DOI 10.1007/s10986-016-9307-2
Lithuanian Mathematical Journal, Vol. 56, No. 1, January, 2016, pp. 107–113

A-Darboux functions*

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Received August 24, 2014; revised April 10, 2015

Abstract. We introduce and investigate the class of A-Darboux functions, namely, the class of functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $a, b \in \mathbb{R}$ with $a < b$ and each y between $f(a)$ and $f(b)$, there is a point $x_0 \in (a, b) \cap A$ (where A is a nonempty fixed subset of $\mathbb{R}$) such that $f(x_0) = y$. Furthermore, we generalize the notion of the A-Darboux property for functions mapping a topological space into a topological space.

MSC: primary 26A21, 54C30; secondary 26A15, 54C08

Keywords: Darboux property, A-Darboux property, strong Świątkowski property, local characterization

1 Preliminaries

By $\mathbb{R}$ and $\mathbb{N}$ we denote the real line and the set of positive integers, respectively. The symbol $\mathrm{I}(a, b)$ denotes the open interval with endpoints a and b. For a set A, we use the symbols $\operatorname{int} A$, $\operatorname{cl} A$, and $\operatorname{card} A$ to denote the interior, closure, and cardinality of A, respectively. The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$. We say that a subset A of $\mathbb{R}$ is $\mathfrak{c}$-dense in $\mathbb{R}$ if $\operatorname{card}(I \cap A) = \mathfrak{c}$ for each open interval $I \subset \mathbb{R}$. Let $x \in \mathbb{R}$. If $x_n \to x$ and $x_n < x_{n+1} < x$ for all $n \in \mathbb{N}$, then we write $x_n \nearrow x$ or $x \nwarrow x_n$. Similarly, if $x_n \to x$ and $x_n > x_{n+1} > x$ for all $n \in \mathbb{N}$, then we write $x_n \searrow x$ or $x \swarrow x_n$.

Let f be a function. The symbols $\mathcal{C}(f)$ and $\mathcal{C}^+(f)$ stand for the sets of points of continuity and right-hand continuity of f, respectively. The symbol $L^+(f, x)$ denotes the cluster set from the right of the function f at the point $x \in \mathbb{R}$[1].

Now for real-valued function defined on $\mathbb{R}$, we introduce the notion of A-Darboux function.

DEFINITION 1. Let $A \subset \mathbb{R}$ be a nonempty set. We say that a function $f : \mathbb{R} \to \mathbb{R}$ has the *A-Darboux property* ($f \in \mathcal{D}_A$) if whenever $a, b \in \mathbb{R}$, $a < b$, and $y \in \mathrm{I}(f(a), f(b))$, there is an $x_0 \in (a, b) \cap A$ such that $f(x_0) = y$.

Sometimes, instead of saying that the function f has the A-Darboux property, we will say that f is an A-Darboux function. We say that $f : \mathbb{R} \to \mathbb{R}$ has the *Darboux property*[2] ($f \in \mathcal{D}$) if Definition 1 holds for $A = \mathbb{R}$. We say that $f : \mathbb{R} \to \mathbb{R}$ has the *strong Świątkowski property* [10] if Definition 1 holds for $A = \mathcal{C}(f)$.

* Supported by Kazimierz Wielki University.

[1] $y \in L^+(f, x)$ iff there is a sequence $(x_n) \subset \mathbb{R}$ such that $x_n \searrow x$ and $f(x_n) \to y$.

[2] If X and Y are topological spaces, then $f : X \to Y$ has the *Darboux property* iff the set $f(S)$ is connected in Y for every connected set $S \subset X$.

2 Introduction

A function $f : \mathbb{R} \to \mathbb{R}$ has the intermediate value property if on each interval $(a, b) \subset \mathbb{R}$, it assumes every real value between $f(a)$ and $f(b)$. In the nineteenth century, some mathematicians believed that this property is equivalent to continuity. In 1875, Darboux showed that this is not true. He proved that every derivative has the intermediate value property and constructed a function with derivative discontinuous on the set of rational numbers [3]. For this reason, the intermediate value property is called the Darboux property, and a function having the intermediate value property is called a Darboux function.

