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Weak max-sum equivalence for dependent heavy-tailed random variables

Lina Dindienė [a] **and Remigijus Leipus** [a,b,1]

[a] Faculty of Mathematics and Informatics, Vilnius University, Naugarduko str. 24, LT-03225 Vilnius, Lithuania
[b] Institute of Mathematics and Informatics, Vilnius University, Akademijos str. 4, LT-08663 Vilnius, Lithuania
(e-mail: lina_dindiene@yahoo.com; remigijus.leipus@mif.vu.lt)

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Abstract. We consider real-valued random variables $X_1, \dots, X_n$ with corresponding distributions $F_1, \dots, F_n$ such that $X_1, \dots, X_n$ admit some dependence structure and $n^{-1}(F_1 + \dots + F_n)$ belongs to the class of dominatedly varying-tailed distributions. We establish weak equivalence relations among $\mathbf{P}(S_n > x)$, $\mathbf{P}(\max\{X_1, \dots, X_n\} > x)$, $\mathbf{P}(\max\{S_1, \dots, S_n\} > x)$, and $\sum_{k=1}^n \overline{F_k}(x)$ as $x \to \infty$, where $S_k := X_1 + \dots + X_k$. Some copula-based examples illustrate the results.

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1 Introduction and main result

Let $X_1, \dots, X_n$ be real-valued random variables (r.v.s) with corresponding distributions $F_1, \dots, F_n$. Denote $\overline{F_k}(x) := 1 - F_k(x)$ and $S_k := X_1 + \dots + X_k$ for $k = 1, \dots, n$.

Li and Tang [12] and Yang et al. [18] investigated the (weak) equivalence relations among the quantities $\mathbf{P}(S_n > x)$, $\mathbf{P}(\max\{X_1, \dots, X_n\} > x)$, $\mathbf{P}(\max\{S_1, \dots, S_n\} > x)$, and $\sum_{k=1}^n \overline{F_k}(x)$ as $x \to \infty$, considering, respectively, the independent or pairwise negative dependent (see the definition below) random variables $X_1, \dots, X_n$ together with the condition that their maximum belongs to a specific class of heavy-tailed distributions. In this paper, we consider the class of dominatedly varying-tailed distributions, denoted by $\mathcal{D}$, and dependence structure given further in (1.4), which covers a wide range of negative and some positive dependent structures. Recall that a distribution function (d.f.) $F(x) = 1 - \overline{F}(x)$ belongs to the class $\mathcal{D}$ if $\limsup \overline{F}(xy)/\overline{F}(x) < \infty$ for any (or for some) $0 < y < 1$. It is well known that $F \in \mathcal{D}$ if and only if

$$L_F := \lim_{y \searrow 1} \liminf_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} > 0.$$

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Recall some related dependence structures. Random variables $X_1, \dots, X_n$ are said to be upper extended negatively dependent (UEND) if there exists a positive constant M such that, for all $x_1, \dots, x_n$,

$$\mathbf{P}(X_1 > x_1, \ \dots, \ X_n > x_n) \leqslant M \prod_{i=1}^{n} \mathbf{P}(X_i > x_i); \tag{1.1}$$

they are said to be lower extended negatively dependent (LEND) if there exists a positive constant M such that, for all $x_1, \dots, x_n$,

$$\mathbf{P}(X_1 \leqslant x_1, \ \dots, \ X_n \leqslant x_n) \leqslant M \prod_{i=1}^{n} \mathbf{P}(X_i \leqslant x_i); \tag{1.2}$$

and they are said to be extended negatively dependent (END) if they are both UEND and LEND (see [13]). When $M = 1$ in (1.1) and (1.2), the r.v.s $X_1, \dots, X_n$ are said to be upper negatively dependent (UND) and lower negatively dependent (LND), respectively, and they are said to be negatively dependent (ND) if (1.1) and (1.2) both hold with $M = 1$; see [4, 6, 17].

Random variables $X_1, \dots, X_n$ are called *pairwise upper extended negatively dependent* (pairwise UEND) if

$$\mathbf{P}(X_i > x_i, \ X_j > x_j) \leqslant M \mathbf{P}(X_i > x_i) \mathbf{P}(X_j > x_j) \tag{1.3}$$

for all $x_i, x_j \in \mathbb{R}$, $i \neq j$, $i, j \in \{1, \dots, n\}$, and some $M > 0$. Similarly, the related positive dependence structures can be introduced.

