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THE RANDOM INTEGRAL REPRESENTATION CONJECTURE: A QUARTER OF A CENTURY LATER

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Abstract. In [Z.J. Jurek, Relations between the s-selfdecomposable and selfdecomposable measures, *Ann. Probab.*, 13(2):592–608, 1985] and [Z.J. Jurek, Random integral representation for classes of limit distributions similar to Lévy class L_0, *Probab. Theory Relat. Fields*, 78:473–490, 1988] the random integral representation conjecture was stated. It claims that (some) limit laws can be written as the probability distributions of random integrals of the form $\int_{(a,b]} h(t)\,\mathrm{d}Y_\nu(r(t))$ for some deterministic functions h, r, and a Lévy process $Y_\nu(t)$, $t \geqslant 0$. Here we review situations where such a claim holds. Each theorem is followed by a remark that gives references to other related papers, results, and historical comments. Moreover, some open questions are stated.

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1 INTRODUCTION

In [15, p. 607] and later on in [16, p. 474], it was conjectured that:

> *Each class of limit distributions derived from sequences of independent random variables is the image of some subset of* ID *by some mapping defined as a random integral.*[1]

More formally, one claims that for a class K of limiting probability distributions on a Banach space E, there exist a function h (a space scaling), a function r (a time change), an interval A (in a positive half-line), and a subset $\mathcal{D}$ of ID (the class of all infinitely divisible distributions) such that

$$K \equiv I_A^{h,r}(\mathcal{D}) := \left\{ I_A^{h,r}(\nu) := \mathcal{L}\left(\int\limits_A h(s)\,\mathrm{d}Y_\nu\big(r(s)\big) \right) \colon \nu \in \mathcal{D} \right\}, \tag{1.1}$$

[1] Chatterji's subsequence principle claiming that: *Given a limit theorem for independent identically distributed random variables under certain moment conditions, there exists an analogous theorem such that an arbitrarily-dependent sequence* (*under the same moment conditions*) *always contains a subsequence satisfying this analogous theorem*, was proved by Aldous [1]. Although we do not expect that the above conjecture and Chatterji's subsequence principle are mathematically related, however, one can see a "philosophical" relation between those two.

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where $Y_\nu(s)$, $s \geqslant 0$, is an E-valued Lévy process with càdlàg paths such that its probability distribution at time 1, $\mathcal{L}(Y_\nu(1)) = \nu$, and $\mathcal{D}$ denotes the domain of existence of the above random integral; cf. http://www.math.uni.wroc.pl/∼zjjurek (The Conjecture).

Note that the notation $I_A^{h,r}$ for the random integral transformation can be simplified as follows:

$$I_A^{h,r}(\nu) = \int_0^\infty \mathbf{1}_A\big(r^*(s)\big)h\big(r^*(s)\big)\,\mathrm{d}Y_\nu(s) = I_{(0,\infty)}^{\tilde{h}(s),s}(\nu) \equiv I^{\tilde{h}}(\nu), \tag{1.1$'$}$$

where $\tilde{h}(s) := \mathbf{1}_A(r^*(s))h(r^*(s))$, and r^* is the inverse function of r.

The term *random integral* emphasizes the fact that the integrand h is a deterministic function. Thus, for $A = (a, b]$, we may define the random integral by the formal integration by parts formula, i.e.,

$$\int_A h(s)\,\mathrm{d}Y_\nu\big(r(s)\big) := h(b)Y_\nu\big(r(b)\big) - h(a)Y_\nu\big(r(a)\big) - \int_A Y_\nu\big(r(s)\big)\,\mathrm{d}h(s), \tag{1.2}$$

provided that h is of bounded variation. Thus, approximating the right-hand side integral by Riemann–Stieltjes sums, we get the following formula for the Fourier transform:

$$\log\big(\widehat{I_A^{h,r}(\nu)}\big)(y) = \int_A \log\widehat{\nu}\big(h(s)y\big)\,\mathrm{d}r(s), \quad \text{where } y \in E' \text{ is the dual Banach space.} \tag{1.3}$$

Random integrals on half-lines (a, ∞) are defined as weak limits of the integrals (1.1) for $(a, b]$ as $b \to \infty$.

Below, we review the old results as well as the more recent ones. In remarks after each theorem, we point out other related facts and papers. This survey is divided into three basic parts, and the last one rephrases the conjecture.