The Darboux property has been studied extensively and in various contexts. In 1995, Maliszewski [10] defined a condition more special than the Darboux property, which was called the strong Świątkowski property. The family of strong Świątkowski functions was examined, among other things, by Maliszewski [10, 11], Kucner and Pawlak [8], and Szczuka [12, 13, 14]. Recently, Grande [5] and Ivanova [6] considered some modifications of strong Świątkowski property changing the continuity with the approximate continuity and with the $\mathcal{I}$-approximate continuity, respectively. In 2014, Ivanova and Wagner-Bojakowska [7, Defs. 5, 7] generalized these definitions replacing continuity with $\mathcal{A}$-continuity. We however notice that the generalization of the Darboux property introduced in [7] boils down to a generalization of the notion of continuity.

In this paper, we introduce the notion of A-Darboux property (Definition 1) for a nonempty fixed set $A \subset \mathbb{R}$ that does not depend on the function f, present a local characterization of A-Darboux functions, and examine some properties concerning such functions. Moreover, in the last section, we generalize the notion of A-Darboux property for functions mapping a topological space into a topological space. Note that almost all results obtained in this paper (with the exception of the results of Section 4) may be also applied to the case where the set A depends on the function f.

3 Local characterization of *A*-Darboux functions

In this section, we present a local characterization of functions with A-Darboux property. We start with a generalization of the definition introduced by Kucner and Pawlak [8, Def. 1].

DEFINITION 2. Let $A \subset \mathbb{R}$ be a nonempty set. We say that a point $x \in \mathbb{R}$ *A-cuts a function* $f : \mathbb{R} \to \mathbb{R}$ if there is $\delta > 0$ such that

$$\emptyset \neq f\big((x-\delta, x) \cap A\big) \subset \big(-\infty, f(x)\big) \quad \text{and} \quad \emptyset \neq f\big((x, x+\delta) \cap A\big) \subset \big(f(x), \infty\big)$$

or

$$\emptyset \neq f\big((x-\delta, x) \cap A\big) \subset \big(f(x), \infty\big) \quad \text{and} \quad \emptyset \neq f\big((x, x+\delta) \cap A\big) \subset \big(-\infty, f(x)\big).$$

Next, we recall a local characterization of Darboux functions (see [1, 9]).

DEFINITION 3. The function $f : \mathbb{R} \to \mathbb{R}$ has the *right-hand Darboux property at a point* $x \in \mathbb{R}$ (briefly $x \in \mathcal{D}^+(f)$) if

$$f^{-1}(\beta) \cap (x, x+\delta) \neq \emptyset \quad \text{for all } \alpha \in L^+(f, x) \setminus \big\{f(x)\big\},\ \beta \in \mathrm{I}\big(f(x), \alpha\big), \text{ and } \delta > 0. \tag{3.1}$$

The left-hand Darboux property of f at a point x is defined analogously (briefly $x \in \mathcal{D}^-(f)$).

DEFINITION 4. The function $f : \mathbb{R} \to \mathbb{R}$ has *the Darboux property at a point* $x \in \mathbb{R}$ (briefly $x \in \mathcal{D}(f)$) if $x \in \mathcal{D}^-(f) \cap \mathcal{D}^+(f)$.

Moreover, the following equivalence is true (see, e.g., [1, Thm. 5.1]).

Theorem 1. *A function $f : \mathbb{R} \to \mathbb{R}$ has the Darboux property if and only if f has the Darboux property at every point $x \in \mathbb{R}$.*

In 2002, Kucner and Pawlak [8, Def. 2] introduced the following local characterization of the strong Świątkowski property.

