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Modeling the beta distribution using multiplicative functions

Gintautas Bareikis and Algirdas Mačiulis

Faculty of Mathematics and Informatics, Vilnius University, Naugarduko str. 24, LT-03225 Vilnius, Lithuania
(e-mail: gintautas.bareikis@mif.vu.lt; algirdas.maciulis@mif.vu.lt)

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Abstract. We prove that any beta distribution can be simulated by means of a sequence of distributions defined via multiplicative functions related to the generalized divisors function. We also estimate the remainder terms.

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1 Introduction and results

In what follows, we assume that p is prime, $d, k, m, n \in \mathbb{N}$, $s = \sigma + \mathrm{i}\tau \in \mathbb{C}$, and $x, u, t, v \in \mathbb{R}$. In the asymptotic relations, we assumed that $x \to \infty$. The letters c and C with or without subscripts denote constants. Either of the notations $f = O(g)$ or $f \ll g$ means that $|f| \leqslant C|g|$ for some positive constant C, which may be absolute or depend upon various parameters.

DEFINITION 1. Let $\varkappa \geqslant 0$ and $0 \leqslant \delta < 1$. We say that a multiplicative function $\varphi : \mathbb{N} \to [0; \infty)$ belongs to the class $\mathcal{G}(\varkappa; \delta)$ if there exists $C_1 \geqslant 0$ such that $\varphi(p^k) \leqslant C_1$ and the function

$$L(s) := \sum_p \frac{\varphi(p) - \varkappa}{p^s}, \quad s = \sigma + \mathrm{i}\tau,\ \sigma > 1,$$

has an analytic continuation $P(s)$ into the region

$$\sigma \geqslant \sigma(\tau) := 1 - \frac{c}{\ln(|\tau| + 3)}$$

for some $0 < c \leqslant 1/2$, where $P(s)$ is holomorphic, and $|P(s)| \leqslant \delta \ln(|\tau| + 1) + c_0$ with some $c_0 \geqslant 0$.

For a multiplicađtive function $f : \mathbb{N} \to [0; \infty)$, we define

$$T_f(m, v) := \sum_{d|m,\, d \leqslant v} f(d), T_f(m, m) =: T_f(m).$$

DEFINITION 2. We say that a pair $(g; f)$ of the multiplicative functions $g, f : \mathbb{N} \to [0, \infty)$ belongs to the class $\mathcal{M}(\varkappa, \alpha; \delta_1, \delta_2)$ if $g \in \mathcal{G}(\varkappa; \delta_1)$ and $g/T_f \in \mathcal{G}(\alpha; \delta_2)$. If $1/T_f \in \mathcal{G}(\alpha; \delta_2)$, then we say that the function f belongs to the class $\mathcal{M}(\alpha; \delta_2)$.

In this paper, we analyze the asymptotic behavior of the distributions

$$F_x(t; g, f) := \frac{1}{G(x)} \sum_{m \leqslant x} g(m) \frac{T_f(m, m^t)}{T_f(m)}, \tag{1.1}$$

where f, g are multiplicative functions, and

$$G(x) := \sum_{d \leqslant x} g(d).$$

We show that, for multiplicative functions $(g; f) \in \mathcal{M}(\varkappa, \alpha; \delta_1, \delta_2)$, distributions (1.1) may have only two types of the limit laws. Namely, it can be either the beta distribution $B(t; a, b)$ concentrated on the interval $t \in [0,\ 1]$ and defined by

$$B(t; a, b) := \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^t \frac{\mathrm{d}v}{v^{1-a}(1-v)^{1-b}}, \quad t \in [0, 1],$$

or the improper distribution concentrated at a single point $t_0 \in \mathbb{R}$,

$$E_{t_0}(t) := \begin{cases} 0 & \text{if } t \leqslant t_0, \\ 1 & \text{if } t > t_0. \end{cases}$$

The first attempt to simulate the arcsine law, that is, $B(t; 1/2, 1/2)$, by means of (1.1) with $f = g \equiv 1$ and, consequently, $(g; f) \in \mathcal{M}(1, 1/2; 0, 0)$ was made by Deshouillers et al. [5] (see also [6, Sect. II.6.2]).

