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On the rate of convergence in the global central limit theorem for random sums of independent random variables

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Abstract. We present upper bounds of the integral $\int_{-\infty}^{\infty}|x|^l|\mathbf{P}\{Z_N < x\} - \Phi(x)|\,\mathrm{d}x$ for $0 \leqslant l \leqslant 1+\delta$, where $0 < \delta \leqslant 1$, $\Phi(x)$ is a standard normal distribution function, and $Z_N = S_N/\sqrt{\mathbf{V}S_N}$ is the normalized random sum with variance $\mathbf{V}S_N > 0$ ($S_N = X_1 + \cdots + X_N$) of centered independent random variables $X_1, X_2, \dots$. The number of summands N is a nonnegative integer-valued random variable independent of $X_1, X_2, \dots$.

MSC: 60F05

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1 Introduction and main results

Let $X_1, X_2, \dots$ be a sequence of independent, not necessarily identically distributed, real random variables (r.v.s) with $\mathbf{E}X_i = 0$ and $\mathbf{E}|X_i|^{2+\delta} < \infty$, where $0 < \delta \leqslant 1$, for all $i = 1, 2, \dots$, and N be a nonnegative integer-valued r.v. independent of $X_1, X_2, \dots$. Let $\Phi(x)$ and $\varphi(x)$ be the standard normal distribution function and density. In what follows, $\mathbb{R}$ is the real line, $\mathbb{N} = \{1, 2, \dots\}$, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Write

$$Z_N = \frac{S_N}{\sqrt{\mathbf{V}S_N}}, \qquad S_N = \sum_{i=1}^{N} X_i, \qquad S_0 = 0,$$

assuming that the variance $\mathbf{V}S_N > 0$.

In this paper, we are interested in estimates of the quantity

$$\mathcal{I}_l = \int_{-\infty}^{\infty} |x|^l \big|\mathbf{P}\{Z_N < x\} - \Phi(x)\big|\,\mathrm{d}x$$

for some $l \geqslant 0$.

The upper bounds of $\mathcal{I}_0$ for random sums of independent r.v.s were obtained in [13]. However, the author has not found any published results on the upper bounds of the quantity $\mathcal{I}_l$ for $l > 0$. In this paper, we fill this vacancy and present the upper bounds of $\mathcal{I}_l$ for all $0 \leqslant l \leqslant 1 + \delta$, where $0 < \delta \leqslant 1$, for centered independent, not necessarily identically distributed, random variables.

For a fixed (nonrandom) number n of summands ($N = n$), we use the corresponding notation

$$Z_n = \frac{S_n - \mathbf{E}S_n}{\sqrt{\mathbf{V}S_n}}, \qquad S_n = \sum_{i=1}^{n} X_i, \qquad \mathbf{V}S_n = B_n^2 = \sum_{i=1}^{n} \mathbf{V}X_i,$$

$$\mathcal{I}_{l,n} = \int_{-\infty}^{\infty} |x|^l \big|\mathbf{P}\{Z_n < x\} - \Phi(x)\big| \,\mathrm{d}x, \qquad L_{2+\delta,n} = \frac{1}{B_n^{2+\delta}} \sum_{i=1}^{n} \mathbf{E}|X_i - \mathbf{E}X_i|^{2+\delta}$$

with variance $B_n^2 > 0$.

By $C(\cdot)$ with an index or without it, we denote a positive finite factor depending only on the quantities indicated in the parentheses (not necessarily the same at different places).

The estimates of the quantity $\mathcal{I}_{l,n}$ for a fixed number of random summands are considered, for example, in [2], for stationary sequences satisfying projective criteria in the style of Gordin or weak dependence conditions with $l = 0$; in [4], for m-dependent r.v.s with $l = 0$; in [5], for independent r.v.s with $l = 0$; in [6], for L_p bounds ($1 \leqslant p \leqslant \infty$) for the remainder in a combinatorial central limit theorem for independent r.v.s; in [9] and [12], for m-dependent r.v.s with $l \geqslant 0$; and in the book [1], for the normal approximation with Lipschitz functions and with the Kolmogorov distance for random sums of independent identically distributed (i.i.d.) r.v.s with nonrandom centering by Stein's method. We would like to mention here the book [7] devoted to general limit theorems for random sums.

Note that in the recent paper [3], new and general Berry–Esseen and Wasserstein bounds in the CLT for nonrandomly centered random sums are given, which are of the correct order in the case of some random indexes.

The main results of this paper are Theorems 1, 2, and 3. To formulate the results, we introduce the additional r.v.s

$$B_N^2 = \sum_{i=1}^{N} \mathbf{V}X_i, \qquad l_{r,N} = \sum_{i=1}^{N} \mathbf{E}|X_i|^r, \quad r = l,\, l+1,\, 2+\delta.$$

First of all, we recall that the variance $\mathbf{V}S_N = \mathbf{E}B_N^2$ under the condition that r.v.s $N, X_1, X_2, \dots$ are independent and $\mathbf{E}X_i = 0$ for all $i = 1, 2, \dots$.

The following statement is valid.

Theorem 1. *Let $X_1, X_2, \dots$ be independent, not necessarily identically distributed, r.v.s with $\mathbf{E}X_i = 0$, $\mathbf{E}|X_i|^{2+\delta} < \infty$, where $0 < \delta \leqslant 1$, for all $i = 1, 2, \dots$, and $\mathbf{V}S_1 = \mathbf{V}X_1 = \mathbf{E}X_1^2 > 0$. Let N be a nonnegative integer-valued r.v. independent of $X_1, X_2, \dots$. Then*

$$\mathcal{I}_l \leqslant C(l,\delta) \frac{\mathbf{E}l_{2+\delta,N}}{(\mathbf{E}B_N^2)^{(2+\delta)/2}} + C(l) \frac{\mathbf{E}|B_N^2 - \mathbf{E}B_N^2|}{\mathbf{E}B_N^2} \tag{1.1}$$

for $0 \leqslant l \leqslant 1$, and

$$\mathcal{I}_l \leqslant C(l) \frac{\mathbf{E}l_{l+1,N}}{(\mathbf{E}B_N^2)^{(l+1)/2}} + C(l,\delta) \frac{\mathbf{E}l_{2+\delta,N}}{(\mathbf{E}B_N^2)^{(2+\delta)/2}} + C(l) \frac{\mathbf{E}|B_N^{l+1} - (\mathbf{E}B_N^2)^{(l+1)/2}|}{(\mathbf{E}B_N^2)^{(l+1)/2}} \tag{1.2}$$

for $1 < l \leqslant 1 + \delta$.

In the case of identically distributed summands $X_1, X_2, \dots$, from Theorem 1 we derive the following result.

Corollary 1. *Let* $X, X_1, X_2, \ldots$ *be i.i.d. r.v.s with* $\mathbf{E}X = 0$*,* $0 < \sigma^2 = \mathbf{V}X$*, and* $\beta_{2+\delta} = \mathbf{E}|X|^{2+\delta} < \infty$*, where* $0 < \delta \leqslant 1$*. Let* N *be a nonnegative integer-valued r.v. with* $0 < \mathbf{E}N < \infty$ *independent of* $X_1, X_2, \ldots$ *. Then* $Z_N = S_N/(\sigma\sqrt{\mathbf{E}N})$*,*

$$\mathcal{I}_l \leqslant C(l,\delta)\frac{\beta_{2+\delta}}{\sigma^{2+\delta}}\frac{1}{(\mathbf{E}N)^{\delta/2}} + C(l)\frac{\mathbf{E}|N - \mathbf{E}N|}{\mathbf{E}N} \tag{1.3}$$

for $0 \leqslant l \leqslant 1$*, and*

$$\mathcal{I}_l \leqslant C(l)\frac{\beta_{l+1}}{\sigma^{l+1}}\frac{1}{(\mathbf{E}N)^{(l-1)/2}} + C(l,\delta)\frac{\beta_{2+\delta}}{\sigma^{2+\delta}}\frac{1}{(\mathbf{E}N)^{\delta/2}} + C(l)\frac{\mathbf{E}|N^{(l+1)/2} - (\mathbf{E}N)^{(l+1)/2}|}{(\mathbf{E}N)^{(l+1)/2}} \tag{1.4}$$

for $1 < l \leqslant 1 + \delta$*.*

Denote

$$\lambda_l = \big|\mathbf{E}|Z_N|^l - \mathbf{E}|Y|^l\big|.$$

(I hope that there will not be any confusion between the absolute value of the difference between the absolute moments λ_l and the Poisson parameter λ.)