Denote the d.f. $H_n(x) := n^{-1}(F_1(x) + \dots + F_n(x))$ and assume that $\overline{H_n}(x) > 0$ for all x. Introduce the following condition:

$$\sum_{1 \leqslant k < l \leqslant n} \mathbf{P}(X_k > x, \ X_l > x) = o(1) \overline{H_n}(x), \quad x \to \infty, \tag{1.4}$$

or, equivalently,

$$\mathbf{P}(X_k > x, \ X_l > x) = o(1) \overline{H_n}(x) \quad \text{for all } k, l = 1, \dots, n, \ k < l. \tag{1.5}$$

Random variables satisfying (1.4) allow a wide range of dependence structures. In particular, they cover the pairwise ND r.v.s and even some positive dependence structures (see Section 3). They also include dependent r.v.s characterized by the condition

$$\mathbf{P}(X_k > x, \ X_l > x) = o(1) \bigl(\overline{F_k}(x) + \overline{F_l}(x) \bigr) \quad \text{with } 1 \leqslant k < l \leqslant n, \tag{1.6}$$

which, in the case where $X_1, \dots, X_n$ are all nonnegative, generate the pairwise quasi-asymptotically independence structure; see [5]. The dependence structure (1.5) strictly contains the structure (1.6). To see this, take the trivial example $n = 3$, $X_1 = X_2$ with distribution $F_1 = F_2 = F$, and independent r.v. X_3 with distribution F_3 such that $\overline{F}(x) = o(\overline{F_3}(x))$. Then X_1, X_2, X_3 do not satisfy (1.6) but satisfy (1.5). Note that under some stronger dependence conditions, related equivalence results for subexponential r.v.s were established by Geluk and Tang [8] and Jiang et al. [11].

When $X_1, \dots, X_n$ are real-valued and identically distributed r.v.s, the dependence structure (1.5) coincides with the bivariate upper tail independence (BUTI) structure. Note that the BUTI is strictly larger than the UEND structure. To see this, consider two positive r.v.s ξ_1 and ξ_2 with the joint tail probability

$$\mathbf{P}(\xi_1 > x, \ \xi_2 > y) = \frac{1}{(x \vee 1)(y \vee 1)(1 + x + y)}, \quad x \geqslant 0, \ y \geqslant 0.$$

Such a pair (ξ_1, ξ_2) is bivariate upper tail independent but not UEND (see Example 3.1 in [14]).

The goal of the paper is to prove some weak max-sum equivalence relations under condition (1.4) and provide some concrete dependence structures satisfying (1.4).

Let $\overline{G_n}(x) = 1 - G_n(x) := \mathbf{P}(\max\{X_1, \dots, X_n\} > x)$, $S_{(n)} := \max\{S_1, \dots, S_n\}$, and $T_n := X_1^+ + \dots + X_n^+$, where $x^+ := \max\{x, 0\}$. All the limit relationships further hold for x tending to ∞. For two positive functions $a(x)$ and $b(x)$, we write $a(x) \lesssim b(x)$ if $\limsup a(x)/b(x) \leqslant 1$ and $a(x) \gtrsim b(x)$ if $\liminf a(x)/b(x) \geqslant 1$.

The main result of the paper is the following theorem.

Theorem 1. *Let r.v.s $X_1, \dots, X_n$ satisfy condition* (1.4)*. If $H_n \in \mathcal{D}$ (or, equivalently, $G_n \in \mathcal{D}$), then*

$$\mathbf{P}(S_{(n)} > x) \leqslant \mathbf{P}(T_n > x) \lesssim L_{H_n}^{-1} n\overline{H_n}(x). \tag{1.7}$$

If, in addition, $H_n(-x) = o(\overline{H_n}(x))$, then

$$\mathbf{P}(S_{(n)} > x) \geqslant \mathbf{P}(S_n > x) \gtrsim L_{H_n} n\overline{H_n}(x). \tag{1.8}$$

Here,

$$L_{H_n} = L_{G_n} \quad \textit{and} \quad n\overline{H_n}(x) \sim \overline{G_n}(x). \tag{1.9}$$

Remark 1. Yang et al. [18] proved that for pairwise ND r.v.s $X_1, \dots, X_n$ such that $G_n \in \mathcal{D}$, we have

$$\mathbf{P}(S_{(n)} > x) \leqslant \mathbf{P}(T_n > x) \lesssim L_{G_n}^{-1} \overline{G_n}(x) \tag{1.10}$$

and, under the condition $G_n(-x) = o(\overline{G_n}(x))$,

$$\mathbf{P}(S_{(n)} > x) \geqslant \mathbf{P}(S_n > x) \gtrsim L_{G_n} \overline{G_n}(x). \tag{1.11}$$

Since pairwise ND r.v.s satisfy condition (1.4), Theorem 1 generalizes the result in [18]; moreover (see Proposition 1), the constant L_{G_n} in (1.10) and (1.11) can be replaced by L_{H_n}.

Remark 2. Clearly, in the case $F_k \in \mathcal{C} \subset \mathcal{D}$, $k = 1, \dots, n$, where $\mathcal{C}$ denotes the consistently varying-tailed class of distributions, characterized by

$$\lim_{y \nearrow 1} \limsup_{x \to \infty} \frac{\overline{F}(xy)}{\overline{F}(x)} = 1,$$

we have $H_n \in \mathcal{C}$ and thus $L_{H_n} = 1$ in Theorem 1. However, there may exist some more relaxed conditions implying $H_n \in \mathcal{C}$; see, for example, a method in Theorem 1.1 of [11].