Theorem 1. (*See* [5].) *Uniformly in* $t \in [0, 1]$,

$$\frac{1}{x} \sum_{m \leqslant x} \frac{T_1(m, m^t)}{T_1(m)} = \frac{2}{\pi} \arcsin \sqrt{t} + O\biggl(\frac{1}{\sqrt{\ln x}}\biggr).$$

Bareikis and Manstavičius [3] generalized this result. They considered multiplicative functions $f \in M(\alpha; \delta_2)$ with $\alpha \in (0; 1)$ such that $0 < f(p^k) \leqslant C$ and proved that

$$\sup_t \bigl| F_x(t; 1, f) - B(t; 1-\alpha, \alpha) \bigr| \ll \ln^{-\alpha} x + \ln^{\alpha-1} x.$$

Later, Bareikis and Mačiulis [1] proved that the increments of the function $F_x(t; 1, f)$ without the assumption $0 < f(p^k) \leqslant C$ approach the increments of the beta distribution. Moreover, when $\alpha \in \{0, 1\}$, it was proved (see [2]) that, in this case, only improper laws concentrated at the points 0 or 1 can occur as limits for (1.1) .

The first result concerning the limit behavior of (1.1) with $g \not\equiv 1$ was obtained by Daoud et al. [4].

Theorem 2. (*See* [4].) *Suppose that* $(g; f) \in \mathcal{M}(\varkappa, \alpha; \delta_1, \delta_2)$ *with* $0 < \alpha < \varkappa \leqslant 1$ *and* $0 < g(p^k) \leqslant 1$, $f(p^k) \leqslant C$. *Then, uniformly for* $x \geqslant 2$ *and* $t \in [0, 1]$,

$$F_x(t; g, f) = B(t; \varkappa - \alpha, \alpha) + O\biggl(\frac{1}{(\ln x)^{\min(\alpha, \varkappa - \alpha)}}\biggr).$$

Theorem 2 shows that the beta distribution $B(t;a,b)$ with parameters $0 < a,\, b < 1$ can be simulated by means of (1.1). We generalize Theorem 2 by showing that if $(g;f) \in M(\varkappa,\alpha;\delta_1,\delta_2)$ with $0 \leqslant \alpha \leqslant \varkappa$, then distributions (1.1) approach some limit law. Moreover, the set of possible limits consists of the beta distributions $B(t;a,b)$ with positive a,b and the improper distributions $E_0(t)$ and $E_1(t)$.

To formulate our main result, we need some additional notation. For all $a,b,u,t \in \mathbb{R}$ and $x > 1$, we set $\eta_x := \ln^{-1} x$,

$$r_x(u,t;a,b) := \frac{\eta_x}{(\eta_x+u)^a(\eta_x+1-u)^b} + \frac{\eta_x}{(\eta_x+t)^a(\eta_x+1-t)^b},$$

$$F_x(u,t;g,f) := F_x(t;g,f) - F_x(u;g,f), \qquad B(u,t;a,b) = B(t;a,b) - B(u;a,b).$$

DEFINITION 3. The Lévy distance between distribution functions F and H is defined by

$$\mathcal{L}(F,H) := \inf\bigl\{\varepsilon > 0 \bigm| F(x-\varepsilon) - \varepsilon \leqslant H(x) \leqslant F(x+\varepsilon) + \varepsilon \ \forall x \in \mathbb{R}\bigr\}.$$

Note that $\mathcal{L}(F_n,F) \to 0$ is necessary and sufficient for $F_n \Rightarrow F$ as $n \to \infty$.

The main result of this paper is the following theorem.

Theorem 3. *Suppose that $x \geqslant 3$ and*

$$(g;f) \in \mathcal{M}(\varkappa,\alpha;\delta_1,\delta_2), \quad \delta_1+\delta_2 < 1.$$

(i) *If $0 < \alpha < \varkappa$ and $0 \leqslant u \leqslant t \leqslant 1$, then*

$$\begin{aligned}&\bigl|F_x(u,t;g,f) - B(u,t;\varkappa-\alpha,\alpha)\bigr| \\ &\quad \ll r_x(u,t;1-\varkappa+\alpha,1-\alpha) + \frac{|\ln(\eta_x+u)|^{\epsilon(\varkappa-\alpha)}}{\ln x} + \frac{|\ln(\eta_x+1-t)|^{\epsilon(\alpha)}}{\ln x},\end{aligned} \tag{1.2}$$

where $\epsilon(v) = 1$ if $v = 1$ and $\epsilon(v) = 0$ otherwise.

(ii) *If $\alpha = \varkappa > 0$, then $F_x \Rightarrow E_0$.*

(iii) *If $\alpha = 0$ and $\varkappa > 0$, then $F_x \Rightarrow E_1$.*

In the last two cases, the convergence rates, estimated by means of the Lévy distance, are

$$\mathcal{L}(F_x,E_j) \ll \frac{(\ln\ln x)^2}{\ln x}$$

with $j = 0$ and $j = 1$, respectively.

Theorem 4 yields a uniform version of estimate (1.2).

