DOI 10.1007/s10986-016-9335-y
Lithuanian Mathematical Journal, Vol. 56, No. 4, October, 2016, pp. 540–551

On the boundedness of hyperbolic Riesz B-potential

Elina L. Shishkina

Voronezh State University, Universitetskaya pl. 1, Voronezh, Russia
(e-mail: shishkina@amm.vsu.ru)

Received December 19, 2015; revised May 21, 2016

Abstract. This paper deals with the hyperbolic Riesz B-potential, which is the negative real power of an operator $B_{\gamma_1} - \sum_{i=2}^{n} B_{\gamma_i}$, where $B_{\gamma_i} = \partial^2/\partial x_i^2 + (\gamma_i/x_i)\partial/\partial x_i$, $i = 1, \dots, n$, is a singular Bessel differential operator. We prove the boundedness of the hyperbolic Riesz B-potential in proper spaces.

MSC: 46E30, 31B99, 47G40

Keywords: hyperbolic Riesz B-potential, fractional power of singular hyperbolic operator, Lorentz distance, singular Bessel differential operator, generalized translation, Marcinkiewicz interpolation theorem, bounded operator

1 Introduction

In this paper, we prove the boundedness for a new type of potential with Lorentz distance in the weighted space L_p^γ. The considered potential $I_{\Box_\gamma}^\alpha$ is the negative real power of the operator

$$\Box_\gamma = B_{\gamma_1} - \sum_{i=2}^{n} B_{\gamma_i},$$

where $B_{\gamma_i} = \partial^2/\partial x_i^2 + (\gamma_i/x_i)\partial/\partial x_i$, $i = 1, \dots, n$, is the singular differential Bessel operator.

The potential $I_\Box^\alpha$ that is the negative real power of the operator

$$\Box = \frac{\partial^2}{\partial x_1^2} - \sum_{i=2}^{n} \frac{\partial^2}{\partial x_i^2}$$

was studied in [11]. In [15], for an operator similar to the operator $I_{\Box_\gamma}^\alpha$, some properties were obtained, but the boundedness of such operators has not yet been proved. This work fills this gap. The results of this paper were announced in [16].

The potential with the Lorentz distance is of the form

$$\left(I_\Box^\alpha f\right)(x) = \frac{1}{H_n(\alpha)} \int\limits_{K_+} \frac{f(x-y)\,\mathrm{d}y}{r^{n-\alpha}(y)}, \quad 2 \leqslant n,\ n-2 < \alpha, \tag{1.1}$$

where

$$H_n(\alpha) = 2^{\alpha-1}\pi^{-1+n/2}\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\alpha+2-n}{2}\right),$$

$$r(x) = \sqrt{x_1^2 - x_2^2 - \cdots - x_n^2}, \qquad K_+(x) = \left\{x\colon x_1^2 \geqslant x_2^2 + \cdots + x_n^2,\ x_1 \geqslant 0\right\}.$$

It was introduced by Riesz [12] (cf. [13, p. 31] and [14, p. 409]).

Potential (1.1) was named the *hyperbolic Riesz potential* in [14, p. 409].

In this paper, we consider a Riesz potential with Lorentz distance connected with generalized translation operator in the following form:

$$\left(I^{\alpha}_{\Box_\gamma} f\right)(x) = \int\limits_{K^+} r^{\alpha-n-|\gamma|}(y)\left(T^y f\right)(x) y^\gamma \, \mathrm{d}y, \quad y^\gamma = \prod_{i=1}^{n} y_i^{\gamma_i}. \tag{1.2}$$

In (1.2), $\gamma = (\gamma_1, \ldots, \gamma_n)$ is a multiindex consisting of positive fixed real numbers γ_i, $i = 1, \ldots, n$, $|\gamma| = \gamma_1 + \cdots + \gamma_n$, $n + |\gamma| - 2 < \alpha < n + |\gamma|$,

$$K^+ = \left\{y \in \mathbb{R}_n\colon y_1^2 \geqslant y_2^2 + \cdots + y_n^2,\ y_1 > 0,\ \ldots,\ y_n > 0\right\},$$

and $(T^y f)(x) = (T^{y_1}_{x_1} \cdots T^{y_n}_{x_n} f)(x)$ is a multidimensional generalized translation. Each of the one-dimensional generalized translations $T^{y_i}_{x_i}$ is defined for $i = 1, \ldots, n$ by the formula (see [4, p. 122, (5.19)]

$$\left(T^{y_i}_{x_i} f\right)(x) = \frac{\Gamma(\frac{\gamma_i+1}{2})}{\Gamma(\frac{\gamma_i}{2})\Gamma(\frac{1}{2})} \int\limits_0^\pi f\left(x_1, \ldots, x_{i-1}, \sqrt{x_i^2 + y_i^2 - 2x_i y_i \cos\varphi_i},\ x_{i+1}, \ldots, x_n\right) \sin^{\gamma_i - 1}\varphi_i \, \mathrm{d}\varphi_i.$$

We will call the operator (1.2) a *hyperbolic Riesz B-potential*. Such potentials are negative real powers of the operator

$$\Box_\gamma = B_{\gamma_1} - \sum_{i=2}^{n} B_{\gamma_i},$$

where $B_{\gamma_i} = \partial^2/\partial x_i^2 + (\gamma_i/x_i)\partial/\partial x_i$ is the singular differential Bessel operator (see [3, p. 3].

Our proof of the boundedness of operator (1.2) is based on applying the appropriate Marcinkiewicz interpolation theorem.

Riesz B-potentials with Euclidian distance (elliptic Riesz B-potentials) are studied in detail (see [5, 6, 7, 8, 9]). Such potentials are negative real powers of the operator $\triangle_\gamma = \sum_{k=1}^{n} B_{\gamma_k}$. But methods of studying elliptic and hyperbolic Riesz B-potentials are different, and we will use techniques for studying the generalized translation developed by Lyakhov [5, 6, 7] and methods of studying hyperbolic potentials (1.1) proposed by Nogin and Sukhinin in [11].

The boundedness of operator (1.2) is essentially used when we construct its inverse, but at the same time, it is of independent interest.

The main result of this paper is a proof of the boundedness for the Riesz potential with Lorentz distance generated by a generalized translation operator T^y in special weighted spaces.

