Lithuanian Mathematical Journal, Vol. 50, No. 4, 2010, pp. 426–446

THIRD-ORDER LINEAR DIFFERENTIAL EQUATION WITH THREE ADDITIONAL CONDITIONS AND FORMULA FOR GREEN'S FUNCTION

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Received May 13, 2010

Abstract. In this paper, we investigate a third-order linear differential equation with three additional conditions. We find a solution to this problem and give a formula and an existence condition for Green's function. We compare two Green's functions for two such problems with different additional conditions: nonlocal and classical boundary conditions. Formula applications are shown by examples.

MSC: 34B05, 34B27, 34B10

Keywords: ordinary differential equations, linear differential equations, Green's function, nonlocal boundary conditions

1 INTRODUCTION

Green's function is used to solve nonhomogeneous differential equations subject to certain boundary conditions. We can find solutions of stationary and nonstationary problems with various boundary conditions. Green's function helps us investigate the existence and uniqueness of the solutions for many boundary problems [12, 27, 32]. For multidimensional stationary problems and nonstationary problems, the formulas for Green's function are more complicated, and Green's functions are represented as functional series even for simple rectangular, spherical, and cylindrical domains [24].

The functional approach to the construction, use, and approximation of Green's functions and the associated ordered exponentials are presented in Fried [14], who discussed new solutions to problems involving particle production in crossed laser fields and nonconstant electric fields. The definition and properties of Green's functions in quantum physics that arise in the response of quantum systems to external fields are given in the book of Rickayzen [26]. In the chapters on superconductivity, superfluidity, and magnetism, he shows how Green's functions are used to describe phase transitions. In this book, it is also shown how Green's functions play a part in the renormalization group method of studying critical phenomena. The formulas of Green's functions for many problems with classical boundary conditions are presented in [13]. In this book, Green's functions are constructed for regular and singular boundary-value problems (BVPs) for ODEs, Helmholtz equation, and linear nonstationary equations (heat equation, wave equation). We can reduce the Sturm–Liouville problem with classical boundary conditions to an integral equation with a symmetric continuous kernel that is Green's function for the Sturm–Liouville differential operator. The investigation of Green's functions for problems with nonclassical boundary conditions is quite a new area. We often have no self-adjoint operators in this case. The problems with nonlocal boundary conditions (NBCs) [9, 10, 21, 23, 30, 31, 33] are good examples in this area.

Green's functions for second-order boundary problems with various NBCs have been constructed in [17, 18, 20, 36, 37, 39]. In these papers, the authors study the existence and multiplicity of solutions. In [34], Green's function has been constructed for a second-order nonhomogeneous differential equation with two additional conditions. Examples in which the expression of Green's function obtained in [34] is used are presented in [28, 29].

In this paper, we consider the third-order ODE

$$\mathcal{L}u := u''' + a_2(x)u'' + a_1(x)u' + a_0(x)u = f(x). \tag{1.1}$$

Third-order BVPs have been considered in many papers. Bai [6] and Yang [38] study the existence of solutions to the third-order BVPs, using the lower and upper solution method. A method based on the Schauder fixed-point theorem is also used in [6]. Yang [38] writes down the expression of Green's function and proves some estimates for positive solutions of the problem. Anderson et al. [1, 2] determine Green's function for a third-order three-point BVP of focal type and prove the existence of positive solutions to the higher-order three-point functional problem. Palamides and Veloni [22] find Green's function for a singular third-order three-point BVP. In [22], Green's function is not positive, but the obtained solution is still positive and increasing. The methods used in the above-mentioned papers are fixed Krasnosel'skii's theorem, fixed-point theorem due to Avery and Peterson [5], Leggett–Williams fixed-point theorem [4], and Ge and Bai's fixed-point theorems [7, 15].

The third-order differential equation with nonlocal boundary-value conditions

$$\begin{aligned} &u''' + h(t)f\big(u(t)\big) = 0, \quad t \in (0,1),\\ &u(0) = u'(0) = 0, \quad u'(1) = \alpha u'(\nu), \end{aligned}$$

where $0 < \nu < 1$ and $1 < \alpha < \frac{1}{\nu}$, is investigated in [16]. In this problem, some existence criteria for at least three positive solutions to the BVP are established by using the well-known Leggett–Williams fixed-point theorem [4], and then, for an arbitrary positive integer m, the existence results of at least $2m - 1$ positive solutions are obtained.

Sun and Zhang [35] investigated the third-order m-point BVP

$$\begin{aligned} &u'''(t) + f\big(t, u(t), u'(t), u''(t)\big) = 0, \quad t \in (0,1),\\ &u(0) = u'(0) = 0, \quad u''(1) = \sum_{i=1}^{m-2} k_i u''(\xi_i). \end{aligned}$$

They have obtained the expression and some useful properties of Green's function and have established the existence results of at least one solution by applying the well-known Leray–Schauder continuation principle.

Green's function is constructed for very simple differential operator $\mathcal{L}u = u'''$ in most of the papers mentioned above (see also [8]).

In this paper, we consider Eq. (1.1). Expressions for Green's functions have been obtained using the method of variation of constants [11, 25]. The advantage of this method is that it is possible to construct Green's function for a nonhomogeneous equation (1.1) with variable coefficients a_2, a_1, a_0, $f \in C[0, l]$ and various additional conditions (for example, NBCs).

The structure of the paper is as follows. In Section 2, we review the properties of functional determinants and linear functionals. We find an expression for the solution of the third-order linear differential equation with three additional conditions in Section 3 and construct Green's function for this problem in Section 4. Finally, we apply the formulas to Green's functions in the problems with NBCs and propose examples on how to construct Green's function for nonclassical problems. The main result of this article is formulated in Theorem 1.

2 FUNCTIONAL DETERMINANTS AND LINEAR FUNCTIONALS

2.1 Functional determinants

We begin this section with simple properties of determinants. Let $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$. Let us define the matrices $\mathbf{A}, \mathbf{B}, \mathbf{I} \in M_3(\mathbb{K})$ as

$$\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}=(\mathbf{A}_1,\mathbf{A}_2,\mathbf{A}_3), \qquad \mathbf{B}=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}=(\mathbf{B}_1,\mathbf{B}_2,\mathbf{B}_3),$$

$\mathbf{I}=(\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}_3)$, where $\mathbf{A}_j$, $\mathbf{B}_j$, $\mathbf{E}_j$ are vectors:

$$\mathbf{A}_j=\begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \end{pmatrix}, \qquad \mathbf{B}_j=\begin{pmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \end{pmatrix}, \quad a_{ij}, b_{ij} \in \mathbb{K},\ i,j=1,2,3,$$

$$\mathbf{E}_1=\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{E}_2=\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{E}_3=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

An adjugate of $\mathbf{A}$ is the matrix $\operatorname{adj}(\mathbf{A}) \in M_3(\mathbb{K})$ whose (i,j) entry is the (j,i) cofactor of $\mathbf{A}$ [19]:

$$\operatorname{adj}(\mathbf{A})=\begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^{\mathrm{T}}=\begin{pmatrix} \det(\mathbf{E}_1,\mathbf{A}_2,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{E}_1,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{A}_2,\mathbf{E}_1) \\ \det(\mathbf{E}_2,\mathbf{A}_2,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{E}_2,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{A}_2,\mathbf{E}_2) \\ \det(\mathbf{E}_3,\mathbf{A}_2,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{E}_3,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{A}_2,\mathbf{E}_3) \end{pmatrix}^{\mathrm{T}}.$$

The adjugate matrix has the property $\det(\operatorname{adj}(\mathbf{A}))=(\det\mathbf{A})^2$. So, we have the equality

$$\begin{vmatrix} \det(\mathbf{E}_1,\mathbf{A}_2,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{E}_1,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{A}_2,\mathbf{E}_1) \\ \det(\mathbf{E}_2,\mathbf{A}_2,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{E}_2,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{A}_2,\mathbf{E}_2) \\ \det(\mathbf{E}_3,\mathbf{A}_2,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{E}_3,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{A}_2,\mathbf{E}_3) \end{vmatrix}=(\det\mathbf{A})^2. \tag{2.1}$$

The function $\mathcal{D}:\mathbb{K}^3\times\mathbb{K}^3\times\mathbb{K}^3\to\mathbb{K}$ (for fixed vectors $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$)

$$\mathcal{D}(\mathbf{B}_1,\mathbf{B}_2,\mathbf{B}_3):=\begin{vmatrix} \det(\mathbf{B}_1,\mathbf{A}_2,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{B}_1,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{A}_2,\mathbf{B}_1) \\ \det(\mathbf{B}_2,\mathbf{A}_2,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{B}_2,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{A}_2,\mathbf{B}_2) \\ \det(\mathbf{B}_3,\mathbf{A}_2,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{B}_3,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{A}_2,\mathbf{B}_3) \end{vmatrix}$$

is an alternating multilinear function. Therefore, the equality $\mathcal{D}(\mathbf{B}_1,\mathbf{B}_2,\mathbf{B}_3)=\det\mathbf{B}\cdot\mathcal{D}(\mathbf{E}_1,\mathbf{E}_2,\mathbf{E}_3)$ is valid. Finally, we derive the formula

$$\begin{vmatrix} \det(\mathbf{B}_1,\mathbf{A}_2,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{B}_1,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{A}_2,\mathbf{B}_1) \\ \det(\mathbf{B}_2,\mathbf{A}_2,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{B}_2,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{A}_2,\mathbf{B}_2) \\ \det(\mathbf{B}_3,\mathbf{A}_2,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{B}_3,\mathbf{A}_3) & \det(\mathbf{A}_1,\mathbf{A}_2,\mathbf{B}_3) \end{vmatrix}=\det\mathbf{B}\cdot(\det\mathbf{A})^2. \tag{2.2}$$