DEFINITION 5. The function $f : \mathbb{R} \to \mathbb{R}$ has the *right-hand strong Świątkowski property at a point* $x \in \mathbb{R}$ if $x \in \mathcal{C}^+(f)$ or the following conditions are satisfied:

1. $f^{-1}(\beta) \cap (x, x+\delta) \cap \mathcal{C}(f) \neq \emptyset$ for all $\alpha \in L^+(f,x) \setminus \{f(x)\}$, $\beta \in \mathrm{I}(f(x), \alpha)$, and $\delta > 0$.
2. For each $\alpha \in \mathbb{R}$, if there are sequences $(x_n), (y_n) \subset \mathbb{R}$ such that $x_n \searrow x \swarrow y_n$ and $f(x_n) \searrow \alpha \nwarrow f(y_n)$, then $f^{-1}(\alpha) \cap (x, x+\delta) \cap \mathcal{C}(f) \neq \emptyset$ for all $\delta > 0$.

The left-hand strong Świątkowski property at a point x is defined analogously (see [8, Def. 3]).

DEFINITION 6. (See [8, Def. 4].) A function $f : \mathbb{R} \to \mathbb{R}$ has the *strong Świątkowski property at a point* $x \in \mathbb{R}$ if f has simultaneously right-hand and left-hand strong Świątkowski properties at a point x, and if the point x $\mathcal{C}(f)$-cuts the function f, then $x \in \mathcal{C}(f)$.

Moreover, Kucner and Pawlak proved the following equivalence.

Theorem 2. (*See* [8, *Thm.* 12].) *A function $f : \mathbb{R} \to \mathbb{R}$ is strong Świątkowski if and only if f has the strong Świątkowski property at every point $x \in \mathbb{R}$.*

Now we present similar results for A-Darboux functions.

DEFINITION 7. Let $A \subset \mathbb{R}$ be a nonempty set. We say that a function $f : \mathbb{R} \to \mathbb{R}$ has the *right-hand A-Darboux property at a point* $x \in \mathbb{R}$ (briefly $x \in \mathcal{D}_A^+(f)$) if condition (3.1) from Definition 3 holds and if $A \neq \mathbb{R}$, then the following condition is satisfied:

($*$) for every $\alpha \in \mathbb{R}$, if there are sequences $(x_n), (y_n) \subset \mathbb{R}$ such that $x_n \searrow x \swarrow y_n$ and $f(x_n) \searrow \alpha \nwarrow f(y_n)$, then $f^{-1}(\alpha) \cap (x, x+\delta) \cap A \neq \emptyset$ for all $\delta > 0$.

We define the left-hand A-Darboux point x of the function f analogously (briefly $x \in \mathcal{D}_A^-(f)$).

DEFINITION 8. Let $A \subset \mathbb{R}$ be a nonempty set. We say that a function $f : \mathbb{R} \to \mathbb{R}$ has the *A-Darboux property at a point* $x \in \mathbb{R}$ (briefly $x \in \mathcal{D}_A(f)$) if $x \in \mathcal{D}_A^-(f) \cap \mathcal{D}_A^+(f)$ and if the point $x \in \mathbb{R}$ A-cuts the function f, then $x \in A$.

Remark 1. Let $f : \mathbb{R} \to \mathbb{R}$ and $A = \mathcal{C}(f)$. In this case, Definition 7 is not equivalent to Definition 5, whence Definition 8 is not equivalent to Definition 6. Indeed, according to Definition 5, if x is a right-hand continuity point of the function f and condition ($*$) of Definition 7 does not hold, then f has the right-hand strong Świątkowski property at x in the sense of Kucner and Pawlak, and f does not have the right-hand A-Darboux property at this point. It is caused by the fact that, in the general case, the continuity may not be correlated with the set A.

Remark 2. Assume that A is a nonempty fixed subset of $\mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$. Then $\mathcal{D}_A(f) \subset \mathcal{D}(f)$.

