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Properties of Gaussian local times*

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Abstract. We investigate properties of local time for one class of Gaussian processes. These processes are called integrators since every function from $L_2([0;1])$ can be integrated over it. Using the white noise representation, we can associate integrators with continuous linear operators in $L_2([0;1])$. In terms of these operators, we discuss the existence and properties of local time for integrators. Also, we study the asymptotic behavior of Brownian self-intersection local time as its end-point tends to infinity.

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1 Introduction

The aim of this article is to study the local time for a certain class of Gaussian processes. Since works of Berman [1, 2], the existence and properties of local time are studied for a wide class of Gaussian processes and fields [25]. In the case of Brownian motion, the local time can be investigated using parabolic equations and potential theory due to the independence of increments and self-similarity. For general Gaussian processes, Berman proposed the notion of local nondeterminism, which in some sense means the almost independence of increments on small intervals. Under some technical assumptions, this property leads to the existence and regularity of the local time with respect to both spatial and time variables. Different authors proposed the version of local nondeterminism property for Gaussian processes and fields and proved not only the existence of local time, but also investigated some its asymptotic properties such as the law of iterated logarithm or small-ball probabilities [19, 25]. Nevertheless, the local nondeterminism property is hard to check for an arbitrary Gaussian process. Simple sufficient conditions were given for processes with stationary increments or for self-similar processes [1, 2]. In the case of planar Gaussian process, the situation is much worse. Namely, for the Brownian motion on the plane, the existence of multiple self-intersection points is well known [11]. The corresponding local time of multiple self-intersection needs to be properly renormalized. Such a renormalization was done by Varadhan [23], Dynkin [12], Rosen [17, 18]. Later, Le Gall [13] obtained an asymptotic expansion for an area of Brownian sausage that contains renormalized self-intersection local times as coefficients. Since all these results are essentially based on the structure of a Brownian motion, the technique used cannot be

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expanded to other Gaussian processes. However, the question of such a generalization is quiet interesting in view of constructing random polymer models using not only Markov processes (in some cases, there are no reasons for molecule to differ the starting point and the end-point) [3, 5, 21]. All mentioned reasons lead to the attempt to find a class of Gaussian processes for which some version of local nondeterminism holds and results related to the existence and renormalization of local time and self–intersection local times can be achieved. Such a class of processes was introduced by Dorogovtsev [8, 9] in connection with anticipating stochastic integration. The original definition is the following.

DEFINITION 1. (See [8].) A centered Gaussian process $x(t)$, $t \in [0;1]$, is said to be an integrator if there exists a constant $c > 0$ such that for an arbitrary partition $0 = t_0 < t_1 < \cdots < t_n = 1$ and real numbers $a_0, \ldots, a_{n-1}$,

$$\mathbf{E}\left(\sum_{k=0}^{n-1} a_k\big(x(t_{k+1}) - x(t_k)\big)\right)^2 \leqslant c\sum_{k=0}^{n-1} a_k^2 \Delta t_k. \tag{1.1}$$

Inequality (1.1) allows us to integrate functions from $L_2([0;1])$ with respect to x. This naturally leads to a definition of Skorokhod-type stochastic integral with respect to x [20]. In [8], the corresponding stochastic calculus, including the Itô formula, for x was considered. The following statement describes the structure of integrators.

Proposition 1. (*See* [8].) *A centered Gaussian process $x(t)$, $t \in [0;1]$, is an integrator iff there exist a Gaussian white noise ξ* [7, 20] *in $L_2([0;1])$ and a continuous linear operator A in the same space such that*

$$x(t) = (A\mathbf{1}_{[0;t]}, \xi), \quad t \in [0;1]. \tag{1.2}$$

In this paper, we use the language of white noise analysis [7, 15, 20, 24]. Note that if A equals identity, then x in expression (1.2) is a Wiener process. For continuously invertible operator A, we can expect that x will inherit some properties of a Wiener process. For example, we will prove in Section 2 that if A is continuously invertible, then x has a local time at any point $u \in \mathbb{R}$. Such a local time can be obtained as the occupation density. Also, we will check that this density is a continuous function in spatial and time variables. In Section 3, we will prove a continuous dependence of local times of integrators on operators generating them. The main method of our investigations is based on the study of functional properties of Hilbert-valued functions. In particular, we obtain some estimations for Gram determinant constructed by increments of such a function. These estimations allow us to investigate conditional moments of Brownian self-intersection local time in dimensions one and two as the end point (the value $w(1)$) tends to infinity. We establish the rate of decreasing of the moments mentioned. The question of conditional behavior of self-intersection local times is inspired by the study of properties of continuous polymer models [5, 21]. Real polymers cannot have self-intersections due to excluded volume effect [21], but the energy of interaction between monomers from different places in polymer molecule influences its form. Flory proposed the evaluation of the size of polymer based on the counting of interaction energy [21]. The Brownian path can be viewed as an ideal Gaussian model of polymer [21]. Applying Flory method, we have to substitute the energy of interaction by the self-intersection local time. So the question of dependence of self-intersection local time on the size of Brownian path is natural. We present corresponding estimations in Section 4. Some necessary facts from geometry of Hilbert-valued functions are proved in Appendix.

2 Existence of local time for Gaussian integrators

Let us recall a definition of local time for one-dimensional Wiener process $w(t)$, $t \in [0;1]$. Put

$$f_\varepsilon(y) = \frac{1}{\sqrt{2\pi\varepsilon}}\mathrm{e}^{-y^2/(2\varepsilon)}, \quad y \in \mathbb{R},\ \varepsilon > 0.$$

DEFINITION 2. For any $t \in [0;1]$ and $u \in \mathbb{R}$,

$$\int_0^t \delta_u\big(w(s)\big)\,\mathrm{d}s := L_2\text{-}\lim_{\varepsilon\to 0}\int_0^t f_\varepsilon\big(w(s)-u\big)\,\mathrm{d}s$$

is said to be the local time of the Wiener process at point u up to time t.