Let $F(X):=\{u \mid u: X\to\mathbb{K}\}$ be a linear space of real (complex) functions, where X can be any set. If we have the vector-function $\boldsymbol{u}=[u_1,\dots,u_n]\in F^n(X):=\prod_{i=1}^n F(X)$ and $\boldsymbol{x}=(x_1,\dots,x_n)\in X^n$ (in this paper $n = 2, 3, 4$), then we consider the matrix-function $[\boldsymbol{u}] : X^n \to M_n(\mathbb{K})$ and its functional determinant $D[\boldsymbol{u}](\boldsymbol{x}) : X^n \to \mathbb{K}$:

$$[\boldsymbol{u}](\boldsymbol{x}) = [u_1, \dots, u_n](x_1, \dots, x_n) := \begin{pmatrix} u_1(x_1) & \cdots & u_1(x_n) \\ \cdots & \cdots & \cdots \\ u_n(x_1) & \cdots & u_n(x_n) \end{pmatrix},$$

$$D[\boldsymbol{u}](\boldsymbol{x}) = \det[\boldsymbol{u}](\boldsymbol{x}) := \begin{vmatrix} u_1(x_1) & \cdots & u_1(x_n) \\ \cdots & \cdots & \cdots \\ u_n(x_1) & \cdots & u_n(x_n) \end{vmatrix}.$$

In the case $n = 3$, we obtain $D[\boldsymbol{u}](\boldsymbol{x}) = D[u_1, u_2, u_3](x_1, x_2, x_3) = \det(\boldsymbol{u}(x_1), \boldsymbol{u}(x_2), \boldsymbol{u}(x_3))$.

In the theory of linear differential equations, a similar Wronskian determinant is often used for $\boldsymbol{u} \in C^{n-1}[0, l]$:

$$W(x) = W[\boldsymbol{u}](x) = \det\bigl(\boldsymbol{u}(x), \boldsymbol{u}'(x), \dots, \boldsymbol{u}^{(n-1)}(x)\bigr) := \begin{vmatrix} u_1(x) & \cdots & u_1^{(n-2)}(x) & u_1^{(n-1)}(x) \\ \cdots & \cdots & \cdots & \cdots \\ u_n(x) & \cdots & u_n^{(n-2)}(x) & u_n^{(n-1)}(x) \end{vmatrix}.$$

We also use the notation [11]

$$\widetilde{W}[\boldsymbol{u}](x, y) = \det\bigl(\boldsymbol{u}(x), \dots, \boldsymbol{u}^{(n-2)}(x), \boldsymbol{u}(y)\bigr) := \begin{vmatrix} u_1(x) & \cdots & u_1^{(n-2)}(x) & u_1(y) \\ \cdots & \cdots & \cdots & \cdots \\ u_n(x) & \cdots & u_n^{(n-2)}(x) & u_n(y) \end{vmatrix}$$

for $\boldsymbol{u} \in C^{n-2}[0, l]$. These two determinants have the same cofactors for last column elements, and the equalities

$$W[\boldsymbol{u}](x) = \sum_{i=1}^{n} u_i^{(n-1)}(x)(-1)^{n-i} W[u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n](x), \tag{2.3}$$

$$\widetilde{W}[\boldsymbol{u}](x, y) = \sum_{i=1}^{n} u_i(y)(-1)^{n-i} W[u_1, \dots, u_{i-1}, u_{i+1}, \dots, u_n](x) \tag{2.4}$$

are valid. If $\boldsymbol{u}$ is fixed, then we omit the notation $[\boldsymbol{u}]$ and write $D(\boldsymbol{x})$, $W(x)$, $\widetilde{W}(x, y)$.

Denote

$$H[\boldsymbol{u}](x, y) := \frac{\widetilde{W}[\boldsymbol{u}](x, y)}{W[\boldsymbol{u}](x)} \tag{2.5}$$

(if $W[\boldsymbol{u}](x) \neq 0$).

If $[\bar{u}_1, \bar{u}_2, \bar{u}_3] = \mathbf{C}[u_1, u_2, u_3]$, where $\mathbf{C} \in M_3(\mathbb{K})$, then

$$D[\bar{\boldsymbol{u}}](\boldsymbol{x}) = D[\boldsymbol{u}](\boldsymbol{x}) \cdot \det \mathbf{C}, \qquad W[\bar{\boldsymbol{u}}](x) = W[\boldsymbol{u}](x) \cdot \det \mathbf{C},$$
$$\widetilde{W}[\bar{\boldsymbol{u}}](x, y) = \widetilde{W}[\boldsymbol{u}](x, y) \cdot \det \mathbf{C}. \tag{2.6}$$

If $W[\boldsymbol{u}](x) \neq 0$ and $\det \mathbf{C} \neq 0$, then from (2.5) and (2.6) we get

$$H[\bar{\boldsymbol{u}}](x, y) = H[\boldsymbol{u}](x, y). \tag{2.7}$$

If $\boldsymbol{u} \in C^{n-1}[0,l]$, then we find the following properties of the function $H(x,y)$:

$$\left.\frac{\partial^k H[\boldsymbol{u}](x,y)}{\partial y^k}\right|_{y=x} = 0, \quad k = 0,\dots,n-2, \qquad \left.\frac{\partial^{n-1} H[\boldsymbol{u}](x,y)}{\partial y^{n-1}}\right|_{y=x} = 1. \tag{2.8}$$

We can also consider $D[\boldsymbol{u}]$ as a map $D : F^n(X) \to F(X^n)$. The rank of this mapping is $\mathcal{D} = \mathcal{D}[\boldsymbol{u}] := \{\varphi \in F(X^n)\colon \varphi = D[\boldsymbol{u}](\boldsymbol{w}) \in F^n(X)\}$. Note that a set $\mathcal{D}$ depends on the functions $u_1,\dots,u_n$, and it is a subset of the linear space $F(X^n)$ but is not a linear subspace in the general case. If the functions $u_1,\dots,u_n$ are linearly dependent, then $\mathcal{D} = \{0\}$.

Let us consider the case $n=3$: $\boldsymbol{x} = (x_1,x_2,x_3)$, $\boldsymbol{y} = (y_1,y_2,y_3) \in X^3$, $\boldsymbol{w} = [w_1,w_2,w_3]$.

Lemma 1. *If $\boldsymbol{w} \in F^3(X)$, then we have the equality*

$$D\bigl[D[\boldsymbol{w}](\cdot,y_2,y_3), D[\boldsymbol{w}](y_1,\cdot,y_3), D[\boldsymbol{w}](y_1,y_2,\cdot)\bigr](\boldsymbol{x}) = D[\boldsymbol{w}](\boldsymbol{x}) \cdot \bigl(D[\boldsymbol{w}](\boldsymbol{y})\bigr)^2. \tag{2.9}$$

Proof. If we take $\mathbf{B}_i = \boldsymbol{w}(x_i)$, $\mathbf{A}_i = \boldsymbol{w}(y_i)$, $i=1,2,3$, in formula (2.2), then we get the equality

$$\begin{vmatrix} D[\boldsymbol{w}](x_1,y_2,y_3) & D[\boldsymbol{w}](y_1,x_1,y_3) & D[\boldsymbol{w}](y_1,y_2,x_1) \\ D[\boldsymbol{w}](x_2,y_2,y_3) & D[\boldsymbol{w}](y_1,x_2,y_3) & D[\boldsymbol{w}](y_1,y_2,x_2) \\ D[\boldsymbol{w}](x_3,y_2,y_3) & D[\boldsymbol{w}](y_1,x_3,y_3) & D[\boldsymbol{w}](y_1,y_2,x_3) \end{vmatrix} = D[\boldsymbol{w}](\boldsymbol{x}) \cdot \bigl(D[\boldsymbol{w}](\boldsymbol{y})\bigr)^2.$$

The left side of this equality is the determinant $D[\tilde{w}_1,\tilde{w}_2,\tilde{w}_3](\boldsymbol{x})$ for the functions $\tilde{w}_1(\cdot) = D[\boldsymbol{w}](\cdot,y_2,y_3)$, $\tilde{w}_2(\cdot) = D[\boldsymbol{w}](y_1,\cdot,y_3)$, $\tilde{w}_3(\cdot) = D[\boldsymbol{w}](y_1,y_2,\cdot)$. Thus, we obtain formula (2.9). □

Lemma 2. *If $X=[0,l]$, $\boldsymbol{w} \in C^2[0,l]$, then we have the equality*

$$W\bigl[D[\boldsymbol{w}](\cdot,y_2,y_3), D[\boldsymbol{w}](y_1,\cdot,y_3), D[\boldsymbol{w}](y_1,y_2,\cdot)\bigr](x) = W[\boldsymbol{w}](x) \cdot \bigl(D[\boldsymbol{w}](\mathbf{y})\bigr)^2. \tag{2.10}$$

Proof. If we take $\mathbf{B}_1 = \boldsymbol{w}(x)$, $\mathbf{B}_2 = \boldsymbol{w}'(x)$, $\mathbf{B}_3 = \boldsymbol{w}''(x)$, $\mathbf{A}_i = \boldsymbol{w}(y_i)$, $i=1,2,3$, in formula (2.2), then we derive the equality

$$\begin{vmatrix} D[\boldsymbol{w}](x,y_2,y_3) & D[\boldsymbol{w}](y_1,x,y_3) & D[\boldsymbol{w}](y_1,y_2,x) \\ \frac{\partial}{\partial x} D[\boldsymbol{w}](x,y_2,y_3) & \frac{\partial}{\partial x} D[\boldsymbol{w}](y_1,x,y_3) & \frac{\partial}{\partial x} D[\boldsymbol{w}](y_1,y_2,x) \\ \frac{\partial^2}{\partial x^2} D[\boldsymbol{w}](x,y_2,y_3) & \frac{\partial^2}{\partial x^2} D[\boldsymbol{w}](y_1,x,y_3) & \frac{\partial^2}{\partial x^2} D[\boldsymbol{w}](y_1,y_2,x) \end{vmatrix} = W[\boldsymbol{w}](x) \cdot \bigl(D[\boldsymbol{w}](\mathbf{y})\bigr)^2.$$