In the case of identically distributed random variables, we obtain the following corollary.

Corollary 1. *Let assumptions of Theorem* 1 *hold, and let $X_1, \dots, X_n$ be identically distributed with common distribution F. Then relations* (1.7) *and* (1.8) *hold with $L_{H_n} = L_{G_n} = L_F$ and $\overline{H_n}(x) = \overline{F}(x)$.*

In Section 2, we formulate an auxiliary proposition and prove the main theorem. In Section 3, we illustrate the result using some copula-based dependence structures.

2 Proof of main result

We start with the following useful proposition.

Proposition 1. *Assume that condition* (1.4) *holds. Then $\overline{G_n}(x) \sim n\overline{H_n}(x)$, and therefore $L_{G_n} = L_{H_n}$.*

Proof. We have

$$\overline{G_n}(x) = \mathbf{P}\left(\bigcup_{k=1}^{n}\{X_k > x\}\right) \leqslant \sum_{k=1}^{n}\mathbf{P}(X_k > x). \tag{2.1}$$

On the other hand,

$$\overline{G_n}(x) \geqslant \sum_{k=1}^{n}\mathbf{P}(X_k > x) - \sum_{1\leqslant k<l\leqslant n}\mathbf{P}(X_k > x,\ X_l > x). \tag{2.2}$$

From (1.4) and (2.1)–(2.2) it follows that $\overline{H_n}(x)$ is positive for all x if and only if $\overline{G_n}(x) > 0$ is positive for all x. Then

$$\limsup \frac{\overline{G_n}(x)}{n\overline{H_n}(x)} \leqslant 1$$

and

$$\liminf \frac{\overline{G_n}(x)}{n\overline{H_n}(x)} \geqslant 1 - \limsup \frac{\sum_{1\leqslant k<l\leqslant n}\mathbf{P}(X_k > x, X_l > x)}{n\overline{H_n}(x)} = 1,$$

implying $\overline{G_n}(x) \sim n\overline{H_n}(x)$ and, thus, $L_{G_n} = L_{H_n}$. □

Proof of Theorem 1. Relations (1.9) hold by Proposition 1, implying the equivalence of $H_n \in \mathcal{D}$ and $G_n \in \mathcal{D}$.

We first show the upper bound (1.7). For any $0 < v < 1$ and $x > 0$, write

$$\begin{aligned}\mathbf{P}(T_n > x) &\leqslant \mathbf{P}\left(\bigcup_{k=1}^{n}\{X_k^+ > (1-v)x\}\right) + \mathbf{P}\left(T_n > x,\ \bigcap_{k=1}^{n}\{X_k^+ \leqslant (1-v)x\}\right)\\ &\leqslant n\overline{H_n}\big((1-v)x\big) + \mathbf{P}\left(T_n > x,\ \bigcup_{i=1}^{n}\left\{X_i^+ > \frac{x}{n}\right\},\ \bigcap_{k=1}^{n}\{X_k^+ \leqslant (1-v)x\}\right)\\ &=: I_1(v,x) + I_2(v,x).\end{aligned}$$

We have by $H_n \in \mathcal{D}$ that

$$\lim_{v\searrow 0}\limsup_{x\to\infty}\frac{I_1(v,x)}{L_{H_n}^{-1}n\overline{H_n}(x)} = L_{H_n}\lim_{v\searrow 0}\limsup_{x\to\infty}\frac{\overline{H_n}((1-v)x)}{\overline{H_n}(x)} = 1.$$

As for $I_2(v,x)$, we have

$$\begin{aligned}I_2(v,x) &\leqslant \sum_{i=1}^{n}\mathbf{P}\left(T_n > x,\ X_i^+ > \frac{x}{n},\ \bigcap_{k=1}^{n}\{X_k^+ \leqslant (1-v)x\}\right)\\ &\leqslant \sum_{i=1}^{n}\mathbf{P}\left(T_n - X_i^+ > vx,\ X_i^+ > \frac{x}{n}\right) \leqslant \sum_{i=1}^{n}\mathbf{P}\left(\bigcup_{\substack{j=1\\ j\neq i}}^{n}\left\{X_j^+ > \frac{vx}{n-1}\right\},\ X_i^+ > \frac{x}{n}\right)\\ &\leqslant \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\mathbf{P}\left(X_j^+ > \frac{vx}{n},\ X_i^+ > \frac{vx}{n}\right).\end{aligned}$$

Hence, by (1.4) and the assumption $H_n \in \mathcal{D}$ we obtain

$$\limsup \frac{I_2(v,x)}{L_{H_n}^{-1} n\overline{H_n}(x)} \leqslant L_{H_n} \limsup \frac{\sum_{i\neq j} \mathbf{P}(X_i > vx/n,\ X_j > vx/n)}{n\overline{H_n}(vx/n)} \limsup \frac{\overline{H_n}(vx/n)}{\overline{H_n}(x)} = 0.$$