The estimates of differences λ_l between the absolute moments of the random sum Z_N and the standard normal r.v. Y follow from estimates of $\mathcal{I}_l$ of Theorem 1.

Namely, the following result is valid.

Theorem 2. *Let the conditions of Theorem* 1 *hold, and let* Y *be a standard normal r.v. Then*

$$\lambda_l \leqslant C(l,\delta)\frac{\mathbf{E}l_{2+\delta,N}}{(\mathbf{E}B_N^2)^{(2+\delta)/2}} + C(l)\frac{\mathbf{E}|B_N^2 - \mathbf{E}B_N^2|}{\mathbf{E}B_N^2} \tag{1.5}$$

for $1 \leqslant l \leqslant 2$*, and*

$$\lambda_l \leqslant C(l)\frac{\mathbf{E}l_{l,N}}{(\mathbf{E}B_N^2)^{l/2}} + C(l,\delta)\frac{\mathbf{E}l_{2+\delta,N}}{(\mathbf{E}B_N^2)^{(2+\delta)/2}} + C(l)\frac{\mathbf{E}|B_N^l - (\mathbf{E}B_N^2)^{l/2}|}{(\mathbf{E}B_N^2)^{l/2}} \tag{1.6}$$

for $2 < l \leqslant 2 + \delta$*.*

In the case of identically distributed summands $X_1, X_2, \ldots$, from Theorem 2 we obtain the following result.

Corollary 2. *Let* $X, X_1, X_2, \ldots$ *be i.i.d. r.v.s with* $\mathbf{E}X = 0$*,* $0 < \sigma^2 = \mathbf{V}X$*, and* $\beta_{2+\delta} = \mathbf{E}|X|^{2+\delta} < \infty$*, where* $0 < \delta \leqslant 1$*. Let* N *be a nonnegative integer-valued r.v. with* $0 < \mathbf{E}N < \infty$ *independent of* $X_1, X_2, \ldots$ *. Then* $Z_N = S_N/(\sigma\sqrt{\mathbf{E}N})$*,*

$$\lambda_l \leqslant C(l,\delta)\frac{\beta_{2+\delta}}{\sigma^{2+\delta}}\frac{1}{(\mathbf{E}N)^{\delta/2}} + C(l)\frac{\mathbf{E}|N - \mathbf{E}N|}{\mathbf{E}N} \tag{1.7}$$

for $1 \leqslant l \leqslant 2$*, and*

$$\lambda_l \leqslant C(l)\frac{\beta_l}{\sigma^l}\frac{1}{(\mathbf{E}N)^{(l-2)/2}} + C(l,\delta)\frac{\beta_{2+\delta}}{\sigma^{2+\delta}}\frac{1}{(\mathbf{E}N)^{\delta/2}} + C(l)\frac{\mathbf{E}|N^{l/2} - (\mathbf{E}N)^{l/2}|}{(\mathbf{E}N)^{l/2}} \tag{1.8}$$

for $2 < l \leqslant 2 + \delta$*.*

Now, we present the results following from inequalities (1.3) of Corollary 1 and (1.7) of Corollary 2 for three concrete random indices N, the definitions of which are as follows. First, introduce a new definition of the τ-shifted $\mathcal{L}$ distributions (τ-shifted Poisson distribution, τ-shifted binomial distribution, τ-shifted negative binomial distribution, and so on).

In the sequel, we write $\xi \sim \mathcal{L}$ if a r.v. ξ is distributed by the $\mathcal{L}$ distribution.

DEFINITION 1. We say that a discrete r.v. N is distributed by the τ-shifted $\mathcal{L}$ distribution ($\tau \geqslant 0$) (for short, $N - \tau \sim \mathcal{L}$), or N is the τ-shifted r.v., if for any discrete r.v. $\xi \sim \mathcal{L}$ taking values $\{x_k\}$,

$$\mathbf{P}\{N = x_k + \tau\} = \mathbf{P}\{\xi = x_k\} = p_k. \tag{1.9}$$

In the particular case, the 0-shifted $\mathcal{L}$ distribution coincides with the $\mathcal{L}$ distribution.

DEFINITION 2. We say that a r.v. N is distributed by the τ-shifted Poisson distribution ($\tau \geqslant 0$) with parameter $\lambda > 0$ (for short, $N - \tau \sim \mathcal{P}(\lambda)$) if

$$\mathbf{P}\{N = k + \tau\} = \mathbf{P}\{\xi = k\} = \frac{\lambda^k}{k!}\mathrm{e}^{-\lambda}, \quad k = 0, 1, 2, \dots . \tag{1.10}$$

DEFINITION 3. We say that a r.v. N is distributed by the τ-shifted binomial distribution ($\tau \geqslant 0$) with parameters $n \in \mathbb{N}$ and $0 < p < 1$ (for short, $N - \tau \sim \mathcal{B}(n, p)$) if

$$\mathbf{P}\{N = k + \tau\} = \mathbf{P}\{\xi = k\} = C_n^k p^k (1 - p)^{n-k}, \quad k = 0, 1, \dots, n. \tag{1.11}$$

DEFINITION 4. We say that a r.v. N is distributed by the τ-shifted negative binomial distribution ($\tau \geqslant 0$) with the parameters $r \in \mathbb{N}$ and $0 < p < 1$ (for short, $N - \tau \sim \mathcal{NB}(r, p)$ if

$$\mathbf{P}\{N = k + \tau\} = \mathbf{P}\{\xi = k\} = \binom{k-1}{r-1} p^r (1 - p)^{k-r}, \quad k = r, r + 1, \dots . \tag{1.12}$$

Now, we present the following statement for three presented τ-shifted $\mathcal{L}$ distributions.

Theorem 3. *Let $X, X_1, X_2, \dots$ be i.i.d. r.v.s with $\mathbf{E}X = 0$, $0 < \sigma^2 = \mathbf{V}X$, and $\beta_{2+\delta} = \mathbf{E}|X|^{2+\delta} < \infty$, where $0 < \delta \leqslant 1$. Let N be a nonnegative integer-valued r.v. independent of $X_1, X_2, \dots$. Then $Z_N = S_N/(\sigma\sqrt{\mathbf{E}N})$, and*:

(i) *If $N - \tau \sim \mathcal{P}(\lambda)$ with $\tau \in \mathbb{N}_0$ and $\lambda > 0$, then*

$$\mathcal{I}_l \leqslant C_1 \frac{1}{(\tau + \lambda)^{\delta/2}} \tag{1.13}$$

for $0 \leqslant l \leqslant 1$, and

$$\lambda_l \leqslant C_2 \frac{1}{(\tau + \lambda)^{\delta/2}} \tag{1.14}$$

for $1 \leqslant l \leqslant 2$.

(ii) *If $N - \tau \sim \mathcal{B}(n, p)$ with $\tau \in \mathbb{N}_0$, $n \in \mathbb{N}$, and $0 < p < 1$, then*

$$\mathcal{I}_l \leqslant C_3 \frac{1}{(\tau + np)^{\delta/2}} \tag{1.15}$$

for $0 \leqslant l \leqslant 1$, and

$$\lambda_l \leqslant C_4 \frac{1}{(\tau + np)^{\delta/2}} \tag{1.16}$$

for $1 \leqslant l \leqslant 2$.