The determinant on the left side of this equality is the Wronskian for the functions $\tilde{w}_1(x) = D[\boldsymbol{w}](x,y_2,y_3)$, $\tilde{w}_2(x) = D[\boldsymbol{w}](y_1,x,y_3)$, $\tilde{w}_3(x) = D[\boldsymbol{w}](y_1,y_2,x)$. Consequently, we have obtained formula (2.10). □

2.2 Linear functionals

Let $F(X) := \{u \mid u : X \to \mathbb{K}\}$ be a linear space of real (complex) functions. We consider the space $F^*(X)$ of linear functionals in the space $F(X)$ and use the notation $\langle f, u\rangle$, $\langle f(\cdot), u(\cdot)\rangle$, $\langle f(x), u(x)\rangle$ for the functional f value of the function u. For example, δ_x is such a functional if $\langle \delta_x, u\rangle = u(x)$. If $f \in F^*(X)$ and $g \in F^*(Y)$, then we can define the linear functional (direct product) $f \cdot g \in F^*(X \times Y)$ by

$$\langle f(x) \cdot g(y), w(x,y)\rangle := \langle f(x), \langle g(y), w(x,y)\rangle\rangle, \quad w(x,y) \in F(X \times Y).$$

Note that if $w(x,y) = u(x) \cdot v(y)$, then

$$\begin{aligned} \langle f(x) \cdot g(y), w(x,y)\rangle &= \langle f(x) \cdot g(y), u(x) \cdot v(y)\rangle = \langle f(x), u(x)\rangle \cdot \langle g(y), v(y)\rangle \\ &= \langle g(y), v(y)\rangle \cdot \langle f(x), u(x)\rangle \\ &= \langle g(y) \cdot f(x), u(x) \cdot v(y)\rangle = \langle g(y) \cdot f(x), w(x,y)\rangle; \end{aligned}$$

if $w(x,y) = u(x)\cdot v(y) - v(x)\cdot u(y)$, $X = Y$, then

$$\begin{aligned}\langle f(x)\cdot g(y), w(x,y)\rangle &= \langle f(x)\cdot g(y), u(x)\cdot v(y)\rangle - \langle f(x)\cdot g(y), v(x)\cdot u(y)\rangle \\ &= \begin{vmatrix}\langle f(x),u(x)\rangle & \langle f(x),v(x)\rangle\\ \langle g(y),u(y)\rangle & \langle g(y),v(y)\rangle\end{vmatrix} = \begin{vmatrix}\langle f,u\rangle & \langle f,v\rangle\\ \langle g,u\rangle & \langle g,v\rangle\end{vmatrix} = -\begin{vmatrix}\langle g,u\rangle & \langle g,v\rangle\\ \langle f,u\rangle & \langle f,v\rangle\end{vmatrix} \\ &= -\begin{vmatrix}\langle g(x),u(x)\rangle & \langle g(x),v(x)\rangle\\ \langle f(y),u(y)\rangle & \langle f(y),v(y)\rangle\end{vmatrix} = -\langle g(x)\cdot f(y), w(x,y)\rangle.\end{aligned}$$

Similarly, we can define the direct product $f_1\cdot f_2\cdot \dots \cdot f_n = f_1\cdot(f_2\cdot\dots\cdot f_n) \in F^*(\prod_{i=1}^n X_i)$ for $\boldsymbol{f} = (f_1,\dots,f_n)$, $f_i \in F^*(X_i)$ and use functionals and their direct product $\dot{\boldsymbol{f}}(\boldsymbol{x}) := f_1(x_1)\cdot\dots\cdot f_n(x_n)$, $\boldsymbol{x} = (x_1,\dots,x_n)$, $\boldsymbol{w} = [w_1,\dots,w_n]$, in the set $\mathcal{D}$:

$$D(\boldsymbol{f})[\boldsymbol{w}] := \langle \dot{\boldsymbol{f}}(\boldsymbol{x}), D[\boldsymbol{w}](\boldsymbol{x})\rangle = \begin{vmatrix}\langle f_1,w_1\rangle & \cdots & \langle f_n,w_1\rangle\\ \cdots & \cdots & \cdots\\ \langle f_1,w_n\rangle & \cdots & \langle f_n,w_n\rangle\end{vmatrix}.$$

For example,

$$\begin{aligned}D(\boldsymbol{f})[w_0,\boldsymbol{w}](x_0) &:= D(\delta_{x_0},\boldsymbol{f})[w_0,\boldsymbol{w}] = \langle \delta_{x_0}(y_0)\cdot\dot{\boldsymbol{f}}(\boldsymbol{y}), D[w_0,\boldsymbol{w}](y_0,\boldsymbol{y})\rangle \\ &= \begin{vmatrix}w_0(x_0) & \langle f_1,w_0\rangle & \cdots & \langle f_n,w_0\rangle\\ w_1(x_0) & \langle f_1,w_1\rangle & \cdots & \langle f_n,w_1\rangle\\ \cdots & \cdots & \cdots & \cdots\\ w_n(x_0) & \langle f_1,w_n\rangle & \cdots & \langle f_n,w_n\rangle\end{vmatrix}, \\ D(\boldsymbol{\delta}_{\boldsymbol{x}})[\boldsymbol{w}] &= \langle \dot{\boldsymbol{\delta}}_{\boldsymbol{x}}(\boldsymbol{y}), D[\boldsymbol{w}](\boldsymbol{y})\rangle = D[\boldsymbol{w}](\boldsymbol{x}), \quad \boldsymbol{y} = (y_1,\dots,y_n).\end{aligned}$$

For the determinant $D(\boldsymbol{f})[\boldsymbol{w}]$, we define the matrix

$$\mathbf{F} = \mathbf{F}[\boldsymbol{w}] = M(\boldsymbol{f})[\boldsymbol{w}] := \begin{pmatrix}\langle f_1,w_1\rangle & \cdots & \langle f_n,w_1\rangle\\ \cdots & \cdots & \cdots\\ \langle f_1,w_n\rangle & \cdots & \langle f_n,w_n\rangle\end{pmatrix}. \tag{2.11}$$

So, $D(\boldsymbol{f})[\boldsymbol{w}] = \det M(\boldsymbol{f})[\boldsymbol{w}] = \det \mathbf{F}[\boldsymbol{w}]$.

Finally, if $A: F(X) \to F(X)$ is a linear operator, then

$$AD(\delta_{x_0},\boldsymbol{f})[w_0,\boldsymbol{w}] = D(A\delta_{x_0},\boldsymbol{f})[w_0,\boldsymbol{w}] = \begin{vmatrix}Aw_0(x_0) & \langle f_1,w_0\rangle & \cdots & \langle f_n,w_0\rangle\\ Aw_1(x_0) & \langle f_1,w_1\rangle & \cdots & \langle f_n,w_1\rangle\\ \cdots & \cdots & \cdots & \cdots\\ Aw_n(x_0) & \langle f_1,w_n\rangle & \cdots & \langle f_n,w_n\rangle\end{vmatrix},$$

and if $f_0 \in F^*(X)$, then $\langle f_0, D(\boldsymbol{f})[w_0,\boldsymbol{w}]\rangle = D(f_0,\boldsymbol{f})[w_0,\boldsymbol{w}]$.

Let $w_1,\dots,w_n \in F(X)$ be linearly independent functions.

Lemma 3. *The functionals $f_1,\dots,f_n$ are linearly independent of* $\operatorname{span}(w_1,\dots,w_n) \subset F(X)$ *if and only if $D(\boldsymbol{f})[\boldsymbol{w}] \neq 0$.*

Proof. We can investigate the case where $F(X) = \operatorname{span}(w_1,\dots,w_n)$. The functionals $f_1,\dots,f_n$ are linearly independent if the equality $\alpha_1 f_1 + \dots + \alpha_n f_n = 0$ is valid only for $\alpha_1 = \dots = \alpha_n = 0$. We can rewrite the equality as $\langle \alpha_1 f_1 + \cdots + \alpha_n f_n, w\rangle = 0$ for all $w \in \mathrm{span}(w_1, \ldots, w_n)$. A system of functions $\{w_1, \ldots, w_n\}$ is a basis of $\mathrm{span}(w_1, \ldots, w_n)$, and the above-mentioned equality is equivalent to the condition

$$\alpha_1 \begin{pmatrix} \langle f_1, w_1\rangle \\ \vdots \\ \langle f_1, w_n\rangle \end{pmatrix} + \cdots + \alpha_n \begin{pmatrix} \langle f_n, w_1\rangle \\ \vdots \\ \langle f_n, w_n\rangle \end{pmatrix} = \begin{pmatrix} \langle \alpha_1 f_1 + \cdots + \alpha_n f_n, w_1\rangle \\ \vdots \\ \langle \alpha_1 f_1 + \cdots + \alpha_n f_n, w_n\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

So, the functionals $f_1, \ldots, f_n$ are linearly independent if and only if the vectors

$$\begin{pmatrix} \langle f_1, w_1\rangle \\ \vdots \\ \langle f_1, w_n\rangle \end{pmatrix}, \quad \ldots, \quad \begin{pmatrix} \langle f_n, w_1\rangle \\ \vdots \\ \langle f_n, w_n\rangle \end{pmatrix}$$

are linearly independent. However, these vectors are linearly independent if and only if

$$\begin{vmatrix} \langle f_1, w_1\rangle & \cdots & \langle f_n, w_1\rangle \\ \cdots & \cdots & \cdots \\ \langle f_1, w_n\rangle & \cdots & \langle f_n, w_n\rangle \end{vmatrix} \neq 0. \qquad \square$$

Corollary 1. *The functionals $f_1, \ldots, f_n$ are linearly dependent on* $\mathrm{span}(w_1, \ldots, w_n) \subset F(X)$ *if and only if* $D(\boldsymbol{f})[\boldsymbol{w}] = 0$.

