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PROBABILITIES OF HIGH EXCURSIONS OF GAUSSIAN FIELDS*

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Abstract. Let $\{\xi(t),\, t \in T\}$ be a differentiable (in the mean-square sense) Gaussian random field with $\mathbf{E}\xi(t) \equiv 0$, $\mathbf{D}\xi(t) \equiv 1$, and continuous trajectories defined on the m-dimensional interval $T \subset \mathbb{R}^m$. The paper is devoted to the problem of large excursions of the random field ξ. In particular, the asymptotic properties of the probability $P = \mathbf{P}\{-v(t) < \xi(t) < u(t),\, t \in T\}$, when, for all $t \in T$, $u(t), v(t) \geqslant \chi$, $\chi \to \infty$, are investigated. The work is a continuation of Rudzkis research started in [R. Rudzkis, Probabilities of large excursions of empirical processes and fields, *Sov. Math., Dokl.*, 45(1):226–228, 1992]. It is shown that if the random field ξ satisfies certain smoothness and regularity conditions, then $P = \mathrm{e}^{-Q} + Qo(1)$, where Q is a certain constructive functional depending on u, v, T, and the matrix function $R(t) = \operatorname{cov}(\xi'(t), \xi'(t))$.

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1 INTRODUCTION

Let $\xi(t)$, $t \in T \subset \mathbb{R}^m$, be a smooth Gaussian random field defined on the m-dimensional interval T, and $\zeta(T) = \sup_{t\in T} \xi(t)$ be its supremum. The study of the probability distribution of the random variable ζ, i.e., the probability $P(u) = \mathbf{P}\{\zeta(T) < u\}$ is a classical problem in probability theory of random fields, first arising in the theoretical radio engineering. In 1945, Rice [12] derived his famous formula for the average number of excursions of a random signal above a certain level used just for the purpose of approximating the probability of achieving this level by the signal. Since that time, a great number of publications has appeared, devoted mostly to the univariate case of Gaussian random functions. Under different assumptions on the process, a review of the methods for studying the behavior of $P(u)$ can be found in [4, 7, 9] and references therein.

During the last twenty years, several new methods have been introduced to obtain more precise results and extend the theory to the multivariate case of Gaussian random fields, which are the objective of this paper. Some examples of these contributions are the double-sum method by Piterbarg [10], the Euler–Poincaré characteristic approximation by Taylor et al. [19] and Adler and Taylor [2], the tube method by Sun [18], and the well-known Rice method, revisited by Azaïs and Delmas [3], Azaïs and Wschebor [4, 5, 6], and

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Piterbarg [11]. A short review of the methods listed above can be also found in [1]. The majority of the cases studied before deal mostly with homogeneous Gaussian random fields and constant by t level u, which is not always justified in some applicative problems. In this work, all these conditions are not assumed.

In this paper, the problem of large excursions of Gaussian field $\xi(t)$, $t \in T$, is investigated. In particular, the behavior of the probability $P = \mathbf{P}\{-v(t) < \xi(t) < u(t),\ t \in T\}$, where $u(t)$ and $v(t)$ are smooth functions, when, for all $t \in T$, $u(t), v(t) \geqslant \chi$, $\chi \to \infty$, is considered. The work is a continuation of Rudzkis research started in [13, 14, 15, 17], where a new method for investigating the distribution of the maximum of Gaussian processes was introduced. We tried here to generalize it to the case of random fields. The main result of this paper without proof was presented in [16].

2 MAIN RESULTS

Let $\{\xi(t),\ t \in \mathbb{R}^m\}$ be a differentiable (in the mean-square sense) Gaussian random field with continuous trajectories, and $\xi'(t)$ be its gradient (vector-row). To simplify the formulation of the results, assume that

$$\mathbf{E}\xi(t) \equiv 0, \qquad \mathbf{D}\xi(t) \equiv 1, \qquad \forall x, t \in \mathbb{R}^m, \quad \big\|R(t)x\big\| > \|x\|, \tag{2.1}$$

where $R(t) = \operatorname{cov}(\xi'(t), \xi'(t))$, and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^m$.

Consider the asymptotics of the probabilities of the form

$$P := \mathbf{P}\big\{-v(t) < \xi(t) < u(t),\ t \in T\big\} \tag{2.2}$$

when

$$\forall t \in T, \quad u(t), v(t) \geqslant \chi, \quad \chi \to \infty, \tag{2.3}$$

and the set $T = T(\chi)$ is an m-dimensional interval:

$$T = \big\{t = (t_1, \dots, t_m)^\top\colon\ a_i \leqslant t_i \leqslant b_i,\ i = \overline{1, m}\big\}, \quad a_{(\cdot)} < b_{(\cdot)}. \tag{2.4}$$

For a certain functional $Q = Q_R(v, u, T)$, which is defined below, the following relationship is proved:

$$P = \mathrm{e}^{-Q} + Qo(1). \tag{2.5}$$

To define the functional Q, let us introduce some additional notation. Let $M = \{1, \dots, m\}$. For any set $D \subset \mathbb{R}$, let $\delta_x(D) = \mathbf{1}_{\{x \in D\}}$,

$$\mu_T(\mathrm{d}t) = \prod_{i \in M} \mu_i(\mathrm{d}t_i) := \mu_1(\mathrm{d}t_1) \times \dots \times \mu_m(\mathrm{d}t_m),$$

where $\mu_i(\mathrm{d}t_i) = \mathrm{d}t_i + \delta_{a_i}(\mathrm{d}t_i) + \delta_{b_i}(\mathrm{d}t_i)$,

$$J = J_t = \{i\colon\ a_i < t_i < b_i,\ i \in M\},$$

$$\mathbb{Y}_{i,t} = \begin{cases} \{0\}, & i \in J, \\ [0, \infty), & t_i = b_i, \\ (-\infty, 0], & t_i = a_i, \end{cases}$$

$$\mathbb{Y}_t = \mathbb{Y}_{1,t} \times \dots \times \mathbb{Y}_{m,t}, \qquad \mu_t^*(\mathrm{d}y) = \prod_{i \in M \setminus J} \mathrm{d}y_i,$$

and $\mu_t^*(\mathbb{Y}_t) = 1$ if $J = M$.

Then $Q_R(v,u,T) = Q_R(v,T) + Q_R(u,T)$, where

$$Q_R(u,T) = \int\limits_T \mu_T(\mathrm{d}t) \int\limits_{\mathbb{Y}_t} \mu_t^*(\mathrm{d}y) \int\limits_{u(t)}^{\infty} \phi(x)\phi\bigl(u'(t)+y\big|R(t)\bigr)\det\bigl(xR(t)\bigr)_J\,\mathrm{d}x. \tag{2.6}$$

Hereinafter, for an arbitrary m-dimensional matrix $B = [B_{i,j}]$, denote $B_J = [B_{i,j}]_{i,j\in J}$ and $B_\emptyset = 1$; by $\phi(\cdot)$ and $\phi(\cdot|R)$ we denote the probability density functions of normal distributions $N(0,1)$ and $N(0,R)$, respectively. The next theorem is the main result of the paper.

Theorem 1. *Let conditions* (2.1)–(2.4) *be satisfied, and, for fixed functions* $\omega(\cdot)$ *and* $\rho(\cdot)$ *and any* $t,s\in\mathbb{R}^m$, *the following conditions be fulfilled*:

$$\max\bigl\{\mathbf{E}\bigl\|\xi'(t)-\xi'(s)\bigr\|, \bigl\|u'(t)-u'(s)\bigr\|, \bigl\|v'(t)-v'(s)\bigr\|\bigr\} \leqslant \omega\bigl(\|t-s\|\bigr), \qquad \lim_{x\to 0}\omega(x) = 0, \tag{2.7}$$

$$\bigl|\mathbf{E}\xi(t)\xi(s)\bigr| \leqslant \rho\bigl(\|t-s\|\bigr), \qquad \lim_{x\to\infty}\rho(x)\log(x) = 0, \qquad \max_{x>\delta}\rho(x) < 1 \quad \forall\delta>0. \tag{2.8}$$

Then asymptotic equality (2.5) *holds with uniform convergence with respect to* ξ, v, u, T *for which the above-mentioned conditions are satisfied.*

From Theorem 1 we get the following well-known result [3, 8].

