{-\mu,-\nu}}+N^{1/2-\nu}\log NK^*), & \frac{1}{2}\leqslant \mu<1,\ 0<\nu<\frac{1}{2}, \\ CN^{-m-\kappa}(N^4\|u\|_{m,\omega_{-\mu,-\nu}}+N^{1/2-\mu}\log NK^*), & 0<\mu<\frac{1}{2},\ \frac{1}{2}\leqslant \nu<1, \end{cases} \tag{4.24}$$

for sufficiently large N and for any κ satisfying (4.19). The desired estimate (4.20) follows from (4.21)–(4.23) and (4.24). □

5 Numerical experiments

In this section, we present some numerical results to illustrate the performance of the Jacobi-collocation method.

Example 1. Consider the following two-dimensional nonlinear weakly singular Volterra integral equation:

$$u(x,y) = \mathrm{e}^{xy} - 2(x+1)^{3/4}(y+1)^{2/3}\sin(x+y) + \int_{-1}^{x}\int_{-1}^{y}(x-\xi)^{-1/4}(y-\eta)^{-1/3}\sin(x+y)\mathrm{e}^{-2\xi\eta}u^2(\xi,\eta)\,\mathrm{d}\eta\,\mathrm{d}\xi. \tag{5.1}$$

This equation has a unique solution $u(x,y) = \mathrm{e}^{xy}$.

Table 1. The L^∞ and $L^2_{\omega_{-\mu,-\nu}}$ errors of $u_N(x,y)$ ($\mu = 1/4$, $\nu = 1/3$)

N	2	4	6
$\lVert u-u_N\rVert_\infty$	$3.20899\mathrm{e}-1$	$1.66576\mathrm{e}-3$	$1.11637\mathrm{e}-5$
$\lVert u-u_N\rVert_{\omega_{-\mu,-\nu}}$	$2.52782\mathrm{e}-1$	$2.90085\mathrm{e}-4$	$5.91151\mathrm{e}-7$
N	8	10	12
$\lVert u-u_N\rVert_\infty$	$7.35348\mathrm{e}-7$	$9.41531\mathrm{e}-11$	$7.32126\mathrm{e}-12$
$\lVert u-u_N\rVert_{\omega_{-\mu,-\nu}}$	$1.29209\mathrm{e}-7$	$2.47146\mathrm{e}-12$	$2.22215\mathrm{e}-12$

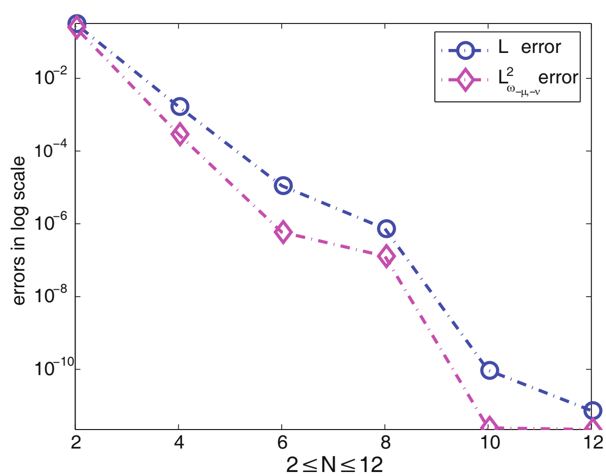

Figure 1. The L^∞ and $L^2_{\omega_{-\mu,-\nu}}$ errors versus N ($\mu = 1/4$, $\nu = 1/3$).

We use the Jacobi-collocation method suggested to resolve (5.1) numerically. Table 1 presents the numerical errors in the L^∞-norm and weighted L^2-norm of (5.1). We also plot the numerical errors versus the number of collocation points in Fig. 1. We observe that the numerical errors decay exponentially as N increases, which illustrates the theoretical results.

6 Conclusion and future work

In this paper, we proposed a spectral collocation method for the two-dimensional nonlinear weakly singular Volterra integral equation with smooth solutions. Under reasonable assumptions on the nonlinearity, we established the converge analysis in the L^∞-norm and weighted L^2-norm. The spectral accuracy of the suggested method was demonstrated by numerical experiments.

In the next work, we will consider the spectral method for the two-dimensional nonlinear weakly singular Volterra integral equations with nonsmooth solutions.

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