Theorem 3. *Let $A \subset \mathbb{R}$ be a nonempty set. The function $f : \mathbb{R} \to \mathbb{R}$ has the A-Darboux property if and only if f has the A-Darboux property at every point $x \in \mathbb{R}$.*

Proof. First, assume that the function f has the A-Darboux property. Fix a point $x \in \mathbb{R}$. We will show that $x \in \mathcal{D}_A(f)$. Since $\mathcal{D}_A \subset \mathcal{D}$, we clearly have

$$x \in \mathcal{D}^-(f) \cap \mathcal{D}^+(f). \tag{3.2}$$

Now fix $\alpha \in \mathbb{R}$ and $\delta > 0$. Assume that there are sequences $(x_n), (y_n) \subset \mathbb{R}$ with $x_n \searrow x \swarrow y_n$ and $f(x_n) \searrow \alpha \nwarrow f(y_n)$. Then there are $a, b \in (x, x+\delta)$ such that $f(a) < \alpha$ and $f(b) > \alpha$. Since $f \in \mathcal{D}_A$,

$f(c) = \alpha$ for some $c \in (a,b) \cap A \subset (x, x+\delta) \cap A$. So, $f^{-1}(\alpha) \cap (x, x+\delta) \cap A \neq \emptyset$. Hence, by condition (3.2) we obtain that $x \in \mathcal{D}_A^+(f)$. In a similar way, we can show that $x \in \mathcal{D}_A^-(f)$.

Further assume that the point x A-cuts the function f. By Definition 2 there are $a \in (x-\delta, x)$ and $b \in (x, x+\delta)$ such that $f(x) \in \mathrm{I}\big(f(a), f(b)\big)$. So, the assumption $f \in \mathcal{D}_A$ implies that $f(c) = f(x)$ for some $c \in (a,b) \cap A \subset (x-\delta, x+\delta) \cap A$. However, $f(z) \neq f(x)$ for each $z \in (x-\delta, x+\delta) \cap (A \setminus \{x\})$, whence $x = c \in A$. Consequently, $x \in \mathcal{D}_A(f)$. This completes the first part of the proof.

Now assume that the function f has the A-Darboux property at every point $x \in \mathbb{R}$. By Remark 2, $\mathcal{D}_A(f) \subset \mathcal{D}(f)$, whence the function f has the Darboux property. We will show that $f \in \mathcal{D}_A$.

Let $a < b$ and $\alpha \in \mathrm{I}(f(a), f(b))$. We can assume that $f(a) < \alpha < f(b)$. (The case $f(a) > \alpha > f(b)$ is analogous.) Define

$$x = \inf\big\{z \in [a,b]\colon\ f(z) > \alpha\big\}. \tag{3.3}$$

We consider two cases.

Case 1. $f(x) < \alpha$. Since $f \in \mathcal{D}$ and condition (3.3) holds, there are sequences $(x_n), (y_n) \subset (x, b)$ such that $x_n \searrow x \swarrow y_n$ and $f(x_n) \searrow \alpha \nwarrow f(y_n)$. So, by condition $(*)$ of Definition 7, $f^{-1}(\alpha) \cap (x,b) \cap A \neq \emptyset$. Hence, $f(c) = \alpha$ for some $c \in (x,b) \cap A \subset (a,b) \cap A$.

Case 2. $f(x) = \alpha$. Then $x > a$. If the point x A-cuts the function f, then $x \in A$. Hence, there is $x \in (a,b) \cap A$ such that $f(x) = \alpha$. So, we can assume that x does not A-cut the function f.

If $f^{-1}(\alpha) \cap (a,x) \cap A \neq \emptyset$, then clearly $f(c) = \alpha$ for some $c \in (a,x) \cap A \subset (a,b) \cap A$. In the other case, $f^{-1}(\alpha) \cap (a,x) \cap A = \emptyset$ and x does not A-cut f. So, by condition (3.3) and since $f \in \mathcal{D}$, there are sequences $(x_n), (y_n) \subset (x,b)$ such that $x_n \searrow x \swarrow y_n$ and $f(x_n) \searrow \alpha \nwarrow f(y_n)$. Therefore, by condition $(*)$ of Definition 7, $f^{-1}(\alpha) \cap (x,b) \cap A \neq \emptyset$. Hence, $f(c) = \alpha$ for some $c \in (x,b) \cap A \subset (a,b) \cap A$. □

The following theorem is an immediate consequence of Theorems 3 and 2.