Corollary 1. Let the random field ξ satisfy the conditions of Theorem 1, and matrix $R(t)$ be constant, $R(t)\equiv R$. Then, for any fixed $\delta>0$,

$$\mathbf{P}\bigl\{\xi(t)<\chi,\ t\in T\bigr\} = \mathrm{e}^{-\widetilde{Q}} + \widetilde{Q}o(1), \quad \chi\to\infty,$$

uniformly for all m-dimensional intervals T for which $b_i - a_i \geqslant \delta$, $i=\overline{1,m}$. Here

$$\widetilde{Q} = |T|(\det R)^{1/2}(2\pi)^{-m/2}\chi^{m-1}\phi(\chi), \quad |T| := \int\limits_T \mathrm{d}t.$$

In particular, we have that, for any $x\in\mathbb{R}$,

$$\begin{aligned}\mathbf{P}\biggl\{\xi(t) < \bigl(2\log|T|\bigr)^{1/2} + \biggl[\frac{m-1}{2}\log\bigl(2\log|T|\bigr) + \log\frac{(\det R)^{1/2}}{(2\pi)^{(m+1)/2}} + x\biggr]\bigl(2\log|T|\bigr)^{-1/2}\biggr\}\\ = \exp\bigl(-\exp(-x)\bigr) + o(1) \quad \text{as } |T|\to\infty.\end{aligned}$$

3 AUXILIARY LEMMAS

In this section, we use the above-mentioned notation and definitions, except condition (2.1), which can be waived; the letters u and v are used to denote univariate variables, but not functions. We assume that the random field ξ is twice continuously differentiable and

$$\xi_i'(t) = \frac{\partial}{\partial t_i}\xi(t), \qquad \xi_{i,j}''(t) = \frac{\partial^2}{\partial t_i\partial t_j}\xi(t), \qquad \xi''(t) = \bigl[\xi_{i,j}''(t)\bigr]_{i,j=\overline{1,m}}.$$

For an arbitrary matrix $B=[B_{i,j}]$, we denote by $\operatorname{vec} B$ a vector-row, consisting of the elements $B_{i,j}$ under condition $i\geqslant j$, $|B| = \max_{i,j} B_{i,j}$. In case $z\in\mathbb{R}^k$, $|z| = \max_i |z_i|$. We denote by f_X the distribution density function of an arbitrary random vector X. Further, we denote by $\varLambda_k$ the set of quadratic symmetric negative definite matrices with dimension k and $\varLambda_0 = \{1\}$.

Let $\zeta := \zeta(T) = \sup_{t\in T} \xi(t)$. For all $u < v$, $t \in T$,

$$\bigl\{\zeta = \xi(t) \in (u,v)\bigr\} \subset \bigl\{\xi(t) \in (u,v),\ \xi'(t) \in \mathbb{Y}_t,\ \xi''_J(t) \in \varLambda_r\bigr\} =: A_t; \tag{3.1}$$

hereinafter $r = r(t) = \operatorname{card} J$. Let us introduce the following functions:

$$\pi_\xi(t,x,y,\lambda) = \mathbf{P}\bigl\{\zeta = \xi(t) \bigm| \xi(t) = x,\ \xi'(t) = y,\ \xi''_J(t) = \lambda\bigr\}, \tag{3.2}$$

$$\psi_\xi(t,x,y,\lambda) = f_{\xi(t),\xi'(t)}(x,y)|\det\lambda|. \tag{3.3}$$

We denote by $F_\xi(\cdot \mid t,x,y)$ the conditional distribution of $\xi''_J(t)$ with respect to the random event $\{\xi(t) = x, \xi'(t) = y\}$ and

$$P_\xi(u,v,T) = \int\limits_T \mu_T(\mathrm{d}t) \int\limits_{\mathbb{Y}_t} \mu_t^*(\mathrm{d}y) \int\limits_u^v \mathrm{d}x \int\limits_{\varLambda_r} \pi_\xi(t,x,y,\lambda)\psi_\xi(t,x,y,\lambda)\, F_\xi(\mathrm{d}\lambda \mid t,x,y). \tag{3.4}$$

Lemma 1. *Let a multivariate Gaussian random field $X(t) = (\xi(t), \xi'(t), \operatorname{vec}\xi''(t))$ have continuous trajectories on T, and, for all $t \neq s$, the distribution of the vector $(X(t), X(s))$ be nondegenerate. Let, for some $\alpha > 0$ and $C_1 < \infty$, ξ'' satisfy the condition*

$$\forall t,s \in T, \quad \mathbf{E}\bigl|\xi''(t) - \xi''(s)\bigr| \leqslant C_1|t-s|^\alpha. \tag{3.5}$$

Then, for all $u < v$,

$$\mathbf{P}\{u < \zeta < v\} = P_\xi(u,v,T). \tag{3.6}$$

Further, in case (3.5) is fulfilled, we write $\xi'' \in \operatorname{Lip}_\alpha(C_1)$.

Corollary 2. If all the conditions of Lemma 1 are fulfilled, then the random variable ζ has an absolutely continuous distribution, and, for almost all u,

$$f_\zeta(u) \leqslant \int\limits_T \mu_T(\mathrm{d}t) \int\limits_{\mathbb{Y}_t} \mu_t^*(\mathrm{d}y) \int\limits_{\varLambda_r} \psi_\xi(t,u,y,\lambda)\, F_\xi(\mathrm{d}\lambda \mid t,u,y). \tag{3.7}$$

Remark 1. If all the conditions of Lemma 1 are fulfilled and the set $D \subset T$ can be presented in the form of a union of a finite number of intervals, then, by Corollary 2, the density function of the random variable $\zeta(D)$ is bounded.

For some twice differentiable function $g(t)$, denote $\overline{\xi}(t) = \max\{\xi(t), g(t) - \xi(t)\}$ and $\overline{\zeta} = \sup_{t\in T}\overline{\xi}(t)$. Let $\overline{P}_\xi(\cdot)$ be defined by formulas (3.2)–(3.4), where the random variable ζ is replaced with $\overline{\zeta}$.

Lemma 2. *Let $g'' \in \operatorname{Lip}_\alpha(C_1)$. Under the conditions of Lemma* 1*, for $v > u > \max_{t\in T} g(t)/2$, the following equality holds*:

$$\mathbf{P}\{u < \overline{\zeta} < v\} = \overline{P}_\xi(u,v,T) + \overline{P}_{g-\xi}(u,v,T). \tag{3.8}$$

4 PROOFS OF THE LEMMAS

Without loss of generality, assume that $T = [0,1]^m$. If there are no additional indications, in different places, we will denote by C different positive finite constants depending only on the covariance of the random field ξ and parameters m, α, C_1. For the proof of Lemma 1, an auxiliary statement is needed.

Lemma 3. *Let $\nu(t)$, $t \in T$, be a multivariate Gaussian random field with continuous trajectories, and let $\nu \in \mathrm{Lip}_\alpha(C_1)$. Then, for all $h > 0$ and $\beta \in (0, \alpha)$, the following inequality holds*:

$$\mathbf{P}\Big\{ \max_{s,t\in T,\,|s-t|\leqslant h} \big|\nu(t) - \nu(s)\big| > h^\beta \Big\} \leqslant Ch^{-m} \exp\Big\{ -\frac{1}{Ch^{2(\alpha-\beta)}} \Big\}. \tag{4.1}$$

Proof. For simplicity, assume that $n = 1/h$ is a natural number and denote $T(n) = \{1/n, 2/n, \dots, 1\}^m$. For $\tau \in T(n)$, let us define the random field $\gamma_\tau(\cdot)$ on T^2 by

$$\gamma_\tau(t,s) := C_1^{-1} h^{-\alpha} \big[\nu(\tau - ht) - \nu(\tau - hs)\big].$$

We have

$$\mathbf{E}\big|\gamma_\tau(\cdot)\big| \leqslant 1, \qquad \mathbf{E}\big|\gamma_\tau(\cdot) - \gamma_\tau(\cdot + \delta)\big| \leqslant 2|\delta|^\alpha.$$

Applying to the components of random field γ_τ Theorem 4.1.1 in [7], we obtain the following estimate:

$$\forall x \geqslant 0, \quad \mathbf{P}\Big\{ \sup_{\lambda\in T^2} \big|\gamma_\tau(\lambda)\big| \geqslant x \Big\} \leqslant C\exp\Big\{ -\frac{x^2}{C} \Big\}, \quad C = C(\alpha, m). \tag{4.2}$$

Since $\operatorname{card} T(n) = n^m$ and

$$\sup_{s,t\in T,\,|s-t|\leqslant h} \big|\nu(t) - \nu(s)\big| = C_1 h^\alpha \max_{\tau\in T(n)} \sup_{\lambda\in T^2} \gamma_\tau(\lambda),$$

(4.1) follows from (4.2). □

Proof of Lemma 1. For an arbitrary natural number k and quadratic matrix λ with dimension k, denote $\Phi_k = \{z\colon\ z \in \mathbb{R}^k,\ |z| \leqslant 1/2\}$, $\lambda\Phi_k = \{\lambda z,\ z \in \Phi_k\}$, and $\Phi_0 = \{1\}$. For $z \in \mathbb{R}^k$ and $G \subset \mathbb{R}^k$, let $\rho(z, G) = \inf_{w\in G} |z - w|$. Further, let the natural number $n \to \infty$ and $h = 1/n$. For $t \in T$, denote $S_t = S_t(h) = \{s\colon\ s \in T,\ |s - t| \leqslant h/2\}$ and $H_t = \{\zeta(S_t) = \zeta \in (u, v)\}$. Recall that $\zeta = \zeta(T)$. We obtain

$$\mathbf{P}\{u < \zeta < v\} \leqslant \sum_{\tau\in T(n)} \mathbf{P}\{H_\tau\}. \tag{4.3}$$