2 FROM A CLASS OF LIMIT LAWS TO A CLASS OF RANDOM INTEGRALS

(a) For the Lévy class L of selfdecomposable probability measures that coincides with the class of limiting distributions of the sequences

$$T_{a_n}(\xi_1 + \xi_2 + \cdots + \xi_n) + x_n, \quad T_a(x) := ax, \quad a > 0,\ x \in E, \tag{2.1}$$

where (ξ_i) are independent E-valued random variables, $x_n \in E$, $a_n > 0$, and the summands in (2.1) are uniformly infinitesimal, we have the following:

Theorem 1. (*See* [26].) *For the class L, we have that*

$$L = \big\{I_{(0,\infty)}^{\mathrm{e}^{-s},s}(\nu)\colon\ \nu \in \mathrm{ID}_{\log}\big\} = \bigg\{\mathcal{L}\bigg(\int_{(0,\infty)} \mathrm{e}^{-s}\,\mathrm{d}Y_\nu(s)\bigg)\colon\ \nu \in \mathrm{ID}_{\log}\bigg\},$$

where $\mathrm{ID}_{\log}$ *is the class of all infinitely divisible measures on E that integrate the function* $\log(1 + \|x\|)$.

Remark 1. (i) Wolfe [37] and Sato with Yamazato [32] had similar characterizations but with proofs valid *only* in Euclidean spaces.

(ii) The processes Y_ν above and, more generally, those in (1.1), are referred to as *background driving Lévy processes*, in short, BDLP; see [18].

(iii) A connection between selfdecomposable distributions and the one-dimensional Ising models in statistical physics was shown in [19].

(iv) Replacing T_a's in (2.1) by *arbitrary linear operators*, we get the so-called *operator-limit distributions theory*; see [24] or [30], and also [34, 36].

(b) If $L_1 \equiv L$ and, for positive integer $m \geqslant 2$, the class L_m is defined as the class of limits of (2.1) with $\mathcal{L}(\xi_i) \in L_{m-1}$, then $L_{m+1} \subset L_m$ and, moreover, we have the following:

Theorem 2. (*See* [12, 13].) *For the classes* L_m, $m = 1, 2, \ldots$, *we have that*

$$L_m = \big\{I_{(0,\infty)}^{\mathrm{e}^{-s}, s^m/m!}(\nu)\colon\ \nu \in \mathrm{ID}_{\log^m}\big\} = \bigg\{\mathcal{L}\bigg(\int\limits_{(0,\infty)} \mathrm{e}^{-s}\, \mathrm{d}Y_\nu\bigg(\frac{s^m}{m!}\bigg)\bigg)\colon\ \nu \in \mathrm{ID}_{\log^m}\bigg\},$$

where $\mathrm{ID}_{\log^m}$ *is the class of all infinitely divisible measures on* E *that integrate the function* $\log^m(1 + \|x\|)$.

Remark 2. (i) The idea of the classes L_m belongs to Urbanik [35] with a different scheme of summation; see also [28]. The iterative approach was proposed in [31] (for Euclidean spaces) and later generalized in [12] in two directions, replacing Euclidean space by an arbitrary separable Banach space E and, *more importantly*, replacing the group $(T_a,\ a > 0)$ of dilations by an arbitrary strongly continuous one-parameter group $\mathbb{U}$ of bounded linear operators on E.

(ii) The particular case of the group $\mathbb{U} := \{\mathrm{e}^{-tQ}\colon\ t \in \mathbb{R}\}$, where Q is a fixed bounded linear operator on a Banach space E, was investigated in [13], where it was shown that $L_m(Q) = \{I_{(0,\infty)}^{\mathrm{e}^{-tQ}, t}(\nu)\colon\ \nu \in \mathrm{ID}_{\log^m}\}$, i.e., in Theorem 2, the scalar function e^{-t} is replaced by the operator-valued function e^{-tQ}. Here one needs a new norm and the polar coordinates in a Banach space; see [14].

(iii) Thu [33] extended the classes L_m, $m = 1, 2, \ldots$, to L_α, $\alpha > 0$, by using the fractional calculus.

(c) Let us replace the linear normalization in (2.1) by the *nonlinear* shrinking s-operation U_r $(r > 0)$ and consider the class $\mathcal{U}$ of limiting distributions in the following scheme:

$$U_{r_n}(\xi_1) + U_{r_n}(\xi_2) + \cdots + U_{r_n}(\xi_n) + x_n, \quad U_r(x) := \max\big(\|x\| - r, 0\big)\frac{x}{\|x\|}, \quad x \neq 0, \tag{2.2}$$

where the summands are uniformly infinitesimal. Limiting distributions of (2.2) are called *s-selfdecomposable distributions*.