Theorem 4. *Suppose that the assumptions of Theorem 3 are satisfied. If $0 < \alpha < \varkappa$, then*

$$\begin{aligned}&\sup_t \bigl|F_x(t;g,f) - B(t;\varkappa-\alpha,\alpha)\bigr| \\ &\quad \ll \frac{1}{(\ln x)^{\min(\varkappa-\alpha;\alpha)}} + \frac{(\ln\ln x)^{\epsilon(\varkappa-\alpha)} + (\ln\ln x)^{\epsilon(\alpha)}}{\ln x}.\end{aligned}$$

Unless otherwise indicated, we assume that the implicit constants in the $\ll$ or $O()$ symbols depend at most on the parameters and constants involved in the definitions of the corresponding classes $\mathcal{M}(\cdot)$ and $\mathcal{G}(\cdot)$.

2 Preliminaries

We need some estimates of the mean values on arithmetical progression for the multiplicative functions $\psi \in \mathcal{G}(\varkappa;\delta)$. Two results of this type yield the following lemmas.

Lemma 1. (*See* [1].) *Let ψ belong to the class $\mathcal{G}(\varkappa;\delta)$ for some $\varkappa > 0$.*
Then, uniformly for all $d \in \mathbb{N}$ and $x \geqslant 1$,

$$\sum_{n \leqslant x} \psi(nd) = \frac{x}{\ln^{1-\varkappa}(\mathrm{e}x)} \left(\frac{L(\psi,\varkappa)\tilde{\psi}(d)}{\Gamma(\varkappa)} + O\left(\frac{\hat{\psi}(d)}{\ln(\mathrm{e}x)} \right) \right),$$

where $L(\psi,\varkappa)$ and multiplicative functions $\tilde{\psi}$ and $\hat{\psi}$ are defined by

$$L(\psi,\varkappa) := \prod_p \left(1 - \frac{1}{p}\right)^{\varkappa} \sum_{k=0}^{\infty} \frac{\psi(p^k)}{p^k},$$

$$\tilde{\psi}(p^m) := \left(\sum_{k=0}^{\infty} \frac{\psi(p^k)}{p^k} \right)^{-1} \sum_{k=0}^{\infty} \frac{\psi(p^{k+m})}{p^k},$$

$$\hat{\psi}(p^m) := \left(1 + \frac{c_1}{p^{\sigma_0}}\right) \sum_{k=0}^{\infty} \frac{\psi(p^{k+m})}{p^{k\sigma_0}}.$$

Here $\sigma_0 = \sigma(0)$ and $c_1 \geqslant 0$ is a constant depending on the parameters c, $\varkappa$, C_1.

Lemma 2. (*See* [2].) *Let $\psi \in \mathcal{G}(\varkappa;\delta)$. Then, uniformly for all $d \in \mathbb{N}$ and $x \geqslant 1$,*

$$\sum_{m \leqslant x} \psi(md) \ll x \cdot \hat{\psi}(d)\eta(x,\varkappa),$$

where

$$\eta(x,\varkappa) := \begin{cases} \mathrm{e}^{-c_2\sqrt{\ln x}} & \text{if } \varkappa = 0, \\ \ln^{\varkappa-1}(\mathrm{e}x) & \text{if } \varkappa > 0, \end{cases}$$

the multiplicative function $\hat{\psi}$ is defined in Lemma 1, and $c_2 = c_2(c,\delta) > 0$.

For $0 \leqslant u \leqslant t \leqslant 1$ and $x \geqslant 1$, we set

$$S(x,u,t,b) := \sum_{x^u < m \leqslant x^t} \frac{a_m}{m \ln^b(\frac{\mathrm{e}x}{m})}, \quad a_m \geqslant 0.$$

This sum may be evaluated in terms of the integral

$$I(u,t;a,b,\eta) := \int_u^t \frac{\mathrm{d}v}{(\eta+v)^a(\eta+1-v)^b},$$

provided that some information about the behavior of the sum

$$M(v) := \sum_{m \leqslant v} a_m$$

is given. A slight modification of the proof of Lemma 3.3 in [1] yields the following result.

Lemma 3. *Assume that $x \geqslant 3$ and*

$$\left|M(v) - \frac{Av}{\ln^a(\mathrm{e}v)}\right| \leqslant \frac{Bv}{\ln^\gamma(\mathrm{e}v)} \tag{2.1}$$

for some $A, a, \gamma \in \mathbb{R}$, $B \geqslant 0$, and all $1 \leqslant v \leqslant x$. Then

$$\begin{aligned}&\left|S(x,u,t,b) - \frac{A}{(\ln x)^{a+b-1}} I(u,t;a,b,\eta_x)\right| \\ &\quad\leqslant \frac{|A\cdot a|}{(\ln x)^{a+b}} I(u,t;a+1,b,\eta_x) + \frac{B}{(\ln x)^{b+\gamma-1}}\bigl(r_x(u,t;\gamma,b) + \bigl(1+|b|\bigr) I(u,t;\gamma,b,\eta_x)\bigr).\end{aligned}$$