For $t \in T$, let

$$\gamma_i = \gamma_i(t) = \begin{cases} \{0\}, & i \in J, \\ 1, & t_i = b_i, \\ -1, & t_i = a_i, \end{cases}$$

$$\xi'^*(t) = \big(\gamma_i \xi'_i(t),\ i \notin J\big), \qquad \xi'_J(t) = \big(\xi'_i(t),\ i \in J\big).$$

Recall that $r = \operatorname{card} J$, $J = J_t$. Let us define the random event $B_t = B_t(h)$ as

$$B_t = \big\{ u < \xi(t) < v,\ \xi'^*(t) \in \mathbb{R}_+^{m-r},\ \xi''_J(t) \in \Lambda_r,\ \xi'_J(t) \in h\xi''_J(t)\Phi_r \big\}, \tag{4.4}$$

where $\mathbb{R}_+ = [0, \infty)$, $\mathbb{R}_+^0 = \{0\}$, $\xi'^*(t) = 0$ if $r = m$, and $\xi'_J(t) = 0$ if $r = 0$. Let us prove the relation

$$\sum_{\tau\in T(n)} \mathbf{P}(H_\tau \setminus B_\tau) = o(1). \tag{4.5}$$

For some $\beta \in (0,\alpha)$, denote $Q_h = \{\max_{t,s\in T,\,|t-s|\leqslant h} |\xi''(t) - \xi''(s)| \leqslant h^\beta\}$. Then Lemma 3 yields the following equality:

$$\mathbf{P}\{Q_h\} = 1 - o\bigl(h^m\bigr). \tag{4.6}$$

If random events Q_h and H_t took place, then, for some $s \in S_t$, a random event A_s defined in (3.1) occurred as well. In this case, from the Taylor formula we derive:

$$\rho\bigl(\xi''_J(t), \varLambda_r\bigr) \leqslant \bigl|\xi''(t) - \xi''(s)\bigr| \leqslant h^\beta, \tag{4.7}$$

$$\rho\bigl(\xi'^*(t), \mathbb{R}_+^{m-r}\bigr) \leqslant \bigl|\xi'(t) - \xi'(s)\bigr| \leqslant Ch\bigl(\bigl|\xi''(t)\bigr| + 1\bigr), \tag{4.8}$$

$$\bigl|\xi(t) - \xi(s)\bigr| = \bigl|\xi(t) - \xi(s) + \xi'(s)(s-t)\bigr| \leqslant Ch^2\bigl(\bigl|\xi''(t)\bigr| + 1\bigr), \tag{4.9}$$

$$\rho\bigl(\xi'_J(t), h\xi''_J(t)\varPhi_r\bigr) \leqslant \bigl|\xi'(t) - \xi'(s) + \xi''(t)(s-t)\bigr| \leqslant Ch^{1+\beta}\bigl(\bigl|\xi''(t)\bigr| + 1\bigr). \tag{4.10}$$

As the probability density function of the random vector $X(t)$ is bounded on T, then, denoting $\overline{B}_t = \{\xi'_J(t) \in h\xi''_J(t)\varPhi_r\}$, it is easy to obtain the following estimates:

$$\mathbf{P}\{\overline{B}_t\} \leqslant Ch^r, \tag{4.11}$$

$$\mathbf{P}\bigl\{0 < \rho\bigl(\xi'_J(t), h\xi''_J(t)\varPhi_r\bigr) < Ch^{1+\beta}\bigl(\bigl|\xi''(t)\bigr| + 1\bigr)\bigr\} \leqslant Ch^{r+\beta}, \tag{4.12}$$

$$\mathbf{P}\bigl\{0 < \rho\bigl(\xi''_J(t), \varLambda_r\bigr) < Ch^\beta,\ \overline{B}_t\bigr\} \leqslant Ch^{r+\beta}, \tag{4.13}$$

$$\mathbf{P}\bigl\{0 < \rho\bigl(\xi(t), (u,v)\bigr) < Ch^2\bigl(1 + \bigl|\xi''(t)\bigr|\bigr),\ \overline{B}_t\bigr\} \leqslant Ch^{r+2}. \tag{4.14}$$

It follows from (4.6)–(4.14) that, for all $t \in T$,

$$\mathbf{P}\{H_t \setminus B_t\} \leqslant Ch^{r+\beta}. \tag{4.15}$$

Note that $\sum_{t\in T(n)} h^r \leqslant C$ and, therefore, (4.15) implies (4.5).

Let us show that

$$\sum_{\tau\in T(n)} \mathbf{P}\{H_\tau B_\tau\} = P_\xi(u,v,T) + o(1). \tag{4.16}$$

For $\tau \in T$, let $T_\tau = \{t\colon\ \gamma_i(t) = \gamma_i(\tau),\ i = \overline{1,m}\}$. Denote $\overline{S}_t = S_t \cap T_\tau$. Under the conditions of Lemma 1, the probability density function $f_{X(t)}(\cdot)$ is uniformly continuous with respect to $t \in T$. Therefore, uniformly for $\tau \in T(n)$ and $t \in \overline{S}_t$, we obtain

$$\mathbf{P}\{B_\tau\} = h^r \int_u^v \mathrm{d}x \int_{\mathbb{Y}_t} \mu_t^*(\mathrm{d}y) \int_{\varLambda_r} \psi_\xi(t,x,y,\lambda)\, F_\xi(\mathrm{d}\lambda \mid t,x,y)\,\bigl(1 + o(1)\bigr). \tag{4.17}$$

Hereinafter, in the proof of (4.16), unless otherwise stated, the notation $o(1)$ means the convergence to zero as $n \to \infty$ uniformly for $\tau \in T(n)$ and $t \in \overline{S}_\tau$. Note that the statement of Corollary 2 follows from (4.3), (4.5), and (4.17). To finish the proof of (4.16), we need to show, for all $t \in \overline{S}_t$, the following equality:

$$\mathbf{P}\{H_\tau \mid B_\tau\} = \mathbf{P}\bigl\{\zeta(T) = \xi(t) \bigm| A_t\bigr\} + o(1). \tag{4.18}$$

Let $Z_t = (\xi(t), \xi'^*(t), \operatorname{vec}\xi''_J(t))$ and $Y_t = (Z_t, \xi'_J(t))$. From Lemma 3, for the process $\xi''(\cdot|t) = \xi''(\cdot) - \mathbf{E}\xi''(\cdot|Y_t)$, we obtain

$$\lim_{\delta\to 0}\max_{t\in T}\mathbf{P}\Big\{\max_{|s-\nu|\leqslant\delta}\big|\xi''(s|t)-\xi''(\nu|t)\big| > \delta^{\alpha/2}\Big\} = 0. \tag{4.19}$$

Let us fix $\delta > 0$ and denote $V_\tau = \{s\colon\ s\in T,\ |s-\tau|\leqslant\delta\}$. Then, using (4.19) and the Taylor formula for all $\tau\in T(n)$ and $t\in\overline{S}_\tau$, we obtain the estimates

$$\begin{aligned}&\mathbf{P}\big\{\zeta(S_\tau) > \zeta(V_\tau\setminus S_\tau)\ \big|\ B_\tau\big\}\geqslant 1-\varepsilon(\delta),\\ &\mathbf{P}\big\{\xi(t) > \zeta(V_\tau\setminus S_\tau)\ \big|\ A_t\big\}\geqslant 1-\varepsilon(\delta),\end{aligned} \tag{4.20}$$

where $\lim_{\delta\to 0}\varepsilon(\delta) = 0$.

Denote $D_\tau = T\setminus V_\tau$, $E_\tau = \{\zeta(D_\tau)\leqslant\zeta(S_\tau)\}$, and $E_{\tau,t} = \{\zeta(D_\tau)\leqslant\xi(t)\}$. By (4.20) and the arbitrariness of δ, to prove (4.18), it is enough to show that

$$\mathbf{P}\{E_\tau\mid B_\tau\} - \mathbf{P}\{E_{\tau,t}\mid A_t\} = o(1). \tag{4.21}$$

By $\mathbb{Z}_t$ denote the set of values of the random vector Z_t for which

$$\{Z_t\in\mathbb{Z}_t\} = \big\{\xi(t)\in(u,v),\ \xi'^*(t)\in\mathbb{R}_+^{m-r},\ \xi''(t)\in\varLambda_r\big\}.$$

We have

$$\mathbf{P}\{E_{\tau,t}\mid A_t\} = \frac{\int_{\mathbb{Z}_\tau}\mathbf{P}\{E_{\tau,t}\mid Y_t=(z,0)\}f_{Y_t}(z,0)\,\mathrm{d}z}{\int_{\mathbb{Z}_\tau}f_{Y_t}(z,0)\,\mathrm{d}z}. \tag{4.22}$$