Therefore,

$$\limsup_{x\to\infty} \frac{\mathbf{P}(T_n > x)}{L_{H_n}^{-1} n\overline{H_n}(x)} \leqslant \lim_{v\searrow 0} \limsup_{x\to\infty} \frac{I_1(v,x)}{L_{H_n}^{-1} n\overline{H_n}(x)} + \lim_{v\searrow 0} \limsup_{x\to\infty} \frac{I_2(v,x)}{L_{H_n}^{-1} n\overline{H_n}(x)} = 1.$$

To obtain the lower bound, note that, for any $v > 0$ and $x > 0$,

$$\begin{aligned} \mathbf{P}(S_n > x) &\geqslant \mathbf{P}\Biggl(S_n > x,\ \bigcup_{k=1}^{n}\{X_k > (1+v)x\}\Biggr) \\ &\geqslant \sum_{k=1}^{n} \mathbf{P}\bigl(S_n > x,\ X_k > (1+v)x\bigr) \\ &\quad - \sum_{1\leqslant i<j\leqslant n} \mathbf{P}\bigl(S_n > x,\ X_i > (1+v)x,\ X_j > (1+v)x\bigr) \\ &=: I_3(v,x) - I_4(v,x). \end{aligned} \tag{2.3}$$

Here, by (1.4),

$$I_4(v,x) \leqslant \sum_{1\leqslant i<j\leqslant n} \mathbf{P}(X_i > x,\ X_j > x) = o\bigl(\overline{H_n}(x)\bigr). \tag{2.4}$$

For $I_3(v,x)$, we have

$$\begin{aligned} I_3(v,x) &\geqslant \sum_{k=1}^{n} \mathbf{P}\bigl(S_n - X_k > -vx,\ X_k > (1+v)x\bigr) \\ &\geqslant \sum_{k=1}^{n} \bigl(\mathbf{P}(S_n - X_k > -vx) + \overline{F_k}\bigl((1+v)x\bigr) - 1\bigr) \\ &= n\overline{H_n}\bigl((1+v)x\bigr) - \sum_{k=1}^{n} \mathbf{P}(S_n - X_k \leqslant -vx) =: I_{31}(v,x) - I_{32}(v,x). \end{aligned} \tag{2.5}$$

Here,

$$\lim_{v\searrow 0} \liminf_{x\to\infty} \frac{I_{31}(v,x)}{L_{H_n} n\overline{H_n}(x)} = 1. \tag{2.6}$$

For the term $I_{32}(v,x)$, since $H_n \in \mathcal{D}$, we have

$$\begin{aligned} I_{32}(v,x) &= \sum_{k=1}^{n} \mathbf{P}\Biggl(\sum_{\substack{i=1\\ i\neq k}}^{n} (-X_i) \geqslant vx\Biggr) \leqslant \sum_{k=1}^{n} \mathbf{P}\Biggl(\bigcup_{\substack{i=1\\ i\neq k}}^{n} \biggl\{-X_i \geqslant \frac{v}{n-1}\, x\biggr\}\Biggr) \\ &\leqslant n^2 H_n\biggl(-\frac{v}{n-1}\, x\biggr) = o(1)\overline{H_n}\biggl(\frac{v}{n-1}\, x\biggr) = o\bigl(\overline{H_n}(x)\bigr). \end{aligned} \tag{2.7}$$

Hence, by (2.3)–(2.7),

$$\liminf_{x\to\infty} \frac{\mathbf{P}(S_n > x)}{L_{H_n} n\overline{H_n}(x)} \geqslant \lim_{v\searrow 0} \liminf_{x\to\infty} \frac{I_{31}(v,x)}{L_{H_n} n\overline{H_n}(x)} - \lim_{v\searrow 0} \limsup_{x\to\infty} \frac{I_{32}(v,x)}{L_{H_n} n\overline{H_n}(x)} - \lim_{v\searrow 0} \limsup_{x\to\infty} \frac{I_4(v,x)}{L_{H_n} n\overline{H_n}(x)} = 1.$$

The proof is completed. □

3 Modeling negative dependence structures with copulas

In this section, we discuss some copula-based examples of dependence structures satisfying (1.4). It is clear that any pairwise ND or pairwise UEND r.v.s $X_1, \dots, X_n$ satisfy (1.4). Moreover, some positive dependent structures can be constructed as well.