Theorem 4. *Let $f : \mathbb{R} \to \mathbb{R}$. If $A = \mathcal{C}(f)$, then the function f has the A-Darboux property at every point $x \in \mathbb{R}$ if and only if f has the strong Świątkowski property in the sense of Kucner and Pawlak at every point $x \in \mathbb{R}$.*

4 Properties of A-Darboux functions

In 1992, Grande defined the Darboux property for restricted functions as follows: if $A \subset \mathbb{R}$ is a nonempty set, then we say that a function $f : A \to \mathbb{R}$ has the Darboux property whenever $f(I \cap A)$ is a connected set for every interval $I \subset \mathbb{R}$ (see [4]). We can extend Grande's notion for real-valued functions on $\mathbb{R}$.

DEFINITION 9. Let $A \subset \mathbb{R}$ be a nonempty set. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is *restrictively A-Darboux* ($f \in \mathcal{D}_A^r$) if for every interval $I \subset \mathbb{R}$, the set $f(A \cap I)$ is connected.

Remark 3. Let $A \subset \mathbb{R}$ be a nonempty set, and $f : \mathbb{R} \to \mathbb{R}$. Then $f \in \mathcal{D}_A^r$ if and only if $f \restriction A$ is Darboux in the sense of Grande.

Now we will compare A-Darboux functions and restrictively A-Darboux functions. Before we will go to the next results, observe that there are restrictively A-Darboux functions without the Darboux property.

Proposition 1. *Let $A \subset \mathbb{R}$ be a fixed nonempty set. Then $\mathcal{D}_A \subset \mathcal{D} \cap \mathcal{D}_A^r$. Moreover, if $\mathbb{R} \setminus A$ is of size $\mathfrak{c}$, then $\mathcal{D}_A \neq \mathcal{D} \cap \mathcal{D}_A^r$.*

Proof. First, we will show that $\mathcal{D}_A \subset \mathcal{D} \cap \mathcal{D}_A^r$. Fix an interval $I \subset \mathbb{R}$ and assume that $f \in \mathcal{D}_A$. Then clearly $f \in \mathcal{D}$. We must show that $f \in \mathcal{D}_A^r$. Since the function f has the Darboux property, the set $f(I)$ is connected. If $f(I)$ is a singleton, then $f(I \cap A)$ is connected. If $f(I)$ is not a singleton, then $f(I) = J$, where J is a nonempty interval (maybe unbounded). So,

$$\operatorname{int} J \subset f(I) \subset \operatorname{cl} J. \tag{4.1}$$

Let $y \in \operatorname{int} J$. There are $y_1, y_2 \in \operatorname{int} J$ with $y_1 < y < y_2$. Hence, by (4.1) there are $x_1, x_2 \in I$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Taking into account that $f \in \mathcal{D}_A$, we have $f(x) = y$ for some $x \in \mathrm{I}(x_1, x_2) \cap A \subset I \cap A$. Consequently, $y \in f(I \cap A)$. Hence, using (4.1), we obtain that

$$\operatorname{int} J \subset f(I \cap A) \subset f(I) \subset \operatorname{cl} J,$$

which proves that $f(I \cap A)$ is connected. So, $f \in \mathcal{D}^r_A$.

Now we will show that $\mathcal{D}_A \neq \mathcal{D} \cap \mathcal{D}^r_A$. Let $\operatorname{card}(\mathbb{R} \setminus A) = \mathfrak{c}$. There is a Darboux function $f : \mathbb{R} \to \mathbb{R}$ such that $f \neq 0$ and $A \subset f^{-1}(0)$. Observe that f is also restrictively A-Darboux. However, $f \notin \mathcal{D}_A$. Indeed, we can choose $a < b$ such that $a \in A$ and $b \in \mathbb{R} \setminus f^{-1}(0)$. By the definition of f it follows that $f(a) = 0$ and $f(b) \neq 0$. Let $y = f(b)/2$. Then $y \in (0, f(b)) = (f(a), f(b))$ and $f(x_0) \neq y$ for each $x_0 \in (a, b) \cap A$. So, $f \in (\mathcal{D} \cap \mathcal{D}^r_A) \setminus \mathcal{D}_A$. □