For $z\in\mathbb{Z}_\tau$, denote by $\mathbb{W}_z$ the set of the values of $\xi'_J(\tau)$ for which the following relationship takes place: $\{\xi'_J(\tau)\in\mathbb{W}_z\}\Leftrightarrow\overline{B}_\tau$, when $Z_\tau = z$. By analogy with (4.22) we get

$$\mathbf{P}\{E_\tau\mid B_\tau\} = \frac{\int_{\mathbb{Z}_\tau}\mathrm{d}z\int_{\mathbb{W}_z}\mathbf{P}\{E_\tau\mid Y_\tau=(z,w)\}f_{Y_\tau}(z,w)\,\mathrm{d}w}{\int_{\mathbb{Z}_\tau}\mathrm{d}z\int_{\mathbb{W}_z}f_{Y_\tau}(z,w)\,\mathrm{d}w}. \tag{4.23}$$

For $t\in\overline{S}_\tau$, we have $\mathbb{Z}_t = \mathbb{Z}_\tau$, and, taking into account the continuity of the distribution density function $f_{Y_t}(y)$ with respect to t and y, we obtain

$$\forall z\in\mathbb{Z}_\tau,\quad \max_{w\in\mathbb{W}_z}\big|f_{Y_\tau}(z,w)-f_{Y_t}(z,0)\big| = o(1). \tag{4.24}$$

Further, let us define the vector-column $\varPsi_{s,t}$ by the equation

$$Y_t\varPsi_{s,t} = \mathbf{E}\big(\xi(s)\ \big|\ Y_t\big) - \mathbf{E}\big(\xi(s)\ \big|\ Y_t = 0\big),$$

and denote $\eta(s|t,y) = \xi(s) + (y - Y_t)\varPsi_{s,t} - y_1$, where y_1 is the first component of the vector y. The random field $\eta(\cdot|t,y)$ does not depend on Y_t, and the following equality holds:

$$\mathbf{P}\{E_{\tau,t}\mid Y_t = y\} = \mathbf{P}\Big\{\max_{s\in D_\tau}\eta(s|t,y)\leqslant 0\Big\}. \tag{4.25}$$

By means of Lemma 3, for any y, it is easy to get the relationship

$$\mathbf{P}\big\{\zeta(S_\tau)-\xi(t) > h\ \big|\ Y_\tau = y\big\} = o(1).$$

Then, for a certain $0 \leqslant \theta \leqslant 1$, uniformly for y and $\tau \in T(n)$, we obtain

$$\mathbf{P}\{E_\tau \mid Y_t = y\} - \mathbf{P}\Big\{\max_{s \in D_\tau} \eta(s|\tau, y) \leqslant \theta h\Big\} = o(1), \tag{4.26}$$

where $\theta = \theta(y, \tau)$. For $\tau \in T(n)$ and $z \in \mathbb{Z}_\tau$, let $\Delta(\tau, z) = \max|\eta(s|t, y) - \eta(s|\tau, \tilde{y})|$, where $y = (z, 0)$, $\tilde{y} = (z, w)$, and the maximum is taken by $s \in K$, $t \in \overline{S}_\tau$, $w \in \mathbb{W}_z$. Since

$$\eta(s|t, y) - \eta(s|\tau, \tilde{y}) = (y - Y_t)\Psi_{s,t} - (\tilde{y} - Y_\tau)\Psi_{s,t},$$

from the relationships $\max_{w \in \mathbb{W}_z}|w| = o(1)$, $\max \mathbf{E}|Y_t - Y_\tau| = o(1)$, and $\max_{s \in K}|\Psi_{s,t} - \Psi_{s,\tau}| = o(1)$ we obtain that

$$\forall z \in \mathbb{Z}_\tau, \quad \mathbf{E}\Delta(\tau, z) = o(1). \tag{4.27}$$

For any y, the random field $\eta(\cdot|\tau, y)$ satisfies the conditions of Lemma 1 on the set D_τ, which is the union of a finite number of intervals. By Remark 1, which follows from the already proved statement of Corollary 2, for all $\tau \in T(n)$ and any $a < b$, we have

$$\mathbf{P}\Big\{\max_{s \in D_\tau} \eta(s|\tau, y) \in (a, b)\Big\} \leqslant C(b - a), \tag{4.28}$$

where C depends only on δ and the distribution of the random field ξ. By (4.25)–(4.28) we get

$$\forall z \in \mathbb{Z}_\tau, \quad \max_{w \in \mathbb{W}_z} \big|\mathbf{P}\big\{E_{\tau,t} \bigm| Y_t = (z, 0)\big\} - \mathbf{P}\big\{E_\tau \bigm| Y_t = (z, w)\big\}\big| = o(1). \tag{4.29}$$

From (4.22)–(4.24) and (4.29) the validity of (4.21) follows, which, in turn, implies (4.18) and (4.16). As a result, letting $n \to \infty$, from (4.3), (4.5), and (4.16) we derive the inequality

$$\mathbf{P}\{u < \zeta < v\} \leqslant P_\xi(u, v, T). \tag{4.30}$$

If we denote $H_t = \{u < \zeta(S_t) < v,\ \zeta(S_t) > \zeta(T \setminus S_t)\}$, then the proofs of Eqs. (4.5) and (4.16) will be still valid, but, instead of (4.3), we will have the inequality

$$\mathbf{P}\{u < \zeta < v\} \geqslant \sum_{\tau \in T(n)} \mathbf{P}\{H_\tau\}.$$

Thus, inequality (4.30) is also valid with the opposite sign, which implies (3.6) and completes the proof of the lemma. □

Proof of Lemma 2. It is analogous to the proof of Lemma 1 with some minor differences. After replacement of the random variable ζ by $\overline{\zeta}$ in the definition of the event H_t, we obtain an analog of Eq. (4.3) for the probability $\mathbf{P}\{u < \overline{\zeta} < v\}$. We denote the events defined in (3.1) and (4.4) by $A_t(\xi)$ and $B_t(\xi)$, and let $B_t = B_t(\xi) \cup B_t(g - \xi)$. Then, under new definitions, we obtain (4.5). Further, $A_t(\xi) \cup A_t(g - \xi) = \emptyset$, and $\mathbf{P}\{B_t(\xi) \cup B_t(g - \xi)\} = \mathbf{P}(B_t)o(1)$. Therefore, to get the analog of Eq. (4.16), it suffices to show that

$$\mathbf{P}\big\{H_\tau \bigm| B_t(\nu)\big\} = \mathbf{P}\big\{\overline{\zeta} = \nu(t) \bigm| A_t(\nu)\big\} + o(1), \quad \nu = \xi, g - \xi. \tag{4.31}$$

The proof of expression (4.31) is the same as that of (4.18) in Lemma 1. Thus,

$$\sum_{\tau \in T(n)} \mathbf{P}(H_\tau B_\tau) = \overline{P}_\xi(u, v, T) + \overline{P}_{g-\xi}(u, v, T) + o(1). \tag{4.32}$$

From (4.32) and newly defined (4.3) and (4.5) an analog of (4.30) follows:

$$\mathbf{P}\{u < \overline{\zeta} < v\} \leqslant \overline{P}_{\xi}(u, v, T) + \overline{P}_{g-\xi}(u, v, T). \tag{4.33}$$

Repeating the same arguments as in the end of the proof of Lemma 1, we obtain (3.8). □

5 PROOF OF THEOREM 1

In the sequel, we will use the notation and definitions introduced in the previous sections. The letters u and v are reserved to denote the functions depending on $t \in \mathbb{R}^m$. In different places, unless stated otherwise, by C we denote positive finite constants, depending only on m, ω, and ρ. The symbols $\wedge$ and $\vee$ are used to denote the minimum and maximum, respectively. Further, we always assume that $|\theta| \leqslant 1$, and L denotes a large enough absolute positive constant. Obviously, it is enough to show that, for any fixed value of L, Eq. (2.5) holds in the case

$$Q = Q_R(v, u, T) \leqslant L, \tag{5.1}$$

which is assumed hereinafter. Let us first prove (2.5) under the additional condition that the functions u, v and the random field ξ are three times differentiable and that the second and third derivatives satisfy the condition

$$\forall t \in \mathbb{R}^m, \quad \max\bigl\{\mathbf{E}\bigl|\xi^{(j)}(t)\bigr|, \bigl|u^{(j)}(t)\bigr|, \bigl|v^{(j)}(t)\bigr|\bigr\} \leqslant æ \min_{s\colon |s-t| \leqslant 1} w_s, \quad j = 2, 3. \tag{5.2}$$

Here $w_t = u(t) \wedge v(t)$, and $æ = æ(\chi) \to 0$ as $\chi \to \infty$. Without loss of generality, assume that, for all $t \neq s$, the distribution of the random vector $(X(t), X(s))$ is not degenerate. If this condition does not hold, the first equality (2.5) is proved for the random field $\tilde{\xi}(t) = (\xi(t) + \varepsilon\hat{\xi}(t))/\sqrt{1+\varepsilon^2}$, then letting ε tend to zero. Here $\hat{\xi}(t)$ is a random field independent of ξ with zero mean and covariance $\operatorname{cov}(\hat{\xi}(t), \hat{\xi}(s)) = \exp\{-\|t-s\|^2/2\}$. Let $\overline{\zeta} = \overline{\zeta}(T) = \max_{t \in T}(\xi(t) - u(t)) \vee (-\xi(t) - v(t))$. Under the stated assumptions, the random field $\eta(t) = \xi(t) - u(t)$ and the function $g(t) = -u(t) - v(t)$ satisfy the conditions of Lemma 2. Denoting

$$\overline{p}_{\eta}(x) = \int\limits_T \mu_T(\mathrm{d}t) \int\limits_{\mathbb{Y}_t} \mu_t^*(\mathrm{d}y) \int\limits_{\Lambda_r} \overline{\pi}_{\eta}(t, x, y, \lambda)\psi_{\eta}(t, x, y, \lambda)\, F_{\eta}(\mathrm{d}\lambda \mid t, x, y), \tag{5.3}$$

where $r = r(t)$, we derive

$$\forall x \geqslant 0, \quad f_{\overline{\zeta}}(x) = \overline{p}_{\xi-u}(x) + \overline{p}_{-\xi-v}(x). \tag{5.4}$$