Consider the occupation measure of w up to time t defined by the formula

$$\mu_t(D) = \int_0^t \mathbf{1}_D\big(w(s)\big)\,\mathrm{d}s, \quad D \in B(\mathbb{R})$$

($B(\mathbb{R})$ is the Borel σ-field on $\mathbb{R}$); $\mu_t(D)$ equals the Lebesgue measure of time a trajectory of the Wiener process spends in the set D up to time t. Levy [16] proved that for almost all trajectories of w and any $t \in [0;1]$, the measure μ_t has a density, that is, there exists a random function $l(u,t)$, $u \in \mathbb{R}$, such that a.s. for any $t \in [0;1]$ and $D \in B(\mathbb{R})$,

$$\mu_t(D) = \int_D l(u,t)\,\mathrm{d}u.$$

Trotter [22] proved that the density of occupation measure of the Wiener process is continuous in u and t. A useful consequence of joint continuity is the next occupation density formula. For every continuous function φ on $\mathbb{R}$ with a compact support,

$$\int_0^t \varphi\big(w(s)\big)\,\mathrm{d}s = \int_{\mathbb{R}} \varphi(u) l(u,t)\,\mathrm{d}u. \tag{2.1}$$

It follows from (2.1) that

$$\int_0^t \delta_u\big(w(s)\big)\,\mathrm{d}s = \lim_{\varepsilon\to 0}\int_0^t f_\varepsilon\big(w(s)-u\big)\,\mathrm{d}s = \lim_{\varepsilon\to 0}\int_{\mathbb{R}} f_\varepsilon(v-u) l(v,t)\,\mathrm{d}v = l(u,t).$$

Therefore, the value of density of occupation measure $l(u,t)$ is the local time of the Wiener process at u up to time t. In this section, we will establish the same properties of local time for Gaussian integrators. To prove the existence of local time for the Gaussian integrator x with representation (1.2), we need the notion of local nondeterminism for Gaussian processes introduced by Berman [1]. Let $\{y(t),\ t \in J\}$ be an $\mathbb{R}$-valued zero-mean Gaussian process on an open interval J. Suppose that there exists $d > 0$ such that:

1) $\mathbf{E}(y(t)-y(s))^2 > 0$ for all $s,t \in J$: $0 \leqslant |t-s| \leqslant d$;
2) $\mathbf{E}y^2(t) > 0$ for all $t \in J$.

For $m \geqslant 2$, $t_1,\dots,t_m \in J$, $t_1 < t_2 < \cdots < t_m$, put

$$V_m = \frac{\mathrm{Var}(y(t_m)-y(t_{m-1}) \mid y(t_1),\dots,y(t_{m-1}))}{\mathrm{Var}(y(t_m)-y(t_{m-1}))},$$

which is the ratio of conditional and unconditional variances.

DEFINITION 3. (See [1].) A Gaussian process y is said to be locally nondetermined on J if for every $m \geqslant 2$,

$$\lim_{c\to 0} \inf_{t_m - t_1 \leqslant c} V_m > 0.$$

The following statement was proved in [1] and demonstrates that the local nondeterminism property can be used as one of sufficient conditions for the existence and smoothness of local time for general Gaussian process.

Theorem 1. (*See* [1].) *Let $y(t)$, $t \in [0;T]$, be a centered Gaussian process satisfying the following three conditions*:

(i) $y(0) = 0$;
(ii) y *is locally nondetermined on* $(0;T)$;
(iii) *there exist positive real numbers γ, δ and a continuous even function $b(t)$ such that $b(0) = 0$, $b(t) > 0$, $t \in (0;T]$,*

$$\lim_{h\to 0} h^{-\gamma} \int_0^h \bigl(b(t)\bigr)^{-1-2\delta}\, \mathrm{d}t = 0,$$

and $\mathbf{E}(y(t) - y(s))^2 \geqslant b^2(t-s)$ *for all* $s, t \in [0;T]$.

Then there exists a version $l(u,t)$, $u \in \mathbb{R}$, $t \in [0;T]$, of local time of the process y that is jointly continuous in (u,t) and satisfies the Hölder condition in t uniformly in u, that is, for every $\gamma_1 < \gamma$, there exist positive and finite random variables η and η_1 such that

$$\sup_u \bigl| l(u, t+h) - l(u,t)\bigr| \leqslant \eta_1 |h|^{\gamma_1}$$

for all $s, t, t+h \in [0;T]$ and all $|h| < \eta$.

To discuss the existence of local time for Gaussian integrator x, we need a reformulation of the notion of local nondeterminism. Denote by $G(e_1, \ldots, e_n)$ the Gram determinant constructed by vectors $e_1, \ldots, e_n$. Let $g \in C([0;1], L_2([0;1]))$, $\Delta g(t_i) = g(t_{i+1}) - g(t_i)$, $i = \overline{1, m-1}$ ($C([0;1], L_2([0;1]))$ is the space of all continuous functions from $[0;1]$ into $L_2([0;1])$).

Lemma 1. *The Gaussian process $y(t) = (g(t), \xi)$, where ξ is a white noise in $L_2([0;1])$, is locally nondetermined on J iff for every $m \geqslant 2$,*

$$\lim_{c\to 0} \inf_{t_m - t_1 \leqslant c} \frac{G(g(t_1), \Delta g(t_1), \ldots, \Delta g(t_{m-1}))}{\|g(t_1)\|^2 \|\Delta g(t_1)\|^2 \cdots \|\Delta g(t_{m-1})\|^2} > 0.$$

Proof. This is a consequence of the definition of V_m and relation

$$\begin{aligned} V_2 \cdots V_m &= \frac{\det \operatorname{cov}(x(t_i), x(t_j))_{ij=1}^m}{\operatorname{Var} x(t_1) \operatorname{Var}(x(t_2) - x(t_1)) \cdots \operatorname{Var}(x(t_m) - x(t_{m-1}))} \\ &= \frac{G(g(t_1), \Delta g(t_1), \ldots, \Delta g(t_{m-1}))}{\|g(t_1)\|^2 \|\Delta g(t_1)\|^2 \cdots \|\Delta g(t_{m-1})\|^2}. \qquad \square \end{aligned}$$

By using Lemma 1 we can establish the following statement.

Theorem 2. (*See* [14].) *Suppose that the operator A in representation* (1.2) *of x is continuously invertible. Then there exists a version $l(u,t)$, $u \in \mathbb{R}$, $t \in [0;1]$, of the local time of x that is jointly continuous in (u,t)*

and satisfies a Hölder condition in t uniformly in u, that is, for every $\gamma < 1/2$, there exist positive and finite random variables η and η_1 such that

$$\sup_u \bigl|l(u,t+h) - l(u,t)\bigr| \leqslant \eta_1 |h|^\gamma$$

for all $s, t, t+h \in [0;1]$ and all $|h| < \eta$.

Note that for $A = I + S$, where S is a compact operator, the integrator x inherits pathwise properties of the Wiener process (see [10]). In the general case, pathwise properties of integrators remain to be unclear even in the case of invertible operator A. The reason is that the constant in Lemma A.1 depends on k. This does not allow us to apply approaches directly based on the independency of increments as for the Wiener process. Of course, for invertible A, we can check that the integrator x does not have a representation

$$x(t) = \int_0^t y(s)\,\mathrm{d}s$$

with the process y satisfying the condition

$$\int_0^1 \bigl(\mathbf{E}y(s)^2\bigr)^{1/2}\,\mathrm{d}s < +\infty.$$

Here the integral is understood in the mean-square sense. Pathwise properties of integrators will be a subject of further consideration.

Theorem 2 was proved in the article of the second author [14]. Here we briefly recall the main steps of the proof, which is based on the following key property of Gram determinant.