(iii) *If* $N-\tau \sim \mathcal{NB}(r,p)$ *with* $\tau \in \mathbb{N}_0$*,* $r \in \mathbb{N}$*, and* $0<p<1$*, then*

$$\mathcal{I}_l \leqslant C_5 \frac{1}{(\tau p + r)^{\delta/2}}, \tag{1.17}$$

for $0 \leqslant l \leqslant 1$*, and*

$$\lambda_l \leqslant C_6 \frac{1}{(\tau p + r)^{\delta/2}}, \tag{1.18}$$

for $1 \leqslant l \leqslant 2$*.*

Here, $C_i = C_i(l, \delta, \sigma, \beta_{2+\delta})$*,* $i = 1,2,3,4,5,6$*.*

Since the 0-shifted $\mathcal{L}$ distribution coincides with the $\mathcal{L}$ distribution, substituting $\tau = 0$ in Theorem 3, we obtain the corresponding estimates of $\mathcal{I}_l$ and λ_l for a Poisson random sum, for a binomial random sum, and for a negative binomial random sum.

2 Auxiliary results

To prove Theorem 1, we use a particular case (Theorem 5) of the following result for the sum with a fixed number n of m-dependent summands $X_1, \dots, X_n$ (Theorem 4).

Let $X_1, \dots, X_n$ be real m-dependent r.v.s (see, e.g., [4]) with $\mathbf{E}|X_i|^\kappa < \infty$ for some $\kappa \geqslant 2$ and all $i = 1, \dots, n$.

Denote

$$Z_n = \sum_{i=1}^n A_i, \qquad A_i = \frac{X_i - \mathbf{E}X_i}{\sqrt{\mathbf{V}(\sum_{i=1}^n X_i)}}, \qquad \overline{A}_i = A_i \mathbf{1}_{\{|A_i| \leqslant t\}}, \qquad \overline{\overline{A}}_i = A_i \mathbf{1}_{\{|A_i| > t\}},$$

$$L_{r,n} = \sum_{i=1}^n \mathbf{E}|A_i|^r, \qquad \overline{L}_{r,n}(t) = \sum_{i=1}^n \mathbf{E}|\overline{A}_i|^r, \qquad \overline{\overline{L}}_{r,n}(t) = \sum_{i=1}^n \mathbf{E}|\overline{\overline{A}}_i|^r,$$

$$\mathcal{I}_{l,n} = \int_{-\infty}^{\infty} |x|^l \big|\mathbf{P}\{Z_n < x\} - \Phi(x)\big| \, \mathrm{d}x,$$

where the truncation level $t > 0$, the variance $\mathbf{V}(\sum_{i=1}^n X_i) > 0$, and $\mathbf{1}_A$ is the indicator of an event A.

Theorem 4. (*See* [12].) *Let* $X_1, \dots, X_n$ *be real* m*-dependent r.v.s with* $\mathbf{E}|X_i|^\kappa < \infty$ *for some* $\kappa \geqslant 2$ *and all* $i = 1, \dots, n$*. Then, for the truncation level* $t_0 = (m+1)^{-1}$ *and any fixed* $n = 1, 2, \dots$*, we have*

$$\mathcal{I}_{l,n} \leqslant C(l)\big((m+1)\overline{\overline{L}}_{2,n}(t_0) + (m+1)^2 \overline{L}_{3,n}(t_0)\big) \tag{2.1}$$

for $0 \leqslant l \leqslant 1$ *and if* $(m+1)L_{2,n} \leqslant C_*$*, then*

$$\mathcal{I}_{l,n} \leqslant C(l, C_*)\big((m+1)^l \overline{\overline{L}}_{l+1,n}(t_0) + (m+1)^2 \overline{L}_{3,n}(t_0)\big) \tag{2.2}$$

for $1 < l \leqslant \kappa - 1$*.*

Theorem 4 in [12] is presented with the additional condition $\mathbf{E}X_i = 0$, $i = 1, \dots, n$, but its proof demonstrates that this restriction is unnecessary.

We recall that to prove Theorem 4, we use a powerful and general direct Stein's method introduced in [11] for estimating the rate of convergence of sums of weakly dependent r.v.s to the normal distribution.

In what follows, for independent ($m = 0$) summands $X_1, \dots, X_n$, we use the same notation as for m-dependent r.v.s but with the truncation level $t = t_0 = 1$. Furthermore, we introduce some additional notation for centered r.v.s:

$$\overline{A}_i = A_i \mathbf{1}_{\{|A_i| \leqslant 1\}}, \qquad \overline{A}_i^{(0)} = \overline{A}_i - \mathbf{E}\overline{A}_i, \qquad \overline{\overline{A}}_i = A_i \mathbf{1}_{\{|A_i| > 1\}}, \qquad \overline{\overline{A}}_i^{(0)} = \overline{\overline{A}}_i - \mathbf{E}\overline{\overline{A}}_i,$$

$$\overline{Z}_n^{(0)} = \sum_{i=1}^n \overline{A}_i^{(0)}, \qquad \overline{\overline{Z}}_n^{(0)} = \sum_{i=1}^n \overline{\overline{A}}_i^{(0)}.$$

The following statement follows from Theorem 4.

Theorem 5. *Let $X_1, \dots, X_n$ be real independent r.v.s with $\mathbf{E}|X_i|^{2+\delta} < \infty$, where $0 < \delta \leqslant 1$, for all $i = 1, \dots, n$. Then, for all $0 \leqslant l \leqslant 1 + \delta$ and any fixed $n = 1, 2, \dots$,*

$$\mathcal{I}_{l,n} \leqslant \frac{C_0(l)}{(\sum_{j=1}^n \mathbf{V}X_j)^{(2+\delta)/2}} \sum_{i=1}^n \mathbf{E}|X_i - \mathbf{E}X_i|^{2+\delta}. \tag{2.3}$$

Proof. Since $L_{2,n} = 1$ for independent r.v.s $X_1, \dots, X_n$, in Theorem 4, we take $m = 0$, $t_0 = 1$, and the evident relations with $1 < l \leqslant 1 + \delta$

$$\overline{\overline{L}}_{2,n}(1) + \overline{L}_{3,n}(1) \leqslant \overline{\overline{L}}_{2+\delta,n}(1) + \overline{L}_{2+\delta,n}(1) = L_{2+\delta,n}, \qquad \overline{\overline{L}}_{l+1,n}(1) \leqslant \overline{\overline{L}}_{2+\delta,n}(1). \qquad \square$$

Note that in the case $l = 0$ and $\delta = 1$, in the paper [5], it was proved that $C_0(0) = 1$, that is, if in Theorem 5, $\mathbf{E}|X_i|^3 < \infty$ for all $i = 1, \dots, n$, then, for any fixed $n = 1, 2, \dots$,

$$\mathcal{I}_{0,n} \leqslant \frac{1}{(\sum_{j=1}^n \mathbf{V}X_j)^{3/2}} \sum_{i=1}^n \mathbf{E}|X_i - \mathbf{E}X_i|^3.$$

To transfer estimate (2.3) for random sums, we need Lemmas 2 and 3. To prove Lemma 2, we use the following statement.

Lemma 1. *Let ξ and η be real r.v.s with $\mathbf{E}|\xi|^{l+1} < \infty$ and $\mathbf{E}|\eta|^{l+1} < \infty$ for some $l \geqslant 0$, respectively. Then*

$$\int_{-\infty}^{\infty} |x|^l \bigl|\mathbf{P}\{\xi < x\} - \mathbf{P}\{\eta < x\}\bigr| \,\mathrm{d}x \leqslant \frac{\mathbf{E}|\xi|^{l+1} + \mathbf{E}|\eta|^{l+1}}{l+1}. \tag{2.4}$$

In a particular case, if $\eta = Y$ is a standard normal r.v. with distribution function $\Phi(x)$, then, for $l \geqslant 0$,

$$\int_{-\infty}^{\infty} |x|^l \bigl|\mathbf{P}\{\xi < x\} - \Phi(x)\bigr| \,\mathrm{d}x \leqslant \begin{cases} \frac{1}{l+1}\bigl(\mathbf{E}|\xi|^{l+1} + \frac{2^{(l+1)/2}\Gamma((l+2)/2)}{\sqrt{\pi}}\bigr) & \text{if } l \geqslant 0, \\ \frac{1}{l+1}\bigl(\mathbf{E}|\xi|^{l+1} + 1\bigr) & \text{if } 0 \leqslant l \leqslant 1, \\ \frac{1}{l+1}\bigl(\mathbf{E}|\xi|^{l+1} + \frac{2\sqrt{2}}{\sqrt{\pi}}\bigr) & \text{if } 1 < l \leqslant 2. \end{cases} \tag{2.5}$$

Moreover, if $\mathbf{E}\xi^2 = 1$, then, for all $0 \leqslant l \leqslant 1$,

$$\int_{-\infty}^{\infty} |x|^l \bigl|\mathbf{P}\{\xi < x\} - \Phi(x)\bigr| \,\mathrm{d}x \leqslant \frac{2}{l+1}. \tag{2.6}$$

Here $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\,\mathrm{d}x < \infty$ with $\alpha > 0$.