Since $\mathbf{D}\xi(t) \equiv 1$, the following equalities hold:

$$\operatorname{cov}\bigl(\xi(t), \xi'(t)\bigr) = 0, \qquad \operatorname{cov}\bigl(\xi(t), \xi''(t)\bigr) = -R(t). \tag{5.5}$$

Therefore, for $\eta = \xi - u$, we have

$$\begin{aligned} &f_{\eta(t),\eta'(t)}(x, y) \det\bigl(\mathbf{E}\bigl(-\xi_J''(t) \bigm| \eta(t) = x\bigr)\bigr) \\ &\quad = \phi\bigl(u'(t) + y \bigm| R(t)\bigr)\phi\bigl(u(t) + x\bigr) \det\bigl(\bigl(u(t) + x\bigr)R(t)\bigr)_J =: q_u(t, y, x). \end{aligned} \tag{5.6}$$

Equality (5.6) remains valid even after replacing u by v and taking $\eta = -\xi - v$. Let $\overline{F}(x) := \mathbf{P}\{\overline{\zeta} < x\}$. By (2.7) and (5.4) we have

$$\overline{F}(x) \leqslant Cq(x), \quad q(x) := \int\limits_T \mu_T(\mathrm{d}t) \int\limits_{\mathbb{Y}_t} \bigl[q_u(t, y, x) + q_v(t, y, x)\bigr]\, \mu_t^*(\mathrm{d}y). \tag{5.7}$$

Let us prove that, for all $x \geqslant 0$,

$$f_{\overline{\zeta}}(x) = \overline{F}(x)q(x)\big(1 + o(1)\big). \tag{5.8}$$

Hereinafter, $o(1)$ means such a convergence to zero, as $\chi \to \infty$, for which the upper bound estimate of the convergence speed depends only on m, ρ, and ω. Since $P = \overline{F}(0)$, $Q_R(u, v, T) = \int_0^\infty q(x)\,\mathrm{d}x$ and $\frac{\mathrm{d}}{\mathrm{d}x}\log \overline{F}(x) = f_{\overline{\zeta}}(x)/\overline{F}(x)$, from (5.8) we obtain the required equality (2.5). To prove (5.8), let us first show that, for some constants $0 < C_1 \leqslant C_2 < \infty$ and the function

$$\hat{q}(x) := \int\limits_T \phi_t(w_t + x)\,\mu_T(\mathrm{d}t), \tag{5.9}$$

where $\phi_t(x) = \phi(x)x^r$, $r = r(t)$, the following estimate is valid:

$$\forall x \geqslant 0, \quad C_1 \leqslant \frac{q(x)}{\hat{q}(x)} \leqslant C_2. \tag{5.10}$$

Note that by (2.1) and (2.7) we have

$$1 \leqslant \det R(\cdot) \leqslant C_3, \tag{5.11}$$

where $C_i = C_i(m, \omega)$, $i = \overline{1,3}$. The upper estimate in (5.10) is trivial. Let us prove the lower estimate. It is easy to see that, for $\beta_t = \beta_t(u) := \min_{y\in\mathbb{Y}_t} |u'(t) + y|$, we get

$$C \geqslant \int\limits_{\mathbb{Y}_t} \phi\big(u'(t) + y\big|R(t)\big)\,\mu_t^*(\mathrm{d}y) \geqslant \frac{\phi(C\beta_t)}{C}. \tag{5.12}$$

Next, denote $\phi_t = \phi_t(u(t))$, $\alpha_D = \int_D \phi_t\,\mu_T(\mathrm{d}t)$. Let us show that, for $D = \{t\colon\ t \in T,\ \beta_t > L\}$,

$$\alpha_D \leqslant \frac{C\alpha_T}{L}. \tag{5.13}$$

For this purpose, denoting $\beta_{t,i} = \min_{y\in\mathbb{Y}_T}(u_i'(t) + y_i)$ and $D_i = \{t\colon\ t \in T,\ \beta_{t,i} > L\}$, we first prove the inequality

$$\alpha_{D_i} \leqslant \frac{C\alpha_T}{L}. \tag{5.14}$$

For simplicity, assume that $i = 1$, $t^* := (t_2, \dots, t_m)$. Let us fix t^*, and let us denote $t = (t_1, t^*)$, $T_1 = \{t_1\colon\ a_1 \leqslant t_1 \leqslant b_1,\ u_1'(t) \leqslant L/2\}$, and $T_2 = \{t_1\colon\ a_1 \leqslant t_1 \leqslant b_1,\ u_1'(t) \geqslant L\}$. Define $l_0 = c_0 = a_1$, $l_k := \max\{c\colon\ a_1 \leqslant c \leqslant b_1,\ [c_{k-1}, c) \cap T_2 = \emptyset\}$, $c_k := \max\{c\colon\ a_1 \leqslant c \leqslant b_1,\ [l_k, c) \cap T_1 = \emptyset\}$, and let n be the smallest natural number for which $a_1 \leqslant l_1 \leqslant c_1 \leqslant \cdots \leqslant l_n \leqslant c_n = b$. For $1 \leqslant k \leqslant n$,

$$u_1'(t) \geqslant \frac{L}{2}, \quad t_1 \in [l_k, c_k], \qquad u_1'(t) \leqslant L, \quad t_1 \in [c_{k-1}, l_k]. \tag{5.15}$$

Denoting $u_k = u(l_k, t^*)$, we have

$$u(t) \geqslant u_k + \frac{L(t_1 - l_k)}{2}, \quad t_1 \in [l_k, c_k]. \tag{5.16}$$

Assume that $l_k < b_1$; in this case, by (2.7), $\mu_1([c_{k-1}, l_k] \cap \{t_1\colon\ u(t) \leqslant u_k\}) \geqslant C$, and, taking into account the equalities

$$r(a_1, t^*) = r(b_1, t^*) = r(t) - 1, \quad t_1 \in (a_1, b_1),$$

from (5.15), (5.16) we derive the inequality

$$\forall L \geqslant 1, \quad \int\limits_{[l_k, c_k]} \phi_t \, \mu_1(\mathrm{d}t_1) \leqslant \frac{C \int_{[c_{k-1}, l_k]} \phi_t \, \mu_1(\mathrm{d}t_1)}{L}, \quad k = 1, \ldots, n. \tag{5.17}$$

Here $\mu_i(\mathrm{d}t_i) := \mathrm{d}t_i + \delta_{a_i}(\mathrm{d}t_i) + \delta_{b_i}(\mathrm{d}t_i)$. Let

$$D_i := \Big\{ t\colon\ t \in T,\ \min_{y \in \mathbb{Y}_t} \big(u_i'(t) + y_i\big) > L \Big\}.$$

From (5.17) and the arbitrariness of t^* it follows that

$$\int\limits_{D_i} \phi_t \, \mu_T(\mathrm{d}t) \leqslant \frac{C}{L} \int\limits_{T} \phi_t \, \mu_T(\mathrm{d}t), \quad i = 1, \ldots, m. \tag{5.18}$$

Estimate (5.18) remains valid if, in the definition of D_i, we replace $u_i'(t) + y_i$ by $-(u_i'(t) + y_i)$. Thus, we obtain (5.14), which implies (5.13). For $x \geqslant 0$, inequalities (5.12) and (5.13) also are valid if we replace the functions $u(\cdot)$ by $u(\cdot) + x$ or $v(\cdot) + x$, which implies (5.10).