Lemma 2. *Suppose that A is a continuously invertible operator in Hilbert space H. Then for all $k \geqslant 1$, there exists a positive constant $c(k)$ depending on k and such that for any $e_1, \dots, e_k \in H$, the following relation holds*:

$$G(Ae_1, \dots, Ae_k) \geqslant c(k) G(e_1, \dots, e_k).$$

Lemma 2 is proved in Appendix (Lemma A.1).

Proof of Theorem 2. To prove the theorem, let us check that x satisfies conditions (i)–(iii) of Theorem 1. It is obvious that $x(0) = 0$. Lemmas 1 and 2 imply that x is locally nondetermined. Let us check that x satisfies condition (iii) of Theorem 1. Let $b(t) = c\sqrt{t}$, $c > 0$. Pick $\delta < 1/2$ and then γ such that $\gamma < 1/2 - \delta$. We can see that

$$\lim_{h \to 0} h^{-\gamma} \int_0^h t^{-1/2-\delta}\,\mathrm{d}t = \frac{2}{1-2\delta} \lim_{h \to 0} h^{1/2-\delta-\gamma} = 0. \qquad \square$$

3 On continuous dependence of local times of integrators on operators generating them

Suppose that A_n and A are continuously invertible operators in $L_2([0;1])$ that generate Gaussian integrators x_n, x, that is,

$$x_n(t) = \bigl(A_n \mathbf{1}_{[0;t]}, \xi\bigr), \quad x(t) = \bigl(A\mathbf{1}_{[0;t]}, \xi\bigr), \quad t \in [0;1].$$

We proved in Section 2 that there exist random variables

$$l_n(u) := l_n(u,1) = \int_0^1 \delta_u\big(x_n(t)\big)\,\mathrm{d}t, \quad l(u) := l(u,1) = \int_0^1 \delta_u\big(x(t)\big)\,\mathrm{d}t, \quad u \in \mathbb{R}.$$

The following statement shows that if a sequence of operators A_n converges strongly to an operator A, then the sequence of local times of integrators x_n converges in mean square to the local time of x. To avoid confusion in the next theorem for the norm in the space $L_2([0;1])$, we use the notation $\|\cdot\|_2$. The operator norm is denoted by $\|\cdot\|$.

Theorem 3. *Suppose that A_n, A are continuously invertible operators in $L_2([0;1])$ such that*:

(i) *for any $y \in L_2([0;1])$, $\|A_n y - Ay\|_2 \to 0$, $n \to \infty$;*
(ii) $\sup_{n\geqslant 1} \|A_n^{-1}\| < \infty$.

Then

$$\mathbf{E}\int_{\mathbb{R}} \big(l_n(u) - l(u)\big)^2\,\mathrm{d}u \to 0, \quad n \to \infty.$$

Proof. To prove the theorem, it suffices to check that

$$\mathbf{E}\int_{\mathbb{R}} l_n^2(u)\,\mathrm{d}u \to \mathbf{E}\int_{\mathbb{R}} l^2(u)\,\mathrm{d}u, \quad \mathbf{E}\int_{\mathbb{R}} l_n(u)l(u)\,\mathrm{d}u \to \mathbf{E}\int_{\mathbb{R}} l^2(u)\,\mathrm{d}u, \quad n \to \infty.$$

It follows from Theorem B.1 that

$$\begin{aligned}\mathbf{E}\int_{\mathbb{R}} l_n(u)l_n(u)\,\mathrm{d}u &= \mathbf{E}\int_0^1\int_0^1 \delta_0\big(x_n(t) - x_n(s)\big)\,\mathrm{d}s\,\mathrm{d}t \\ &= \lim_{\varepsilon\to 0}\mathbf{E}\int_0^1\int_0^1 f_\varepsilon\big(x_n(t) - x_n(s)\big)\,\mathrm{d}s\,\mathrm{d}t = \frac{2}{\sqrt{2\pi}}\int_{\Delta_2}\frac{\mathrm{d}s\,\mathrm{d}t}{\|A_n\mathbf{1}_{[s;t]}\|_2},\end{aligned}$$

where $\Delta_2 = \{0 \leqslant s \leqslant t \leqslant 1\}$. It follows from the invertibility of operators A_n and from condition (ii) that

$$\frac{1}{\|A_n\mathbf{1}_{[s;t]}\|_2} \leqslant \sup_{n\geqslant 1}\|A_n^{-1}\|\frac{1}{\sqrt{t-s}}.$$

The Lebesgue dominated convergence theorem implies that

$$\mathbf{E}\int_{\mathbb{R}} l_n^2(u)\,\mathrm{d}u \to \mathbf{E}\int_{\mathbb{R}} l^2(u)\,\mathrm{d}u, \quad n \to \infty.$$

Let us check that

$$\mathbf{E}\int_{\mathbb{R}} l_n(u)l(u)\,\mathrm{d}u \to \mathbf{E}\int_{\mathbb{R}} l^2(u)\,\mathrm{d}u, \quad n \to \infty.$$

Again by using Theorem B.1 we can write

$$\mathbf{E}\int\limits_{\mathbb{R}} l_n(u)l(u)\,\mathrm{d}u = \mathbf{E}\int\limits_0^1\int\limits_0^1 \delta_0\bigl(x_n(t)-x(s)\bigr)\,\mathrm{d}s\,\mathrm{d}t = \frac{2}{\sqrt{2\pi}}\int\limits_{\Delta_2}\frac{\mathrm{d}s\,\mathrm{d}t}{\|A_n\mathbf{1}_{[0;t]}-A\mathbf{1}_{[0;s]}\|_2}$$
$$= \frac{2}{\sqrt{2\pi}}\int\limits_{\Delta_2}\frac{\mathrm{d}s\,\mathrm{d}t}{\|A_n(\mathbf{1}_{[0;t]}-A_n^{-1}A\mathbf{1}_{[0;s]})\|_2} \leqslant \frac{2}{\sqrt{2\pi}}\sup_{n\geqslant 1}\|A_n^{-1}\|\int\limits_{\Delta_2}\frac{\mathrm{d}s\,\mathrm{d}t}{\|\mathbf{1}_{[0;t]}-\varkappa_n(s)\|_2},$$

where $\varkappa_n(s) = A_n^{-1}A\mathbf{1}_{[0;s]}$. It follows from Lemma A.4 (see Appendix A) that the sequence $\{1/\|\mathbf{1}_{[0;t]} - \varkappa_n(s)\|_2\}_{n\geqslant 1}$ is uniformly integrable. Consequently,

$$\mathbf{E}\int\limits_{\mathbb{R}} l_n(u)l(u)\,\mathrm{d}u \to \mathbf{E}\int\limits_{\mathbb{R}} l^2(u)\,\mathrm{d}u, \quad n\to\infty. \qquad \square$$

4 Conditional moments of Brownian self-intersection local time

In this section, we discuss relationships between the norm of end-point of Brownian path and its self-intersection local time. As it was mentioned in Introduction, such relation reflects the fact that real polymers have a greater Flory number than ideal polymers due to excluded volume effect. Here we will study the conditional distribution of the self-intersection local time for the Brownian motion under the condition that its end-point tends to infinity. Let us begin with one-dimensional Brownian motion w. As it was discussed, for example, in [4], the self-intersection local time for w exists. Denote it by

$$T_2 = \int\limits_{\Delta_2}\delta_0\bigl(w(t_2)-w(t_1)\bigr)\,\mathrm{d}t_1\,\mathrm{d}t_2.$$

Let us check the following statement.