3.1 Generalized FGM copula

Consider the class of generalized Farlie–Gumbel–Morgenstern (GFGM) copulas given by the formula

$$Q^{\mathrm{GFGM}}(u_1, \dots, u_n) = \prod_{i=1}^n u_i \biggl(1 + \sum_{1\leqslant k<l\leqslant n} \theta_{kl}\bigl(1-u_k^\alpha\bigr)\bigl(1-u_l^\alpha\bigr)\biggr)^m \tag{3.1}$$

with $\alpha > 0$, $m \in \{0, 1, 2, \dots\}$, and θ_{kl} taking values from a corresponding admissible region. Obviously, if θ_{kl} are all nonpositive and take values from a corresponding admissible region, then $Q^{\mathrm{GFGM}}(u_1, \dots, u_n) \leqslant u_1 \dots u_n$, that is, we obtain the LND structure.

The following particular cases of (3.1) are well known:

- If $m = 0$, then we get the independence copula.
- If $m = 1$ and $\alpha = 1$, then we get the classical multivariate FGM copula

$$Q^{\mathrm{FGM}}(u_1, \dots, u_n) = \prod_{i=1}^n u_i \biggl(1 + \sum_{1\leqslant k<l\leqslant n} \theta_{kl}(1-u_k)(1-u_l)\biggr),$$

  which was introduced in [7, 9] and [16] in the case $n = 2$. This copula was widely investigated and used in practice. The well-known limitation to FGM copula is that it does not allow the modeling of high dependencies. For example, if $n = 2$, then the admissible region for the parameter θ_{12} is $[-1, 1]$, and the correlation ρ between the corresponding uniformly distributed random variables is $\rho = \theta_{12}/3$; thus, the range for correlation ρ is $[-1/3, 1/3]$.
- If $m = 1$, $n = 2$, and $\alpha > 0$, then we get the copula introduced by Huang and Kotz [10]. It was shown that the admissible range of θ_{12} is $-\min\{1, \alpha^{-2}\} \leqslant \theta_{12} \leqslant \alpha^{-1}$ and the correlation ρ between the corresponding uniformly distributed random variables is $\rho = 3\theta_{12}\alpha^2(\alpha+2)^{-2}$; thus, the range for correlation ρ is $-3(\alpha+2)^{-2}\min\{1, \alpha^2\} \leqslant \rho \leqslant 3\alpha(\alpha+2)^{-2}$.
- If $m \geqslant 1$, $n = 2$, and $\alpha > 0$, then we get the copula introduced by Bekrizadeh et al. [3]. They have shown that the admissible range of θ_{12} is $-\min\{1, (m\alpha^2)^{-1}\} \leqslant \theta_{12} \leqslant (m\alpha)^{-1}$ and the correlation between the corresponding uniformly distributed random variables is given by the formula

$$\rho = 12\int_0^1\int_0^1 Q^{\mathrm{GFGM}}(u,v)\,\mathrm{d}u\,\mathrm{d}v - 3 = 12\sum_{k=1}^m \binom{m}{k}\theta_{12}^k\biggl(\frac{\Gamma(k+1)\Gamma(2/\alpha)}{\alpha\Gamma(k+1+2/\alpha)}\biggr)^2.$$

Because of the weak dependence generated by the FGM family, many authors considered the modifications of this class. Examples of modified FGM copula can be found in [1, 2], among others.

The finding of the admissible region for parameters θ_{kl} in (3.1) is technical, although straightforward, task. Essentially, it requires the verification that the corresponding copula density (if exists) $q^{\mathrm{GFGM}}(u_1,\dots,u_n) = \partial^n Q^{\mathrm{GFGM}}(u_1,\dots,u_n)/\partial u_1\dots\partial u_n$ is nonnegative for all $u_1,\dots,u_n$. In the case of copula (3.1) with $m=1$,

$$q^{\mathrm{GFGM}}(u_1,\dots,u_n) = 1 + \sum_{1\leqslant k<l\leqslant n} \theta_{kl}\bigl(1-(1+\alpha)u_k^\alpha\bigr)\bigl(1-(1+\alpha)u_l^\alpha\bigr),$$

and these conditions can be obtained by considering the 2^n cases for $u_k = 0$ or 1, $k=1,\dots,n$, and verifying that $q^{\mathrm{GFGM}}(u_1,\dots,u_n)\geqslant 0$. For example, if $m=1$ and $n=3$, then these conditions are the following:

$$1+\alpha^2\theta \geqslant 0, \quad \theta_{kl} \geqslant \begin{cases} \frac{\alpha\theta-1}{1+\alpha} & \text{if } \alpha\theta>1, \\ \frac{1}{\alpha}\frac{\alpha\theta-1}{1+\alpha} & \text{if } \alpha\theta\leqslant 1, \end{cases} \quad 1\leqslant k<l\leqslant 3, \text{ for } \alpha>1$$

and

$$1+\theta \geqslant 0, \quad \theta_{kl} \geqslant \begin{cases} \frac{1}{\alpha}\frac{\alpha\theta-1}{1+\alpha} & \text{if } \alpha\theta>1, \\ \frac{\alpha\theta-1}{1+\alpha} & \text{if } \alpha\theta\leqslant 1, \end{cases} \quad 1\leqslant k<l\leqslant 3, \text{ for } 0<\alpha\leqslant 1$$

with $\theta := \theta_{12}+\theta_{13}+\theta_{23}$.