Further, let us slightly change the functional $\overline{p}_\eta(\cdot)$ defined in (5.3). Let $\eta = \xi - u$. Denote $\psi_\eta(t, x, y) = f_{\eta(t), \eta'(t)}(x, y) \det(xR(t))_J$,

$$\widehat{\mathbb{Y}}_t = \widehat{\mathbb{Y}}_{t,\eta} = \big\{ y\colon\ y \in \mathbb{Y}_t,\ \big|\mathbf{E}\eta'(t) - y\big| \leqslant L \big\},$$

and let $\overline{\Lambda}_r$ be the set of symmetric matrices of dimension $r = r(t)$. By (5.2), (5.5), and (5.11), for $x \geqslant 0$, we achieve

$$\begin{aligned} &\int\limits_{T} \mu_T(\mathrm{d}t) \int\limits_{\mathbb{Y}_t} \mu_t^*(\mathrm{d}y) \int\limits_{\overline{\Lambda}_r} \big| \psi_\eta(t, x, y, \lambda) \mathbf{1}_{\lambda \in \Lambda_r} - \psi_\eta(t, x, y) \mathbf{1}_{y \in \widehat{\mathbb{Y}}_t} \big| \, F_\eta(\mathrm{d}\lambda \mid t, x, y) \\ &\quad \leqslant C \hat{q}(x) \big(\text{æ} + \mathrm{e}^{-L^2/C} \big), \end{aligned} \tag{5.19}$$

where $\psi_\eta(t, x, y, \lambda)$ and F_η are defined in (3.3), and $r = r(t)$. In the sequel, we need the following notation:

$$\underline{u}_t := \min_{s \in T} \big(u(s) + |t - s| \big). \tag{5.20}$$

It is obvious that

$$|\underline{u}_t - \underline{u}_\tau| \leqslant |t - \tau|. \tag{5.21}$$

Denoting $T_\eta = \{t\colon\ t \in T,\ u(t) \leqslant (4\underline{u}_t) \wedge (v(t) + \sqrt{\chi})\}$, it is easy to show that

$$\forall x \geqslant 0, \quad \int\limits_{T \setminus T_\eta} \mu_T(\mathrm{d}t) \int\limits_{\widehat{\mathbb{Y}}_t} \psi_\eta(t, x, y) \, \mu_t^*(\mathrm{d}y) = o(1) \hat{q}(x). \tag{5.22}$$

In particular, one can use inequality (5.54), which is proved below. Finally, it is clear that

$$\int\limits_{\overline{\Lambda}_r} \overline{\pi}_\eta(t, x, y, \lambda) \, F_\eta(\mathrm{d}\lambda \mid t, x, y) = \mathbf{P}\big\{ \overline{\zeta} = x \bigm| \eta(t) = x,\ \eta'(t) = y \big\} =: \overline{\pi}_\eta(t, x, y). \tag{5.23}$$

Denote

$$\hat{p}_\eta(x) = \int\limits_T \mu_T(\mathrm{d}t) \int\limits_{\widehat{\mathbb{Y}}_t} \overline{\pi}_\eta(t,x,y)\psi_\eta(t,x,y)\,\mu_t^*(\mathrm{d}y). \tag{5.24}$$

From (5.20)–(5.22) we obtain the estimate

$$\left|f_{\overline{\zeta}}(x) - \hat{p}_{\xi-u}(x) - \hat{p}_{-\xi-v}(x)\right| \leqslant C\hat{q}(x)\big(o(1) + \mathrm{e}^{-L^2/C}\big) \quad \forall x \geqslant 0. \tag{5.25}$$

Taking into account (5.10) and the arbitrariness of L, to prove (5.8), it remains to show that

$$\int\limits_T \mu_T(\mathrm{d}t) \int\limits_{\mathbb{Y}_t} \big[\overline{F}(x) - \overline{\pi}_\eta(t,x,y)\big]\psi_\eta(t,x,y)\,\mu_t^*(\mathrm{d}y) = C(L)\hat{q}(x)o(1), \tag{5.26}$$

where $C(L)$ is a finite positive function. Clearly, it suffices to prove (5.26) for $x = 0$ since if $x > 0$, instead of the functions u and v, we can consider the functions $u(\cdot) + x$ and $v(\cdot) + x$. Let us fix $\tau \in T_\eta$, $y \in \mathbb{Y}_\tau$ and denote $\overline{t} = t - \tau$, $z = (u(\tau), u'(\tau) + y)$, $Z = (\xi(\tau), \xi'(\tau))$. For some $\delta > 0$, let $S = S_\tau(\delta) = \{t\colon |\overline{t}| \leqslant \delta\} \cap T$. Let us show that

$$\mathbf{P}\big\{\overline{\zeta}(S) > 0 \bigm| Z = z\big\} = o(1). \tag{5.27}$$

Denoting $\epsilon(t) = \eta(t) - \eta(\tau) - \eta'(\tau)\overline{t} = \int_0^1 \mathrm{d}\alpha \int_0^\alpha \overline{t}^\top \eta''(\tau + \beta\overline{t})\overline{t}\,\mathrm{d}\beta$ and using (5.2) and (5.5), we obtain

$$\mathbf{E}\big(\epsilon(t) \bigm| Z = z\big) = -u(\tau)\overline{t}^\top R(\tau)\overline{t}\left(1 + \frac{\theta}{2}\right) \tag{5.28}$$

if χ is large and δ is small enough. Applying Lemma 3 to the matrix random field $\tilde{\eta}''(t) = \eta''(t) - \mathbf{E}(\eta''(t) \mid Z)$ and using (5.2), we obtain the equality

$$\mathbf{P}\left\{\max_{t\in S}\|\tilde{\eta}''(t)\| \geqslant \frac{u(\tau)}{3}\right\} = o(1). \tag{5.29}$$

From (5.28) and (5.29) we get (5.27). Further, by (5.7) and (5.10) we find

$$f_{\overline{\zeta}(T)}(x) \leqslant C\int\limits_T \phi_t(w_t + x)\,\mu_T(\mathrm{d}t). \tag{5.30}$$

Replacing in (5.30) the set T by S, we have

$$\mathbf{P}\big\{\overline{\zeta}(S) > 0\big\} < C\phi(\chi)\chi^{m-1} = o(1). \tag{5.31}$$

Denote

$$\rho_0(t) = \mathbf{E}\xi(t)\xi(\tau), \qquad \rho_1(t) = \mathbf{E}\xi(t)\xi'(\tau).$$

For convenience, assume that, in (2.8), the norm $\|\cdot\|$ is replaced by $|\cdot|$. Furthermore, assume that

$$\forall t,\tau \in T, \quad \big|\rho_1(t)\big| \leqslant C\big(\big(\rho\big(|\overline{t}|\big)w_\tau\big) \wedge 1\big) =: \hat{\rho}_1\big(|\overline{t}|\big). \tag{5.32}$$

Without loss of generality, assume that the derivative of the function ρ satisfies the inequality

$$\forall x \geqslant 0, \quad \big|\rho'(x)\big| \leqslant \rho(x). \tag{5.33}$$

Otherwise, we can construct a function $\hat\rho(\cdot) \geqslant \rho(\cdot)$ satisfying conditions (2.8) and (5.33). We have

$$\mathbf{E}\bigl(\xi(t) \bigm| Z\bigr) = \rho_0(t)\xi(\tau) + \rho_1(t)R^{-1}(\tau)\xi'^{\top}(\tau) =: Z\gamma(t),$$
$$\mathbf{E}\bigl(\xi(t) \bigm| Z = z\bigr) = z\gamma(t),$$
$$\overline{\pi}(\tau, 0, y) = \mathbf{P}\bigl\{-v(t) \leqslant \xi(t) + (z - Z)\gamma(t) \leqslant u(t),\ t \in T\bigr\}.$$

If $|Z| \leqslant L$ for $t \in T_\eta \setminus S$ and all $y \in \mathbb{Y}_\tau$, then

$$\bigl|(z - Z)\gamma(t)\bigr| \leqslant \rho\bigl(|\bar t|\bigr)(w_\tau + \sqrt{\chi}) + CL\bigl(\rho\bigl(|\bar t|\bigr)w_\tau \wedge 1\bigr) =: g_\tau\bigl(|\bar t|\bigr). \tag{5.34}$$

Let $g_\tau(x) = g_\tau(\delta)$ for $x \leqslant \delta$. Denote

$$\varDelta_u(\tau) = \mathbf{P}\biggl\{\max_{t\in T} \frac{|\xi(t) - u(t)|}{g_\tau(|\bar t|)} \leqslant 1\biggr\}.$$

By (5.27), (5.31), and (5.34), we have

$$\max_{y\in\mathbb{Y}_\tau} \bigl|\overline{F}(0) - \overline{\pi}_\eta(\tau, 0, y)\bigr| \leqslant o(1) + \mathbf{P}\bigl\{|Z| > L\bigr\} + \varDelta_u(\tau) + \varDelta_v(\tau). \tag{5.35}$$

Taking into account (5.35), the arbitrariness of L, and the estimate

$$\mathbf{P}\bigl\{|Z| > L\bigr\} < C\phi\biggl(\frac{L}{C}\biggr),$$

to finish the proof of (5.26), it remains to obtain the relation

$$\int_{T_\eta} \varDelta_u(\tau)\phi_\tau(w_\tau)\,\mu_\tau(\mathrm{d}\tau) = o(1)C(L)\hat q(0). \tag{5.36}$$

To use Lemma 1 for this purpose, we first need to smooth the function g. Let K_1 be a three-times continuously differentiable nonnegative even function satisfying the conditions $\int_{-1}^{1} K_1(x)\,\mathrm{d}x = 1$ and $K_1(x) = 0$ for $|x| \geqslant 1$. Denote

$$a(t) = a_\tau(t) = \hat g_\tau\bigl(|\bar t| - 1\bigr), \quad \hat g_\tau(x) = \int_{-1}^{1} K_1(\alpha)g_\tau(x + \alpha)\,\mathrm{d}\alpha.$$