Theorem 4. *For any $p>0$ and $\beta\in(0;1)$,*

$$\mathbf{E}\bigl(T_2^p \mid w(1)=a\bigr) = O\bigl(|a|^{-\beta}\bigr), \quad a\to\infty.$$

Proof. It suffices to consider integer p. Then

$$\mathbf{E}\bigl(T_2^p \mid w(1)=a\bigr) = \mathbf{E}\int\limits_{\Delta_2^p}\prod_{i=1}^p\delta_0\bigl(\eta(t_2^i)-\eta(t_1^i)\bigr)\,\mathrm{d}\vec{t},$$

where $\eta(t) = w(t) - tw(1) + at$, $t\in[0;1]$. In terms of white noise $\xi=\dot{w}$, the process η has a representation

$$\eta(t) = \bigl(Qg_0(t),\xi\bigr) + at.$$

Here $g_0(t) = \mathbf{1}_{[0;t]}$, and Q is the projection onto the orthogonal complement to $g_0(1) = \mathbf{1}_{[0;1]}$. To estimate the conditional expectation for T_2^p, let us use the following lemma from Appendix (Lemma A.5).

Lemma 3. *For elements $e_1, \dots, e_n$ of $L_2([0;1])$ and a projection Q, let $Qe_1, \dots, Qe_n$ be linearly independent. Suppose that elements f, g satisfy the following relationships for all $i = 1, \dots, n$:*

$$(f, e_i) = (g, Qe_i).$$

Then

$$\|P_1 f\| \leqslant \|P_2 g\|,$$

where P_1 and P_2 are theorthogonal projections on linear spans of $e_1, \dots, e_n$ and $Qe_1, \dots, Qe_n$, respectively.

To apply this lemma for our situation, denote by $\Gamma^Q_{\vec{t}}$ and $P^Q_{\vec{t}}$ the Gram determinant for $Qe_1, \dots, Qe_p$ and the projection on its linear span, where $e_i = \mathbf{1}_{[t^i_1;t^i_2]}$, $i = 1, \dots, p$, and Q is the projection onto $\mathbf{1}^{\perp}_{[0;1]}$. Then

$$\mathbf{E} \int_{\Delta^p_2} \prod_{i=1}^p \delta_0\big(\eta\big(t^i_2\big) - \eta\big(t^i_1\big)\big)\, d\vec{t} = \int_{\Delta^p_2} \frac{e^{-\|P^Q_{\vec{t}} h_{\vec{t}}\|^2 a^2/2}}{\Gamma^Q_{\vec{t}}}\, d\vec{t}.$$

Here $h_{\vec{t}}$ is taken in a such way that for all $i = 1, \dots, p$,

$$(h_{\vec{t}}, Qe_i) = t^i_2 - t^i_1.$$

It follows from the previous lemma that

$$\big\|P^Q_{\vec{t}} h_{\vec{t}}\big\| \geqslant \big\|P_{\vec{t}} \mathbf{1}_{[0;1]}\big\|,$$

where $P_{\vec{t}}$ is the projection onto the linear span of $\mathbf{1}_{[t^1_1;t^1_2]}, \dots, \mathbf{1}_{[t^p_1;t^p_2]}$. Consequently, for arbitrary $k = 1, \dots, p$,

$$e^{-\|P^Q_{\vec{t}} h_{\vec{t}}\|^2 a^2/2} \leqslant e^{-(t^k_2 - t^k_1) a^2/2}.$$

To find the estimation for $\Gamma^Q_{\vec{t}}$, note that

$$\Gamma^Q_{\vec{t}} = \Gamma\big(\mathbf{1}_{[0;1]}, \mathbf{1}_{[t^1_1;t^1_2]}, \dots, \mathbf{1}_{[t^p_1;t^p_2]}\big).$$

Let us use the following lemma (Lemma A.6).

Lemma 4. *Let $\Delta_0 = \emptyset$, and $\Delta_1, \dots, \Delta_n$ be subsets of* $[0;1]$. *Then*

$$\Gamma(\mathbf{1}_{\Delta_1}, \dots, \mathbf{1}_{\Delta_n}) \geqslant \prod_{k=1}^n \left|\Delta_k \setminus \bigcup_{j=1}^{k-1} \Delta_j\right|.$$

As a consequence of this lemma, we can obtain the following estimate for Gram determinant:

$$\Gamma^Q_{\vec{t}} \geqslant \prod_{j=1}^N |\widetilde{\Delta}_j|,$$

where $\widetilde{\Delta}_j$, $j = 1, \dots, N$, are intervals from the partition of $[0;1]$ by end-points of intervals $[t^1_k, t^2_k]$, $k = 1, \dots, p$. Now using the previous estimation for $\|P^Q{}_{\vec{t}} h_{\vec{t}}\|$, we can get that

$$\mathbf{E} \int_{\Delta^p_2} \prod_{i=1}^p \delta_0\big(\eta\big(t^i_2\big) - \eta\big(t^i_1\big)\big)\, d\vec{t} \leqslant (2p)! \frac{1}{\sqrt{2\pi}^p} \int_{\Delta_{2p}} \frac{e^{-a^2(t_{2p} - t_{2p-1})/2}}{(\prod_{j=0}^{2p}(t_{j+1} - t_j))^{1/2}}\, d\vec{t},$$

where $t_0 = 0$ and $t_{2p+1} = 1$. Consider the following integral with respect to the last variable t_{2p}:

$$\int_{2p-1}^{1} \frac{\mathrm{e}^{-a^2(t_{2p}-t_{2p-1})/2}}{\sqrt{(t_{2p}-t_{2p-1})(1-t_{2p})}}\,\mathrm{d}t_{2p}.$$

Using the expression $\delta = 1 - t_{2p-1}$ and changing the variable, we can rewrite the last integral as

$$\int_0^{\delta} \frac{\mathrm{e}^{-a^2 s/2}}{\sqrt{s(\delta - s)}}\,\mathrm{d}s = \int_0^1 \frac{\mathrm{e}^{-a^2\delta s'/2}}{\sqrt{s'(1-s')}}\,\mathrm{d}s'.$$