Proposition 2. *Let $A \subset \mathbb{R}$ be a fixed nonempty set. Then $\mathcal{D}_A = \mathcal{D} \cap \mathcal{D}^r_A$ if and only if $\operatorname{card}(\mathbb{R} \setminus A) < \mathfrak{c}$.*

Proof. By Proposition 1, if $\mathcal{D}_A = \mathcal{D} \cap \mathcal{D}^r_A$, then $\operatorname{card}(\mathbb{R} \setminus A) < \mathfrak{c}$. So, assume that the set $\mathbb{R} \setminus A$ has cardinality less then $\mathfrak{c}$. Since $\mathcal{D}_A \subset \mathcal{D} \cap \mathcal{D}^r_A$, it suffices to show the opposite inclusion. Let $f \in \mathcal{D} \cap \mathcal{D}^r_A$. Fix $a, b \in \mathbb{R}$ such that $a < b$ and $y \in \mathrm{I}(f(a), f(b))$. Since f has the Darboux property, there is $x \in (a, b)$ with $f(x) = y$, and the sets $(a, x) \cap f^{-1}(\mathrm{I}(f(a), y))$ and $(x, b) \cap f^{-1}(\mathrm{I}(y, f(b)))$ are of size $\mathfrak{c}$. Hence, there are $c \in A \cap (a, x) \cap f^{-1}(\mathrm{I}(f(a), y))$ and $d \in A \cap (x, b) \cap f^{-1}(\mathrm{I}(y, f(b)))$. Observe that the condition $f \in \mathcal{D}^r_A$ implies that the set $f([c, d] \cap A)$ is connected. Moreover, since $y \in \mathrm{I}(f(c), f(d))$, we have $y \in f([c, d] \cap A)$. So, there is $x_0 \in [c, d] \cap A \subset (a, b) \cap A$ such that $f(x_0) = y$, and, consequently, $f \in \mathcal{D}_A$. □

In the next two theorems, we present other interesting properties of A-Darboux functions. Note that if $f \in \mathcal{D}_A$ and there is an interval $I \subset \mathbb{R}$ such that $\operatorname{card}(I \cap A) < \mathfrak{c}$, then the function f is constant on $\operatorname{cl} I$. So, from now on we can assume that A is a $\mathfrak{c}$-dense subset of $\mathbb{R}$.

Theorem 5. *Let A be a fixed $\mathfrak{c}$-dense subset of $\mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$. Then there is a sequence (f_n) of A-Darboux functions such that $f_n : \mathbb{R} \to \mathbb{R}$ for each $n \in \mathbb{N}$ and f is the pointwise limit of (f_n).*

Proof. Since A is $\mathfrak{c}$-dense subset of $\mathbb{R}$, by [2, Lemma 4.1] $A = \bigcup_{n \in \mathbb{N}} A_n$, where all sets A_n are nonempty, pairwise disjoint, and $\mathfrak{c}$-dense in $\mathbb{R}$. For each $n \in \mathbb{N}$, define the function $f_n : \mathbb{R} \to \mathbb{R}$ as follows: $f_n = f$ on $(\mathbb{R} \setminus A) \cup \bigcup_{k=1}^{n} A_k$, and f_n takes every value on $I \cap \bigcup_{k=n+1}^{\infty} A_k$ for each interval $I \subset \mathbb{R}$. Then clearly $f_n \in \mathcal{D}_A$ for each $n \in \mathbb{N}$, and f is the pointwise limit of (f_n). □

Theorem 6. *Let A be a fixed $\mathfrak{c}$-dense subset of $\mathbb{R}$, and let $f : \mathbb{R} \to \mathbb{R}$. Then there are A-Darboux functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $f = g + h$.*