Proof. First, we observe that, for any $u, v, x \in \mathbb{R}$ and $l \geqslant 0$,

$$\int_{-\infty}^{\infty} |x|^l |\mathbf{1}_{\{u<x\}} - \mathbf{1}_{\{v<x\}}| \,\mathrm{d}x \leqslant \left| \int_u^v |x|^l \,\mathrm{d}x \right| \leqslant |u-v| \max\{|u|^l; |v|^l\}, \tag{2.7}$$

and therefore

$$\begin{aligned} \int_{-\infty}^{\infty} |x|^l \big|\mathbf{P}\{\xi < x\} - \mathbf{P}\{\eta < x\}\big| \,\mathrm{d}x &\leqslant \int_{-\infty}^{\infty} |x|^l \mathbf{E}|\mathbf{1}_{\{\xi<x\}} - \mathbf{1}_{\{\eta<x\}}| \,\mathrm{d}x \\ &\leqslant \mathbf{E}|\xi-\eta| \max\{|\xi|^l; |\eta|^l\}. \end{aligned} \tag{2.8}$$

Thus, (2.4) with $l = 0$ follows from (2.8).

Let now $l > 0$. Since, for $l > 0$ (see [10, p. 208, Cor. 2] or [8, p. 61, Lemma 2.4]),

$$\mathbf{E}|\xi|^l = l \int_0^{\infty} x^{l-1} \mathbf{P}\{|\xi| \geqslant x\} \,\mathrm{d}x, \tag{2.9}$$

we have that, for $l > 0$,

$$\begin{aligned} \int_{-\infty}^{\infty} |x|^l \big|\mathbf{P}\{\xi < x\} - \mathbf{P}\{\eta < x\}\big| \,\mathrm{d}x &\leqslant \int_0^{\infty} x^l \mathbf{P}\{|\xi| \geqslant x\} \,\mathrm{d}x + \int_0^{\infty} x^l \mathbf{P}\{|\eta| \geqslant x\} \,\mathrm{d}x \\ &\leqslant (l+1)^{-1} \big(\mathbf{E}|\xi|^{l+1} + \mathbf{E}|\eta|^{l+1}\big). \end{aligned} \tag{2.10}$$

Thus, (2.4) is proved for all $l \geqslant 0$.

To prove (2.5), we substitute the well-known expression of the absolute moment $\mathbf{E}|Y|^{l+1}$ of the standard normal r.v. $\eta = Y$ into (2.4) (see, e.g., [10, p. 245, Prob. 7]): for $p \geqslant 0$,

$$\mathbf{E}|Y|^p = \frac{2^{p/2}}{\sqrt{\pi}} \Gamma\left(\frac{p+1}{2}\right), \tag{2.11}$$

where $\Gamma(t) = \int_0^{\infty} x^{t-1} \mathrm{e}^{-x} \,\mathrm{d}x$ is the gamma function, and we observe that $\mathbf{E}|Y|^{l+1} \leqslant 1$ for $0 \leqslant l \leqslant 1$, and $\mathbf{E}|Y|^{l+1} = (2^{(l+1)/2}/\sqrt{\pi})\Gamma((l+2)/2) \leqslant 2\sqrt{2}/\sqrt{\pi}$ for $1 < l \leqslant 2$.

Estimate (2.6) follows from (2.5).

Lemma 1 is proved. □

Lemma 2. *Let $X_1, \dots, X_n$ be real independent r.v.s with $\mathbf{E}|X_i|^{\kappa} < \infty$ for some $\kappa \geqslant 2$ and all $i = 1, \dots, n$. Then, for all $0 \leqslant l \leqslant \kappa - 1$ and any fixed $n = 1, 2, \dots,$*

$$\mathbf{E}\big|\overline{Z}_n^{(0)}\big|^{l+1} \leqslant C(l), \qquad \mathbf{E}\big|\overline{\overline{Z}}_n^{(0)}\big|^{l+1} \leqslant C(l)\overline{\overline{L}}_{l+1,n}, \tag{2.12}$$

$$\mathcal{I}_{l,n} \leqslant C_*(l)\left(1 + \frac{1}{(\mathbf{V}S_n)^{(l+1)/2}} \sum_{i=1}^n \mathbf{E}|\widehat{X}_i|^{l+1} \mathbf{1}_{\{|\widehat{X}_i| \geqslant \sqrt{\mathbf{V}S_n}\}}\right), \tag{2.13}$$

where $\widehat{\xi} = \xi - \mathbf{E}\xi$.

Proof. Since $\mathbf{E}(\overline{Z}_n^{(0)})^2 = \sum_{i=1}^n \mathbf{E}(\overline{A}_i^{(0)})^2 \leqslant \sum_{i=1}^n \mathbf{E}\overline{A}_i^2 = \overline{L}_2$, by the Lyapunov inequality we have that, for $0 \leqslant l \leqslant 1$,

$$\mathbf{E}\big|\overline{Z}_n^{(0)}\big|^{l+1} \leqslant \big(\mathbf{E}\big(\overline{Z}_n^{(0)}\big)^2\big)^{(l+1)/2} \leqslant \overline{L}_2^{(l+1)/2} \leqslant L_2^{(l+1)/2} = 1.$$

Let now $1 < l \leqslant \kappa - 1$. Then $\mathbf{E}|\overline{A}_i^{(0)}|^{l+1} \leqslant 2^{l+1}\mathbf{E}|\overline{A}_i|^{l+1}$ and $\overline{L}_{l+1} \leqslant \overline{L}_2 \leqslant L_2 = 1$. Therefore, by the Rosenthal inequality (see [8, p. 59, Thm. 2.9])

$$\mathbf{E}\big|\overline{Z}_n^{(0)}\big|^{l+1} \leqslant C(l)\max\left\{\sum_{i=1}^n \mathbf{E}\big|\overline{A}_i^{(0)}\big|^{l+1}, \left(\sum_{i=1}^n \mathbf{E}\big(\overline{A}_i^{(0)}\big)^2\right)^{(l+1)/2}\right\} \leqslant C(l).$$

The first inequality of (2.12) is proved.

Since $\mathbf{E}|\overline{\overline{A}}_i^{(0)}|^{l+1} \leqslant 2^{l+1}\mathbf{E}|\overline{\overline{A}}_i|^{l+1}$, we have by the Bahr and Esseen (1965) inequality (see [8, p. 82]) that, for $0 \leqslant l \leqslant 1$,

$$\mathbf{E}\big|\overline{\overline{Z}}_n^{(0)}\big|^{l+1} \leqslant \left(2 - \frac{1}{n}\right)\sum_{i=1}^n \mathbf{E}\big|\overline{\overline{A}}_i^{(0)}\big|^{l+1} \leqslant 2^{l+2}\overline{\overline{L}}_{l+1}.$$

Let now $1 < l \leqslant \kappa - 1$. Then $\mathbf{E}|\overline{\overline{A}}_i^{(0)}|^{l+1} \leqslant 2^{l+1}\mathbf{E}|\overline{\overline{A}}_i|^{l+1}$ and $\overline{\overline{L}}_2 \leqslant \overline{\overline{L}}_{l+1}$. Since $\overline{\overline{L}}_2 \leqslant L_2 = 1$, we have by the Rosenthal inequality that

$$\begin{aligned}\mathbf{E}\big|\overline{\overline{Z}}_n^{(0)}\big|^{l+1} &\leqslant C(l)\max\left\{\sum_{i=1}^n \mathbf{E}\big|\overline{\overline{A}}_i^{(0)}\big|^{l+1}, \left(\sum_{i=1}^n \mathbf{E}\big(\overline{\overline{A}}_i^{(0)}\big)^2\right)^{(l+1)/2}\right\}\\ &\leqslant C(l)\max\big\{\overline{\overline{L}}_{l+1}, \overline{\overline{L}}_2^{(l+1)/2}\big\} \leqslant C(l)\max\big\{\overline{\overline{L}}_{l+1}, \overline{\overline{L}}_2\big\} \leqslant C(l)\overline{\overline{L}}_{l+1}.\end{aligned}$$

The second inequality of (2.12) is also proved.