Using the Hölder inequality, we get, for $\alpha \in (1;2)$,

$$\int_0^1 \frac{\mathrm{e}^{-a^2\delta s'/2}}{\sqrt{s'(1-s')}}\,\mathrm{d}s' \leqslant c_\alpha \left(\int_0^1 \mathrm{e}^{-a^2\delta(\alpha/(\alpha-1))s'/2}\,\mathrm{d}s' \right)^{\alpha/(\alpha-1)} \leqslant \widetilde{c}_\alpha \left(\frac{\alpha-1}{\alpha\delta a^2} \right)^{\alpha/(\alpha-1)},$$

where c_α and $\widetilde{c}_\alpha$ are positive constants depending on α. Finally, for any $\alpha \in (1;2)$,

$$\mathbf{E}\int_{\Delta_2^p} \prod_{i=1}^{p} \delta_0\big(\eta\big(t_2^i\big) - \eta\big(t_1^i\big)\big)\,\mathrm{d}\vec{t} \leqslant \widetilde{\widetilde{c}}_\alpha \int_{\Delta_{2p-1}} \prod_{j=0}^{2p-2} \frac{1}{\sqrt{t_{j+1}-t_j}}\,\mathrm{d}\vec{t}\cdot a^{-2\alpha/(\alpha-1)}. \qquad \square$$

For a planar Wiener process w on the interval $[0;1]$, consider the trajectories with $w(1) = a$. We can expect that if $\|a\|$ is large, then the trajectory of w has a small number of self-intersections. The conditional distribution of the Wiener process under the condition $w(1) = a$ coincides with the distribution of the Brownian bridge

$$y_a(t) = w(t) - tw(1) + at, \quad t \in [0;1].$$

Let us investigate the dependence of the self-intersection local time of the process $y_a(t),\ t \in [0;1]$, on $\|a\|$. Denote

$$T_2(a,\alpha) = \int_{\Delta_2(a,\alpha)} \delta_0\big(y_a(t_2) - y_a(t_1)\big)\,\mathrm{d}t_1\,\mathrm{d}t_2,$$

where

$$\Delta_2(a,\alpha) = \big\{(t_1,t_2)\colon\ 0 \leqslant t_1 \leqslant 1 - \|a\|^{-\alpha},\ t_1 + \|a\|^{-\alpha} \leqslant t_2 \leqslant 1\big\}.$$

The self-intersection local time

$$\int_{\Delta_2(a,\alpha)} \delta_0\big(w(t_2) - w(t_1)\big)\,\mathrm{d}t_1\,\mathrm{d}t_2$$

exists (see [4]). As before, we can check that

$$T_2(a,\alpha) = \mathbf{E}\left(\frac{\int_{\Delta_2(a,\alpha)} \delta_0(w(t_2) - w(t_1))\,\mathrm{d}t_1\,\mathrm{d}t_2}{w(1)} = a \right).$$

Let us prove the following statement.

Theorem 5.

(i) *For* $\alpha = 2$, $\lim_{\|a\|\to+\infty} \mathbf{E}T_2(a,\alpha) = 1/(2\pi)\int_1^{+\infty}(1/y)\mathrm{e}^{-y/2}\,\mathrm{d}y$;

(ii) *For* $\alpha < 2$, $\lim_{\|a\|\to+\infty} \mathbf{E}T_2(a,\alpha) = 0$;

(iii) *For* $\alpha > 2$, $\lim_{\|a\|\to\infty} \mathbf{E}T_2(a,\alpha) = +\infty$.

Proof. Let $\Delta t_1 = t_2 - t_1$. Then

$$\mathbf{E}T_2(a,\alpha) = \int\limits_{\Delta_2(a,\alpha)} \frac{1}{2\pi\Delta t_1(1-\Delta t_1)}\mathrm{e}^{\Delta t_1\|a\|^2/(2(1-\Delta t_1))}\,\mathrm{d}\vec{t}. \tag{4.1}$$

Let $t_1 = s_1$ and $\|a\|^2\Delta t_1 = s_2$. Then (4.1) equals

$$\begin{aligned}
&\int\limits_0^{1-\|a\|^{-\alpha}}\int\limits_{\|a\|^{-\alpha+2}}^{(1-s_1)\|a\|^2} \frac{\|a\|^2}{2\pi s_2(\|a\|^2-s_2)}\mathrm{e}^{-\|a\|^2 s_2/(2(\|a\|^2-s_2))}\,\mathrm{d}s_2\,\mathrm{d}s_1\\
&= \int\limits_{\|a\|^{-\alpha+2}}^{\|a\|^2}\int\limits_0^{1-s_2/\|a\|^2} \frac{\|a\|^2}{2\pi s_2(\|a\|^2-s_2)}\mathrm{e}^{-\|a\|^2 s_2/(2(\|a\|^2-s_2))}\,\mathrm{d}s_1\,\mathrm{d}s_2\\
&= \int\limits_{\|a\|^{-\alpha+2}}^{\|a\|^2}\left(1-\frac{s_2}{\|a\|^2}\right)\frac{\|a\|^2}{2\pi s_2(\|a\|^2-s_2)}\mathrm{e}^{-\|a\|^2 s_2/(2(\|a\|^2-s_2))}\,\mathrm{d}s_2\\
&= \frac{1}{2\pi}\int\limits_{\|a\|^{-\alpha+2}}^{\|a\|^2}\frac{1}{s_2}\mathrm{e}^{-\|a\|^2 s_2/(2(\|a\|^2-s_2))}\,\mathrm{d}s_2.
\end{aligned} \tag{4.2}$$

Put $\|a\|^2 s_2/\|a\|^2 - s_2 = y$. Then (4.2) equals

$$\frac{1}{2\pi}\int\limits_{\|a\|^{2-\alpha}/(1-\|a\|^{-\alpha})}^{+\infty}\frac{\|a\|^2}{y(\|a\|^2+y)}\mathrm{e}^{-y/2}\,\mathrm{d}y. \tag{4.3}$$

Note that, for $\alpha = 2$,

$$\frac{1}{2\pi}\int\limits_{1/(1-\|a\|^{-2})}^{+\infty}\frac{\|a\|^2}{y(\|a\|^2+y)}\mathrm{e}^{-y/2}\,\mathrm{d}y \to \frac{1}{2\pi}\int\limits_1^{+\infty}\frac{1}{y}\mathrm{e}^{-y/2}\,\mathrm{d}y, \quad \|a\|\to+\infty.$$

We can see that, for $\alpha < 2$,

$$\frac{1}{2\pi}\int\limits_{\|a\|^{2-\alpha}/(1-\|a\|^{-2})}^{+\infty}\frac{\|a\|^2}{y(\|a\|^2+y)}\mathrm{e}^{-y/2}\,\mathrm{d}y \leqslant \frac{1}{2\pi}\frac{(1-\|a\|^{-\alpha})^2}{\|a\|^{2-\alpha}}\mathrm{e}^{-\|a\|^{2-\alpha}/(1-\|a\|^{-\alpha})}. \tag{4.4}$$

Estimate (4.4) implies that, for $\alpha < 2$,

$$\lim_{\|a\|\to+\infty} \frac{1}{2\pi} \int\limits_{\|a\|^{2-\alpha}/(1-\|a\|^{-\alpha})}^{+\infty} \frac{\|a\|^2}{y(\|a\|^2+y)} \mathrm{e}^{-y/2}\, \mathrm{d}y = 0.$$