By (5.33) and properties of the kernel K_1, the derivatives of the function $a(\cdot)$ satisfy the inequality

$$\bigl|a^{(j)}(t)\bigr| \leqslant Ca(t), \quad j = \overline{1,3},\ t \in T. \tag{5.37}$$

Since $a(t) \geqslant g_\tau(|\bar t|)$, (5.35) remains true after the replacement of $g_\tau(|\bar t|)$ by $a(t)$ in the definition of $\varDelta_u(\tau)$. By (2.8), for some $\beta = \beta(\delta, \rho) > 0$, we have

$$\frac{a(t)}{w_\tau} \leqslant (1 - \beta) \wedge \bigl(CL\rho\bigl(|t - \tau| - 1\bigr)\bigr) =: \alpha\bigl(|\bar t|\bigr). \tag{5.38}$$

Applying Corollary 2 to the random field $\nu(t) = \eta(t)/a(t)$, we obtain

$$\varDelta_u(\tau) \leqslant \int\limits_T \mu_T(\mathrm{d}t) \int\limits_{-1}^{1} \mathrm{d}x \int\limits_{\mathbb{Y}_t} \mu_t^*(\mathrm{d}y) \int\limits_{\varLambda_r} f_{\nu(t),\nu'(t)}(x,y) |\det\lambda|\, F_\nu(\mathrm{d}\lambda \mid t,x,y). \tag{5.39}$$

Denoting $\tilde{x} = x + u(t)$, $\tilde{y} = u'(t) + y + xa^{-1}(t)a'(t)$, and

$$\tilde{\lambda} = u''(t) + \lambda + 2a^{-1}(t)a'(t)y^\top + a^{-1}(t)xa''(t),$$

from (5.39) we obtain the estimate

$$\varDelta_u(\tau) \leqslant \int\limits_T \mu_T(\mathrm{d}t) \int\limits_{-a(t)}^{a(t)} \mathrm{d}x \int\limits_{\mathbb{Y}_t} \mu_t^*(\mathrm{d}y) \int\limits_{\lambda\in\varLambda_r} \phi(\tilde{x})\phi\big(\tilde{y}\big|R(t)\big)|\det\lambda|\, F_\xi(\mathrm{d}\tilde{\lambda} \mid t,\tilde{x},\tilde{y}). \tag{5.40}$$

Recall that $F_\xi(\cdot \mid t,\tilde{x},\tilde{y})$ is the conditional distribution of $\xi_J''(t)$ with respect to $\xi(t) = \tilde{x}$, $\xi'(t) = \tilde{y}$. Denoting $q_{\tau,t}^* = \alpha(|\bar{t}|)w_\tau w_t^r \phi(w_t - \alpha(|\bar{t}|)w_\tau)$, from (2.1), (5.2), (5.37), (5.38), and (5.40) we derive

$$\varDelta_u(\tau) \leqslant Cq_{\tau,t}^*. \tag{5.41}$$

Let

$$q_{\tau,t} = \alpha\big(|\bar{t}|\big)w_t^{r+1}\phi\big(\big(1-\alpha\big(|\bar{t}|\big)\big)w_t\big). \tag{5.42}$$

Since $\phi_\tau(w_\tau)q_{\tau,t}^* \leqslant \phi_\tau(w_\tau)q_{\tau,t} + \phi_t(w_t)q_{t,\tau}$, from (5.41) we get the estimate

$$\int\limits_{T_\eta} \phi_\tau(w_\tau)\varDelta_u(\tau)\,\mu_T(\mathrm{d}\tau) \leqslant C\int\limits_T \phi_\tau(w_\tau)\bigg[\int\limits_T q_{\tau,t}\,\mu_T(\mathrm{d}t)\bigg]\mu_T(\mathrm{d}\tau).$$

It remains to show that

$$\max_{\tau\in T}\int\limits_T q_{\tau,t}\,\mu_T(\mathrm{d}t) = o(1). \tag{5.43}$$

This relation could, in turn, be derived from (2.3), (5.38), the property $\lim_{x\to\infty}\rho(x)\log(x) = 0$, and the estimate

$$\int\limits_T \frac{\phi_t(w_t)}{w_t}\,\mu_T(\mathrm{d}t) \leqslant CL, \tag{5.44}$$

which follows from (5.2) and (5.10).

Let $T_\tau = \{t\colon\ w_t \geqslant 5m\log(|\bar{t}|+1)\} \cap T$. It is obvious that

$$\int\limits_{T_\tau} q_{\tau,t}\,\mu_T(\mathrm{d}t) = o(1). \tag{5.45}$$

Further, as $|\bar{t}| \to \infty$, we have $\alpha(|\bar{t}| = o(1)/\log(|\bar{t}|))$ and, therefore,

$$\int\limits_{T\setminus T_\tau} q_{\tau,t}\,\mu_T(\mathrm{d}t) = o(1)\bigg(1 + \int\limits_{T\setminus T_\tau}\frac{\phi_t(w_t)}{w_t}\,\mu_T(\mathrm{d}t)\bigg). \tag{5.46}$$

From (5.44)–(5.46) we get (5.43). Finally, relation (5.8) is proved, which implies (2.5). Let us prove (2.5) without conditions (5.2) and (5.32).

To this end, let us smooth the random field ξ and functions u, v. Set $K(s) = \prod_{i=1}^m K_1(s_i)$ and $h = h(t) = \int_{\mathbb{R}^m} K(t-s)(\underline{u}_s^{-1} + \underline{v}_s^{-1})\,\mathrm{d}s$, where $\underline{u}_t$ is defined in (5.20). Denote $\xi_h(t) := \int_{\mathbb{R}^m} K(s)\xi(t+hs)\,\mathrm{d}s$, $\sigma^2(t) = \mathbf{D}\xi_h(t)$, $\xi^*(t) = \xi_h(t)/\sigma(t)$, $u^*(t) = u_h(t)/\sigma(t)$, and analogously define v^*. Using condition (2.7) and properties of the kernel K_1, it is easy to obtain the relations

$$h = \left(\frac{1}{\underline{u}_t} + \frac{1}{\underline{v}_t}\right)\bigl(1+o(1)\bigr), \qquad \bigl|h'(t)\bigr| \vee \bigl|h''(t)\bigr| \leqslant Ch^2, \tag{5.47}$$

$$u^*(t) - u(t) = o(1)h, \qquad \mathbf{E}\bigl|\xi(t) - \xi^*(t)\bigr| = o(1)h, \tag{5.48}$$

$$\bigl|u'^*(t) - u'(t)\bigr| = o(1), \qquad \mathbf{E}\bigl|\xi'(t) - \xi'^*(t)\bigr| = o(1). \tag{5.49}$$

It is evident that, after the replacement of ξ, u, v, ω, ρ by ξ^*, u^*, v^*, ω^*, ρ^*, conditions (5.2) and the conditions of Theorem 1 are fulfilled, where

$$\omega^*(\cdot) \leqslant C\omega(\cdot), \qquad \rho^*(\cdot) \leqslant \bigl(1+o(1)\bigr)\rho\bigl(\cdot - o(1)\bigr). \tag{5.50}$$

Finally, taking into account the equality

$$\xi'^*(\tau) = \left(\frac{1}{\sigma(\tau)}\right)'\xi_h(\tau) + \frac{1}{\sigma(\tau)}\int\limits_{\mathbb{R}^m} \frac{\mathrm{d}}{\mathrm{d}\tau}\left[\frac{1}{h(\tau)}K\left(\frac{1}{h(\tau)}(\tau - s)\right)\right]\xi(s)\,\mathrm{d}s,$$

we obtain (5.32) by replacing ξ, ρ with ξ^*, ρ^*. Thus, the following equality is valid:

$$P^* = \mathrm{e}^{-Q^*} + Q^* o(1). \tag{5.51}$$

Hereinafter, the addition symbol "$*$" in the notation of P, Q, q means the replacement of the variables ξ, u, v by ξ^*, u^*, v^* in the corresponding definitions. Let us show that (5.51) implies (2.5). First, we prove the relation

$$Q - Q^* = Qo(1). \tag{5.52}$$

Denote $\Phi_t(u) = \phi(u_t)u_t^{r-1}$. By (5.10) we have

$$Q = \int\limits_0^\infty q(x)\,\mathrm{d}x \geqslant \frac{\int_T \Phi_t(w)\,\mu_T(\mathrm{d}t)}{C}. \tag{5.53}$$

By means of (5.47)–(5.49) and the definition of q_u in (5.6) we obtain the estimate

$$\int\limits_0^\infty \mathrm{d}x \int\limits_{\mathbb{Y}_t} \bigl|q_u(t,x,y) - q^*(t,x,y)\bigr|\,\mu_t^*(\mathrm{d}y) \leqslant o(1)\Phi_t(w) + C\phi\left(\frac{L}{h}\right). \tag{5.54}$$