The rest of the paper is organized as follows. In Section 2, we present necessary preliminary definitions and theorems. In Section 3, we prove our main theorem on the boundedness of the hyperbolic Riesz B-potential from L_p^γ to L_q^γ for functions from the Schwartz space. The last section contains further study of the potential (1.2), which implies its absolute convergence.

2 Preliminaries

We consider functions $f = f(x)$ defined on

$$\mathbb{R}_n^+ = \{x = (x_1, \ldots, x_n) \in \mathbb{R}_n\colon\ x_1 > 0,\ \ldots,\ x_n > 0\}.$$

We call a function defined on $\mathbb{R}_n^+$ to be even with respect to x_i, $i = 1, \ldots, n$, if it can be extended on $\mathbb{R}_n$ as an even function with respect to x_i preserving the considered class of functions.

The weighted $L_p^\gamma(\mathbb{R}_n^+) = L_p^\gamma$ space, $p \geqslant 1$, is the set of all measurable functions from $\mathbb{R}_n^+$ to $\mathbb{R}$ that are even with respect to each variable and, moreover, the absolute value of such a function raised to the pth power and multiplied by $x^\gamma = \prod_{i=1}^n x_i^{\gamma_i}$ is integrable, that is,

$$\int\limits_{\mathbb{R}_n^+} |f(x)|^p x^\gamma \, \mathrm{d}x < \infty.$$

For a real number $p \geqslant 1$, the L_p^γ-norm of f is defined by

$$\|f\|_{L_p^\gamma(\mathbb{R}_n^+)} = \|f\|_{p,\gamma} = \biggl(\int\limits_{\mathbb{R}_n^+} |f(x)|^p x^\gamma \, \mathrm{d}x\biggr)^{1/p}, \quad x^\gamma = \prod_{i=1}^n x_i^{\gamma_i}.$$

Let $\Omega \subset \mathbb{R}_n^+ \cup \{x_i = 0,\ i = 1, \ldots, n\}$, and $\mathrm{mes}_\gamma\, \Omega$ be the *weighted measure* of Ω defined by

$$\mathrm{mes}_\gamma\, \Omega = \int\limits_\Omega x^\gamma \, \mathrm{d}x.$$

For every measurable function $f(x)$ defined on $\mathbb{R}_n^+$, we consider

$$\mu_\gamma(f, t) = \mathrm{mes}_\gamma \{x \in \mathbb{R}_n^+\colon\ |f(x)| > t\} = \int\limits_{\{x\colon |f(x)| > t\}^+} x^\gamma \, \mathrm{d}x,$$

where $\{x\colon\ |f(x)| > t\}^+ = \{x \in \mathbb{R}_n^+\colon\ |f(x)| > t\}$. The function $\mu_\gamma = \mu_\gamma(f, t)$ is called the *weighted distribution function* of $|f(x)|$.

The space $L_\infty^\gamma(\mathbb{R}_n^+) = L_\infty^\gamma$ is defined as the set of measurable functions f on $\mathbb{R}_n^+$ that are even with respect to each variable and such that

$$\|f\|_{L_\infty^\gamma(\mathbb{R}_n^+)} = \|f\|_{\infty,\gamma} = \operatorname*{ess\,sup_\gamma}_{x \in \mathbb{R}_n^+} |f(x)| = \inf_{a \in \mathbb{R}} \{\mu_\gamma(f, a) = 0\} < \infty.$$

We have the following inequality [10]:

$$\mu_\gamma(f, t) \leqslant \left(\frac{\|f\|_{p,\gamma}}{t}\right)^p. \tag{2.1}$$

The norms of the spaces L_p^γ and L_∞^γ are connected by the following equality:

$$\|f\|_{\infty,\gamma} = \lim_{p \to \infty} \|f\|_{p,\gamma}. \tag{2.2}$$

The space $S_{\mathrm{ev}}(\mathbb{R}_n^+) = S_{\mathrm{ev}}$ consists of all functions on $\mathbb{R}_n^+$ that are even with respect to each variable and belong to the space of Schwartz functions.

We denote by $SL_p^\gamma(\mathbb{R}_n^+) = SL_p^\gamma$ the set of all even with respect to each variable functions for which the norm

$$\|f\|_{SL_p^\gamma(\mathbb{R}_n^+)} = \|f\|_{SL_p^\gamma} = \sup_{0<t<\infty} t\big(\mu_\gamma(f,t)\big)^{1/p} < \infty, \quad 1 \leqslant p < \infty.$$

The operator A is said to be *quasilinear* (see [2, p. 41]) if $A(f_1+f_2)$ is uniquely defined, Af_1 and Af_2 are defined, and if there exists a constant κ such that for all f_1 and f_2 , the following inequality is valid pointwise:

$$\big|A(f_1+f_2)\big| \leqslant \kappa\big(|Af_1| + |Af_2|\big).$$

A quasilinear operator A from L_p^γ to L_q^γ is of *strong type* $(p,q)_\gamma$, $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, if the following inequality is valid:

$$\|Af\|_{q,\gamma} \leqslant h\|f\|_{p,\gamma} \quad \forall f \in L_p^\gamma, \tag{2.3}$$

where the constant h does not depend on f.

We say that a quasilinear operator A is an operator of *weak type* $(p,q)_\gamma$ $(1 \leqslant p \leqslant \infty, 1 \leqslant q < \infty)$ if

$$\mu_\gamma(Af,\lambda) \leqslant \left(\frac{h\|f\|_{p,\gamma}}{\lambda}\right)^q \quad \forall f \in L_p^\gamma, \tag{2.4}$$

where h does not depend on f and $\lambda > 0$.

If $q = \infty$, then a quasilinear operator A is an operator of weak type $(p,q)_\gamma$ if it is of strong type $(p,q)_\gamma$.

A generalized convolution is defined by

$$(f * g)_\gamma(x) = \int\limits_{\mathbb{R}_n^+} f(y)\big(T^y g\big)(x) y^\gamma \,\mathrm{d}y$$

(see [17] formula (14) for one-dimensional convolution and [3, p. 19] for the general case).