To prove (2.13), observing that $Z_n = \overline{Z}_n^{(0)} + \overline{\overline{Z}}_n^{(0)}$ and using the so-called c_r-inequality and inequalities (2.12), we obtain that, for all $0 \leqslant l \leqslant \kappa - 1$,

$$\mathbf{E}|Z_n|^{l+1} \leqslant 2^l\big(\mathbf{E}\big|\overline{Z}_n^{(0)}\big|^{l+1} + \mathbf{E}\big|\overline{\overline{Z}}_n^{(0)}\big|^{l+1}\big) \leqslant C(l)(1 + \overline{\overline{L}}_{l+1}).$$

It only remains to take $\xi = Z_n$ in inequality (2.5) and use the last inequality.

Lemma 2 is proved. □

Lemma 3. *For any $a > 0$ and all $l \geqslant 0$,*

$$\int_{-\infty}^{\infty} |x|^l\big|\Phi(xa) - \Phi(x)\big|\,\mathrm{d}x = \frac{2^{(l+2)/2}\Gamma((l+2)/2)}{\sqrt{2\pi}}|a-1|\int_0^1 \frac{\mathrm{d}t}{(\gamma(t))^{(l+2)/2}} \tag{2.14}$$

$$\leqslant \frac{2^{(l+2)/2}\Gamma((l+2)/2)}{\sqrt{2\pi}(l+1)}\left|1 - \frac{1}{a^{l+1}}\right|, \tag{2.15}$$

where $\gamma(t) = [1 + t(a-1)]^2 > 0$.

Proof. By Taylor's theorem with the remainder term of the integral form

$$\Phi(xa)-\Phi(x)=(xa-x)\int_0^1\varphi\bigl(x+t(xa-x)\bigr)\,\mathrm{d}t=\frac{1}{\sqrt{2\pi}}(a-1)x\int_0^1\mathrm{e}^{-(1/2)x^2\gamma(t)}\,\mathrm{d}t. \tag{2.16}$$

Therefore, for all $l\geqslant 0$,

$$\int_{-\infty}^{\infty}|x|^l\bigl|\Phi(xa)-\Phi(x)\bigr|\,\mathrm{d}x=\frac{2}{\sqrt{2\pi}}|a-1|\int_0^1\int_0^{\infty}x^{l+1}\mathrm{e}^{-(1/2)x^2\gamma(t)}\,\mathrm{d}x\,\mathrm{d}t. \tag{2.17}$$

Observing that, for all $l\geqslant 0$ and any $b>0$,

$$\int_0^{\infty}x^{l+1}\mathrm{e}^{-bx^2}\,\mathrm{d}x=\frac{\Gamma((l+2)/2)}{2b^{(l+2)/2}}, \tag{2.18}$$

we get (2.14).

Estimate (2.15) follows from (2.14) and the evident equality

$$\int_0^1\frac{\mathrm{d}t}{(\gamma(t))^{(l+2)/2}}=\frac{1}{(l+1)(1-a)}\left(\frac{1}{a^{l+1}}-1\right). \tag{2.19}$$

Lemma 3 is proved. □

To prove Theorem 3, we need three lemmas.

The following statement is valid for the any τ-shifted r.v. N.

Lemma 4. *Let $N-\tau\sim\mathcal{L}$ with $\tau\geqslant 0$. Then the moment-generating function (m.g.f.) $M_N(t)=\mathbf{E}\mathrm{e}^{tN}$ of the τ-shifted r.v. N*

$$M_N(t)=\mathrm{e}^{\tau t}M_\xi(t), \tag{2.20}$$

and

$$\mathbf{E}N=\tau+\mathbf{E}\xi,\qquad \mathbf{E}N^2=(\tau+\mathbf{E}\xi)^2+\mathbf{V}\xi,\qquad \mathbf{V}N=\mathbf{V}\xi. \tag{2.21}$$

Moreover, if $\mathbf{E}N>0$, then

$$\frac{\mathbf{E}|N-N|}{\mathbf{E}N}\leqslant\frac{\sqrt{\mathbf{V}\xi}}{\tau+\mathbf{E}\xi}. \tag{2.22}$$

Proof. First, we observe that the m.g.f. of the τ-shifted r.v. N is as follows:

$$M_N(t)=\sum_k\mathrm{e}^{t(x_k+\tau)}\mathbf{P}\{\xi=x_k\}=\mathrm{e}^{\tau t}\sum_k\mathrm{e}^{tx_k}p_k=\mathrm{e}^{\tau t}M_\xi(t).$$

Equalities (2.21) of $\mathbf{E}N$, $\mathbf{E}N^2$, and $\mathbf{V}N$ follow directly using their definitions or using the well-known equalities $\mathbf{E}N^k=(\mathrm{d}^k/\mathrm{d}t^k)M_N(t)|_{t=0}$, $k=1,2$.

The inequality of Lemma 4 follows from the estimate $\mathbf{E}|N-\mathbf{E}N|\leqslant\sqrt{\mathbf{V}N}$ and the equality of $\mathbf{E}N$ presented above.

Lemma 4 is proved. □

The following statement is valid for a τ-shifted Poisson r.v. N.

Lemma 5. *Let $N-\tau \sim \mathcal{P}(\lambda)$ with $\tau \geqslant 0$ and $\lambda > 0$. Then*

$$\mathbf{E}N = \tau+\lambda, \qquad \mathbf{E}N^2 = (\tau+\lambda)^2+\lambda, \qquad \mathbf{V}N = \lambda. \tag{2.23}$$

Moreover,

$$\frac{\mathbf{E}|N-\mathbf{E}N|}{\mathbf{E}N} \leqslant \frac{1}{\sqrt{\tau+\lambda}}. \tag{2.24}$$

Proof. The lemma follows from Lemma 4 and the fact that $\mathbf{E}\xi = \mathbf{V}\xi = \lambda$ for the r.v. $\xi \sim \mathcal{P}(\lambda)$. □

Analogously, it is easy to see that the corresponding statements are valid for τ-shifted binomial and τ-shifted negative binomial r.v.s N, the proofs of which we omit, but we recall that $\mathbf{E}\xi = np$, $\mathbf{V}\xi = np(1-p)$ for a r.v. $\xi \sim \mathcal{B}(n,p)$ and $\mathbf{E}\xi = r/p$, $\mathbf{V}\xi = r(1-p)/p^2$ for a r.v. $\xi \sim \mathcal{NB}(r,p)$.