On the other hand, for $\alpha > 2$,

$$\begin{aligned}\frac{1}{2\pi} \int\limits_{\|a\|^{2-\alpha}/(1-\|a\|^{-\alpha})}^{+\infty} \frac{\|a\|^2}{y(\|a\|^2+y)} \mathrm{e}^{-y/2}\, \mathrm{d}y &\geqslant \frac{1}{2\pi} \int\limits_{\|a\|^{2-\alpha}/(1-\|a\|^{-\alpha})}^{1/m} \frac{\|a\|^2}{y(\|a\|^2+y)} \mathrm{e}^{-y}\, \mathrm{d}y \\ &\geqslant \frac{1}{2\pi} \frac{m\|a\|^2}{\|a\|^2+\frac{1}{m}} \bigl(-\mathrm{e}^{1/m} + \mathrm{e}^{-\|a\|^{2-\alpha}/(1-\|a\|^{-\alpha})}\bigr). \end{aligned} \tag{4.5}$$

It follows from (4.5) that, for $\alpha > 2$,

$$\lim_{\|a\|\to+\infty} \frac{1}{2\pi} \int\limits_{\|a\|^{2-\alpha}/(1-\|a\|^{-\alpha})}^{+\infty} \frac{\|a\|^2}{y(\|a\|^2+y)} \mathrm{e}^{-y/2}\, \mathrm{d}y = +\infty. \qquad \square$$

Appendix A: On some geometry of Hilbert-valued functions

In this appendix, we collect some useful estimates for a Gramian matrix and Gram determinant that describe the changing of geometry of Hilbert-valued functions under the action of a linear continuous operator. Also, for $0 \leqslant \alpha < 1$, we check that

$$\sup_{y\in L_2([0;1])} \int\limits_0^1 \frac{\mathrm{d}t}{\|\mathbf{1}_{[0;t]} - y\|^{1+\alpha}} < +\infty.$$

Let $B(e_1,\dots,e_n)$ be the Gramian matrix constructed from vectors $e_1,\dots,e_n$ in Hilbert space H.

Lemma A.1. *Suppose that A is a continuously invertible operator in a Hilbert space H. Then, for all $k \geqslant 1$, there exists a positive constant $c(k)$, depending on k and A, such that for any $e_1,\dots,e_k \in H$, we have the following relation*:

$$G(Ae_1,\dots,Ae_k) \geqslant c(k)G(e_1,\dots,e_k).$$

Proof. It suffices to check that

$$\inf G\biggl(\frac{Af_1}{\|Af_1\|},\dots,\frac{Af_k}{\|Af_k\|}\biggr) > 0,$$

where the infimum is taking over all orthonormal systems $(f_1,\dots,f_k)$. Using the Gram–Schmidt orthogonalization procedure build the orthogonal system $q_1,\dots,q_k$ from $Af_1/\|Af_1\|,\dots,Af_k/\|Af_k\|$. Here

$$q_j = \frac{Af_j}{\|Af_j\|} - \sum_{i=1}^{j-1} a_{ij} \frac{Af_i}{\|Af_i\|}$$

with some a_{ij}. Let us prove that

$$\inf_{(f_1,\ldots,f_k)} G\left(\frac{Af_1}{\|Af_1\|},\ldots,\frac{Af_k}{\|Af_k\|}\right)=\inf_{(f_1,\ldots,f_k)}\prod_{i=1}^{k}\|q_i\|^2>0.$$

If it is not so, then there exist the sequence $\{f_1^n,\ldots,f_k^n\}_{n\geqslant 1}$ and $j=\overline{1,k}$ such that $\|q_j^n\|\to 0,\ n\to\infty$. The invertibility of the operator A implies that

$$\left\|\frac{f_j^n}{\|Af^n)j\|}-\sum_{i=1}^{j-1}a_{ij}^n\frac{f_i^n}{\|Af_i^n\|}\right\|\to 0,\quad n\to\infty.$$

But

$$\left\|\frac{f_j^n}{\|Af_j^n\|}-\sum_{i=1}^{j-1}a_{ij}^n\frac{f_i^n}{\|Af_i^n\|}\right\|\geqslant\frac{1}{\|Af_j^n\|}>0.\qquad\square$$

Lemma A.2. *Suppose that A is a continuously invertible operator in a Hilbert space H. Then, for any $e_0=0$, $e_1,\ldots,e_n\in H$ such that $e_{i+1}-e_i\perp e_{j+1}-e_j$, $i,j=\overline{1,n-1}$, $i\neq j$, there exists a positive constant c such that for all $\vec{u}\in\mathbb{R}^n$ with $u_0=0$, we have the following relation*:

$$\big(B^{-1}(Ae_1,\ldots,Ae_n)\vec{u},\vec{u}\big)\geqslant c\sum_{i=0}^{n-1}\frac{(u_{i+1}-u_i)^2}{\|e_{i+1}-e_i\|^2}.$$

Proof. It was proved in [6] that in the case $\vec{u}=((h_0,Ae_1),\ldots,(h_0,Ae_n)),\ h_0\in H$, we have the following relation:

$$\big(B^{-1}(Ae_1,\ldots,Ae_n)\vec{u},\vec{u}\big)=\|P_{Ae_1\ldots Ae_n}h_0\|^2,$$

where $P_{e_1\ldots e_n}$ is the projection onto $LS\{e_1,\ldots,e_n\}$ (the linear span generated by elements $e_1,\ldots,e_n$). Note that

$$\big((h_0,Ae_1),\ldots,(h_0,Ae_n)\big)=\big((A^*h_0,e_1),\ldots,(A^*h_0,e_n)\big).$$

Since $(A^*h_0,e_1)=u_1,\ \ldots,\ (A^*h_0,e_n)=u_n$, we have

$$A^*h_0=\sum_{i=0}^{n-1}\frac{e_{i+1}-e_i}{\|e_{i+1}-e_i\|^2}(u_{i+1}-u_i)+r,\tag{A.1}$$

where $r\perp e_i$, $i=\overline{0,n}$.

Consequently,

$$h_0=\sum_{i=0}^{n-1}A^{*-1}\left(\frac{e_{i+1}-e_i}{\|e_{i+1}-e_i\|^2}(u_{i+1}-u_i)\right)+A^{*-1}r.$$

Let us remark that the continuous invertibility of the operator A implies the existence of A^{*-1}. It follows from (A.1) that

$$\big(B^{-1}(Ae_1,\ldots,Ae_n)\vec{u},\vec{u}\big)=\left\|A^{*-1}\left(\sum_{i=0}^{n-1}\frac{(e_{i+1}-e_i)(u_{i+1}-u_i)}{\|e_{i+1}-e_i\|^2}+r\right)\right\|^2,$$

consequently,

$$\big(B^{-1}(Ae_1,\dots,Ae_n)\vec{u},\vec{u}\big) \geqslant c\sum_{i=0}^{n-1}\frac{(u_{i+1}-u_i)^2}{\|e_{i+1}-e_i\|^2} + c\|r\|^2 \geqslant c\sum_{i=0}^{n-1}\frac{(u_{i+1}-u_i)^2}{\|e_{i+1}-e_i\|^2}. \qquad \square$$

The following statement describes a direct application of Lemmas A.1 and A.2.