Let $p,q,r \in [1,\infty]$ and

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}. \tag{2.5}$$

If $f \in L_p^\gamma$, $g \in L_q^\gamma$, $1 \leqslant p,q,r \leqslant \infty$, $1/q = 1/p + 1/r - 1$, then a generalized convolution$(f*g)_\gamma$ is bounded almost everywhere, and the Hausdorf–Young inequality is valid:

$$\big\|(f*g)_\gamma\big\|_{r,\gamma} \leqslant \|f\|_{p,\gamma}\|g\|_{q,\gamma}. \tag{2.6}$$

We obtain the inequality

$$\big\|(f*g)_\gamma\big\|_{\infty,\gamma} \leqslant \|f\|_{p,\gamma}\|g\|_{q,\gamma} \tag{2.7}$$

from (2.6) by tending to the limit as $r\to\infty$ using (2.2) (with p and q such that $1/p+1/q=1$).

We present the Marcinkiewicz interpolation theorem in the following form (see [1] and [10]).

Theorem 1. *Let* $1 \leqslant p_i \leqslant q_i < \infty$ $(i=1,2)$, $q_1 \neq q_2$, $0 < \tau < 1$, $1/p = (1-\tau)/p_1 + \tau/p_2$, *and* $1/q = (1-\tau)/q_1 + \tau/q_2$. *If a quasilinear operator A has simultaneously weak types* $(p_1,q_1)_\gamma$ *and* $(p_2,q_2)_\gamma$, *then A has a strong type* $(p,q)_\gamma$, *and*

$$\|Af\|_{q,\gamma} \leqslant M\|f\|_{p,\gamma}, \tag{2.8}$$

where the constant $M = M(\gamma,\tau,\kappa,p_1,p_2,q_1,q_2)$ *does not depend on f and A.*

3 The boundedness of the hyperbolic Riesz B-potential with density function from the Schwartz space

Along with potential (1.2), we consider the operator

$$\big(I^\alpha_{\square_\gamma,\delta}f\big)(x)=\int\limits_{\delta y_1^2\geqslant y_2^2+\dots+y_n^2} r^{\alpha-n-|\gamma|}(y)\big(T^y f\big)(x)y^\gamma\,\mathrm{d}y,\quad 0<\delta<1. \tag{3.1}$$

Lemma 1. *If $f\in L_s^\gamma$, $1\leqslant s<(n+|\gamma|)/\alpha$, $n+|\gamma|-2<\alpha<n+|\gamma|$, then we have the estimate*

$$\mu_\gamma\big((1-\delta)^{(n+|\gamma|-\alpha)/2}\big(I^\alpha_{\square_\gamma,\delta}f\big)(x),\lambda\big)\leqslant C_{\alpha,n,\gamma,s}(1-\delta)^{(n+|\gamma|-\alpha)/2}\left(\frac{\|f\|_{s,\gamma}}{\lambda}\right)^t, \tag{3.2}$$

where $t=(n+|\gamma|)s/(n+|\gamma|-\alpha s)$, $0<\delta<1$, $\lambda>0$, and $C_{\alpha,n,\gamma,s}$ does not depend on λ, δ, and f.

Proof. Assume, without loss of generality, that $\|f\|_{s,\gamma}=1$. Let ω be a fixed real number. We define

$$G^0_{\delta,\omega}=\{x\in\mathbb{R}^+_n\colon\ \delta x_1^2\geqslant x_2^2+\dots+x_n^2,\ 0\leqslant x_1\leqslant\omega\},$$
$$G^\infty_{\delta,\omega}=\{x\in\mathbb{R}^+_n\colon\ \delta x_1^2\geqslant x_2^2+\dots+x_n^2,\ \omega<x_1\},$$

$$K^+_{0,\delta}(x)=\begin{cases} r^{\alpha-n-|\gamma|}(x), & x\in G^0_{\delta,\omega},\\ 0, & x\in\mathbb{R}^+_n\setminus G^0_{\delta,\omega},\end{cases}\qquad K^+_{\infty,\delta}(x)=\begin{cases} r^{\alpha-n-|\gamma|}(x), & x\in G^\infty_{\delta,\omega},\\ 0, & x\in\mathbb{R}^+_n\setminus G^\infty_{\delta,\omega}.\end{cases}$$

Using these notations, we obtain

$$\big(I^\alpha_{\square_\gamma,\delta}f\big)(x)=\big(K^+_{0,\delta}*f\big)_\gamma+\big(K^+_{\infty,\delta}*f\big)_\gamma. \tag{3.3}$$

Let $x'=(x_2,\cdots,x_n)$, $|x'|=\sqrt{x_2^2+\dots+x_n^2}$, $(x')^{\gamma'}=x_2^{\gamma_2}\cdots x_n^{\gamma_n}$. Then we have

$$\begin{aligned}
\|K^+_{0,\delta}\|_{1,\gamma}&=\int\limits_{\mathbb{R}^+_n}K^+_{0,\delta}(x)x^\gamma\,\mathrm{d}x=\int\limits_{G^0_{\delta,\omega}}\big(x_1^2-x_2^2-\dots-x_n^2\big)^{(\alpha-n-|\gamma|)/2}x^\gamma\,\mathrm{d}x\\
&=\int\limits_0^\omega x_1^{\gamma_1}\,\mathrm{d}x_1\int\limits_{|x'|^2\leqslant\delta x_1^2}\big(x_1^2-|x'|^2\big)^{(\alpha-n-|\gamma|)/2}(x')^{\gamma'}\,\mathrm{d}x'\\
&=\big\{x'=x_1y',\ y'\in\mathbb{R}^+_{n-1}\big\}\\
&=\int\limits_0^\omega x_1^{\alpha-1}\,\mathrm{d}x_1\int\limits_{|y'|^2\leqslant\delta}\big(1-|y'|^2\big)^{(\alpha-n-|\gamma|)/2}(y')^{\gamma'}\,\mathrm{d}y'\\
&\leqslant\int\limits_0^\omega x_1^{\alpha-1}\,\mathrm{d}x_1\int\limits_{|y'|^2\leqslant 1}\big(1-|y'|^2\big)^{(\alpha-n-|\gamma|)/2}(y')^{\gamma'}\,\mathrm{d}y'\\
&=\frac{\omega^\alpha}{\alpha}\int\limits_{|y'|\leqslant 1}\big(1-|y'|^2\big)^{(\alpha-n-|\gamma|)/2}(y')^{\gamma'}\,\mathrm{d}y'=C^1_{\alpha,n,\gamma}\omega^\alpha,
\end{aligned}$$

where $C^1_{\alpha,n,\gamma} = 2^{1-n}\Gamma((\alpha-n-|\gamma|+2)/2)\prod_{i=2}^n \Gamma((\gamma_i+1)/2))/(\alpha\Gamma((\alpha-\gamma_1+1)/2))$ does not depend on δ. Consequently,

$$\big\|K^+_{0,\delta}\big\|_{1,\gamma} \leqslant C^1_{\alpha,n,\gamma}\ \omega^\alpha, \tag{3.4}$$

which means that $K^+_{0,\delta} \in L^\gamma_1$.