Lemma 6. *Let $N-\tau \sim \mathcal{B}(n,p)$ with $\tau \geqslant 0$, $n \in \mathbb{N}$, and $0<p<1$. Then*

$$\mathbf{E}N = \tau+np, \qquad \mathbf{E}N^2 = (\tau+np)^2+np(1-p), \qquad \mathbf{V}N = np(1-p). \tag{2.25}$$

Moreover,

$$\frac{\mathbf{E}|N-\mathbf{E}N|}{\mathbf{E}N} < \frac{1}{\sqrt{\tau+np}}. \tag{2.26}$$

Lemma 7. *Let $N-\tau \sim \mathcal{NB}(r,p)$ with $\tau \geqslant 0$, $r \in \mathbb{N}$, and $0<p<1$. Then*

$$\mathbf{E}N = \tau+\frac{r}{p}, \quad \mathbf{V}N = \frac{r(1-p)}{p^2}, \tag{2.27}$$

and

$$\frac{\mathbf{E}|N-\mathbf{E}N|}{\mathbf{E}N} < \frac{1}{\sqrt{\tau p+r}}. \tag{2.28}$$

3 Proof of Theorem 1

In addition, denote

$$\Delta(x) = \mathbf{P}\{S_N < x\sqrt{\mathbf{V}S_N}\} - \Phi(x), \qquad \xi_k = \frac{S_k}{\sqrt{\mathbf{V}S_k}}, \qquad a_k = \frac{\sqrt{\mathbf{V}S_N}}{\sqrt{\mathbf{V}S_k}},$$

$$\mathcal{I}_{l,k} = \int_{-\infty}^{\infty} |x|^l \big|\mathbf{P}\{\xi_k < x\} - \Phi(x)\big| \,\mathrm{d}x, \qquad p_k = \mathbf{P}\{N=k\},$$

where for independent r.v.s $X_1,\dots,X_k$, $\mathbf{V}S_k = \sum_{i=1}^k \mathbf{V}X_i = B_k^2 > 0$ for all $k = 1,2,\dots$ (it suffices to require $\mathbf{V}S_1 = \mathbf{V}X_1 = \mathbf{E}X_1^2 > 0$). It is clear that, for all $x \in \mathbb{R}$,

$$\Delta(x) = \sum_{k=0}^{\infty} \left[\mathbf{P}\left\{\frac{S_k}{\sqrt{\mathbf{V}S_N}} < x\right\} - \Phi(x)\right] p_k.$$

Let $K(\alpha)=\{k\in\mathbb{N}\colon |\mathbf{V}S_k-\mathbf{V}S_N|\leqslant(1-\alpha)\mathbf{V}S_N\}$, the complement $\overline{K}(\alpha)=\{k\in\mathbb{N}\colon |\mathbf{V}S_k-\mathbf{V}S_N|>(1-\alpha)\mathbf{V}S_N\}$ of $K(\alpha)$, $\overline{K}_0=\{k=0\colon |\mathbf{V}S_k-\mathbf{V}S_N|>(1-\alpha)\mathbf{V}S_N\}$, and let α be an arbitrary number from the interval $(0,1)$. Since

$$\begin{aligned}&\sum_{k=1}^{\infty}\biggl[\mathbf{P}\biggl\{\frac{S_k}{\sqrt{\mathbf{V}S_N}}<x\biggr\}-\Phi(x)\biggr]p_k\\&\quad=\sum_{k\in K(\alpha)}\bigl[\mathbf{P}\{\xi_k<xa_k\}-\Phi(xa_k)\bigr]p_k+\sum_{k\in K(\alpha)}\bigl[\Phi(xa_k)-\Phi(x)\bigr]p_k\\&\qquad+\sum_{k\in\overline{K}(\alpha)}\bigl[\mathbf{P}\{\xi_k<xa_k\}-\Phi(x)\bigr]p_k,\end{aligned}$$

we have the following decomposition of $\Delta(x)$: for all $x\in\mathbb{R}$,

$$\begin{aligned}\Delta(x)&=\sum_{k\in K(\alpha)}\bigl[\mathbf{P}\{\xi_k<xa_k\}-\Phi(xa_k)\bigr]p_k\\&\quad+\sum_{k\in K(\alpha)}\bigl[\Phi(xa_k)-\Phi(x)\bigr]p_k+\sum_{k\in\overline{K}(\alpha)\cup\overline{K}_0}\bigl[\mathbf{P}\{\xi_k<xa_k\}-\Phi(x)\bigr]p_k\\&=\Sigma_1(x)+\Sigma_2(x)+\Sigma_4(x).\end{aligned}\tag{3.1}$$

First, we observe that the variance $\mathbf{V}S_k\geqslant\alpha\mathbf{V}S_N>0$ if $k\in K(\alpha)$ because $\alpha\in(0,1)$ and $\mathbf{V}S_N>0$.

Substituting (3.1) into the expression of

$$\mathcal{I}_l=\int_{-\infty}^{\infty}|x|^l\bigl|\mathbf{P}\{S_N<x\sqrt{\mathbf{V}S_N}\}-\Phi(x)\bigr|\,\mathrm{d}x,$$

we obtain the following decomposition of $\mathcal{I}_l$.

Proposition 1. *Let the conditions of Theorem* 1 *be satisfied. Let, in addition,* $\mathbf{V}S_k>0$ *for all* $k=1,2,\dots$ *. Then, for all* $\alpha\in(0,1)$,

$$\mathcal{I}_l\leqslant\Sigma_1+\Sigma_2+\Sigma_4,\tag{3.2}$$

where

$$\begin{aligned}\Sigma_1&=\sum_{k\in K(\alpha)}\int_{-\infty}^{\infty}|x|^l\bigl|\mathbf{P}\{\xi_k<xa_k\}-\Phi(xa_k)\bigr|\,\mathrm{d}xp_k,\\\Sigma_2&=\sum_{k\in K(\alpha)}\int_{-\infty}^{\infty}|x|^l\bigl|\Phi(xa_k)-\Phi(x)\bigr|\,\mathrm{d}xp_k,\\\Sigma_4&=\sum_{k\in\overline{K}(\alpha)\cup\overline{K}_0}\int_{-\infty}^{\infty}|x|^l\bigl|\mathbf{P}\{\xi_k<xa_k\}-\Phi(x)\bigr|\,\mathrm{d}xp_k.\end{aligned}$$

Estimation of Σ_1. First, we observe that

$$\Sigma_1=\sum_{k\in K(\alpha)}\frac{1}{a_k^{l+1}}\mathcal{I}_{l,k}p_k.\tag{3.3}$$

Since $1/a_k^{l+1} \leqslant (2-\alpha)^{(l+1)/2}$ and $\mathbf{V}S_k \geqslant \alpha \mathbf{V}S_N$ for $k \in K(\alpha)$, using (2.3), we obtain from (3.3) that, for all $0 \leqslant l \leqslant 1+\delta$,

$$\begin{aligned} \Sigma_1 &\leqslant (2-\alpha)^{(l+1)/2} \sum_{k \in K(\alpha)} \frac{C_0(l)}{(\mathbf{V}S_k)^{(2+\delta)/2}} \sum_{i=1}^{k} \mathbf{E}|X_i|^{2+\delta} p_k \\ &\leqslant \frac{(2-\alpha)^{(l+1)/2} C_0(l)}{\alpha^{(2+\delta)/2}} \frac{\mathbf{E} l_{2+\delta,N} \mathbf{1}_{\{N \in K(\alpha)\}}}{(\mathbf{V}S_N)^{(2+\delta)/2}}. \end{aligned} \tag{3.4}$$

Estimation of Σ_2. To estimate Σ_2, we use (2.14) of Lemma 3, according to which, for all $l \geqslant 0$,

$$\int_{-\infty}^{\infty} |x|^l \big|\Phi(xa_k) - \Phi(x)\big| \, \mathrm{d}x = \frac{2^{(l+2)/2} \Gamma((l+2)/2)}{\sqrt{2\pi}} |a_k - 1| \int_0^1 \frac{\mathrm{d}t}{(\gamma_k(t))^{(l+2)/2}}, \tag{3.5}$$

where $\gamma_k(t) = [1 + t(a_k - 1)]^2$. Now observe that, for $k \in K(\alpha)$,

$$\frac{1}{\sqrt{2-\alpha}} \leqslant a_k \leqslant \frac{1}{\sqrt{\alpha}}. \tag{3.6}$$