Note that any pair of variables $X_1,\dots,X_n$, linked by copula (3.1), satisfies $\mathbf{P}(X_k\leqslant x,\ X_l\leqslant y) = Q_{kl}^{\mathrm{GFGM}}(F_k(x),F_l(y))$, $k\neq l$, where

$$Q_{kl}^{\mathrm{GFGM}}(u,v) = uv\bigl(1+\theta_{kl}\bigl(1-u^\alpha\bigr)\bigl(1-v^\alpha\bigr)\bigr)^m. \tag{3.2}$$

Obviously, (3.2) implies $Q_{kl}^{\mathrm{GFGM}}(u,v)\leqslant uv$, $k<l$, whenever θ_{kl} are all nonpositive. Hence, the generalized FGM copula (3.1) provides a pairwise negative dependence structure if $\theta_{kl}\leqslant 0$, $1\leqslant k<l\leqslant n$. The following proposition shows that this copula also captures the pairwise UEND structure, which contains some positive dependence structures too.

Proposition 2. *Let the distribution of* $(X_1,\dots,X_n)$ *be generated by copula in* (3.1)*. Then*

$$\mathbf{P}(X_k>x,\ X_l>y) \leqslant C_{kl}\overline{F_k}(x)\overline{F_l}(y), \tag{3.3}$$

where $C_{kl} := 1+\max\{\alpha,1\}((|\theta_{kl}|+1)^m-1)$.

Proof. For every $k<l$, by (3.2) we have that

$$\mathbf{P}(X_k\leqslant x,\ X_l\leqslant y) = F_k(x)F_l(y)\bigl[1+\theta_{kl}\bigl(1-F_k^\alpha(x)\bigr)\bigl(1-F_l^\alpha(y)\bigr)\bigr]^m.$$

Hence,

$$\begin{aligned}
\mathbf{P}(X_k>x,\ X_l>y) &= 1-F_k(x)-F_l(y)+\mathbf{P}(X_k\leqslant x,\ X_l\leqslant y)\\
&= 1-F_k(x)-F_l(y)+F_k(x)F_l(y)\bigl(1+\theta_{kl}\bigl(1-F_k^\alpha(x)\bigr)\bigl(1-F_l^\alpha(y)\bigr)\bigr)^m\\
&= \overline{F_k}(x)+\overline{F_l}(y)-1+\bigl(1-\overline{F_k}(x)-\overline{F_l}(y)+\overline{F_k}(x)\overline{F_l}(y)\bigr)\\
&\quad\times\left(1+\sum_{i=1}^m\binom{m}{i}\theta_{kl}^i\bigl(\overline{F_k^\alpha}(x)\bigr)^i\bigl(\overline{F_l^\alpha}(y)\bigr)^i\right)
\end{aligned}$$

$$= \overline{F_k}(x)\overline{F_l}(y) + \big(1 - \overline{F_k}(x) - \overline{F_l}(y) + \overline{F_k}(x)\overline{F_l}(y)\big) \times \sum_{i=1}^{m} \binom{m}{i} \theta_{kl}^{i} \big(\overline{F_k^{\alpha}}(x)\big)^{i} \big(\overline{F_l^{\alpha}}(y)\big)^{i},$$

where $\overline{F_k^{\alpha}}(x) := 1 - F_k^{\alpha}(x)$. Using the inequality $1 - u^{\alpha} \leqslant \max\{\alpha, 1\}(1 - u)$, $u \in [0, 1]$, we get

$$\begin{aligned}\mathbf{P}(X_k > x,\ X_l > y) &\leqslant \overline{F_k}(x)\overline{F_l}(y) + \overline{F_k^{\alpha}}(x)\overline{F_l^{\alpha}}(y) \sum_{i=1}^{m} \binom{m}{i} |\theta_{kl}|^{i} \\ &\leqslant \big(1 + \max\{\alpha, 1\}\big(\big(|\theta_{kl}| + 1\big)^{m} - 1\big)\big)\overline{F_k}(x)\overline{F_l}(y). \qquad \square\end{aligned}$$

Obviously, if θ_{kl} in (3.1) are all nonnegative, then $X_1, \dots, X_n$ are both lower positive dependent and pairwise positive dependent. By (3.3) this is also a pairwise UEND structure.