Proof. Integration by parts yields

$$S(x,u,t,b) = \frac{M(v)}{v\ln^b(\frac{\mathrm{e}x}{v})}\bigg|_{x^u}^{x^t} + \int_{x^u}^{x^t} \frac{M(v)}{v^2\ln^b(\frac{\mathrm{e}x}{v})}\left(1 - \frac{b}{\ln(\frac{\mathrm{e}x}{v})}\right)\mathrm{d}v.$$

Since

$$\int_{x^u}^{x^t} \frac{\mathrm{d}v}{v\ln^a(\mathrm{e}v)\ln^b(\frac{\mathrm{e}x}{v})} = (\ln x)^{1-a-b} I(u,t;a,b,\eta_x),$$

setting $\Delta_1(v) := M(v) - Av\ln^{-a}(\mathrm{e}v)$ and

$$\Delta_2 := \ln^{-a}(\mathrm{e}v)\ln^{-b}\left(\frac{\mathrm{e}x}{v}\right)\bigg|_{x^u}^{x^t} - b(\ln x)^{-a-b} I(u,t;a,b+1,\eta_x),$$

we have

$$\begin{aligned}&S(x,u,t,b) - \frac{A}{(\ln x)^{a+b-1}} I(u,t;a,b,\eta_x) \\ &\quad= \frac{\Delta_1(v)}{v\ln^b(\frac{\mathrm{e}x}{v})}\bigg|_{x^u}^{x^t} + A\cdot\Delta_2 + \int_{x^u}^{x^t} \frac{\Delta_1(v)}{v^2\ln^b(\frac{\mathrm{e}x}{v})}\left(1 - \frac{b}{\ln(\frac{\mathrm{e}x}{v})}\right)\mathrm{d}v.\end{aligned} \tag{2.2}$$

Integrating the integral $I(u,t;a,b+1,\eta_x)$ by parts, we get

$$\Delta_2 = -a(\ln x)^{-a-b} I(u,t;a+1,b,\eta_x)$$

if $b \neq 0$. For $b = 0$, the same equality follows immediately. Thus, in view of (2.1) and (2.2), we arrive at the desired inequality. □

Tthe following simplified version of Lemma 3 will be useful.

Lemma 4. *Assume that the conditions of Lemma 3 are satisfied and $\gamma = a+1$. Then*

$$\begin{aligned}&(\ln x)^{a+b}\left|S(x,u,t,b) - \frac{A}{(\ln x)^{a+b-1}} I(u,t;a,b,\eta_x)\right| \\ &\quad\ll r_x(u,t;a,b-1)\ln x + \bigl|\ln(\eta_x+u)\bigr|^{\epsilon(a+1)} + \bigl|\ln(\eta_x+1-t)\bigr|^{\epsilon(b)}.\end{aligned} \tag{2.3}$$

The implicit constant in $\ll$ symbol depends on A, B, a, and b only.

Proof. Setting $\rho(a,b) := r_x(u,t;a,b)\ln x$, we note that

$$r_x(u,t;a+1,b) \leqslant \frac{\eta_x\cdot\rho(a,b-1)}{(\eta_x+u)(\eta_x+1-u)} + \frac{\eta_x\cdot\rho(a,b-1)}{(\eta_x+t)(\eta_x+1-t)} \leqslant 4\rho(a,b-1).$$

To estimate the integral $I := I(u,t;a+1,b,\eta_x)$, we consider three cases.

1. If $0\leqslant u\leqslant t\leqslant 1/2$, then

$$I \leqslant 2^b\int_u^t \frac{\mathrm{d}v}{(\eta_x+v)^{a+1}} \ll \bigl(1-\epsilon(a+1)\bigr)\rho(a,b-1)+\epsilon(a+1)\bigl|\ln(\eta_x+u)\bigr|. \tag{2.4}$$

2. If $1/2\leqslant u\leqslant t\leqslant 1$, then, similarly,

$$I \ll \bigl(1-\epsilon(b)\bigr)\rho(a,b-1)+\epsilon(b)\bigl|\ln(\eta_x+1-t)\bigr|. \tag{2.5}$$