Therefore, for $0 < t < 1$ and $k \in K(\alpha)$,

$$\gamma_k(t) \geqslant a_k^2 \geqslant \frac{1}{2-\alpha}$$

if $1/\sqrt{2-\alpha} \leqslant a_k \leqslant 1$, and

$$\gamma_k(t) \geqslant 1 > \frac{1}{2-\alpha}$$

if $1 < a_k \leqslant 1/\sqrt{\alpha}$. Thus, we obtained that with $0 < t < 1$ (immediately it can be extended to $0 \leqslant t \leqslant 1$) and $\alpha \in (0,1)$, for $k \in K(\alpha)$,

$$\gamma_k(t) \geqslant \frac{1}{2-\alpha}. \tag{3.7}$$

The upper bound of $|a_k - 1|$ for $k \in K(\alpha)$ easily follows:

$$|a_k - 1| = \frac{|\mathbf{V}S_k - \mathbf{V}S_N|}{\sqrt{\mathbf{V}S_k}(\sqrt{\mathbf{V}S_k} + \sqrt{\mathbf{V}S_N})} \leqslant \frac{1}{\alpha + \sqrt{\alpha}} \frac{|\mathbf{V}S_k - \mathbf{V}S_N|}{\mathbf{V}S_N}. \tag{3.8}$$

Substituting (3.8) and (3.7) into (3.5), we obtain that, for all $l \geqslant 0$ and $k \in K(\alpha)$,

$$\int_{-\infty}^{\infty} |x|^l \big|\Phi(xa_k) - \Phi(x)\big| \, \mathrm{d}x \leqslant \frac{[2(2-\alpha)]^{(l+2)/2} \Gamma((l+2)/2)}{\sqrt{2\pi}(\alpha + \sqrt{\alpha})} \frac{|\mathbf{V}S_k - \mathbf{V}S_N|}{\mathbf{V}S_N}. \tag{3.9}$$

It only remains to substitute (3.9) into the expression of Σ_2. We obtain that, for all $l \geqslant 0$,

$$\Sigma_2 \leqslant \frac{[2(2-\alpha)]^{(l+2)/2} \Gamma((l+2)/2)}{\sqrt{2\pi}(\alpha + \sqrt{\alpha})} \frac{\mathbf{E}|B_N^2 - \mathbf{V}S_N| \mathbf{1}_{\{N \in K(\alpha)\}}}{\mathbf{V}S_N}. \tag{3.10}$$

Estimation of Σ_4. To estimate Σ_4, we rewrite the complement $\overline{K}(\alpha) = \{k \in \mathbb{N}\colon |\mathbf{V}S_k - \mathbf{V}S_N| > (1-\alpha)\mathbf{V}S_N\}$ of $K(\alpha)$ as the union of two sets $\overline{K}(\alpha) = \overline{K}^-(\alpha) \cup \overline{K}^+(\alpha)$, where $\overline{K}^-(\alpha) = \{k \in \mathbb{N}\colon \mathbf{V}S_k < \alpha\mathbf{V}S_N\}$ and $\overline{K}^+(\alpha) = \{k \in \mathbb{N}\colon \mathbf{V}S_k > (2-\alpha)\mathbf{V}S_N\}$. It is easy to see that

$$\Sigma_4 \leqslant \Sigma_{40} + \Sigma_{41} + \Sigma_{42} + \Sigma_{43}, \tag{3.11}$$

where

$$\Sigma_{40} = \int_{-\infty}^{\infty} |x|^l \left|\mathbf{P}\left\{\frac{S_0}{\sqrt{\mathbf{V}S_N}} < x\right\} - \Phi(x)\right| \mathrm{d}x p_0,$$

$$\Sigma_{41} = \sum_{k\in\overline{K}^-(\alpha)} \frac{1}{a_k^{l+1}} \mathcal{I}_{l,k} p_k, \qquad \Sigma_{42} = \sum_{k\in\overline{K}^+(\alpha)} \frac{1}{a_k^{l+1}} \mathcal{I}_{l,k} p_k,$$

$$\Sigma_{43} = \sum_{k\in\overline{K}(\alpha)} \int_{-\infty}^{\infty} |x|^l \big|\Phi(xa_k) - \Phi(x)\big| \,\mathrm{d}x p_k.$$

Estimation of $\Sigma_{40} + \Sigma_{41}$. First, we observe that $1/a_k \leqslant \sqrt{\alpha}$ for $k \in \overline{K}^-(\alpha)$.

From (2.5) of Lemma 1 an estimate of Σ_{40} for $0 \leqslant l \leqslant 2$ follows:

$$\Sigma_{40} \leqslant \frac{p_0}{l+1} \cdot \begin{cases} 1 & \text{if } 0 \leqslant l \leqslant 1, \\ \frac{2\sqrt{2}}{\sqrt{\pi}} & \text{if } 1 < l \leqslant 2. \end{cases} \tag{3.12}$$

Since by (2.6) of Lemma 1, $\mathcal{I}_{l,k} \leqslant 2/(l+1)$ for $0 \leqslant l \leqslant 1$, using (3.12), we obtain that, for $0 \leqslant l \leqslant 1$,

$$\begin{aligned} \Sigma_{40} + \Sigma_{41} &\leqslant \frac{p_0}{l+1} + \frac{2\alpha^{(l+1)/2}}{l+1} \sum_{k\in\overline{K}^-(\alpha)} p_k \leqslant \frac{\max\{1, 2\alpha^{(l+1)/2}\}}{l+1} \sum_{k\in\overline{K}^-(\alpha)\cup\overline{K}_0} p_k \\ &\leqslant \frac{\max\{1, 2\alpha^{(l+1)/2}\}}{(l+1)(1-\alpha)} \frac{\mathbf{E}|B_N^2 - \mathbf{V}S_N|\mathbf{1}_{\{N\in\overline{K}(\alpha)\cup\overline{K}_0\}}}{\mathbf{V}S_N}. \end{aligned} \tag{3.13}$$

Now, let $1 < l \leqslant 1+\delta$, $0 < \delta \leqslant 1$. In this case, instead of (2.6) of Lemma 1, we use (2.13) of Lemma 2, whereby, for any fixed $k = 1, 2, \dots,$

$$\mathcal{I}_{l,k} \leqslant C_*(l)\left(1 + \frac{1}{(\mathbf{V}S_k)^{(l+1)/2}} \sum_{i=1}^k \mathbf{E}|X_i|^{l+1}\mathbf{1}_{\{|X_i|>\sqrt{\mathbf{V}S_k}\}}\right). \tag{3.14}$$

Thus, using (3.14) and (3.12), we obtain that, for $1 < l \leqslant 1+\delta$, $0 < \delta \leqslant 1$,

$$\begin{aligned} \Sigma_{40} + \Sigma_{41} \leqslant{}& \frac{2\sqrt{2}p_0}{\sqrt{\pi}(l+1)} + C_*(l)\alpha^{(l+1)/2} \sum_{k\in\overline{K}^-(\alpha)} p_k \\ &+ C_*(l)\frac{1}{(\mathbf{V}S_N)^{(l+1)/2}} \sum_{k\in\overline{K}^-(\alpha)} \sum_{i=1}^k \mathbf{E}|X_i|^{l+1}\mathbf{1}_{\{|X_i|>\sqrt{\mathbf{V}S_k}\}} p_k \end{aligned}$$

$$
\begin{aligned}
&\leqslant \max\biggl\{\frac{2\sqrt{2}}{\sqrt{\pi}(l+1)},\, C_*(l)\alpha^{(l+1)/2}\biggr\} \sum_{k\in\overline{K}^-(\alpha)\cup\overline{K}_0} p_k + C_*(l)\frac{\mathbf{E}l_{l+1,N}\mathbf{1}_{\{N\in\overline{K}^-(\alpha)\}}}{(\mathbf{V}S_N)^{(l+1)/2}} \\
&\leqslant \max\biggl\{\frac{2\sqrt{2}}{\sqrt{\pi}(l+1)},\, C_*(l)\alpha^{(l+1)/2}\biggr\}\frac{1}{1-\alpha}\frac{\mathbf{E}|B_N^2-\mathbf{V}S_N|\mathbf{1}_{\{N\in\overline{K}(\alpha)\cup\overline{K}_0\}}}{\mathbf{V}S_N} \\
&\quad + C_*(l)\frac{\mathbf{E}l_{l+1,N}\mathbf{1}_{\{N\in\overline{K}^-(\alpha)\}}}{(\mathbf{V}S_N)^{(l+1)/2}}. \qquad (3.15)
\end{aligned}
$$