If $(\bar{f}_1, \ldots, \bar{f}_n) = (f_1, \ldots, f_n)\mathbf{C}_f$, $[\bar{w}_1, \ldots, \bar{w}_n] = \mathbf{C}_w[w_1, \ldots, w_n]$, where $\mathbf{C}_f, \mathbf{C}_w \in M_n(\mathbb{K})$, then

$$\begin{aligned} M(\bar{\boldsymbol{f}})[\bar{\boldsymbol{w}}] &= \mathbf{C}_w\big(M(\boldsymbol{f})[\boldsymbol{w}]\big)\mathbf{C}_f, \\ D(\bar{\boldsymbol{f}})[\bar{\boldsymbol{w}}] &= \det \mathbf{C}_w \cdot D(\boldsymbol{f})[\boldsymbol{w}] \cdot \det \mathbf{C}_f, \\ M(\bar{\boldsymbol{f}}, f_0)[\bar{\boldsymbol{w}}, w_0] &= \tilde{\mathbf{C}}_w\big(M(\boldsymbol{f}, f_0)[\boldsymbol{w}, w_0]\big)\tilde{\mathbf{C}}_f, \\ D(\bar{\boldsymbol{f}}, f_0)[\bar{\boldsymbol{w}}, w_0] &= \det \mathbf{C}_w \cdot D(\boldsymbol{f}, f_0)[\boldsymbol{w}, w_0] \cdot \det \mathbf{C}_f, \end{aligned} \tag{2.12}$$

where

$$\tilde{\mathbf{C}}_w = \begin{pmatrix} \mathbf{C}_w & \mathbf{O}^{\mathrm{T}} \\ \mathbf{O} & 1 \end{pmatrix}, \qquad \tilde{\mathbf{C}}_f = \begin{pmatrix} \mathbf{C}_f & \mathbf{O}^{\mathrm{T}} \\ \mathbf{O} & 1 \end{pmatrix}, \qquad \mathbf{O} = (0, \ldots, 0).$$

2.3 Special basis in a three-dimensional space of solutions

Let us consider the solutions of homogeneous linear differential equation (1.1). Geometrically all the solutions of this equation are in the three-dimensional linear subspace $S \subset C^3[0, l]$ with $\dim S = 3$. We investigate three additional equations:

$$\langle L_1, u\rangle = 0, \qquad \langle L_2, u\rangle = 0, \qquad \langle L_3, u\rangle = 0, \quad u \in S, \tag{2.13}$$

where $L_1, L_2, L_3 \in S^*$ are linearly independent linear functionals. Let $\boldsymbol{u} = \{u_1, u_2, u_3\}$ be a fixed basis of the linear space S. We will use the shorter notation $D(\boldsymbol{y}) := D[\boldsymbol{u}](\boldsymbol{y})$, $\boldsymbol{y} = (y_1, y_2, y_3)$. Denote $\boldsymbol{L} = (L_1, L_2, L_3)$. We introduce three functions

$$\begin{aligned} \bar{v}_1(x) &:= \langle L_2(y_2) \cdot L_3(y_3), D(x, y_2, y_3)\rangle = \langle L_2(\cdot) \cdot L_3(\cdot), D(x, \cdot, \cdot)\rangle = D(\delta_x, L_2, L_3), \\ \bar{v}_2(x) &:= \langle L_1(y_1) \cdot L_3(y_3), D(y_1, x, y_3)\rangle = \langle L_1(\cdot) \cdot L_3(\cdot), D(\cdot, x, \cdot)\rangle = D(L_1, \delta_x, L_3), \\ \bar{v}_3(x) &:= \langle L_1(y_1) \cdot L_2(y_2), D(y_1, y_2, x)\rangle = \langle L_1(\cdot) \cdot L_2(\cdot), D(\cdot, \cdot, x)\rangle = D(L_1, L_2, \delta_x). \end{aligned}$$

For these functions, $\langle L_i, \bar{v}_j \rangle = \delta_{ij} D(\boldsymbol{L})$, $i, j = 1, 2, 3$, i.e., $\bar{v}_j \in \operatorname{Ker} L_i$ for $i \neq j$. So, the function $\bar{v}_1$ satisfies Eqs. (2.13b) and (2.13c), the function $\bar{v}_2$ satisfies Eqs. (2.13a) and (2.13c), and the function $\bar{v}_3$ satisfies Eqs. (2.13a) and (2.13b). If we denote

$$\mathbf{A}_i = \begin{pmatrix} \langle L_i, u_1 \rangle \\ \langle L_i, u_2 \rangle \\ \langle L_i, u_3 \rangle \end{pmatrix}, \quad i = 1, 2, 3, \tag{2.14}$$

then the components of the functions $\bar{v}_1$, $\bar{v}_2$, and $\bar{v}_3$ in the basis $\{u_1, u_2, u_3\}$ are

$$\begin{pmatrix} \det(\mathbf{E}_1, \mathbf{A}_2, \mathbf{A}_3) \\ \det(\mathbf{E}_2, \mathbf{A}_2, \mathbf{A}_3) \\ \det(\mathbf{E}_3, \mathbf{A}_2, \mathbf{A}_3) \end{pmatrix}, \qquad \begin{pmatrix} \det(\mathbf{A}_1, \mathbf{E}_1, \mathbf{A}_3) \\ \det(\mathbf{A}_1, \mathbf{E}_2, \mathbf{A}_3) \\ \det(\mathbf{A}_1, \mathbf{E}_3, \mathbf{A}_3) \end{pmatrix}, \qquad \begin{pmatrix} \det(\mathbf{A}_1, \mathbf{A}_2, \mathbf{E}_1) \\ \det(\mathbf{A}_1, \mathbf{A}_2, \mathbf{E}_2) \\ \det(\mathbf{A}_1, \mathbf{A}_2, \mathbf{E}_3) \end{pmatrix},$$

respectively. Thus, the functions $\bar{v}_1$, $\bar{v}_2$, and $\bar{v}_3$ are linearly independent if and only if the determinant on the left side of Eq. (2.1) is nonzero. However, the determinant is zero if and only if $\det(\mathbf{A}) = D(\boldsymbol{L}) = 0$. We combine Corollary 1 and these results in the next lemma.

Lemma 4. *Let $\{u_1, u_2, u_3\}$ be a basis of the linear space S. Then the following propositions are equivalent*:

1. *The functionals L_1, L_2, and L_3 are linearly independent*;
2. *The functions $\bar{v}_1(x)$, $\bar{v}_2(x)$, and $\bar{v}_3(x)$ are linearly independent*;
3. $D(\boldsymbol{L}) \neq 0$.

Corollary 2. *Let $\{u_1, u_2, u_3\}$ be a basis of the linear space S. Then the following three propositions are equivalent*:

1. *The functionals L_1, L_2, and L_3 are linearly dependent*;
2. *The functions $\bar{v}_1(x)$, $\bar{v}_2(x)$, and $\bar{v}_3(x)$ are linearly dependent*;
3. $D(\boldsymbol{L}) = 0$.

Let us take $\mathbf{B}_1 = \boldsymbol{u}(x)$, $\mathbf{B}_2 = \boldsymbol{u}'(x)$, $\mathbf{B}_3 = \boldsymbol{u}''(x)$, and $\mathbf{A}_i$, $i = 1, 2, 3$, as in (2.14). We get the equality (see formula (2.2))

$$\begin{aligned} W[\bar{\boldsymbol{v}}](x) &= \begin{vmatrix} D(\delta_x, L_2, L_3)[\boldsymbol{u}] & D(L_1, \delta_x, L_3)[\boldsymbol{u}] & D(L_1, L_2, \delta_x)[\boldsymbol{u}] \\ \frac{\mathrm{d}}{\mathrm{d}x} D(\delta_x, L_2, L_3)[\boldsymbol{u}] & \frac{\mathrm{d}}{\mathrm{d}x} D(L_1, \delta_x, L_3)[\boldsymbol{u}] & \frac{\mathrm{d}}{\mathrm{d}x} D(L_1, L_2, \delta_x)[\boldsymbol{u}] \\ \frac{\mathrm{d}^2}{\mathrm{d}x^2} D(\delta_x, L_2, L_3)[\boldsymbol{u}] & \frac{\mathrm{d}^2}{\mathrm{d}x^2} D(L_1, \delta_x, L_3)[\boldsymbol{u}] & \frac{\mathrm{d}^2}{\mathrm{d}x^2} D(L_1, L_2, \delta_x)[\boldsymbol{u}] \end{vmatrix} \\ &= W[\boldsymbol{u}](x) \cdot \bigl(D(\boldsymbol{L})[\boldsymbol{u}]\bigr)^2. \end{aligned}$$

Thus, we have proved the equality

$$W[\bar{\boldsymbol{v}}](x) = W(x) \cdot \bigl(D(\boldsymbol{L})\bigr)^2. \tag{2.15}$$

Comparing this equality with (2.6), we get that $\det \mathbf{C} = (D(\boldsymbol{L}))^2$. So, the equalities

$$D[\bar{\boldsymbol{v}}](\boldsymbol{x}) = D(\boldsymbol{x}) \cdot \bigl(D(\boldsymbol{L})\bigr)^2, \qquad \widetilde{W}[\bar{\boldsymbol{v}}](x, y) = \widetilde{W}(x, y) \cdot \bigl(D(\boldsymbol{L})\bigr)^2 \tag{2.16}$$

are also valid.

The ODE theory (see [11, 25]) implies that the functions $\{v_1, v_2, v_3\}$ are a fundamental system (e.g., a basis of the space of solutions) if and only if $W[\boldsymbol{v}](x) \neq 0$. Consequently, $W[\boldsymbol{u}](x) \neq 0$, and the following lemma is valid.