3. For $0\leqslant u\leqslant 1/2\leqslant t\leqslant 1$, we have

$$\begin{aligned} I &\leqslant 2^b\int_u^{1/2}\frac{\mathrm{d}v}{(\eta_x+v)^{a+1}} + 2^{a+1}\int_{1/2}^t \frac{\mathrm{d}v}{(\eta_x+1-v)^b}\\ &\ll \bigl(2-\epsilon(a+1)-\epsilon(b)\bigr)\bigl(1+\rho(a,b-1)\bigr)\\ &\quad + \epsilon(a+1)\bigl|\ln(\eta_x+u)\bigr| + \epsilon(b)\bigl|\ln(\eta_x+1-t)\bigr|. \end{aligned} \tag{2.6}$$

These inequalities and Lemma 3 imply estimate (2.3). □

3 Proofs of Theorems

Proof of Theorem 3. (i) We first consider the case $0<\alpha<\varkappa$. Set

$$S_x(t;g,f) := \frac{1}{G(x)}\sum_{m\leqslant x} g(m)\frac{T_f(m,x^t)}{T_f(m)}.$$

Then

$$R_x(t) := S_x(t;g,f) - F_x(t;g,f) = \frac{1}{G(x)}\sum_{m\leqslant x} g(m)\frac{T_f(m,x^t)-T_f(m,m^t)}{T_f(m)}.$$

Setting

$$\varPhi_x(u,t) := S_x(t;g,f) - S_x(u;g,f),$$

we have

$$F_x(u,t;g,f) = \varPhi_x(u,t) + O\bigl(R_x(t)+R_x(u)\bigr). \tag{3.1}$$

The remainder and the main terms in (3.1) have the representations

$$R_x(t) = \frac{1}{G(x)}\sum_{l\leqslant x^t} f(l)\sum_{n<l^{(1-t)/t}} h(nl), \tag{3.2}$$

and

$$\Phi_x(u,t) = \frac{1}{G(x)} \sum_{x^u < l \leqslant x^t} f(l) \sum_{n \leqslant x/l} h(nl), \tag{3.3}$$

where h is the multiplicative function defined by $h(n) = g(n)/T_f(n)$. Note that $h \in \mathcal{G}(\alpha; \delta_2)$ since $(g; f) \in \mathcal{M}(\varkappa, \alpha; \delta_1, \delta_2)$. Thus, Lemma 1 with $\psi := h$ yields

$$\sum_{n \leqslant x/l} h(nl) = \frac{x}{l \ln^{1-\alpha}(\mathrm{e}\frac{x}{l})} \biggl(\frac{\tilde{h}(l) L(h, \alpha)}{\Gamma(\alpha)} + O\biggl(\frac{\hat{h}(l)}{\ln(\mathrm{e}\frac{x}{l})} \biggr) \biggr). \tag{3.4}$$

Since $g(p^k) \leqslant C_1$, we have

$$\tilde{h}(p^k) = h(p^k) + O\biggl(\frac{1}{pT_f(p^k)} \biggr) \quad \text{and} \quad \hat{h}(p^k) = h(p^k) + O\biggl(\frac{1}{p^{\sigma_0} T_f(p^k)} \biggr). \tag{3.5}$$

The assumptions of the theorem and (3.5) yield that

$$\sum_{p \leqslant z} \frac{f(p)\hat{h}(p)}{p} = (\varkappa - \alpha) \ln\ln z + O(1) \tag{3.6}$$

and

$$\sum_{p \leqslant z} \frac{f(p)\tilde{h}(p)}{p} = (\varkappa - \alpha) \ln\ln z + O(1).$$

By Lemma 2 the inner sum in (3.2) can be estimated as follows:

$$\sum_{n \leqslant l^{(1-t)/t}} h(nl) \ll \hat{h}(l) \frac{l^{(1-t)/t}}{\ln^{1-\alpha}(\mathrm{e} l^{(1-t)/t})}. \tag{3.7}$$

Further, Lemma 1 with $\psi := g$ and $d = 1$ yields

$$G(x) = \frac{x}{\ln^{1-\varkappa}(\mathrm{e}x)} \biggl(\frac{L(g, \varkappa)}{\Gamma(\varkappa)} + O\biggl(\frac{1}{\ln(\mathrm{e}x)} \biggr) \biggr). \tag{3.8}$$

Combining the last estimate with (3.2) and (3.7), we get that

$$R_x(t) \ll \frac{1}{G(x)} \frac{x}{\ln^{1-\alpha}(\mathrm{e}x^{1-t})} \frac{1}{x^t} \sum_{l \leqslant x^t} f(l)\hat{h}(l). \tag{3.9}$$

Applying (3.6), we deduce that $\psi = f\hat{h} \in \mathcal{G}(\varkappa - \alpha; \delta_1 + \delta_2)$. Thus, Lemma 2 with $d = 1$ and (3.8) yield

$$R_x(t) + R_x(u) \ll r_x(u, t; 1 - \varkappa + \alpha, 1 - \alpha)$$

for $0 \leqslant u \leqslant t \leqslant 1$.