To estimate Σ_{42}, we use (2.3) of Theorem 5. Since $\mathbf{V}S_k > (2-\alpha)\mathbf{V}S_N$ for $k \in \overline{K}^+(\alpha)$, we obtain that, for all $0 \leqslant l \leqslant 1+\delta$, $0<\delta\leqslant 1$,

$$
\begin{aligned}
\Sigma_{42} &\leqslant C_0(l)\sum_{k\in\overline{K}^+(\alpha)} \frac{1}{(\mathbf{V}S_N)^{(l+1)/2}(\mathbf{V}S_k)^{(1+\delta-l)/2}}\sum_{i=1}^{k}\mathbf{E}|X_i|^{2+\delta}p_k \\
&\leqslant \frac{C_0(l)}{(2-\alpha)^{(1+\delta-l)/2}}\frac{\mathbf{E}l_{2+\delta,N}\mathbf{1}_{\{N\in\overline{K}^+(\alpha)\}}}{(\mathbf{V}S_N)^{(2+\delta)/2}}. \qquad (3.16)
\end{aligned}
$$

To estimate Σ_{43}, we use (2.15) of Lemma 3 and obtain that, for all $l \geqslant 0$,

$$
\begin{aligned}
\Sigma_{43} &\leqslant \frac{2^{(l+2)/2}\Gamma((l+2)/2)}{\sqrt{2\pi}(l+1)}\sum_{k\in\overline{K}(\alpha)}\biggl|1-\frac{1}{a_k^{l+1}}\biggr|p_k \\
&\leqslant \frac{2^{(l+2)/2}\Gamma((l+2)/2)}{\sqrt{2\pi}(l+1)}\frac{\mathbf{E}|B_N^{l+1}-(\mathbf{V}S_N)^{(l+1)/2}|\mathbf{1}_{\{N\in\overline{K}(\alpha)\}}}{(\mathbf{V}S_N)^{(l+1)/2}}. \qquad (3.17)
\end{aligned}
$$

Substituting (3.13) in the case $0\leqslant l\leqslant 1$ ((3.15) in the case $1<l\leqslant 1+\delta$), (3.16), and (3.17) into (3.11) and observing that the function $f(l) = |1-1/a^{l+1}|$, where $0<a<\infty$, is nondecreasing for $l\in[-1,\infty)$, we obtain that

$$
\begin{aligned}
\Sigma_4 &\leqslant \frac{1}{l+1}\biggl(\frac{\max\{1, 2\alpha^{(l+1)/2}\}}{1-\alpha}+\frac{2^{(l+2)/2}\Gamma((l+2)/2)}{\sqrt{2\pi}}\biggr)\frac{\mathbf{E}|B_N^2-\mathbf{V}S_N|\mathbf{1}_{\{N\in\overline{K}(\alpha)\cup\overline{K}_0\}}}{\mathbf{V}S_N} \\
&\quad + \frac{C_0(l)}{(2-\alpha)^{(1+\delta-l)/2}}\frac{\mathbf{E}l_{2+\delta,N}\mathbf{1}_{\{N\in\overline{K}^+(\alpha)\}}}{(\mathbf{V}S_N)^{(2+\delta)/2}} \qquad (3.18)
\end{aligned}
$$

for $0\leqslant l\leqslant 1$ and

$$
\begin{aligned}
\Sigma_4 &\leqslant \biggl(\max\biggl\{\frac{2\sqrt{2}}{\sqrt{\pi}(l+1)}, C_*(l)\alpha^{(l+1)/2}\biggr\}\frac{1}{1-\alpha}+\frac{2^{(l+2)/2}\Gamma((l+2)/2)}{\sqrt{2\pi}(l+1)}\biggr) \\
&\quad\times\frac{\mathbf{E}|B_N^{l+1}-(\mathbf{V}S_N)^{(l+1)/2}|\mathbf{1}_{\{N\in\overline{K}(\alpha)\cup\overline{K}_0\}}}{(\mathbf{V}S_N)^{(l+1)/2}} \\
&\quad + C_*(l)\frac{\mathbf{E}l_{l+1,N}\mathbf{1}_{\{N\in\overline{K}^-(\alpha)\}}}{(\mathbf{V}S_N)^{(l+1)/2}} + \frac{C_*(l)}{(2-\alpha)^{(1+\delta-l)/2}}\frac{\mathbf{E}l_{2+\delta,N}\mathbf{1}_{\{N\in\overline{K}^+(\alpha)\}}}{(\mathbf{V}S_N)^{(2+\delta)/2}} \qquad (3.19)
\end{aligned}
$$

for $1<l\leqslant 1+\delta$, $0<\delta\leqslant 1$.

Substituting (3.4), (3.10), and (3.18) for $0 \leqslant l \leqslant 1$ ((3.19) for $1 < l \leqslant 1 + \delta$) into (3.2) and taking a concrete $\alpha \in (0, 1)$, for example, $\alpha = 1/2$, we obtain estimates (1.1) and (1.2) of Theorem 1.

Theorem 1 is proved.

The proof of Corollary 1 immediately follows from Theorem 1.

The proof of Theorem 2 immediately follows from Theorem 1 since, for all $l \geqslant 1$,

$$\lambda_l \leqslant l\mathcal{I}_{l-1}. \tag{3.20}$$

The proof of Corollary 2 immediately follows from Theorem 2.

The proof of Theorem 3 immediately follows from inequalities (1.3) of Corollary 1, (1.7) of Corollary 2, and Lemmas 5, 6, and 7.

References

1. L.H.Y. Chen, L. Goldstein, and Q.-M. Shao, *Normal Approximation by Stein's Method*, Springer-Verlag, Berlin, Heidelberg, New York, 2011.
2. J. Dedecker and E. Rio, On mean central limit theorems for stationary sequences, *Ann. Inst. Henri Poincaré, Probab. Stat.*, **44**(4):693–726, 2008.
3. Ch. Döbler, New Berry–Esseen and Wasserstein bounds in the CLT for non-randomly centered random sums by probabilistic method, *ALEA, Lat. Am. J. Probab. Math. Stat.*, **12**(2):863–902, 2015.
4. R.V. Erickson, L_1 bounds for asymptotic normality of m-dependent sums using Stein's technique, *Ann. Probab.*, **2**(3):522–529, 1974.
5. L. Goldstein, Bounds on the constant in the mean central limit theorem, *Ann. Probab.*, **38**(4):1672–1689, 2010.
6. S.-T. Ho and L.H.Y. Chen, An L_p bound for the remainder in a combinatorial central limit theorem, *Ann. Probab.*, **6**(2):231–249, 1978.
7. V.M. Kruglov and V.Yu. Korolev, *Limit Theorems for Random Sums*, Moscow Univ. Press, Moscow, 1990 (in Russian).
8. V.V. Petrov, *Limit Theorems of Probability Theory. Sequences of Independent Random Variables*, Clarendon Press, Oxford, 1995.
9. V.V. Shergin, On the global version of the central limit theorem for m-dependent random variables, *Teor. Veroyatn. Mat. Stat.*, **29**:122–128, 1983 (in Russian).
10. A.N. Shiryaev, *Probability,* 2nd ed., Springer, New York, 1996.
11. Ch. Stein, A bound for the error in the normal approximation to the distribution of a sum of dependent random variables, in *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Vol. 2*, Univ. California Press, Berkeley, CA, 1972, pp. 583–602.
12. J. Sunklodas, On the global central limit theorem for all m-dependent random variables, *Lith. Math. J.*, **34**(2):208–213, 1994.
13. J. Sunklodas, L_1 bounds for asymptotic normality of random sums of independent random variables, *Lith. Math. J.*, **53**(4):438–447, 2013.