Proof. Take a set $B \subset A$ such that both B and $A \setminus B$ are $\mathfrak{c}$-dense in $\mathbb{R}$. There are functions g_1 and h_1 such that g_1 takes every value on $I \cap B$ and h_1 takes every value on $I \cap (A \setminus B)$ for each interval $I \subset \mathbb{R}$. Now define the functions $g, h : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in \mathbb{R} \setminus A, \\ g_1(x) & \text{if } x \in B, \\ f(x) - h_1(x) & \text{if } x \in A \setminus B, \end{cases}$$

$$h(x)=\begin{cases}0 & \text{if } x\in\mathbb{R}\setminus A,\\ f(x)-g_1(x) & \text{if } x\in B,\\ h_1(x) & \text{if } x\in A\setminus B.\end{cases}$$

Then clearly $f=g+h$, and since $B\subset A$, it is easy to see that $g,h\in\mathcal{D}_A$. □

Remark 4. Theorems 5 and 6 can be applied only if A is a fixed subset of $\mathbb{R}$. If the set A depends on the function, Theorems 5 and 6 cannot be used (see, e.g., [10, Cor. 6] concerning pointwise limits of strong Świątkowski functions and [11, Cor. II.3.4] concerning sums of strong Świątkowski functions).

5 Topological definition of A-Darboux functions

In the natural way, we can generalize the notion of A-Darboux property for functions mapping a topological space into a topological space.

DEFINITION 10. Let X and Y be topological spaces, and $A\subset X$ be a nonempty set. We say that the function $f:X\to Y$ has the A-Darboux property if $\operatorname{int} f(S)=\operatorname{int} f(S\cap A)$ and the set $f(S)$ is connected in Y for every connected set $S\subset X$.

Note that if $A=X$ and the function $f:X\to Y$ has the A-Darboux property, then f is Darboux in the general sense.

Now we will show that Definition 1 is equivalent to Definition 10 for each real-valued function defined on $\mathbb{R}$.

Theorem 7. *Let $A\subset\mathbb{R}$ be a nonempty set, and let $f:\mathbb{R}\to\mathbb{R}$. The following conditions are equivalent*:

1. *For all $a,b\in\mathbb{R}$, $a<b$, and $y\in \mathrm{I}(f(a),f(b))$, there is an $x_0\in(a,b)\cap A$ such that $f(x_0)=y$,*
2. $\operatorname{int} f(S)=\operatorname{int} f(S\cap A)$, *and $f(S)$ is connected in $\mathbb{R}$ for each connected set $S\subset\mathbb{R}$.*

Proof. Fix a connected set $S\subset\mathbb{R}$. Since $f\in\mathcal{D}$, the set $f(S)$ is connected in $\mathbb{R}$. Moreover, since $\operatorname{int} f(S\cap A)\subset \operatorname{int} f(S)$, it suffices to show the opposite inclusion. Let $y\in\operatorname{int} f(S)$. Then there is an open interval (c,d) such that $y\in(c,d)\subset f(S)$. Without loss of generality, we can assume that $f(a)=c$ and $f(b)=d$ for some $a,b\in S$. By Assumption 1, for each $z\in(f(a),f(b))=(c,d)$, there is an $x_0\in\mathrm{I}(a,b)\cap A$ with $f(x_0)=z$. However, S is connected in $\mathbb{R}$, whence if $a,b\in S$, then $\mathrm{I}(a,b)\subset S$. So, $x_0\in S\cap A$. Consequently, $(c,d)\subset f(S\cap A)$, whence $y\in\operatorname{int} f(S\cap A)$. This completes the first part of the proof.

Now let $a<b$ and $y\in\mathrm{I}(f(a),f(b))$. Since the function f maps connected sets onto connected sets, $\mathrm{I}(f(a),f(b))\subset f((a,b))$. Observe that the open interval (a,b) is connected in $\mathbb{R}$. So, by assumption $\operatorname{int} f((a,b))=\operatorname{int} f((a,b)\cap A)$. Consequently,

$$\mathrm{I}\big(f(a),f(b)\big)\subset\operatorname{int} f\big((a,b)\big)=\operatorname{int} f\big((a,b)\cap A\big)\subset f\big((a,b)\cap A\big).$$

Therefore, $y\in f((a,b)\cap A)$, whence $f(x_0)=y$ for some $x_0\in(a,b)\cap A$. □

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