To continue with the proof of the theorem, we consider relation (3.3). An application of (3.4) shows that relation (3.3) can be written as

$$\Phi_x(u,t) = \frac{x}{G(x)}\biggl(\frac{L(h,\alpha)}{\Gamma(\alpha)}\Psi_x(u,t;1-\alpha) + O\bigl(R_x(u,t;2-\alpha)\bigr)\biggr), \tag{3.10}$$

where

$$\Psi_x(u,t;a) = \sum_{x^u<l\leqslant x^t} \frac{f(l)\tilde{h}(l)}{l\ln^a(\mathrm{e}\frac{x}{l})} \quad \text{and} \quad R_x(u,t;b) = \sum_{x^u<l\leqslant x^t} \frac{f(l)\hat{h}(l)}{l\ln^b(\mathrm{e}\frac{x}{l})}.$$

Set

$$M(z) := \sum_{n\leqslant z} f(n)\hat{h}(n).$$

Using Lemma 2, we obtain that

$$M(z) \ll \frac{z}{\ln^{1-\varkappa+\alpha} z}.$$

Then in the notations of Lemma 4, taking $A=0$, $b=2-\alpha$, $a=\alpha-\varkappa$, and $S(x,u,t,b)=R_x(u,t;b)$, we deduce

$$R_x(u,t;2-\alpha) \ll \frac{1}{\ln^{1-\varkappa} x}\biggl(r_x(u,t;\alpha-\varkappa,1-\alpha) + \frac{|\ln(\eta_x+1-t)|^{\epsilon(2-\alpha)}}{\ln x}\biggr). \tag{3.11}$$

Now we are in a position to estimate the main term of (3.10). From (3.5) it follows that $\psi = f\tilde{h} \in \mathcal{G}(\varkappa-\alpha;\delta_1+\delta_2)$. So we can employ Lemma 1 with $d=1$ and then Lemma 4 again with

$$A = \frac{L(f\tilde{h},\varkappa-\alpha)}{\Gamma(\varkappa-\alpha)}, \qquad a = 1-\varkappa+\alpha, \qquad b = 1-\alpha,$$

and $S(x,u,t,b) = \Psi_x(u,t;1-\alpha)$. This yields

$$\begin{aligned}\Psi_x(u,t;1-\alpha) = {} & \frac{A}{\ln^{1-\varkappa} x} I(u,t;1-\varkappa+\alpha,1-\alpha,\eta_x) \\ & + O\biggl(\frac{1}{\ln^{2-\varkappa} x}\Bigl(\ln x\cdot r_x(u,t;1-\varkappa+\alpha,-\alpha) + \bigl|\ln(\eta_x+u)\bigr|^{\epsilon(2-\varkappa+\alpha)}\Bigr)\biggr).\end{aligned}$$

Combining the last relation, (3.8), and (3.11), we conclude that (3.10) becomes

$$\begin{aligned}\Phi_x(u,t) = {} & \frac{\Gamma(\varkappa)}{\Gamma(\varkappa-\alpha)\Gamma(\alpha)}\frac{L(f\tilde{h},\varkappa-\alpha)L(h,\alpha)}{L(g,\varkappa)} I(u,t;1-\varkappa+\alpha,1-\alpha,\eta_x) \\ & + O\biggl(r_x(u,t;1-\varkappa+\alpha,1-\alpha) + \frac{|\ln(\eta_x+u)|^{\epsilon(2-\varkappa+\alpha)} + |\ln(\eta_x+1-t)|^{\epsilon(2-\alpha)}}{\ln x}\biggr)\end{aligned}$$

since

$$r_x(u,t;\alpha-\varkappa,1-\alpha) + r_x(u,t;1-\varkappa+\alpha,-\alpha) \ll r_x(u,t;1-\varkappa+\alpha,1-\alpha).$$

We have

$$L(h,\alpha)L(f\tilde{h},\varkappa-\alpha) = \prod_p \biggl(1-\frac{1}{p}\biggr)^{\varkappa} \sum_{i=0}^{\infty} \frac{f(p^i)\tilde{h}(p^i)}{p^i}\cdot K(p), \tag{3.12}$$

where

$$K(p) := \sum_{i=0}^{\infty} \frac{h(p^i)}{p^i}.$$

By the definition of $\tilde{h}$ the inner sum in (3.12) is

$$\left(1 + \frac{1}{K(p)} \sum_{i=1}^{\infty} \frac{f(p^i)}{p^i} \sum_{j=0}^{\infty} \frac{h(p^{i+j})}{p^j}\right) K(p) = K(p) + \sum_{j=1}^{\infty} \frac{g(p^j)}{T(p^j)p^j} \sum_{i=0}^{j-1} f\bigl(p^{j-i}\bigr) = \sum_{j=0}^{\infty} \frac{g(p^j)}{p^j}.$$