Let us take s' such that $1/s+1/s'=1$. We will estimate $\|K^+_{\infty,\delta}\|_{s',\gamma}$. Suppose first that $s \neq 1$ (i.e., $s' \neq \infty$). Then

$$\begin{aligned}
\big\|K^+_{\infty,\delta}\big\|_{s',\gamma} &= \biggl(\int\limits_{\mathbb{R}^+_n} \big|K^+_{0,\delta}(x)\big|^{s'} x^\gamma\,\mathrm{d}x\biggr)^{1/s'} \\
&= \biggl(\int\limits_{G^\infty_{\delta,\omega}} \bigl(x_1^2 - x_2^2 - \cdots - x_n^2\bigr)^{(\alpha-n-|\gamma|)/2\cdot s'} x^\gamma\,\mathrm{d}x\biggr)^{1/s'} \\
&= \biggl(\int\limits_\omega^\infty x_1^{\gamma_1}\,\mathrm{d}x_1 \int\limits_{|x'|^2\leqslant \delta x_1^2} \bigl(x_1^2 - |x'|^2\bigr)^{(\alpha-n-|\gamma|)/2\cdot s'} (x')^{\gamma'}\,\mathrm{d}x'\biggr)^{1/s'} \\
&= \bigl\{x' = x_1 y',\ y' \in \mathbb{R}^+_{n-1}\bigr\} \\
&= \biggl(\int\limits_\omega^\infty x_1^{(\alpha-n-|\gamma|)s'+n+|\gamma|-1}\,\mathrm{d}x_1 \int\limits_{|y'|^2\leqslant\delta} \bigl(1-|y'|^2\bigr)^{(\alpha-n-|\gamma|)/2\cdot s'} (y')^{\gamma'}\,\mathrm{d}y'\biggr)^{1/s'} \\
&\leqslant \frac{\prod_{i=2}^n \Gamma(\frac{\gamma_i+1}{2})}{2^n\Gamma(\frac{n+|\gamma'|+1}{2})}(1-\delta)^{(\alpha-n-|\gamma|)/2}\biggl(\int\limits_\omega^\infty x_1^{(\alpha-n-|\gamma|)s'+n+|\gamma|-1}\,\mathrm{d}x_1\biggr)^{1/s'} \\
&= C^2_{\alpha,n,\gamma,s}(1-\delta)^{(\alpha-n-|\gamma|)/2}\omega^{-(n+|\gamma|)/q},
\end{aligned}$$

$$C^2_{\alpha,n,\gamma,s} = \frac{2^{-n}\prod_{i=2}^n \Gamma(\frac{\gamma_i+1}{2})}{((n+|\gamma|-\alpha)s'-n-|\gamma|)^{1/s'}\Gamma(\frac{n+|\gamma'|+1}{2})}.$$

Here we take into account that $\alpha-n-|\gamma|<0$, $s'=s/(s-1)$, $s<(n+|\gamma|)/\alpha$ and $t=(n+|\gamma|)s/(n+|\gamma|-\alpha s)$. Then we derive

$$\big\|K^+_{\infty,\delta}\big\|_{s',\gamma} \leqslant C^2_{\alpha,n,\gamma,s}(1-\delta)^{(\alpha-n-|\gamma|)/2}\omega^{-(n+|\gamma|)/t}, \quad \frac{1}{s}+\frac{1}{s'}=1, \tag{3.5}$$

which means that $K^+_{\infty,\delta} \in L^\gamma_{s'}$, $s'<\infty$.

Taking the limit in (3.5) as $s'\to\infty$, we get

$$\big\|K^+_{\infty,\delta}\big\|_{\infty,\gamma} \leqslant C^2_{\alpha,n,\gamma,1}(1-\delta)^{(\alpha-n-|\gamma|)/2}\omega^{-(n+|\gamma|)/t}, \quad C^2_{\alpha,n,\gamma,1} = \frac{2^{-n}\prod_{i=2}^n \Gamma(\frac{\gamma_i+1}{2})}{\Gamma(\frac{n+|\gamma'|+1}{2})}. \tag{3.6}$$

Now combining (3.5) and (3.6), we derive

$$\big\|K^+_{\infty,\delta}\big\|_{s',\gamma} \leqslant C^2_{\alpha,n,\gamma,s}(1-\delta)^{(\alpha-n-|\gamma|)/2}\omega^{-(n+|\gamma|)/t}, \quad 1\leqslant s<\frac{n+|\gamma|}{\alpha},\ \frac{n+|\gamma|}{n+|\gamma|-\alpha}<s'\leqslant\infty. \tag{3.7}$$

Then, for any $\lambda > 0$, (3.3) implies that

$$\begin{aligned}&\mu_\gamma\big((1-\delta)^{(n+|\gamma|-\alpha)/2}\big(I^\alpha_{\square_\gamma,\delta} f\big)(x), 2\lambda\big)\\ &\quad\leqslant \mu_\gamma\big((1-\delta)^{(n+|\gamma|-\alpha)/2}\big(K^+_{0,\delta} * f\big)_\gamma, \lambda\big) + \mu_\gamma\big((1-\delta)^{(n+|\gamma|-\alpha)/2}\big(K^+_{\infty,\delta} * f\big)_\gamma, \lambda\big).\end{aligned} \tag{3.8}$$

From (2.7) and (3.7) it follows that

$$\begin{aligned}&(1-\delta)^{(n+|\gamma|-\alpha)/2}\big\|\big(K^+_{\infty,\delta} * f\big)_\gamma\big\|_{\infty,\gamma}\\ &\quad\leqslant (1-\delta)^{(n+|\gamma|-\alpha)/2}\|f\|_{s,\gamma}\big\|K^+_{\infty,\delta}\big\|_{s',\gamma} \leqslant C^2_{\alpha,n,\gamma,s}\,\omega^{-(n+|\gamma|)/t}.\end{aligned}$$