3.2 Ali–Mikhail–Haq copula

Consider the copula of the form

$$Q^{\mathrm{AMH}}(u_1, \dots, u_n) = \frac{u_1 \cdots u_n}{1 - \theta(1 - u_1) \cdots (1 - u_n)}, \quad -1 \leqslant \theta < 1, \tag{3.4}$$

and let $\mathbf{P}(X_1 \leqslant x_1,\ \dots,\ X_n \leqslant x_n) = Q^{\mathrm{AMH}}(F_1(x_1), \dots, F_n(x_n))$. Then, for $k \neq l$,

$$\mathbf{P}(X_k \leqslant x,\ X_l \leqslant y) = \frac{F_k(x)F_l(y)}{1 - \theta\overline{F_k}(x)\overline{F_l}(y)},$$

and hence

$$\begin{aligned}\mathbf{P}(X_k > x,\ X_l > y) &= 1 - F_k(x) - F_l(y) + \frac{F_k(x)F_l(y)}{1 - \theta\overline{F_k}(x)\overline{F_l}(y)} \\ &\leqslant \overline{F_k}(x)\overline{F_l}(y) \end{aligned}\tag{3.5}$$

if $-1 \leqslant \theta \leqslant 0$. In the case $0 < \theta < 1$, we have

$$\mathbf{P}(X_k \leqslant x,\ X_l \leqslant y) \leqslant \frac{1}{1 - \theta} F_k(x)F_l(y), \tag{3.6}$$

$$\mathbf{P}(X_k > x,\ X_l > y) \leqslant \frac{1}{1 - \theta} \overline{F_k}(x)\overline{F_l}(y). \tag{3.7}$$

Inequality (3.6) is obvious. In order to show (3.7), it suffices to verify that

$$1 - u - v + \frac{uv}{1 - \theta(1 - u)(1 - v)} \leqslant \frac{(1 - u)(1 - v)}{1 - \theta}, \quad 0 \leqslant u, v \leqslant 1,\ 0 < \theta < 1.$$

The proof is straightforward, and we omit it.

By (3.5)–(3.7) the copula in (3.4) generates a pairwise ND structure if $-1 \leqslant \theta \leqslant 0$ and a pairwise END structure (which is also lower positive dependent and pairwise positive dependent) if $0 < \theta < 1$.

3.3 Frank copula

Consider the copula of the form

$$Q^{\mathrm{F}}(u_1,\dots,u_n) = -\frac{1}{\theta}\log\biggl(1+\frac{(\mathrm{e}^{-\theta u_1}-1)\cdots(\mathrm{e}^{-\theta u_n}-1)}{(\mathrm{e}^{-\theta}-1)^{n-1}}\biggr), \quad \theta>0, \tag{3.8}$$

and assume that $\mathbf{P}(X_1\leqslant x_1,\dots,X_n\leqslant x_n) = Q^{\mathrm{F}}(F_1(x_1),\dots,F_n(x_n))$. Then $\mathbf{P}(X_k\leqslant x,\, X_l\leqslant y) = Q^{\mathrm{F}}(F_k(x),F_l(y))$, $k\neq l$, where

$$Q^{\mathrm{F}}(u,v) := -\frac{1}{\theta}\log\biggl(1+\frac{(\mathrm{e}^{-\theta u}-1)(\mathrm{e}^{-\theta v}-1)}{\mathrm{e}^{-\theta}-1}\biggr).$$

In this case, the copula density is bounded:

$$q^{\mathrm{F}}(u,v) = \frac{-\theta(\mathrm{e}^{-\theta}-1)\mathrm{e}^{-\theta(u+v)}}{((\mathrm{e}^{-\theta}-1)+(\mathrm{e}^{-\theta u}-1)(\mathrm{e}^{-\theta v}-1))^2} \leqslant \frac{\theta}{(1-\mathrm{e}^{-\theta})\mathrm{e}^{-2\theta}} =: c_\theta.$$

Thus, denoting the corresponding marginal densities $f_k(x)$, we have

$$\begin{aligned}\mathbf{P}(X_k>x,\ X_l>y) &= \int\limits_{w>x,z>y} q^{\mathrm{F}}\bigl(F_k(w),F_l(z)\bigr)f_k(w)f_l(z)\,\mathrm{d}w\,\mathrm{d}z\\ &\leqslant c_\theta\overline{F_k}(x)\overline{F}_l(y), \quad k\neq l,\end{aligned}$$

that is, the Frank copula generates a pairwise UEND structure.

3.4 Clayton copula

Consider the copula

$$Q^{\mathrm{C}}(u_1,\dots,u_n) = \bigl(u_1^{-\theta}+\dots+u_n^{-\theta}-n+1\bigr)^{-1/\theta}, \quad \theta>0, \tag{3.9}$$

and assume that $\mathbf{P}(X_1\leqslant x_1,\dots,X_n\leqslant x_n) = Q^{\mathrm{C}}(F_1(x_1),\dots,F_n(x_n))$. Then $\mathbf{P}(X_k\leqslant x,\, X_l\leqslant y) = Q^{\mathrm{C}}(F_k(x),F_l(y))$, where

$$Q^{\mathrm{C}}(u,v) = \bigl(u^{-\theta}+v^{-\theta}-1\bigr)^{-1/\theta}.$$

Note that if $\theta\to0$, then $Q^{\mathrm{C}}(u,v)$ tends to uv, that is, we obtain the independence copula, whereas if $\theta\to\infty$, then $Q^{\mathrm{C}}(u,v)$ tends to $\min(u,v)$, that is, the comonotonicity copula.