Lemma 5. *Let* $\{u_1, u_2, u_3\} \in C^3[0, l]$ *be a fundamental system of homogeneous equation* (1.1). *Then Eq.* (2.15) *is valid, and*

$$W[\bar{\boldsymbol{v}}](x) \neq 0 \quad \Leftrightarrow \quad D(\boldsymbol{L}) \neq 0, \qquad W[\bar{\boldsymbol{v}}](x) \equiv 0 \quad \Leftrightarrow \quad D(\boldsymbol{L}) = 0.$$

Corollary 3. *Propositions in Lemma* 4 *are equivalent to the condition* $W[\bar{\boldsymbol{v}}](x) \neq 0$, *and propositions in Corollary* 2 *are equivalent to the condition* $W[\bar{\boldsymbol{v}}](x) \equiv 0$.

Thus, we can reformulate the known result on the fundamental system not in terms of Wronskian but rather in terms of linear functionals, i.e., additional conditions.

Corollary 4. *If the functionals* L_1, L_2, L_3 *are linearly independent, i.e.,* $D(\boldsymbol{L}) \neq 0$, *and* $\boldsymbol{v}(x) = \bar{\boldsymbol{v}}(x)/D(\boldsymbol{L})$, *then two bases* $\{v_1, v_2, v_3\}$ *and* $\{L_1, L_2, L_3\}$ *are biorthogonal*:

$$\langle L_i, v_j \rangle = \delta_{ij}, \quad i, j = 1, 2, 3, \tag{2.17}$$

and

$$D[\boldsymbol{v}](\boldsymbol{x}) = \frac{D(\boldsymbol{x})}{D(\boldsymbol{L})}, \qquad W[\boldsymbol{v}](x) = \frac{W(x)}{D(\boldsymbol{L})},$$
$$\widetilde{W}[\boldsymbol{v}](x, y) = \frac{\widetilde{W}(x, y)}{D(\boldsymbol{L})}, \qquad H[\boldsymbol{v}](x, y) = H(x, y). \tag{2.18}$$

3 A THIRD-ORDER LINEAR DIFFERENTIAL EQUATION WITH THREE ADDITIONAL CONDITIONS

Let $\boldsymbol{u} = [u_1, u_2, u_3] \in C^3[0, l]$ be such that $W[\boldsymbol{u}](x) \neq 0$ for all $x \in [0, l]$. These functions are solutions of the third-order linear homogeneous differential equation

$$\mathcal{L}u := u''' + a_2 u'' + a_1 u' + a_0 u = 0, \tag{3.1}$$

where

$$a_0[\boldsymbol{u}](x) = -\frac{\det(\boldsymbol{u}'(x), \boldsymbol{u}''(x), \boldsymbol{u}'''(x))}{W[\boldsymbol{u}](x)},$$
$$a_1[\boldsymbol{u}](x) = \frac{\det(\boldsymbol{u}(x), \boldsymbol{u}''(x), \boldsymbol{u}'''(x))}{W[\boldsymbol{u}](x)},$$
$$a_2[\boldsymbol{u}](x) = -\frac{\det(\boldsymbol{u}(x), \boldsymbol{u}'(x), \boldsymbol{u}'''(x))}{W[\boldsymbol{u}](x)}.$$

Note that the functions $a_0, a_1, a_2 \in C[0, l]$ define all the solutions, i.e., the linear space $S := \{u \in C^3[0, l]\colon \mathcal{L}u = 0\}$, which can be described by the fundamental system $\{u_1, u_2, u_3\}$; conversely, the fundamental system $\{u_1, u_2, u_3\}$ defines a_0, a_1, and a_2. If $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ is another fundamental system and if $[\bar{u}_1, \bar{u}_2, \bar{u}_3] = \mathbf{C}[u_1, u_2, u_3]$, where $\mathbf{C} \in M_3(\mathbb{K})$, $\det \mathbf{C} \neq 0$, then $a_i[\bar{\boldsymbol{u}}] = a_i[\boldsymbol{u}](x)$, $i = 1, 2, 3$.

We see that Abel's formula is valid ($x, x_0, \bar{x} \in [0, l]$):

$$W[\boldsymbol{u}](x) \exp\left(\int_{\bar{x}}^{x} a_2[\boldsymbol{u}](\xi)\, \mathrm{d}\xi\right) = W[\boldsymbol{u}](x_0) \exp\left(\int_{\bar{x}}^{x_0} a_2[\boldsymbol{u}](\xi)\, \mathrm{d}\xi\right). \tag{3.2}$$

In this section, we consider the third-order inhomogeneous differential equation

$$\mathcal{L}u := u''' + a_2(x)u'' + a_1(x)u' + a_0(x)u = f(x), \quad f \in C[0,l], \tag{3.3}$$

with three additional conditions

$$\langle L_i, u\rangle = f_i \in \mathbb{K}, \quad i = 1,2,3, \tag{3.4}$$

where L_1, L_2, L_3 are linearly independent functionals in $C^3[0,l]$. We denote $\boldsymbol{L} = (L_1, L_2, L_3)$, $\boldsymbol{f} = (f_1, f_2, f_3)$.

3.1 The solution to a nonhomogeneous problem with homogeneous additional conditions

Let $\{u_1, u_2, u_3\}$ be a fixed fundamental system of homogeneous equation (3.1), and let $\boldsymbol{u} = [u_1, u_2, u_3]$. Then the general solution of this equation is

$$u_h(x) = C_1u_1(x) + C_2u_2(x) + C_3u_3(x),$$

where C_1, C_2, C_3 are arbitrary constants. We replace the constants C_1, C_2, C_3 by the functions $c_1(x)$, $c_2(x)$, $c_3(x)$, respectively (variation of parameters [11]). Then we substitute

$$u(x) = c_1(x)u_1(x) + c_2(x)u_2(x) + c_3(x)u_3(x) \tag{3.5}$$

into the nonhomogeneous equation (3.3) and determine these functions $c_1(x)$, $c_2(x)$, $c_3(x)$ by solving the system

$$\begin{aligned} c_1'(x)u_1(x) + c_2'(x)u_2(x) + c_3'(x)u_3(x) &= 0, \\ c_1'(x)u_1'(x) + c_2'(x)u_2'(x) + c_3'(x)u_3'(x) &= 0, \\ c_1'(x)u_1''(x) + c_2'(x)u_2''(x) + c_3'(x)u_3''(x) &= f(x). \end{aligned}$$

Since $W(x) \neq 0$, we derive

$$\begin{aligned} c_1(x) &= \int_0^x f(s)\frac{W[u_2,u_3](s)}{W[\boldsymbol{u}](s)}\,\mathrm{d}s + A_1, \\ c_2(x) &= -\int_0^x f(s)\frac{W[u_1,u_3](s)}{W[\boldsymbol{u}](s)}\,\mathrm{d}s + A_2, \\ c_3(x) &= \int_0^x f(s)\frac{W[u_1,u_2](s)}{W[\boldsymbol{u}](s)}\,\mathrm{d}s + A_3. \end{aligned}$$

Thus, we get (see [11])

$$\begin{aligned} u(x) = {} & \int_0^x f(s)\frac{u_1(x)W[u_2,u_3](s) - u_2(x)W[u_1,u_3](s) + u_3(x)W[u_1,u_2](s)}{W[\boldsymbol{u}](s)}\,\mathrm{d}s \\ & + A_1u_1(x) + A_2u_2(x) + A_3u_3(x). \end{aligned}$$

Since $u_1(x)W[u_2,u_3](s) - u_2(x)W[u_1,u_3](s) + u_3(x)W[u_1,u_2](s) = \widetilde{W}[\boldsymbol{u}](s,x)$ (see Eq. (2.4)), we consequently derive

$$u(x) = \langle \theta_f^x(\cdot), H[\boldsymbol{u}](\cdot,x)\rangle + A_1u_1(x) + A_2u_2(x) + A_3u_3(x), \tag{3.6}$$

where

$$\theta(x) := \begin{cases} 1, & x > 0,\\ 0, & x \leqslant 0, \end{cases} \qquad \langle \theta_f^x, w\rangle := \int_0^l \theta(x-s)f(s)w(s)\,\mathrm{d}s.$$

Formula (3.6) is known in the literature (see [11]). We use this formula for the special basis $\{v_1, v_2, v_3\}$. In this case, we have

$$u(x) = \langle \theta_f^x(\cdot), H[\boldsymbol{u}](\cdot,x)\rangle + A_1v_1(x) + A_2v_2(x) + A_3v_3(x). \tag{3.7}$$

Let us take the homogeneous conditions

$$\langle L_1, u\rangle = 0, \qquad \langle L_2, u\rangle = 0, \qquad \langle L_3, u\rangle = 0. \tag{3.8}$$

By substituting the general solution (3.7) into the homogeneous additional conditions, we find

$$A_i = -\langle L_i(y), \langle \theta_f^y(s), H[\boldsymbol{u}](s,y)\rangle\rangle, \quad i = 1,2,3.$$

Then we obtain a formula for the solution in the case of the third-order LDE with three additional homogeneous conditions

$$\begin{aligned} u_f(x) = {} & -\langle L_1(y), \langle \theta_f^y(s), H(s,y)\rangle\rangle v_1(x) - \langle L_2(y), \langle \theta_f^y(s), H(s,y)\rangle\rangle v_2(x)\\ & - \langle L_3(y), \langle \theta_f^y(s), H(s,y)\rangle\rangle v_3(x) + \langle \theta_f^x(s), H(s,x)\rangle, \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} v_1(x) &= \frac{D(\delta_x, L_2, L_3)}{D(\boldsymbol{L})} = \frac{\langle L_2(\cdot)\cdot L_3(\cdot), D(x,\cdot,\cdot)\rangle}{D(\boldsymbol{L})},\\ v_2(x) &= \frac{D(L_1, \delta_x, L_3)}{D(\boldsymbol{L})} = \frac{\langle L_1(\cdot)\cdot L_3(\cdot), D(\cdot,x,\cdot)\rangle}{D(\boldsymbol{L})},\\ v_3(x) &= \frac{D(L_1, L_2, \delta_x)}{D(\boldsymbol{L})} = \frac{\langle L_1(\cdot)\cdot L_2(\cdot), D(\cdot,\cdot,x)\rangle}{D(\boldsymbol{L})}. \end{aligned}$$

3.2 A homogeneous equation with additional conditions

Let us consider a homogeneous equation (3.3) with additional conditions (3.4),

$$\mathcal{L}u = 0, \qquad \langle L_i, u\rangle = f_i, \quad i = 1,2,3.$$

We can find a solution

$$u_n(x) = f_1\cdot v_1(x) + f_2\cdot v_2(x) + f_3\cdot v_3(x) \tag{3.10}$$

to this problem if the general solution is inserted into the additional conditions.