Putting $\omega = (C^2_{\alpha,n,\gamma,s})^{t/(n+|\gamma|)}\lambda^{-t/(n+|\gamma|)}$, it follows that

$$\mu_\gamma\big((1-\delta)^{(n+|\gamma|-\alpha)/2}\big(K^+_{\infty,\delta} * f\big)_\gamma, \lambda\big) = 0. \tag{3.9}$$

Combining (2.1) and (2.6) with (3.4), we obtain

$$\begin{aligned}&\mu_\gamma\big((1-\delta)^{(n+|\gamma|-\alpha)/2}\big(K^+_{0,\delta} * f\big)_\gamma, \lambda\big)\\ &\quad\leqslant \frac{(1-\delta)^{(n+|\gamma|-\alpha)s/2}\|(K^+_{0,\delta} * f)_\gamma\|^s_{s,\gamma}}{\lambda^s} \leqslant \frac{(1-\delta)^{(n+|\gamma|-\alpha)s/2}\|K^+_{0,\delta}\|^s_{1,\gamma}\|f\|^s_{s,\gamma}}{\lambda^s}\\ &\quad\leqslant \frac{(C^1_{\alpha,n,\gamma})^s(1-\delta)^{s(n+|\gamma|-\alpha)/2}\,\omega^{s\alpha}}{\lambda^s}.\end{aligned}$$

Since $\omega = (C^2_{\alpha,n,\gamma,s})^{t/(n+|\gamma|)}\lambda^{-t/(n+|\gamma|)}$ and $t = (n+|\gamma|)s/(n+|\gamma|-\alpha s)$, denoting

$$C_{\alpha,n,\gamma,s} = \big(C^1_{\alpha,n,\gamma}\big)^s \times \big(C^2_{\alpha,n,\gamma,s}\big)^{\alpha^2/(n+|\gamma|-\alpha s)},$$

we have

$$\mu_\gamma\big((1-\delta)^{(n+|\gamma|-\alpha)/2}\big(K^+_{0,\delta} * f\big)_\gamma, \lambda\big) \leqslant \frac{C_{\alpha,n,\gamma,s}(1-\delta)^{(n+|\gamma|-\alpha)/2}}{\lambda^t}. \tag{3.10}$$

From (3.8), (3.9), and (3.10) it is clear that

$$\mu_\gamma\big((1-\delta)^{(n+|\gamma|-\alpha)/2}\big(I^\alpha_{\square_\gamma,\delta} f\big)(x), \lambda\big) \leqslant \frac{C_{\alpha,n,\gamma,s}(1-\delta)^{(n+|\gamma|-\alpha)/2}}{\lambda^t},$$

where $C_{\alpha,n,\gamma,s}$ does not depend on λ, δ, and f. This completes the proof. □

Theorem 2. *Let* $n+|\gamma|-2 < \alpha < n+|\gamma|$, $1 \leqslant p < (n+|\gamma|)/\alpha$. *Then the estimate*

$$\big\|I^\alpha_{\square_\gamma} f\big\|_{q,\gamma} \leqslant C_{n,\gamma,p}\|f\|_{p,\gamma}, \quad f \in S_{\text{ev}}, \tag{3.11}$$

holds if and only if $q = (n+|\gamma|)p/(n+|\gamma|-\alpha p)$.

Proof. *Necessity.* Let $n+|\gamma|-2 < \alpha < n+|\gamma|$, $1 < p < (n+|\gamma|)/\alpha$ and suppose that for some q,

$$\big\|I^\alpha_{\square_\gamma} f\big\|_{q,\gamma} \leqslant C_{n,\gamma,p}\|f\|_{p,\gamma}, \quad f \in S_{\text{ev}}. \tag{3.12}$$

Lets us show that inequality (3.12) holds only for $q = (n+|\gamma|)p/(n+|\gamma|-\alpha p)$.

Lets consider the extension operator τ_δ: $(\tau_\delta f)(x) = f(\delta x)$, $\delta > 0$. We have

$$\|\tau_\delta f\|_{p,\gamma} = \delta^{-(n+|\gamma|)/p}\|f\|_{p,\gamma}, \tag{3.13}$$

$$I^\alpha_{\square_\gamma} f(x) = \delta^\alpha \tau_\delta^{-1} I^\alpha_{\square_\gamma} \tau_\delta f(x), \tag{3.14}$$

and

$$\big\|\tau_\delta^{-1} I^\alpha_{\square_\gamma} f(x)\big\|_{q,\gamma} = \delta^{(n+|\gamma|)/q}\big\|I^\alpha_{\square_\gamma} f(x)\big\|_{q,\gamma}. \tag{3.15}$$

From (3.13), (3.14), and (3.15) we immediately obtain

$$\begin{aligned}\big\|I^\alpha_{\square_\gamma} f(x)\big\|_{q,\gamma} &= \delta^\alpha\big\|\tau_\delta^{-1} I^\alpha_{\square_\gamma}\tau_\delta f(x)\big\|_{q,\gamma} = \delta^{(n+|\gamma|)/q+\alpha}\big\|I^\alpha_{\square_\gamma}\tau_\delta f(x)\big\|_{q,\gamma}\\ &\leqslant C_{n,\gamma,p}\delta^{(n+|\gamma|)/q+\alpha}\big\|\tau_\delta f(x)\big\|_{p,\gamma}\\ &= C_{n,\gamma,p}\delta^{(n+|\gamma|)/q-(n+|\gamma|)/p+\alpha}\big\|f(x)\big\|_{p,\gamma}\end{aligned}$$

or

$$\big\|I^\alpha_{\square_\gamma} f(x)\big\|_{q,\gamma} \leqslant C_{n,\gamma,p}\delta^{(n+|\gamma|)/q-(n+|\gamma|)/p+\alpha}\big\|f(x)\big\|_{p,\gamma}. \tag{3.16}$$

If $(n+|\gamma|)/q-(n+|\gamma|)/p+\alpha > 0$ and $(n+|\gamma|)/q-(n+|\gamma|)/p+\alpha < 0$, then taking the limit in (3.16) as $\delta \to 0$ or $\delta \to \infty$, respectively, we get that

$$\big\|I^\alpha_{\square_\gamma} f(x)\big\|_{q,\gamma} = 0$$

for all functions $f \in L_p^\gamma$, and this is obviously false. This means that inequality (3.16) is possible only if $(n+|\gamma|)/q-(n+|\gamma|)/p+\alpha = 0$, that is, if $q = (n+|\gamma|)p/(n+|\gamma|-\alpha p)$. Thus, the necessity is proved.