From this and from (3.12) it follows that

$$L(h, \alpha) L(f\tilde{h}, \varkappa - \alpha) = L(g, \varkappa).$$

Thus,

$$\begin{aligned} \Phi_x(u,t) = {} & \frac{\Gamma(\varkappa)}{\Gamma(\varkappa - \alpha)\Gamma(\alpha)} I(u,t; 1-\varkappa+\alpha, 1-\alpha, \eta_x) \\ & + O\left(r_x(u,t; 1-\varkappa+\alpha, 1-\alpha) + \frac{|\ln(\eta_x + u)|^{\epsilon(\varkappa-\alpha)} + |\ln(\eta_x + 1 - t)|^{\epsilon(\alpha)}}{\ln x} \right) \end{aligned}$$

since $\epsilon(2-\beta) = \epsilon(\beta)$.

To complete the proof of case (i), we need to estimate the difference

$$\Delta_x(u,t) := I(u,t; a, b, 0) - I(u,t; a, b, \eta_x)$$

with $a = 1 - \varkappa + \alpha < 1$ and $b = 1 - \alpha < 1$. First, assume that

$$\eta_x \leqslant u \leqslant t \leqslant 1 - \eta_x. \tag{3.13}$$

By the Lagrange mean value theorem

$$\Delta_x(u,t) = -\eta_x \frac{\mathrm{d}I(u,t,a,b,\eta)}{\mathrm{d}\eta} = \eta_x \bigl(a \cdot I(u,t; a+1, b, \eta) + b \cdot I(u,t; a, b+1, \eta)\bigr) \tag{3.14}$$

with some $\eta \in (0, \eta_x)$. Integrating $I(u,t; a, b+1, \eta)$ by parts, for $b \neq 0$, we have

$$\Delta_x(u,t) = \eta_x \bigl(2a \cdot I(u,t; a+1, b, \eta) + (\eta + v)^{-a}(\eta + 1 - v)^{-b}\big|_u^t\bigr).$$

From this and from (3.14), having in mind (3.13), we deduce that

$$\bigl|\Delta_x(u,t)\bigr| \ll \eta_x |a| \cdot I(u,t; a+1, b, \eta_x) + r_x(u,t; a, b)$$

for any $b < 1$. The integral $I(u,t; a+1, b, \eta_x)$ was considered in the proof of Lemma 4. Namely, for $a \neq 0$, inequalities (2.4), (2.5), and (2.6) yield

$$I(u,t; a+1, b, \eta_x) \ll 1 + r_x(u,t; a, b-1) \ln x.$$

Thus,

$$\bigl|\Delta_x(u,t)\bigr| \ll \eta_x + r_x(u,t; a, b) \tag{3.15}$$

for all $a < 1$ and $b < 1$, provided that (3.13) holds.

If $0 \leqslant u \leqslant \eta_x$ or $1-\eta_x \leqslant t \leqslant 1$, then, in addition, we have to estimate $\Delta_x(u,\eta_x)$ and $\Delta_x(1-\eta_x,t)$, respectively. So, for $0 \leqslant u \leqslant \eta_x$, we have

$$\left|\Delta_x(u,\eta_x)\right| \ll \int\limits_u^{\eta_x} \left(\frac{1}{v^a}+\frac{1}{(\eta_x+v)^a}\right) \mathrm{d}v \ll \eta_x^{1-a} \ll r_x(u,t;a,b).$$

Similarly, for $1-\eta_x \leqslant t \leqslant 1$, we obtain

$$\left|\Delta_x(1-\eta_x,t)\right| \ll \eta_x^{1-b} \ll r_x(u,t;a,b).$$

Thus, estimate (3.15) holds for all $0 \leqslant u \leqslant t \leqslant 1$, and the validity of (1.2) is proved for $0<\alpha<\varkappa$.