The solution of nonhomogeneous problems is of the form $u(x) = u_f(x) + u_n(x)$ (see (3.9) and (3.10)). So, we get a simple formula for solving problem (3.3)–(3.4).

Proposition 1. *The solution of problem* (3.3)–(3.4) *can be expressed by the formula*

$$\begin{aligned} u(x) = \langle \theta_f^x(s), H(s,x)\rangle &+ \big(f_1 - \langle L_1(y), \langle \theta_f^y(s), H(s,y)\rangle\rangle\big) v_1(x) \\ &+ \big(f_2 - \langle L_2(y), \langle \theta_f^y(s), H(s,y)\rangle\rangle\big) v_2(x) \\ &+ \big(f_3 - \langle L_3(y), \langle \theta_f^y(s), H(s,y)\rangle\rangle\big) v_3(x). \end{aligned} \tag{3.11}$$

Formula (3.11) is known (in more classical forms) for very different cases of classical boundary conditions of the first, second, and third types. The investigation of linear and quasi-linear problems with NBCs (see [35, 38]) requires a general theory for constructing the solutions with very different boundary conditions. Formula (3.11) can be effectively employed to get the solutions to the linear third-order differential equation, where $a_0(x)$, $a_1(x)$, and $a_2(x)$ are not constants and with any right-hand side function $f(x)$, any functionals L_1, L_2, L_3, and any f_1, f_2, f_3, provided that the general solution of the homogeneous equation is known. In this paper, we also use (3.11) to get formulas for Green's function.

3.3 Relation between two solutions

Next, let us consider two problems with the same third-order nonhomogeneous differential equation with a differential operator as in the previous subsection:

$$\begin{cases} \mathcal{L}u = f, \\ \langle l_i, u\rangle = f_i, \quad i = 1,2,3, \end{cases} \qquad \begin{cases} \mathcal{L}v = f, \\ \langle L_i, v\rangle = F_i, \quad i = 1,2,3, \end{cases} \tag{3.12}$$

and $D(\boldsymbol{L}) \neq 0$. The difference $w = v - u$ satisfies the problem

$$\begin{cases} \mathcal{L}w = 0, \\ \langle L_i, w\rangle = F_i - \langle L_i, u\rangle, \quad i = 1,2,3. \end{cases}$$

Thus, from formula (3.11) it follows that

$$w(x) = \big(F_1 - \langle L_1, u\rangle\big) v_1(x) + \big(F_2 - \langle L_2, u\rangle\big) v_2(x) + \big(F_3 - \langle L_3, u\rangle\big) v_3(x) \tag{3.13}$$

or

$$\begin{aligned} v(x) = u(x) &+ \big(F_1 - \langle L_1, u\rangle\big) \frac{D(\delta_x, L_2, L_3)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]} + \big(F_2 - \langle L_2, u\rangle\big) \frac{D(L_1, \delta_x, L_3)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]} \\ &+ \big(F_3 - \langle L_3, u\rangle\big) \frac{D(L_1, L_2, \delta_x)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]}, \end{aligned} \tag{3.14}$$

and we can express the solution of the second problem (3.12) via the solution of the first problem.

Corollary 5. *The relation*

$$v(x) = \frac{1}{D(\boldsymbol{L})[\boldsymbol{u}]} \begin{vmatrix} \langle L_1, u_1\rangle & \langle L_2, u_1\rangle & \langle L_3, u_1\rangle & u_1(x) \\ \langle L_1, u_2\rangle & \langle L_2, u_2\rangle & \langle L_3, u_2\rangle & u_2(x) \\ \langle L_1, u_3\rangle & \langle L_2, u_3\rangle & \langle L_3, u_3\rangle & u_3(x) \\ \langle L_1, u\rangle - F_1 & \langle L_2, u\rangle - F_2 & \langle L_3, u\rangle - F_3 & u(x) \end{vmatrix} \tag{3.15}$$

between the two solutions of problems (3.12) *is valid.*

Proof. Expanding determinant (3.15) according to the last row, we get formula (3.14). □

Remark 1. The determinant in formula (3.15) is equal to

$$\begin{vmatrix} \langle L_1,u_1\rangle & \langle L_2,u_1\rangle & \langle L_3,u_1\rangle & u_1(x) \\ \langle L_1,u_2\rangle & \langle L_2,u_2\rangle & \langle L_3,u_2\rangle & u_2(x) \\ \langle L_1,u_3\rangle & \langle L_2,u_3\rangle & \langle L_3,u_3\rangle & u_3(x) \\ \langle L_1,u\rangle & \langle L_2,u\rangle & \langle L_3,u\rangle & u(x) \end{vmatrix} - \begin{vmatrix} \langle L_1,u_1\rangle & \langle L_2,u_1\rangle & \langle L_3,u_1\rangle & u_1(x) \\ \langle L_1,u_2\rangle & \langle L_2,u_2\rangle & \langle L_3,u_2\rangle & u_2(x) \\ \langle L_1,u_3\rangle & \langle L_2,u_3\rangle & \langle L_3,u_3\rangle & u_3(x) \\ F_1 & F_2 & F_3 & 0 \end{vmatrix}.$$

In this way, we can rewrite (3.15) as

$$v(x) = \frac{D(\boldsymbol{L},\delta_x)[\boldsymbol{u},u]}{D(\boldsymbol{L})[\boldsymbol{u}]} + \frac{F_1 D(\delta_x,L_2,L_3)[\boldsymbol{u}] + F_2 D(L_1,\delta_x,L_3)[\boldsymbol{u}] + F_3 D(L_1,L_2,\delta_x)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]}. \qquad (3.16)$$

Note that, in this formula, the function u is only in the first term.

4 GREEN'S FUNCTIONS

We propose a definition of Green's function (see [3, 11, 13, 27, 32]). Let U, F be linear subspaces in $L^\infty[0,l]$. Each function $G \in L^1(0,l)$ defines the linear functional $g \in U^*$: $\langle g,\varphi\rangle = \int_0^l G(y)\varphi(y)\,\mathrm{d}y$, where U^* is the dual space for U. We call such a functional a *regular* functional and call the function G its *generator*. For each $x \in [0,l]$, we consider the singular functional $\delta_x \in U^*$: $\langle \delta_x,\varphi\rangle = \varphi(x)$. Let $A : U \to F$ be a linear operator that has a well-defined inverse $A^{-1} : F \to U$. Consider the operator equation

$$Au = f, \qquad (4.1)$$

where $u \in U$ is unknown and $f \in F$ is given. Also, suppose that the problem is regular, i.e., there exists only a trivial solution to the homogeneous problem. We have $u(x) = \langle \delta_x, u\rangle = \langle \delta_x, A^{-1}f\rangle = \langle (A^{-1})^*\delta_x, f\rangle$. If the functional $(A^{-1})^*\delta_x$ is regular with generator $G(x,y) \in L^1(0,l)$ for all $x \in [0,l]$, then G is called *Green's function of operator* A, and the solution of (4.1) can be expressed by the following integral representation:

$$u(x) = \int_0^l G(x,y) f(y)\,\mathrm{d}y. \qquad (4.2)$$

Let $\mathcal{L}$ be the operator (3.3) with classical boundary conditions, and such a problem is regular. If

$$U = \left\{ u \in C^3[0,l]\colon \sum_{k=1}^{3} \big(M_{jk} u^{(k-1)}(0) + N_{jk} u^{(k-1)}(l) \big) = 0,\ j = 1,2,3 \right\}$$

and $F = C[0,l]$, then there is a unique solution $u(x)$ that satisfies Eq. (3.3), and it is given by formula (4.2), where $G(x,s)$ is Green's function satisfying the following conditions (see [3, 11]):

1. $G(x,s)$, $\frac{\partial G(x,s)}{\partial x}$, and $\frac{\partial G(x,s)}{\partial s}$ are continuous in $[0,l] \times [0,l]$;
2. $\mathcal{L}G(x,s) = 0$ for $x \neq s$;
3. The derivative "jump" satisfies $\frac{\partial^2 G(s+0,s)}{\partial x^2} - \frac{\partial^2 G(s-0,s)}{\partial x^2} = 1$;
4. As a function of x, $G(x,s)$ satisfies BC.

Such "good" properties are valid only for the classical boundary conditions. Problems with NBC are not self-adjoint, and we have Green's functions that can be discontinuous.

4.1 Green's functions for the third-order linear differential equation with three additional conditions

Let us consider a nonhomogeneous equation with operator (3.3), $\mathcal{L} : U \to F$, where $F = C[0,l]$. Additional homogeneous conditions define the subspace $U = \{u \in C^3[0,l]\colon \langle L_1, u\rangle = 0,\ \langle L_2, u\rangle = 0,\ \langle L_3, u\rangle = 0\}$.