Sufficiency. Assume, without loss of generality, that $f(x)\geqslant 0$, $x \in \mathbb{R}_n^+$.

The quasilinear operator A has a weak type $(s,t)_\gamma$ if

$$\mu_\gamma(Af,\lambda) \leqslant \left(\frac{h\|f\|_{s,\gamma}}{\lambda}\right)^t \quad \forall f \in L_p^\gamma$$

(see (2.4)).

If we put $s = 1$ in Lemma 1, then $t = (n+|\gamma|)/(n+|\gamma|-\alpha)$, and inequality (3.2) will be of the form

$$\mu_\gamma\big((1-\delta)^{(n+|\gamma|-\alpha)/2}\big(I^\alpha_{\square_\gamma,\delta} f\big)(x),\lambda\big) \leqslant C_{\alpha,n,\gamma,1}(1-\delta)^{(n+|\gamma|-\alpha)/2}\left(\frac{\|f\|_{1,\gamma}}{\lambda}\right)^{(n+|\gamma|)/(n+|\gamma|-\alpha)}.$$

This means that the quasilinear operator $I^\alpha_{\square_\gamma,\delta}$ (see (3.1)) has a weak type $(1,(n+|\gamma|)/(n+|\gamma|-\alpha))_\gamma$. Similarly, assuming that $s = p_1$ and $t = (n+|\gamma|)p_1/(n+|\gamma|-\alpha p_1)$ in Lemma 1, we obtain that the quasilinear operator $I^\alpha_{\square_\gamma,\delta}$ has a weak type $(p_1,(n+|\gamma|)p_1/(n+|\gamma|-\alpha p_1))_\gamma$, where $1 < p_1 < (n+|\gamma|/\alpha$.

Let us take $p_1 = p(1-\tau)/(1-\tau p)$, $\tau \in (0,1)$, so that $1 < p_1 < (n+|\gamma|)/\alpha$. Then the operator $I^\alpha_{\square_\gamma,\delta}$ has weak types $(p_1,q_1)_\gamma$ and $(p_2,q_2)_\gamma$, where $p_1 = p(1-\tau)/(1-\tau p)$, $q_1 = p(n+|\gamma|)(1-\tau)/((n+|\gamma|)(1-\tau p)-\alpha p(1-\tau))$, and $p_2 = 1$, $q_2 = (n+|\gamma|)/(n+|\gamma|-\alpha$. Therefore, by Theorem 1 the operator $I^\alpha_{\square_\gamma,\delta}$ has a strong type $(p,(n+|\gamma|)p/(n+|\gamma|-\alpha p))_\gamma$, and by (2.8)

$$\big\|\big(I^\alpha_{\square_\gamma,\delta} f\big)(x)\big\|_{q,\gamma} \leqslant M\|f\|_{p,\gamma}, \tag{3.17}$$

where $1 \leqslant p < (n+|\gamma|)/\alpha$, $q = (n+|\gamma|)p/(n+|\gamma|-\alpha p)$, and $n+|\gamma|-2 < \alpha < n+|\gamma|$, where $M = M(\gamma,\tau,\kappa,p_1,p_2,q_1,q_2)$ does not depend on f and $I^\alpha_{\square_\gamma,\delta}$.

Since $f(x) \geqslant 0$, for $0 < \delta_1 \leqslant \delta_2 \leqslant \cdots \leqslant \delta_m \leqslant \cdots < 1$, we have

$$\big(I^\alpha_{\square_\gamma,\delta_1} f\big)(x) \leqslant \big(I^\alpha_{\square_\gamma,\delta_2} f\big)(x) \leqslant \cdots \leqslant \big(I^\alpha_{\square_\gamma,\delta_m} f\big)(x) \leqslant \cdots .$$

Since

$$\lim_{\delta\to 1} \big(I^\alpha_{\square_\gamma,\delta} f\big)(x) = \big(I^\alpha_{\square_\gamma} f\big)(x),$$

taking the limit in (3.17) as $\delta \to 1$, we obtain

$$\big\|\big(I^\alpha_{\square_\gamma} f\big)(x)\big\|_{q,\gamma} \leqslant M\|f\|_{p,\gamma}, \quad 1\leqslant p < \frac{n+|\gamma|}{\alpha},\ n+|\gamma|-2 < \alpha < n+|\gamma|.$$

This proves the desired result. $\square$

4 On absolute convergence of the hyperbolic Riesz B-potential

Let $\overline{p} = (p_1,\ldots,p_n)$, $1 \leqslant p_i \leqslant \infty$, and $L^\gamma_{\overline{p}}(\mathbb{R}^+_n) = L^\gamma_{\overline{p}}$ be the space of measurable functions f on $\mathbb{R}^+_n$ that are even with respect to each variable for which the norm

$$\begin{aligned}\|f\|_{\overline{p},\gamma} &= \big\|\cdots\big\|\|f\|_{p_1,\gamma_1}\big\|_{p_2,\gamma_2}\cdots\big\|_{p_n,\gamma_n}\\ &= \left(\int\limits_0^\infty \left(\cdots\left(\int\limits_0^\infty \left(\int\limits_0^\infty \big|f(x)\big|^{p_1} x_1^{\gamma_1}\,\mathrm{d}x_1\right)^{p_2/p_1} x_2^{\gamma_2}\,\mathrm{d}x_2\right)^{p_3/p_2}\cdots\right)^{p_n/p_{n-1}} x_n^{\gamma_n}\,\mathrm{d}x_n\right)^{1/p_n}\end{aligned}$$

is finite.