Theorem 3. (*See* [15].) *For the class* $\mathcal{U}$ *of* s*-selfdecomposable distributions, we have that*

$$\mathcal{U} = \big\{I_{(0,1)}^{s,s}(\nu)\colon\ \nu \in \mathrm{ID}\big\} = \bigg\{\mathcal{L}\bigg(\int\limits_{(0,1)} s\, \mathrm{d}Y_\nu(s)\bigg)\colon\ \nu \in \mathrm{ID}\bigg\},$$

where ID *is the class of all infinitely divisible measures.*

Remark 3. (i) Note that $U_r(U_s(x)) = U_{r+s}(x)$ (semigroup of nonlinear transformations) and for positive random variable $\xi > 0$, we get $U_r(\xi) = (\xi - r)^+$ (the positive part), i.e., it coincides with the famous financial derivative *European call option*.

(ii) Characterizations of the s-selfdecomposable distributions in terms of the Lévy–Khintchine formula were presented during the *Second Vilnius Conference*; see [10]. Complete proofs were given in [11].

(iii) The CLT for s-operations U_r was proved by Housworth and Shao [8].

(iv) The classes $\mathcal{U}_\beta := I_{(0,1)}^{s,s^\beta}(\mathrm{ID})$ were investigated in a series of papers: [16] for $\beta > 0$, [17] for $-1 \leqslant \beta < 0$, and [25] for $-2 < \beta \leqslant -1$. In the last two cases, the stable distributions appeared as convolution factors of the limiting distributions. Measures from $\mathcal{U}_\beta$ are called *generalized s-selfdecomposable distributions*.

In a similar way as the classes L_m were introduced in Theorem 2, one may iterate the random integral mapping $I^{s,s}_{(0,1)}$ from Theorem 3 and get the classes $\mathcal{U}^{\langle m\rangle}$, for which we have the following:

Theorem 4. (*See* [20].) *For the class* $\mathcal{U}^{\langle m\rangle}$ (*with* $m = 1, 2, \ldots$) *of* m*-times* s*-selfdecomposable distributions we have that*

$$\mathcal{U}^{\langle m\rangle} = \bigl\{ I^{s,\tau_m(s)}_{(0,1)}(\nu) \colon \nu \in \mathrm{ID} \bigr\} = \biggl\{ \mathcal{L}\biggl(\int\limits_{(0,1)} s \,\mathrm{d}Y_\nu\bigl(\tau_m(s)\bigr) \biggr) \colon \nu \in \mathrm{ID} \biggr\},$$

$$\tau_m(s) := \frac{1}{(m-1)!} \int\limits_0^s (-\log u)^{m-1} \,\mathrm{d}u, \ \ 0 < s \leqslant 1,$$

where ID *is the class of all infinitely divisible measures.*

Although the classes L_m and $\mathcal{U}^{\langle m\rangle}$ originated in two different limiting schemes (via the linear dilations T_a and the nonlinear s-operations U_r, respectively), they still admit some unexpected relations.

Corollary 1. (See [20].)

(a) We have the inclusions

$$L_{m+1} \subset \mathcal{U}^{\langle m+1\rangle} \subset \mathcal{U}^{\langle m\rangle} \subset \mathrm{ID}, \quad m = 1, 2, \ldots.$$

(b) $L_\infty := \bigcap_{m=1}^\infty L_m = \mathcal{U}^{\langle\infty\rangle} := \bigcap_{m=1}^\infty \mathcal{U}^{\langle m\rangle} =$ the smallest closed convolution semigroup that contains all stable measures.

Remark 4. (i) Maejima and Sato [29] proved that, besides the two instances described in Corollary 1(b), there are other three classes for which infinitely many integral iterations lead to the smallest closed convolution semigroup that contains all stable measures.

(ii) *Still an open question* is to describe $L_\infty(\mathbb{U}) := \bigcap_{m=1}^\infty L_m(\mathbb{U})$, where $\mathbb{U}$ is a one-parameter group of bounded linear operators on a Banach space E; cf. Remark 2(i) and [12].