Recalling the definition of the functional θ_f^y and using the formula

$$\langle L(y), \langle \theta_f^y(s), H(s,y)\rangle\rangle = \int_0^l \langle L(y), \theta(y-s)H(s,y)\rangle f(s)\,\mathrm{d}s,$$

we derive the following formula for the solution (see (3.11)):

$$u(x) = \int_0^l \theta(x-s)H(s,x)f(s)\,\mathrm{d}s - \sum_{k=1}^3 \int_0^l v_k(x)\langle L_k(y), \theta(y-s)H(s,y)\rangle f(s)\,\mathrm{d}s.$$

Obviously, we have proved the next lemma on Green's function. We denote $H_\theta(y,s) := \theta(y-s)H(s,y)$.

Lemma 6. *Green's function for problem* (3.3) *with three homogeneous additional conditions* $\langle L_1, u\rangle = 0$, $\langle L_2, u\rangle = 0$, $\langle L_3, u\rangle = 0$ *is equal to*

$$\begin{aligned}
G(x,s) &= H_\theta(x,s) - \sum_{i=1}^3 v_i(x)\langle L_i(y), H_\theta(y,s)\rangle \\
&= H_\theta(x,s) - \langle L_1(y), H_\theta(y,s)\rangle \frac{D(\delta_x, L_2, L_3)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]} \\
&\quad - \langle L_2(y), H_\theta(y,s)\rangle \frac{D(L_1, \delta_x, L_3)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]} - \langle L_3(y), H_\theta(y,s)\rangle \frac{D(L_1, L_2, \delta_x)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]} \\
&= \frac{1}{D(\boldsymbol{L})[\boldsymbol{u}]} \begin{vmatrix} \langle L_1, u_1\rangle & \langle L_2, u_1\rangle & \langle L_3, u_1\rangle & u_1(x) \\ \langle L_1, u_2\rangle & \langle L_2, u_2\rangle & \langle L_3, u_2\rangle & u_2(x) \\ \langle L_1, u_3\rangle & \langle L_2, u_3\rangle & \langle L_3, u_3\rangle & u_3(x) \\ \langle L_1(\cdot), H_\theta(\cdot,s)\rangle & \langle L_2(\cdot), H_\theta(\cdot,s)\rangle & \langle L_3(\cdot), H_\theta(\cdot,s)\rangle & H_\theta(x,s) \end{vmatrix} \\
&= \frac{\langle L_1 \cdot L_2 \cdot L_3, D[u_1(\cdot), u_2(\cdot), u_3(\cdot), H_\theta(\cdot,s)](\cdot,\cdot,\cdot,x)\rangle}{D(\boldsymbol{L})[\boldsymbol{u}]} \\
&= \frac{D(\boldsymbol{L}, \delta_x)[\boldsymbol{u}, H_\theta(\cdot,s)]}{D(\boldsymbol{L})[\boldsymbol{u}]}.
\end{aligned} \tag{4.3}$$

If $[\bar u_1, \bar u_2, \bar u_3] = \mathbf{C}[u_1, u_2, u_3]$, where $\mathbf{C} \in M_3(\mathbb{K})$, $\det \mathbf{C} \neq 0$, then (see Eq. (2.12))

$$D(\boldsymbol{L}, \delta_x)\big[\bar{\boldsymbol{u}}, H_\theta(\cdot,s)\big] = D(\boldsymbol{L}, \delta_x)\big[\boldsymbol{u}, H_\theta(\cdot,s)\big] \cdot \det \mathbf{C}.$$

By (4.3) and (2.6) we get that Green's function

$$G(x,s) = G[\bar{\boldsymbol{u}}](x,s) = G[\boldsymbol{u}](x,s), \tag{4.4}$$

i.e., it is invariant with respect to the basis $\{u_1, u_2, u_3\}$.

The function $H_\theta(x,s)$ has the same property (see (2.7)). It is easy to check (see Eq. (2.8)) that $H_\theta(x,s)$ is Green's function for problem (3.3) with special additional (in this case, initial) conditions $u(0) = 0$, $u'(0) = 0$, $u''(0) = 0$.

For theoretical investigation of problems with nonlocal boundary conditions, the next result about the relations between Green's functions $G_u(x,s)$ and $G_v(x,s)$ of two inhomogeneous problems with the same f,

$$\begin{cases} Lu = f, \\ \langle l_i, u\rangle = 0, \quad i=1,2,3, \end{cases} \qquad \begin{cases} Lv = f, \\ \langle L_i, v\rangle = 0, \quad i=1,2,3, \end{cases} \tag{4.5}$$

is useful.

Theorem 1. *We have the following relations between Green's functions $G_v(x,s)$ and $G_u(x,s)$ for problems* (4.5):

$$\begin{aligned} G_v(x,s) &= G_u(x,s) - \sum_{i=1}^{3} v_i(x)\langle L_i(y), G_u(y,s)\rangle \\ &= G_u(x,s) - \langle L_1(y), G_u(y,s)\rangle \frac{D(\delta_x, L_2, L_3)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]} \\ &\quad - \langle L_2(y), G_u(y,s)\rangle \frac{D(L_1, \delta_x, L_3)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]} - \langle L_3(y), G_u(y,s)\rangle \frac{D(L_1, L_2, \delta_x)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]} \\ &= \frac{1}{D(\boldsymbol{L})[\boldsymbol{u}]} \begin{vmatrix} \langle L_1, u_1\rangle & \langle L_2, u_1\rangle & \langle L_3, u_1\rangle & u_1(x) \\ \langle L_1, u_2\rangle & \langle L_2, u_2\rangle & \langle L_3, u_2\rangle & u_2(x) \\ \langle L_1, u_3\rangle & \langle L_2, u_3\rangle & \langle L_3, u_3\rangle & u_3(x) \\ \langle L_1(\cdot), G_u(\cdot,s)\rangle & \langle L_2(\cdot), G_u(\cdot,s)\rangle & \langle L_3(\cdot), G_u(\cdot,s)\rangle & G_u(x,s) \end{vmatrix} \\ &= \frac{\langle L_1 \cdot L_2 \cdot L_3, D[u_1(\cdot), u_2(\cdot), u_3(\cdot), G_u(\cdot,s)](\cdot,\cdot,\cdot,x)\rangle}{D(\boldsymbol{L})[\boldsymbol{u}]} \\ &= \frac{D(\boldsymbol{L}, \delta_x)[\boldsymbol{u}, G_u(\cdot,s)]}{D(\boldsymbol{L})[\boldsymbol{u}]}. \end{aligned} \tag{4.6}$$

Proof. The proof of these relations follows from Eq. (3.16) (the case $F_1 = F_2 = F_3 = 0$) and integral representation (4.2) of the solutions u and v. □

Formulas (4.6) allow us to easily find Green's function for an equation with three additional conditions if we know Green's function for the same equation but with other additional conditions. The formula

$$u(x) = \int_0^l G(x,s)f(s)\,\mathrm{d}s + v_1(x)f_1 + v_2(x)f_2 + v_3(x)f_3 \tag{4.7}$$

can be used to get solutions of the third-order equations with a differential operator (where $a_0(x)$, $a_1(x)$, and $a_2(x)$ are not constants) with any three linear additional (initial, or boundary, or nonlocal boundary) conditions if the general solution of a homogeneous equation is known [11].

4.2 Applications to problems with nonlocal boundary conditions

Let us investigate Green's function for the problem with NBCs

$$\mathcal{L}u := u''' + a_2(x)u'' + a_1(x)u' + a_0(x)u = f(x), \tag{4.8}$$

$$\begin{gathered} \langle L_1, u\rangle := \langle k_1, u\rangle - \gamma_1\langle n_1, u\rangle = 0, \quad \langle L_2, u\rangle := \langle k_2, u\rangle - \gamma_2\langle n_2, u\rangle = 0, \\ \langle L_3, u\rangle := \langle k_3, u\rangle - \gamma_3\langle n_3, u\rangle = 0, \end{gathered} \tag{4.9}$$

where $a_0, a_1, a_2 \in C[0,1]$, and $f \in C[0,1]$. We can write many problems with NBCs in this form, where $\langle k_i, u\rangle := \langle k_i(x), u(x)\rangle$ is a classical part, and $\langle n_i, u\rangle := \langle n_i(x), u(x)\rangle$, $i = 1,2,3$, is a nonlocal part of the boundary conditions. For example, the functionals n_i, $i = 1,2,3$, can describe the multipoint ($\xi_j \in [0,1]$, $j = 1,\dots,n$) or integral NBCs

$$\langle n, u\rangle = \sum_{j=1}^{n}\sum_{k=1}^{3}\bigl(\varkappa_j^k u^{(k-1)}(\xi_j)\bigr), \qquad \langle n, u\rangle = \int_0^1 \varkappa(t)u(t)\,\mathrm{d}t,$$

and the functional k_i, $i = 1,2,3$, can describe the local (classical) boundary conditions.