For $s_1,\dots,s_n \in \Delta_n$ and $u_1,\dots,u_n \in \mathbb{R}$, let $p_{s_1\dots s_n}(u_1,\dots,u_n)$ be the density of the Gaussian vector $(x(s_1),\dots,x(s_n))$ in $\mathbb{R}^n$. Here x is the Gaussian integrator with representation (1.2).

Lemma A.3. (*See* [14].) *Suppose that A in representation* (1.2) *is continuously invertible. Then there exist positive constants $c_1(n), c_2$ such that we have the following relation*:

$$p_{s_1\dots s_n}(u_1,\dots,u_n) \leqslant \frac{c_1(n)}{\sqrt{s_1(s_2-s_1)\dots(s_n-s_{n-1})}}\exp\left\{-\frac{1}{2}c_2\sum_{j=0}^{n-1}\frac{(u_{j+1}-u_j)^2}{s_{j+1}-s_j}\right\}.$$

For a proof, see [14].

Lemma A.4. *For any $0 \leqslant \alpha < 1$,*

$$\sup_{y\in L_2([0;1])}\int_0^1 \frac{1}{\|\mathbf{1}_{[0;t]}-y\|^{1+\alpha}}\,\mathrm{d}t < +\infty.$$

Proof. Put $g_0(t) := \mathbf{1}_{[0;t]}$. Note that

$$\int_0^1 \frac{1}{\|g_0(t)-y\|^{1+\alpha}}\,\mathrm{d}t = \int_0^{+\infty}\lambda\big\{t\colon \big\|g_0(t)-y\big\|^{-(1+\alpha)} \geqslant z\big\}\,\mathrm{d}z,$$

where λ is the Lebesgue measure on $[0;1]$. Then to prove the lemma, it suffices to check that, for $b>0$,

$$\sup_{y\in L_2([0;1])}\int_b^{+\infty}\lambda\big\{t\colon \big\|g_0(t)-y\big\|^{-(1+\alpha)} \geqslant z\big\}\,\mathrm{d}z < +\infty.$$

For any $g_0(t_0)$ and $g_0(t_1)$ from the closed ball $\overline{B}(y, 1/z^{1/(1+\alpha)})$, we have the following relation:

$$|t_0-t_1| = \big\|g_0(t_0)-g_0(t_1)\big\|^2 \leqslant \frac{4}{z^{2/(1+\alpha)}}. \tag{A.2}$$

Inequality (A.2) implies that

$$\left\{t\colon \big\|g_0(t)-y\big\| \leqslant \frac{1}{z^{1/(1+\alpha)}}\right\} \subset \left[t_0-\frac{4}{z^{2/(1+\alpha)}};\, t_0+\frac{4}{z^{2/(1+\alpha)}}\right]. \tag{A.3}$$

It follows from (A.3) that, for $0\leqslant\alpha<1$,

$$\int_b^{\infty}\lambda\left\{t\colon \big\|g_0(t)-y\big\| \leqslant \frac{1}{z^{1/(1+\alpha)}}\right\}\mathrm{d}z \leqslant 8\int_b^{+\infty}\frac{\mathrm{d}z}{z^{2/(1+\alpha)}} < +\infty.$$

Lemma A.5. *For elements* $e_1, \ldots, e_n$ *of* $L_2([0;1])$ *and a projection* Q*, let* $Qe_1, \ldots, Qe_n$ *be linearly independent. Suppose that elements* f*,* g *satisfy the following relationships for all* $i = 1, \ldots, n$*:*

$$(f, e_i) = (g, Qe_i).$$

Then

$$\|P_1 f\| \leqslant \|P_2 g\|,$$

where P_1 *and* P_2 *are the orthogonal projections onto linear spans of* $e_1, \ldots, e_n$ *and* $Qe_1, \ldots, Qe_n$*, respectively.*

Proof. Note that, for all $i = 1, \ldots, n$,

$$(g, Qe_i) = (P_2 g, Qe_i) = (QP_2 g, e_i) = (P_1 f, e_i).$$

Consequently,

$$P_1 f = P_1 Q P_2 g = P_1 P_2 g.$$

Hence,

$$\|P_1 f\| \leqslant \|P_2 g\|. \qquad \square$$

Lemma A.6. *Let* $\Delta_0 = \emptyset$*, and* $\Delta_1, \ldots, \Delta_n$ *be subsets of* $[0;1]$*. Then*

$$\varGamma(\mathbf{1}_{\Delta_1}, \ldots, \mathbf{1}_{\Delta_n}) \geqslant \prod_{k=1}^{n} \left| \Delta_k \setminus \bigcup_{j=1}^{k-1} \Delta_j \right|.$$

Proof. Since

$$\varGamma(\mathbf{1}_{\Delta_1}, \ldots, \mathbf{1}_{\Delta_n}) = |\Delta_1| \prod_{k=2}^{n} \|h_k\|^2,$$

where h_k is the orthogonal component of $\mathbf{1}_{\Delta_k}$ with respect to the linear span of $\mathbf{1}_{\Delta_1}, \ldots, \mathbf{1}_{\Delta_{k-1}}$, it suffices to prove that, for $k = 2, \ldots, n$,

$$\|h_k\|^2 \geqslant \left| \Delta_k \setminus \bigcup_{j=1}^{k-1} \Delta_j \right|.$$

For the set Δ such that $|\Delta| > 0$, denote by P_Δ the orthogonal projection onto $\mathbf{1}_\Delta$. Then

$$\|P_{\Delta_j} \mathbf{1}_{\Delta_k}\|^2 = \frac{|\Delta_k \cap \Delta|^2}{|\Delta|} \leqslant |\Delta_k \cap \Delta|.$$

Consider the representation

$$\bigcup_{j=1}^{k-1} \Delta_j = \bigcup_{i=1}^{l} H_i,$$

where $|H_i| > 0$, $H_i \cap H_j = \emptyset$, $i \neq j$. All H_i belong to the algebra generated by $\{\Delta_j\}$ and every Δ_j can be obtained as the union of certain H_i. Then the linear span of $\mathbf{1}_{\Delta_1}, \ldots, \mathbf{1}_{\Delta_{k-1}}$ is a subset of the linear span of $\mathbf{1}_{H_1}, \ldots, \mathbf{1}_{H_l}$. Hence,

$$\|h_k\|^2 \geqslant |\Delta_k| - \sum_{i=1}^{l} |\Delta_k \cap H_i| = \left| \Delta_k \setminus \bigcup_{j=1}^{k-1} \Delta_j \right|. \qquad \square$$

Appendix B: On some relations between generalized functionals from white noise

Consider linearly independent elements $f_1, \dots, f_n \in L_2([0;1])$. Here we investigate conditions on elements $r_j \in L_2([0;1])$, $j = \overline{1, n-1}$, that allow us to establish the following relation:

$$\int_{\mathbb{R}} \prod_{k=1}^{n} \delta_0\big((f_k, \xi) - u\big)\,\mathrm{d}u = \prod_{j=1}^{n-1} \delta_0\big((r_j, \xi)\big), \tag{B.1}$$

which is understood as equality of the generalized functionals from white noise [24] and will be checked using the Fourier–Wiener transform.