We will show that for any $k\neq l$ and $x,y\in\mathbb{R}$,

$$\mathbf{P}(X_k>x,\ X_l>y)\leqslant(1+\theta)\overline{F_k}(x)\overline{F_l}(y).$$

This implies the pairwise UEND property and, hence, relation (1.4). The proof of this inequality follows from the identity $\mathbf{P}(X_k>x,X_l>y) = 1-F_k(x)-F_l(y)+\mathbf{P}(X_k\leqslant x,\ X_l\leqslant y)$ and the following lemma.

Lemma 1. *For any* $(u,v)\in[0,1]^2$ *and* $\theta>0$, *we have*

$$\bigl(u^{-\theta}+v^{-\theta}-1\bigr)^{-1/\theta}\leqslant uv+\theta(1-u)(1-v).$$

Proof. Denote, for convenience, $Q_\theta(u,v) := (u^{-\theta} + v^{-\theta} - 1)^{-1/\theta}$. Take any small $\epsilon > 0$ and write

$$\begin{aligned} Q_\theta(u,v) - Q_\epsilon(u,v) &= \int_\epsilon^\theta \frac{\partial Q_t(u,v)}{\partial t}\,\mathrm{d}t = \int_\epsilon^\theta \frac{(u^{-t}+v^{-t}-1)\log(u^{-t}+v^{-t}-1) - u^{-t}\log u^{-t} - v^{-t}\log v^{-t}}{t^2(u^{-t}+v^{-t}-1)^{1+1/t}}\,\mathrm{d}t \\ &= \int_\epsilon^\theta Q_t(u,v)\frac{(u^{-t}+v^{-t}-1)\log(u^{-t}+v^{-t}-1) - u^{-t}\log u^{-t} - v^{-t}\log v^{-t}}{t^2(u^{-t}+v^{-t}-1)}\,\mathrm{d}t. \end{aligned}$$

For all $(u,v) \in [0,1]^2$ and $t > 0$, we have

$$Q_t(u,v) \leqslant \sqrt{uv} \tag{3.10}$$

and

$$\frac{(u^{-t}+v^{-t}-1)\log(u^{-t}+v^{-t}-1) - u^{-t}\log u^{-t} - v^{-t}\log v^{-t}}{t^2(u^{-t}+v^{-t}-1)} \leqslant \frac{(1-u)(1-v)}{\sqrt{uv}}. \tag{3.11}$$

Bound (3.10) is due to the inequality $(u^{-t/2} - v^{-t/2})^2 + u^{-t/2}v^{-t/2} \geqslant 1$. In order to prove (3.11), we use the following inequality:

$$(x+y-1)\log(x+y-1) - x\log x - y\log y \leqslant (x+y-1)\log x \log y \tag{3.12}$$

for any $x \geqslant 1$, $y \geqslant 1$. Denote

$$f(x,y) := (x+y-1)\log(x+y-1) - x\log x - y\log y - (x+y-1)\log x\log y.$$

Then (3.12) follows by noting that $f(1,y) = 0$ for any $y \geqslant 1$ and

$$\frac{\partial f(x,y)}{\partial x} = -\left(\log x\log y + \frac{y-1}{x}\log y + \log\frac{xy}{x+y-1}\right) \leqslant 0, \quad x,y \geqslant 1.$$

By (3.12) we have

$$\frac{(u^{-t}+v^{-t}-1)\log(u^{-t}+v^{-t}-1) - u^{-t}\log u^{-t} - v^{-t}\log v^{-t}}{t^2(u^{-t}+v^{-t}-1)} \leqslant \log u\log v,$$

where, by the inequality $\log x \leqslant (x-1)/\sqrt{x}$, $x \geqslant 1$ (see [15, p. 272]),

$$-\log u = \log\frac{1}{u} \leqslant \frac{1/u - 1}{1/\sqrt{u}} = \frac{1-u}{\sqrt{u}}.$$

Inequalities (3.10) and (3.11) imply

$$Q_\theta(u,v) \leqslant Q_\epsilon(u,v) + (\theta - \epsilon)(1-u)(1-v).$$

Taking $\epsilon \to 0$, we obtain the desired inequality. □

Summarizing, we have the following corollary.

Corollary 2. *Let r.v.s $X_1, \dots, X_n$ have corresponding univariate distributions $F_1, \dots, F_n$ such that $H_n \in \mathcal{D}$, and let the dependence structure be generated by either of the copulas in* (3.1)*,* (3.4)*,* (3.8)*, or* (3.9)*. Then the asymptotic relation* (1.7) *holds. If, in addition, $H_n(-x) = o(\overline{H_n}(x))$, then* (1.8) *also holds.*

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