Let $x, y \in \mathbb{R}^+_n$. We have the generalized Minkowski inequality

$$\bigg\|\int\limits_{\mathbb{R}^+_n} \varphi(x,y)y^\gamma\,\mathrm{d}y\bigg\|_{\overline{p},\gamma} \leqslant \bigg\|\int\limits_0^\infty y_n^{\gamma_n}\,\mathrm{d}y_n\cdots\bigg\|\int\limits_0^\infty y_2^{\gamma_2}\,\mathrm{d}y_2\bigg\|\int\limits_0^\infty \varphi(x,y)y_1^{\gamma_1}\,\mathrm{d}y_1\bigg\|_{p_1,\gamma_1}\bigg\|_{p_2,\gamma_2}\cdots\bigg\|_{p_n,\gamma_n}. \tag{4.1}$$

Lemma 2. *Let* $\varphi(x) \in L^\gamma_p$, $1 < p < (n+|\gamma|-1)/\alpha$, $n+|\gamma|-2 < \alpha < n+|\gamma|-1$, *and* $1/q = 1/p - \alpha/(n+|\gamma|-1)$. *Then*

$$\big\|I^\alpha_{\square_\gamma}\varphi\big\|_{\overline{q},\gamma} \leqslant c_{n,\gamma,p}\|\varphi\|_{p,\gamma}, \quad \overline{q} = (p,\underbrace{q,\ldots,q}_{n-1}).$$

Proof. We derive

$$\begin{aligned}\big\|I^\alpha_{\square_\gamma}\varphi\big\|_{\overline{q},\gamma} &= \bigg\|\int\limits_{K^+} r^{\alpha-n-|\gamma|}(y)\big(T^y f\big)(x)y^\gamma\,\mathrm{d}y\bigg\|_{\overline{q},\gamma}\\ &= \Bigg\|\Bigg\|\int\limits_0^{+\infty} y_n^{\gamma_n}\,\mathrm{d}y_n \int\limits_{y_1^2-y_2^2-\cdots-y_{n-1}^2\geqslant y_n^2} r^{\alpha-n-|\gamma|}(y)\big(T^y f\big)(x)(y')^{\gamma'}\,\mathrm{d}y'\Bigg\|_{\overline{q}\,'}\Bigg\|_{q,\gamma_n},\end{aligned}$$

where $y=(y_1,\ldots,y_n)\in\mathbb{R}^+_n$, $y'=(y_1,\ldots,y_{n-1})$, $\overline{q}'=(p,\underbrace{q,\ldots,q}_{n-2})$, and $(y')^{\gamma'}\,\mathrm{d}y'=y_1^{\gamma_1}\cdots y_{n-1}^{\gamma_{n-1}}\mathrm{d}y_1\cdots\mathrm{d}y_{n-1}$.

Next we apply inequality (4.1):

$$\left\|I^\alpha_{\square_\gamma}\varphi\right\|_{\overline{q},\gamma} \leqslant \left\| \int\limits_0^{+\infty} y_n^{\gamma_n}\,\mathrm{d}y_n \left\| \int\limits_{y_1^2-y_2^2-\cdots-y_{n-1}^2\geqslant y_n^2} r^{\alpha-n-|\gamma|}(y)\big(T^y f\big)(x)(y')^{\gamma'}\,\mathrm{d}y' \right\|_{\overline{q}\,'} \right\|_{q,\gamma_n}.$$

Let us consider the expression

$$\int\limits_{y_1^2-y_2^2-\cdots-y_{n-1}^2\geqslant y_n^2} r^{\alpha-n-|\gamma|}(y)\big(T^y f\big)(x)(y')^{\gamma'}\,\mathrm{d}y' = \int\limits_{\mathbb{R}^+_{n-1}} \widetilde{r}^{\,\alpha-n-|\gamma|}(y',y_n)\big(T^{y'}\big(T^{y_n}f\big)\big)(x',x_n)(y')^{\gamma'}\,\mathrm{d}y',$$

where

$$\widetilde{r}^{\,\alpha-n-|\gamma|}(y) = \widetilde{r}^{\,\alpha-n-|\gamma|}(y',y_n) = \begin{cases} r^{\alpha-n-|\gamma|}(y), & y'\in\mathbb{R}^+_{n-1},\ y_1^2-y_2^2-\cdots-y_{n-1}^2\geqslant y_n^2,\\ 0, & y'\in\mathbb{R}^+_{n-1},\ y_1^2-y_2^2-\cdots-y_{n-1}^2< y_n^2. \end{cases}$$

Then

$$\int\limits_{y_1^2-y_2^2-\cdots-y_{n-1}^2\geqslant y_n^2} r^{\alpha-n-|\gamma|}(y)\big(T^y f\big)(x)(y')^{\gamma'}\,\mathrm{d}y' = \big(T^{y_n}f(\cdot,y_n) * \widetilde{r}^{\,\alpha-n-|\gamma|}(\cdot,y_n)\big)_{\gamma'}.$$

Applying the Hausdorf–Young inequality (2.6) to this generalized convolution, we get

$$\big\|\big(T^{y_n}f(\cdot,y_n) * \widetilde{r}^{\,\alpha-n-|\gamma|}(\cdot,y_n)\big)_{\gamma'}\big\|_{\overline{q}\,'} \leqslant \big\|T^{y_n}f(\cdot,y_n)\big\|_p \big\|\widetilde{r}^{\,\alpha-n-|\gamma|}(\cdot,y_n)_{\gamma'}\big\|_{\overline{s}},$$

where $\overline{s} = (1,\underbrace{s,\dots,s}_{n-2})$, $1/s = 1-1/p+1/q$.

Lets estimate the norm $\|\widetilde{r}^{\,\alpha-n-|\gamma|}(\cdot,y_n))_{\gamma'}\|_{\overline{s}}$. First, we have

$$\big\|\widetilde{r}^{\,\alpha-n-|\gamma|}(\cdot,y_n)_{\gamma'}\big\|_{\overline{s}} = \big\|\big\|\widetilde{r}^{\,\alpha-n-|\gamma|}(\cdot,y_n)\big\|_{1,\gamma_1}\big\|_{s,\gamma_2,\dots,\gamma_{n-1}}.$$

Further, it follows that

$$\big\|\widetilde{r}^{\,\alpha-n-|\gamma|}(\cdot,y_n)\big\|_{1,\gamma_1} = \int\limits_{K_+} \big(y_1^2-y_2^2-\cdots-y_n^2\big)^{(\alpha-n-|\gamma|/2} y_1^{\gamma_1}\,\mathrm{d}y_1.$$