3 FROM A CLASS OF RANDOM INTEGRALS TO…

The original aim (in the 1980s) was to identify *a given class* K *of limit distributions* as a collection of probability distributions of some random integrals; cf. Section 2. Later on, the question of whether *a given class of distributions* (or Fourier transforms or Lévy spectral measures) can be described in terms of some random integrals was often asked. In this section, we discuss only two such examples.

(a) Voiculescu and others, when studying the so-called *free-probability*, introduced new binary operations on probability measures and termed them *free-convolutions*; see [5] and references therein. For the *additive free-convolution* $\boxplus$, the Voiculescu transform $V_\nu(z)$, $z \in \mathbb{C}$ (an analogue of the characteristic function $\hat{\nu}(t)$, $t \in \mathbb{R}$) is additive. Namely, $V_{\nu_1 \boxplus \nu_2}(z) = V_{\nu_1}(z) + V_{\nu_2}(z)$. This property allowed introducing the notion of free-infinite divisibility.

Theorem 5. (*See* [21].) *A probability measure* ν *is* $\boxplus$*-infinitely divisible if and only if there exists a unique* $*$*-infinitely divisible probability measure* μ *such that*

$$(\mathrm{i}t)V_\nu\bigl((\mathrm{i}t)^{-1}\bigr) = \log\bigl(I^{s,1-\mathrm{e}^{-s}}_{(0,\infty)}(\mu)\bigr)\widehat{\ }(t) = \log\Biggl(\mathcal{L}\Biggl(\int\limits_0^\infty s \,\mathrm{d}Y_\mu\bigl(1-\mathrm{e}^{-s}\bigr)\Biggr)\Biggr)\widehat{\ }(t), \quad t \neq 0,$$

where $(Y_\mu(t),\, t \geqslant 0)$ *is a Lévy process such that* $\mathcal{L}(Y_\mu(1)) = \mu$*.*

Remark 5. Using Theorem 5, one can easily see that we have the integral mapping

$$\mathcal{K}(\mu) := I^{s,1-\mathrm{e}^{-s}}_{(0,\infty)}(\mu) = I^{-\log s, s}_{(0,1)}(\mu), \quad \mu \in \mathrm{ID}. \tag{3.1}$$

The mapping $\mathcal{K}$ was called the Υ (upsilon) transform and studied from a different point of view by Barndorff-Nielsen et al. [3], Barndorff-Nielsen et al. [4], and Maejima and Sato [29].

(b) The Thorin class $T(\mathbb{R}^d)$ is an example of a class of infinitely divisible distributions defined by properties of their Lévy spectral measures; later on, it was characterized by some random integrals. For more details, see [29, p. 121]; for related results, see [7]. For the class G of distributions, see [2].

Theorem 6. (*See* [29].)

$$T(\mathbb{R}^d) = \left\{ \mathcal{L}\left(\int_0^\infty e^*(t)\,\mathrm{d}Y_\mu(t) \right):\ \mu \in \mathrm{ID}_{\log} \right\} = \left\{ I^{s,e(s)}_{(0,\infty)}(\mu):\ \mu \in \mathrm{ID}_{\log} \right\},$$

where $e(s) := \int_s^\infty u^{-1}\mathrm{e}^{-u}\,\mathrm{d}u$, $s > 0$, *and* $e^*(t)$, $t > 0$, *is its inverse function.*

Remark 6. In [21, Prop. 4], the class $\mathcal{TS}_\alpha$ of tempered stable distributions with index $0 < \alpha < 1$ was identified as the class of random integrals

$$\mathcal{TS}_\alpha = \left\{ I^{s,\Gamma(-\alpha,s)}_{(0,\infty)}(\mu):\ \mu \in \mathrm{ID}_\alpha \right\} \quad \text{and} \quad \Gamma(-\alpha, s) := \int_s^\infty w^{-\alpha-1}\mathrm{e}^{-w}\,\mathrm{d}w, \quad s > 0,$$

where ID_α denotes the set of all infinitely measures whose Lévy spectral measures integrate $\|x\|^\alpha$ over the space E.