(ii) Assume that $\varkappa=\alpha>0$. Then $g \in G(\alpha;\delta_1)$, $h=g/T_f \in G(\alpha;\delta_2)$. Consider the difference

$$1-F_x(u;g,f)=\Phi_x(u,1)+R_x(u). \tag{3.16}$$

Having in mind that $h \in G(\alpha;\delta_2)$ and applying Lemma 2, we obtain

$$\Phi_x(u,1) \ll \frac{x}{G(x)} \sum_{x^u<l\leqslant x} \frac{f(l)\hat{h}(l)}{l \ln^{1-\alpha}(\mathrm{e}\frac{x}{l})}. \tag{3.17}$$

Note that $f\hat{h} \in G(0;\delta_1+\delta_2)$. Therefore, by Lemma 2 we get

$$M(v)=\sum_{m\leqslant v} f(m)\hat{h}(m) \ll v\exp(-c_3\sqrt{\ln v}). \tag{3.18}$$

From (3.17) and (3.8) we have

$$\Phi_x(u,1) \ll \ln^{1-\alpha} x \int\limits_{x^u}^{x} \frac{\mathrm{d}M(v)}{v\ln^{1-\alpha}(\mathrm{e}\frac{x}{v})}.$$

Partial integration with respect to (3.18) yields

$$\Phi_x(u,1) \ll \exp(-c_4\sqrt{u\ln x}). \tag{3.19}$$

Further, (3.9), (3.8), and (3.18) imply

$$R_x(u) \ll \frac{1}{(\eta_x+(1-u))^{1-\alpha}x^u} \sum_{m\leqslant x^u} f(m)\hat{h}(m) \ll \exp(-c_5\sqrt{u\ln x}). \tag{3.20}$$

Let

$$\varepsilon=c_6\frac{\ln\ln^2 x}{\ln x}.$$

From (3.20), (3.16), and (3.19) it follows that

$$E_0(y-\varepsilon)-\varepsilon \leqslant F_x(y;g,f) \leqslant E_0(y+\varepsilon)+\varepsilon$$

for some $c_6>0$ and any $y \in \mathbb{R}$. Hence,

$$\mathcal{L}(F_x,E_0) \ll \frac{\ln\ln^2 x}{\ln x}.$$

(iii) Assume that $\varkappa > 0$ and $\alpha = 0$. In this case, we have that

$$g \in G(\varkappa; \delta_1), \qquad h \in G(0; \delta_2), \qquad f\hat{h} \in G(\varkappa; \delta_1 + \delta_2).$$

It is easy to see that

$$F_x(t; g, f) \leqslant S_x(t; g, f) = \frac{1}{G(x)} \sum_{m \leqslant x^t} f(m) \sum_{l \leqslant x/m} h(ml).$$

Applying Lemma 2 and (3.8), we get

$$\begin{aligned} F_x(t; g, f) &\ll (\ln x)^{1-\varkappa} \sum_{m \leqslant x^t} \frac{f(m)\hat{h}(m)}{m} \exp\biggl(-c_7 \sqrt{\ln \frac{x}{m}}\biggr) \\ &= (\ln x)^{1-\varkappa} \int\limits_{1^-}^{x^t} v^{-1} \exp\bigl(-c_7 \sqrt{\ln x - \ln v}\,\bigr) \,\mathrm{d}M(v). \end{aligned} \tag{3.21}$$

Moreover, by the same Lemma 2

$$M(v) = \sum_{m \leqslant v} f(m)\hat{h}(m) \ll \frac{v}{\ln^{1-\varkappa}(\mathrm{e}v)}.$$

Therefore, partial integration in (3.21) yields

$$F_x(t; g, f) \ll \exp\bigl(-c_8 \sqrt{(1-t)\ln x}\,\bigr).$$

The arguments above and the last estimate yield

$$\mathcal{L}(F_x, E_1) \ll \frac{\ln^2 \ln x}{\ln x}.$$

Theorem 3 is proved. □

Proof of Theorem 4. We have

$$F_x(t; g, f) = F_x(0, t; g, f) + F_x(0; g, f). \tag{3.22}$$

By Theorem 3

$$\begin{aligned} &\bigl|F_x(0, t; g, f) - B(t; \varkappa - \alpha, \alpha)\bigr| \\ &\quad \ll r_x(0, t; 1 - \varkappa + \alpha, 1 - \alpha) + \frac{|\ln(\eta_x)|^{\epsilon(\varkappa-\alpha)}}{\ln x} + \frac{|\ln(\eta_x)|^{\epsilon(\alpha)}}{\ln x}. \end{aligned} \tag{3.23}$$

We recall that $g/T_f \in \mathcal{G}(\alpha; \delta_2)$. Therefore, Lemma 2 and (3.8) yield

$$F_x(0; g, f) = \frac{1}{G(x)} \sum_{m \leqslant x} \frac{g(m)}{T_f(m)} \ll \frac{1}{\ln^{\varkappa-\alpha} x}. \tag{3.24}$$

Since

$$r_x(0, t; 1 - \varkappa + \alpha, 1 - \alpha) \ll \frac{1}{\ln^{\varkappa-\alpha} x} + \frac{1}{\ln^{\alpha} x} + \frac{1}{\ln x},$$

the proof of Theorem 4 now follows from (3.22), (3.23), and (3.24). □

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