Let us recall that for random variable α that has a finite second moment and is measurable with respect to the white noise ξ, its Fourier–Wiener transform is

$$\mathcal{T}(\alpha)(h) = \mathbf{E}\alpha \mathrm{e}^{(h,\xi) - \|h\|^2/2}.$$

It is well known that the Fourier–Wiener transform uniquely determines a random variable α. The definition of the action of δ-function on the random vector $((f_1, \xi) - u, \dots, (f_n, \xi) - u)$ or $((r_1, \xi), \dots, (r_{n-1}, \xi))$ can be found in [24]. For example, to consider the Fourier–Wiener transform of $\prod_{j=1}^{n-1} \delta_0((r_j, \xi))$, we can substitute δ_0 by the Gaussian density

$$f_\varepsilon(u) = \frac{1}{\sqrt{2\pi\varepsilon}} \mathrm{e}^{-u^2/(2\varepsilon)}$$

and pass to the limit as $\varepsilon \to 0$.

The following statement describes a possible choice for r_j, $j = \overline{1, n-1}$.

Theorem B.1. *Let $f_1, \dots, f_n$ be linearly independent elements in $L_2([0;1])$. Then*

$$\int_{\mathbb{R}} \prod_{k=1}^{n} \delta_0\big((f_k, \xi) - u\big)\,\mathrm{d}u = \prod_{k=1}^{n-1} \delta_0\big((f_{k+1} - f_k, \xi)\big). \tag{B.2}$$

Proof. To prove the statement let us calculate the Fourier–Wiener transform of the left-hand side and the right-hand side of equality (B.2). Denote by $\mathcal{T}(\alpha)(h)$ the Fourier–Wiener transform of random variable α. We can check that

$$\mathcal{T}\Bigg(\prod_{j=1}^{n-1} \delta_0\big((r_j, \xi)\big)\Bigg)(h) = \frac{1}{(2\pi)^{(n-1)/2}\sqrt{G(r_1, \dots, r_{n-1})}} \mathrm{e}^{-\|P_{r_1 \dots r_{n-1}} h\|^2/2}. \tag{B.3}$$

Let us find the Fourier–Wiener transform of $\int_{\mathbb{R}} \prod_{k=1}^{n} \delta_0((f_k, \xi) - u)\,\mathrm{d}u$:

$$\mathcal{T}\Bigg(\int_{\mathbb{R}} \prod_{k=1}^{n} \delta_0\big((f_k, \xi) - u\big)\,\mathrm{d}u\Bigg)(h) = \int_{\mathbb{R}} \frac{1}{(2\pi)^{n/2}\sqrt{G(f_1, \dots, f_n)}} \mathrm{e}^{-(B^{-1}(f_1, \dots, f_n)(u\vec{e} - \vec{a}),\, u\vec{e} - \vec{a})/2}\,\mathrm{d}u, \tag{B.4}$$

where $\vec{e} = (1, \dots, 1)^{\mathrm{T}}$ and $\vec{a} = ((f_1, h), \dots, (f_n, h))^{\mathrm{T}}$. By integrating (B.4) over u we can get

$$\frac{1}{(2\pi)^{(n-1)/2}\sqrt{G(f_1, \dots, f_n)}\sqrt{(B^{-1}(f_1, \dots, f_n)\vec{e}, \vec{e})}} \times \exp\Bigg\{-\frac{1}{2}\Bigg(\big(B^{-1}(f_1, \dots, f_n)\vec{a}, \vec{a}\big) - \frac{(B^{-1}(f_1, \dots, f_n)\vec{a}, \vec{e})^2}{(B^{-1}(f_1, \dots, f_n)\vec{e}, \vec{e})}\Bigg)\Bigg\}. \tag{B.5}$$

It is not difficult to check that

$$\big(B^{-1}(f_1,\dots,f_n)\vec{a},\vec{a}\big)=\|P_{f_1\dots f_n}h\|^2.$$

Consider the function $f\in LS\{f_1,\dots,f_n\}$ such that $(f,f_k)=1$, $k=\overline{1,n}$. Then

$$\big(B^{-1}_{f_1\dots f_n}\vec{e},\vec{e}\big)=\|P_{f_1\dots f_n}f\|^2=\|f\|^2,\qquad \big(B^{-1}_{f_1\dots f_n}\vec{a},\vec{e}\big)=(P_{f_1\dots f_n}h,f).$$

Therefore, (B.5) equals

$$\frac{1}{(2\pi)^{(n-1)/2}\sqrt{G(f_1,\dots,f_n)}\|f\|}\mathrm{e}^{-(\|P_{f_1\dots f_n}h\|^2-\|P_fP_{f_1\dots f_n}h\|^2)/2}.$$

Denote by $f^{\perp}=\{v\in LS\{f_1,\dots,f_n\}\colon\ (v,f)=0\}$. Then

$$\mathcal{T}\Bigg(\int_{\mathbb{R}}\prod_{k=1}^{n}\delta_0\big((f_k,\xi)-u\big)\,\mathrm{d}u\Bigg)(h)=\frac{1}{(2\pi)^{(n-1)/2}\sqrt{G(f_1,\dots,f_n)}\|f\|}\mathrm{e}^{-\|P_{f^\perp}h\|^2/2}.\tag{B.6}$$

By comparing (B.3) and (B.6) we obtain the following conditions on elements r_k, $k=\overline{1,n-1}$:

1) $LS\{r_1,\dots,r_{n-1}\}=f^{\perp}$;
2) $G(r_1,\dots,r_{n-1})=G(f_1,\dots,f_n)\|f\|^2$.

Let us check that $r_j:=f_{j+1}-f_j$ satisfy conditions 1) and 2). Indeed, put $M=LS\{f_2-f_1,\dots,f_n-f_{n-1}\}$. Then $f\perp M$. Denote by r the distance from f_1 to M. W can see that

$$G(f_1,\dots,f_n)=G(f_1,\,f_2-f_1,\,\dots,\,f_n-f_{n-1})=r^2G(f_2-f_1,\,\dots,\,f_n-f_{n-1}).$$

Since

$$\left(f_1,\frac{f}{\|f\|}\right)=\|f_1\|\cos\alpha=r,$$

it follows that $r=1/\|f\|$. Consequently,

$$\|f\|^2G(f_1,\dots,f_{n-1})=G(f_2-f_1,\,\dots,\,f_n-f_{n-1}).\qquad\square$$

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