4 A CALCULUS ON (LÉVY EXPONENTS OF) ID DISTRIBUTIONS

On the random integrals (1.1) (viewed as mappings defined on (some) infinitely divisible probability measures μ), one can perform transformations such as compositions or the arithmetic operations. Because of the formula (1.3), all those operations have natural generalizations to the Lévy exponents, that is, the logarithms $\Phi := \log \hat{\mu}$ of Fourier transforms of ID measures μ. Such a calculus may lead to new factorization properties. For simplicity of notation, let us put

$$\mathcal{I}(\mu) \equiv I^{\mathrm{e}^{-s},s}_{(0,\infty)}(\mu) \quad \text{for } \mu \in \mathrm{ID}_{\log} \quad \text{and} \quad \mathcal{J}(\mu) \equiv I^{s,s}_{(0,1)}(\mu) \quad \text{for } \mu \in \mathrm{ID}. \tag{4.1}$$

Then we have the following:

Theorem 7. (*See* [15, 22].) (i) *For the mappings $\mathcal{I}$ and $\mathcal{J}$ and for $\nu \in \mathrm{ID}_{\log}$, we have the identity*

$$\mathcal{I}\big(\nu * \mathcal{J}(\nu)\big) = \mathcal{I}(\nu) * \mathcal{I}\big(\mathcal{J}(\nu)\big) = \mathcal{I}(\nu) * \mathcal{J}\big(\mathcal{I}(\nu)\big) = \mathcal{J}(\nu);$$

(ii) *For each selfdecomposable measure $\mu \in L$, there exists a unique s-selfdecomposable measure $\tilde{\mu} \in \mathcal{U}$ such that*

$$\mu = \tilde{\mu} * \mathcal{I}(\tilde{\mu}) \quad \textit{and} \quad \mathcal{J}(\mu) = \mathcal{I}(\tilde{\mu}). \tag{4.2}$$

Remark 7. (i) When one considers $\mathcal{I}$ and $\mathcal{J}$ as the mappings defined on the Lévy exponents Φ (via Eq. (1.3)), one gets $\mathcal{I}(\mathcal{J}) = \mathcal{I} - \mathcal{J}$ or $\mathcal{J}(I + \mathcal{I}) = \mathcal{I}$ or $\mathcal{I}(I - \mathcal{J}) = \mathcal{J}$ or $(I - \mathcal{J})(I + \mathcal{I}) = I$.

(ii) Iterating property (4.2), we get a convolution decomposition of selfdecomposable distributions with m-times s-selfdecomposable distributions (from the class $\mathcal{U}^{\langle m \rangle}$) as the factors; see [22].

From Theorem 1 and formula (4.1) we have that $L = \mathcal{I}(\mathrm{ID}_{\log})$. We say that a selfdecomposable $\mu = \mathcal{I}(\rho)$ has *the factorization property* if $\mathcal{I}(\rho) * \rho \in L$ and denote by L^f the class of all class L distributions having the factorization property.

Theorem 8. (*See* [9] *and* [6].) (i) $L^f = \mathcal{I}(\mathcal{J}(\mathrm{ID}_{\log}))$;

$$\text{(ii)}\quad L^f = I_{(0,\infty)}^{\mathrm{e}^{-s}, s+\mathrm{e}^{-s}-1}(\mathrm{ID}_{\log}) = \left\{\mathcal{L}\left(\int_0^\infty \mathrm{e}^{-s}\,\mathrm{d}Y_\nu\big(s+\mathrm{e}^{-s}-1\big)\right):\ \nu \in \mathrm{ID}_{\log}\right\}.$$

Remark 8. (i) Note that $I_{(0,\infty)}^{\mathrm{e}^{-s}, s+\mathrm{e}^{-s}-1} = I_{(0,1)}^{s, s-\log s-1}$.

(ii) One of the most important examples of the class L^f distribution is the Lévy's stochastic area integral (the hyperbolic sine characteristic function). In [27], BDLPs were identified as Bessel squared processes.

(iii) Above, we have that $I_{(0,1)}^{s,s}(I_{(0,\infty)}^{\mathrm{e}^{-s},s}) = I_{(0,\infty)}^{\mathrm{e}^{-s}, s+\mathrm{e}^{-s}-1}$, which means that the composition of two random integrals is again a random integral.

5 CONCLUDING REMARKS

Taking into account the above historical survey:

(α) Can we still hope for a general proof of the *random integral representation conjecture*?

Even without settling the previous question:

(β) Can we develop "an abstract theory" of a calculus on random integral mappings $I_A^{h,r}$ (or on the corresponding Lévy exponents or Lévy spectral measures)?

Since (many) classes of probability distributions of random integral mappings discussed above naturally form convolution semigroups:

(γ) Can we find structural descriptions of *all* (closed) convolution subsemigroups of the semigroup ID (of all infinitely divisible distributions)?

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