Using inequalities (5.10), (5.53), and (5.54), together with the inequality $\phi(L/h) \leqslant C\phi(L\underline{u}_t/3) + C\phi(L\underline{v}_t/3)$, to obtain relation (5.52), it suffices to show that

$$\int\limits_T \phi(2\underline{u}_t)\,\mu_T(\mathrm{d}t) = o(1)Q. \tag{5.55}$$

Denoting by N the set of integers, for $k \in N^m$, let $T_k = \{t\colon\ t \in T,\ k_i \leqslant t_i \leqslant k_i + 1,\ i = \overline{1, m}\}$ and $W_k = \{t\colon\ t \in T,\ \underline{u}_t = |t-s| + u(s),\ s \in T_k\}$. If $W_k \neq \emptyset$, then due to (2.7) and the definition of μ_T, we have

$$\int\limits_{T_k} \Phi_t(u)\, \mu_T(\mathrm{d}t) > \frac{\phi(\hat{u}_k)}{C\hat{u}_k}, \quad \hat{u}_k := \min_{t \in T_k} u(t). \tag{5.56}$$

On the other hand, for all $t \in W_k$ and fixed $s \in T_k$, we have $\underline{u}_t \geqslant \hat{u}_k + |t-s| - 1$; therefore,

$$\int\limits_{W_k} \phi(2\underline{u}_t)\, \mu_T(\mathrm{d}t) \leqslant C\phi(2\hat{u}_k - 1). \tag{5.57}$$

Since $T = \bigcup_k T_k = \bigcup_k W_k$, (5.53), (5.56), and (5.57) imply (5.55). It is obvious that relations (5.54) and (5.55) will be valid, and, after replacing u by v, Eq. (5.52) is proved. It remains to show that

$$P - P^* = o(1)Q. \tag{5.58}$$

Let $\underline{w}_t = \underline{u}_t \wedge \underline{v}_t$, $\eta = \xi - u$. For an arbitrary closed set $D \subset \mathbb{R}^m$, denote $\tau = \tau(D) = \arg\min_{t \in D} u(t)$, $h = h(\tau)$, and $\eta_D = \max_{t \in D} \eta(t)$. Assume that, for some $c = c(D) \in \mathbb{R}^m$, the following conditions are satisfied:

$$D = \prod_{i=1}^{m} [c_i, c_i + \hbar_i] \subset T, \quad \hbar_i = \hbar_i(D) = \left(\left(1 + \frac{\theta}{2}\right)h\right) \wedge (b_i - a_i), \quad i = \overline{1, m}. \tag{5.59}$$

By (5.47), (5.53), and (5.55), there exists a complex $\mathfrak{D}$ of sets D of the form (5.59) such that

$$T = \bigcup_{D \in \mathfrak{D}} D, \qquad \sum_{D \in \mathfrak{D}} \widehat{Q}_D \leqslant CQ, \tag{5.60}$$

where $\widehat{Q}_D := \int_D [\Phi_t(u) + \phi(2\underline{w}_t)]\, \mu_T(\mathrm{d}t)$. Denote $H_D = \{\eta_D > 0\} \Delta \{\eta^*_D > 0\}$. It is obvious that

$$\mathbf{P}(H_T) \leqslant \sum_{D \in \mathfrak{D}} \mathbf{P}(H_D). \tag{5.61}$$

By (5.60) and (5.61), to obtain the relation

$$\mathbf{P}(H_T) = o(1)Q, \tag{5.62}$$

it suffices to show that

$$\mathbf{P}(H_D) = o(1)\widehat{Q}_D, \quad D \in \mathfrak{D}. \tag{5.63}$$

Let us first get a similar estimate for the probability of the event $\widehat{H}_D := \{\eta_D > 0\} \Delta \{\hat{\eta}_D > 0\}$, where $\hat{\eta}(t) := \eta(\tau) + \eta'(\tau)^\top (t - \tau)$. We have

$$0 \leqslant \hat{\eta}_D - \eta(\tau) \leqslant \hat{\xi}_D - \xi(\tau) \leqslant \sum_{i=1}^{m} \left|\xi'_i(\tau)\hbar_i\right|. \tag{5.64}$$

By (2.1), (5.59), and (5.64), we have

$$\forall \alpha \geqslant 0, \quad \mathbf{P}\bigl\{\hat{\eta}_D - \eta(\tau) \geqslant \alpha\bigr\} \leqslant C\phi\left(\frac{\alpha}{Ch}\right). \tag{5.65}$$

Let $\nu(t) = \eta(t) - \hat{\eta}(t)$. Let us show that

$$\forall \alpha \geqslant x \geqslant 0, \quad \mathbf{P}\bigl\{\nu_D \geqslant \alpha \bigm| \xi(\tau) = x\bigr\} \leqslant C\phi\biggl(\frac{\alpha - x}{C\omega(h)h}\biggr). \tag{5.66}$$

By (2.1), (2.7), and the Taylor formula we get

$$\mathbf{E}\bigl|\nu(t) = \nu(s)\bigr| \leqslant C\omega\bigl(|t-s|\bigr)|t-s|, \qquad \max_{x\in D}\bigl|\mathbf{E}\bigl(\nu(t) \bigm| \xi(\tau) = x\bigr)\bigr| \leqslant C\omega(h)hx. \tag{5.67}$$

Next, let us estimate the distribution of the random variable $\overline{\nu}_D$, where $\overline{\nu}(t) = \nu(t) - \mathbf{E}(\nu(t) \mid \xi(\tau))$. Let $D_n = \{t\colon\ t \in D,\ |t_i - \tau_i| = \hbar_i j/4^n,\ i = \overline{1,m},\ j \in \{1,\dots,4^n\}\}$, $D_0 = D$. For $t \in D$, let $\hat{t}(n) = \arg\min_{s\in D_n}|t-s|$. We have

$$\mathbf{E}\bigl|\overline{\nu}(t) - \overline{\nu}\bigl(\hat{t}(n)\bigr)\bigr| \leqslant \frac{C\omega(h)h}{4^n}. \tag{5.68}$$

Denoting $\epsilon_k = \max_{t\in D_k}|\overline{\nu}(t) - \overline{\nu}(\hat{t}(k-1))|$, we obtain

$$\forall t \in D_n, \quad \bigl|\overline{\nu}(t)\bigr| \leqslant \sum_{k=1}^{n}\epsilon_k. \tag{5.69}$$

Taking into account the continuity of the trajectories of $\overline{\nu}$, (5.68), (5.69), and the inequality $\operatorname{card} D_k \leqslant 2^m 4^k$, we derive

$$\forall \alpha \geqslant 0, \quad \mathbf{P}\{\overline{\nu}_D \geqslant \alpha\} \leqslant \sum_{k=1}^{\infty}\mathbf{P}\biggl\{\epsilon_k \geqslant \frac{\alpha}{2^k}\biggr\} \leqslant C\sum_{k=1}^{\infty}4^k\phi\biggl(\frac{2^k\alpha}{C\omega(h)h}\biggr). \tag{5.70}$$

Finally, from (5.67) and (5.70) we obtain (5.66). Further, by means of (5.47) and the inequality $\mu_T(D) \geqslant h^m/C$, we obtain the inequality

$$\frac{\phi(u(\tau))}{u(\tau)} \leqslant C\widehat{Q}_D. \tag{5.71}$$

Using (5.47), (5.65), (5.66), and (5.71), we have

$$\mathbf{P}\bigl\{\widehat{H}_D,\ \eta(\tau) \leqslant -Lh\bigr\} \leqslant \epsilon(L)\widehat{Q}_D, \tag{5.72}$$

where $\lim_{L\to\infty}\epsilon(L) = 0$.

Then by (5.47), (5.66), and (5.1) we have

$$\mathbf{P}\bigl\{0 \geqslant \eta(\tau) \geqslant -Lh,\ \nu_D \geqslant \sqrt{\omega(h)}h\bigr\} = o(1)\widehat{Q}_D. \tag{5.73}$$

Finally, taking into account that $\hat{\eta}_D - \eta(\tau)$ is independent of $\xi(\tau)$, we get

$$\mathbf{P}\bigl\{0 \geqslant \eta(\tau) \geqslant -Lh,\ |\hat{\eta}_D| \leqslant \sqrt{\omega(h)}h\bigr\} = o(1)\widehat{Q}_D. \tag{5.74}$$

From (5.72)–(5.74) we have the relation

$$\mathbf{P}\{\widehat{H}_D\} = o(1)\widehat{Q}_D. \tag{5.75}$$

Denoting $\widehat{H}_D^* = \{\eta_D^* > 0\}\varDelta\{\hat{\eta}_D > 0\}$ and following the same procedure, we obtain the relation

$$\mathbf{P}\bigl\{\widehat{H}_D^*\bigr\} = o(1)\widehat{Q}_D,$$

which, together with (5.75), leads to (5.63), which in turn yields (5.62). It is obvious that (5.62) will still be valid after replacing the random field $\eta(\cdot)$ by $-\xi(\cdot) - v(\cdot)$ in the definition of H_τ and the function u by v. Thus, relation (5.58) is proved, and this completes the proof of the theorem.

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