Since

$$\alpha = \left(\frac{1}{p}-\frac{1}{q}\right)\big(n+|\gamma|-1\big) \quad\text{and}\quad \frac{1}{s} = 1-\frac{1}{p}+\frac{1}{q},$$

we have

$$\begin{aligned} \alpha-n-|\gamma| = \alpha-n-|\gamma|+1-1 &= \left(\frac{1}{p}-\frac{1}{q}\right)\big(n+|\gamma|-1\big) - \big(n+|\gamma|-1\big) - 1 \\ &= -\big(n+|\gamma|-1\big)\left(1-\frac{1}{p}+\frac{1}{q}\right) - 1 = -\frac{n+|\gamma|-1}{s} - 1 \end{aligned}$$

and

$$\left\|\widetilde{r}^{\,\alpha-n-|\gamma|}(\cdot,y_n)\right\|_{1,\gamma_1}=\int\limits_{y_1^2-y_2^2-\cdots-y_{n-1}^2\geqslant y_n^2}\left(y_1^2-|y'|^2\right)^{-(n+|\gamma|-1)/(2s)-1/2}y_1^{\gamma_1}\,\mathrm{d}y_1.$$

By the change of variable $y_1=\rho|y'|$ we derive

$$\begin{aligned}
&\left\|\widetilde{r}^{\,\alpha-n-|\gamma|}(\cdot,y_n)\right\|_{1,\gamma_1}\\
&=|y'|^{-(n+|\gamma|-1)/s+\gamma_1}\int\limits_1^{+\infty}\left(\rho^2-1\right)^{-(n+|\gamma|-1)/(2s)-1/2}\rho^{\gamma_1}\,\mathrm{d}\rho\\
&=|y'|^{-(n+|\gamma|-1)/s+\gamma_1}\int\limits_1^{+\infty}(\rho-1)^{-(n+|\gamma|-1)/(2s)-1/2}\rho^{(\gamma_1-1)/2}\,\mathrm{d}\rho\\
&=|y'|^{-(n+|\gamma|-1)/s+\gamma_1}\int\limits_0^{1}(1-\rho)^{-(n+|\gamma|-1)/(2s)-1/2}\rho^{(n+|\gamma|-1-\gamma_1)/(2s)-1}\,\mathrm{d}\rho\\
&=|y'|^{-(n+|\gamma|-1)/s+\gamma_1}B\left(-\frac{n+|\gamma|-1}{2s}+\frac{1}{2},\frac{n+|\gamma|-1-\gamma_1}{2s}\right)\\
&=C_1|y'|^{-(n+|\gamma|-1)/s+\gamma_1},
\end{aligned}$$

where $-(n+|\gamma|-1)/(2s)+1/2>0$ and $(n+|\gamma|-1-\gamma_1)/(2s)>0$.

This completes the proof of Lemma 2. $\square$

By (3.11) there is a unique extension of $I^{\alpha}_{\square_\gamma}$ to all L_p^{γ}, $1<p<(n+|\gamma|)/\alpha$, preserving the boundedness. It follows that this extension is introduced by the integral (1.2) from its absolute convergence, and the absolute convergence of (1.2) is a consequence of Lemma 2 when $n+|\gamma|-2<\alpha<n+|\gamma|-1$.

References

1. J. Bergh and J. Löfström, *Interpolation Spaces: An Introduction*, Springer-Verlag, Berlin, Heidelberg, New York, 1976.
2. O.V. Besov, V.P. Il'in, and S.M. Nikol'skii, *Integral Representation of Functions and Imbedding Theorems, Vol. I*, V.H. Winston, Washington, 1978.
3. I.A. Kipriyanov, *Singular Elliptic Boundary Value Problems*, Nauka, Moscow, 1997 (in Russian).
4. B.M. Levitan, Expansion in Fourier series and integrals with Bessel functions, *Usp. Mat. Nauk*, **6**(2(42)):102–143, 1951 (in Russian).
5. L.N. Lyakhov, Inversion of Riesz B-potentials, *Dokl. Akad. Nauk SSSR*, **321**(3):466–469, 1991 (in Russian).
6. L.N. Lyakhov, Spaces of Riesz B-potentials, *Dokl. Akad. Nauk SSSR*, **334**(3):278–280, 1994 (in Russian).
7. L.N. Lyakhov, Symbol of the integral operator of Riesz B-potential with single characteristic, *Dokl. Akad. Nauk*, **351**(2):164–168, 1996 (in Russian).
8. L.N. Lyakhov and E.L. Shishkina, Generalized Riesz B-potentials of the mixed type, *Dokl. Math.*, **73**(1):42–45, 2006.

9. L.N. Lyakhov and E.L. Shishkina, Inversion of general Riesz B-potentials, in M.V. Dubatovskaya and S.V. Rogosin (Eds.), *The 7th International Workshop on Analytical Methods of Analysis and Differential Equations, Vol. 10, Minsk, Belarus, September 10–15, 2012*, Cambridge Scientific Publishers, Cottenham, 2013, pp. 115–126.

10. L.N. Lyakhov and E.L. Shishkina, Marcinkiewicz interpolation theorem for weighted Lebesgue classes, *Vestn. Voronezh. Gos. Univ., Ser. Fiz. Mat.*, **10**(1):159–169, 2015 (in Russian).

11. V.A. Nogin and E.V. Sukhinin, Inversion and characterization of hyperbolic potentials in L_p-spaces, *Dokl. Akad. Nauk, Ross. Akad. Nauk*, **329**(5):550–552, 1993 (in Russian).

12. M. Riesz, Intégrale de Riemann–Liouville et solution invariantive du problème de Cauchy pour l'équation de sondes, in *Comptes Rendus du Congrès International des Mathematiciens, Oslo 1936, Vol. II*, A.W. Brøggers Boktrykkeri A/S, Oslo, 1937, pp. 44–45.

13. M. Riesz, L'intégrale de Riemann–Liouville et le probléme de Cauchy, *Acta Math.*, **81**:1–223, 1949.

14. S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon & Breach, Yverdon, 1993.

15. M.Z. Sarikaya, H. Yildirim, and Ö. Akin, On generalized Riesz type potential with Lorentz distance, *Lobachevskii J. Math.*, **29**(1):32–39, 2008.

16. E.L. Shishkina, Boundedness of potential operators with hyperbolic distance, in *Abstracts of Reports of the 8th International Workshop AMADE-2015, Minsk, Belarus, September 14–19, 2015*, Institute of Mathematics, NASB, 2015, p. 90.

17. Ya.I. Zhitomirskii, Cauchy's problem for systems of linear partial differential equations with differential operators of Bessel type, *Mat. Sb., Nov. Ser.*, **36(78)**(2):299–310, 1955.