If $\gamma_1, \gamma_2, \gamma_3 = 0$, then problem (4.8)–(4.9) becomes classical. Suppose that there exists Green's function $G^{\mathrm{cl}}(x,s)$ for the classical case. Then Green's function exists for problem (4.8)–(4.9) if $\vartheta = D(\boldsymbol{L})[\boldsymbol{u}] \neq 0$. For $L_i = k_i - \gamma_i n_i$, $i = 1,2,3$, we derive

$$\begin{aligned}\vartheta &= D(k_1\cdot k_2\cdot k_3) - \gamma_1 D(n_1\cdot k_2\cdot k_3) - \gamma_2 D(k_1\cdot n_2\cdot k_3) - \gamma_3 D(k_1\cdot k_2\cdot n_3)\\ &\quad + \gamma_1\gamma_2 D(n_1\cdot n_2\cdot k_3) + \gamma_1\gamma_3 D(n_1\cdot k_2\cdot n_3) + \gamma_2\gamma_3 D(k_1\cdot n_2\cdot n_3)\\ &\quad - \gamma_1\gamma_2\gamma_3 D(n_1\cdot n_2\cdot n_3). \end{aligned} \tag{4.10}$$

Defining the matrices

$$\mathbf{K} := \begin{pmatrix}\langle k_1,u_1\rangle & \langle k_2,u_1\rangle & \langle k_3,u_1\rangle\\ \langle k_1,u_2\rangle & \langle k_2,u_2\rangle & \langle k_3,u_2\rangle\\ \langle k_1,u_3\rangle & \langle k_2,u_3\rangle & \langle k_3,u_3\rangle\end{pmatrix}, \qquad \mathbf{\Gamma} := \begin{pmatrix}\gamma_1 & 0 & 0\\ 0 & \gamma_2 & 0\\ 0 & 0 & \gamma_3\end{pmatrix},$$

$$\mathbf{N} := \begin{pmatrix}\langle n_1,u_1\rangle & \langle n_2,u_1\rangle & \langle n_3,u_1\rangle\\ \langle n_1,u_2\rangle & \langle n_2,u_2\rangle & \langle n_3,u_2\rangle\\ \langle n_1,u_3\rangle & \langle n_2,u_3\rangle & \langle n_3,u_3\rangle\end{pmatrix},$$

we see that the condition $\vartheta \neq 0$ is equivalent to

$$\det\bigl(\mathbf{I} - \mathbf{\Gamma}\mathbf{N}\mathbf{K}^{-1}\bigr) \neq 0. \tag{4.11}$$

This result generalizes a similar condition for the second-order equation (see [9, 10]). Since $\langle k_i(\cdot), G^{\mathrm{cl}}(\cdot,s)\rangle = 0$, $i = 1,2,3$, we can rewrite formula (4.6) as

$$\begin{aligned} G(x,s) &= G^{\mathrm{cl}}(x,s) + \sum_{i=1}^{3}\gamma_i v_i(x)\langle n_i(y), G^{\mathrm{cl}}(y,s)\rangle\\ &= G^{\mathrm{cl}}(x,s) + \gamma_1\langle n_1(y), G^{\mathrm{cl}}(y,s)\rangle\frac{D(\delta_x, L_2, L_3)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]}\\ &\quad + \gamma_2\langle n_2(y), G^{\mathrm{cl}}(y,s)\rangle\frac{D(L_1,\delta_x, L_3)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]} + \gamma_3\langle n_3(y), G^{\mathrm{cl}}(y,s)\rangle\frac{D(L_1, L_2,\delta_x)[\boldsymbol{u}]}{D(\boldsymbol{L})[\boldsymbol{u}]}\\ &= \frac{\begin{vmatrix}\langle L_1,u_1\rangle & \langle L_2,u_1\rangle & \langle L_3,u_1\rangle & u_1(x)\\ \langle L_1,u_2\rangle & \langle L_2,u_2\rangle & \langle L_3,u_2\rangle & u_2(x)\\ \langle L_1,u_3\rangle & \langle L_2,u_3\rangle & \langle L_3,u_3\rangle & u_3(x)\\ -\gamma_1\langle n_1(\cdot), G^{\mathrm{cl}}(\cdot,s)\rangle & -\gamma_2\langle n_2(\cdot), G^{\mathrm{cl}}(\cdot,s)\rangle & -\gamma_3\langle n_3(\cdot), G^{\mathrm{cl}}(\cdot,s)\rangle & G^{\mathrm{cl}}(x,s)\end{vmatrix}}{D(\boldsymbol{L})[\boldsymbol{u}]}. \end{aligned} \tag{4.12}$$

Example 1. Let us consider the problem

$$u''' = f(x), \quad x \in (0,1),$$
$$u(0) = 0, \quad u'(0) = 0, \quad u'(1) = \gamma u'(\xi),$$

where $\xi \in (0,1)$. This problem was investigated by Guo et al. [16]. We can take $u_1(x) = 1$, $u_2(x) = x$, and $u_3(x) = x^2$. Then

$$\widetilde{W}(s,x) = \begin{vmatrix} 1 & 0 & 1 \\ s & 1 & x \\ s^2 & 2s & x^2 \end{vmatrix} = (x-s)^2, \qquad W(s) = \begin{vmatrix} 1 & 0 & 0 \\ s & 1 & 0 \\ s^2 & 2s & 2 \end{vmatrix} = 2.$$

So, we obtain

$$H_\theta(x,s) = \theta(x-s)\frac{(x-s)^2}{2}. \tag{4.13}$$

For the problem with the boundary conditions $u(0) = u'(0) = u'(1) = 0$, we have

$$D(L,\delta_x)\big[\boldsymbol{u}, H_\theta(\cdot,s)\big] = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & x \\ 0 & 0 & 2 & x^2 \\ 0 & 0 & (1-s) & \theta(x-s)\frac{(x-s)^2}{2} \end{vmatrix} = \theta(x-s)(x-s)^2 - x^2(1-s),$$
$$D(L)[\boldsymbol{u}] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2,$$

with “classical” Green’s function

$$G^{\mathrm{cl}}(x,s) = \theta(x-s)\frac{(x-s)^2}{2} - (1-s)\frac{x^2}{2} = -\frac{1}{2}\begin{cases} 2xs - x^2s - s^2, & s \leqslant x, \\ x^2(1-s), & x \leqslant s. \end{cases}$$

For the “nonlocal” problem with boundary conditions $u(0) = u'(0) = 0$, $u'(1) - \gamma u'(\xi) = 0$, we have

$$D(L)[\boldsymbol{u}] = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1-\gamma \\ 0 & 0 & 2(1-\gamma\xi) \end{vmatrix} = 2(1-\gamma\xi), \qquad D(L_1 \cdot L_2 \cdot \delta_x)[\boldsymbol{u}] = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & x \\ 0 & 0 & x^2 \end{vmatrix} = x^2.$$

It follows from Eq. (4.12) that

$$G(x,s) = G^{\mathrm{cl}}(x,s) + \gamma \frac{\mathrm{d}}{\mathrm{d}x} G^{\mathrm{cl}}(x,s)\bigg|_{x=\xi} \cdot \frac{x^2}{2(1-\gamma\xi)} \tag{4.14}$$

if $1 - \gamma\xi \neq 0$. Thus,

$$G(x,s) = \frac{1}{2(\gamma\xi - 1)} \cdot \begin{cases} x^2 s(\gamma-1) + (1-\gamma\xi)(2xs - s^2), & s \leqslant \min\{\xi, x\}, \\ x^2(1-\gamma\xi) + x^2 s(\gamma - 1), & x \leqslant s \leqslant \xi, \\ x^2(\gamma\xi - s) + (1-\gamma\xi)(2xs - s^2), & \xi \leqslant s \leqslant x, \\ x^2(1-s), & \max\{\xi, x\} \leqslant s, \end{cases}$$

if $\gamma\xi \neq 1$ (see [16, Lemma 2.1]).

Example 2. Let us consider the problem

$$u''' = f(x), \quad x \in (0,1),$$
$$u(0) = 0, \quad u''(0) = 0, \quad u(1) = \gamma \int_0^1 u(t)\,\mathrm{d}t.$$

We take a fundamental system $\{u_1, u_2, u_3\}$ as in the previous example. Therefore, the function $H_\theta(x,s)$ is the same (see Eq. (4.13)). For the problem with boundary conditions $u(0) = u''(0) = u(1) = 0$, Green's function $G^{\mathrm{cl}}(x,s)$ in Eq. (4.6) is of the form

$$G^{\mathrm{cl}}(x,s) = \theta(x-s)\frac{(x-s)^2}{2} - \frac{x(1-s)^2}{2} = -\frac{1}{2}\begin{cases}(1-x)(x-s^2), & s \leqslant x,\\ x(1-s)^2, & x \leqslant s.\end{cases}$$

For the problem with boundary conditions $u(0) = u''(0) = 0$, $u(1) = \gamma\int_0^1 u(t)\,\mathrm{d}t$, we calculate $D(\boldsymbol{L}) = \gamma - 2$, $D(L_1, L_2, \delta_x) = -2x$. It follows from Eq. (4.12) that

$$G(x,s) = G^{\mathrm{cl}}(x,s) + \gamma \int_0^1 G^{\mathrm{cl}}(t,s)\,\mathrm{d}t \cdot \frac{x}{1-\gamma/2} \tag{4.15}$$

if $\gamma \neq 2$. Finally, we get

$$\begin{aligned} G(x,s) &= \theta(x-s)\frac{(x-s)^2}{2} + \frac{x(1-s)^2}{3(2-\gamma)}\bigl(\gamma(1-s) - 3\bigr)\\ &= \frac{x(1-s)^2}{3(2-\gamma)}\bigl(\gamma(1-s)-3\bigr) + \frac{1}{2}\begin{cases}(x-s)^2, & s \leqslant x,\\ 0, & x \leqslant s,\end{cases}\end{aligned}$$

for $\gamma \neq 2$.

Example 3. Let us consider the problem

$$\begin{gathered} u'''(x) = f(x), \quad x \in (x_1, x_3),\\ \alpha u(x_1) - \beta u'(x_1) = 0,\\ \gamma u(x_2) + \delta u'(x_2) = 0, \quad x_2 \in (x_1, x_3),\\ u''(x_3) = 0. \end{gathered} \tag{4.16}$$

Such a problem was considered by Anderson et al. [2]. For this problem, we find

$$\begin{aligned} D(\boldsymbol{L})[\boldsymbol{u}] &= \begin{vmatrix} \alpha & \gamma & 0\\ \alpha x_1 - \beta & \gamma x_2 + \delta & 0\\ \alpha x_1^2 - 2\beta x_1 & \gamma x_2^2 + 2\delta x_2 & 2\end{vmatrix} = 2(\alpha\gamma x_2 + \alpha\delta - \alpha\gamma x_1 + \beta\gamma)\\ &= 2\alpha\gamma(x_2 - x_1) + 2(\alpha\delta + \beta\gamma).\end{aligned}$$

For $x, s \in [x_1, x_3]$, there exists Green's function if $k = \alpha\delta + \beta\gamma + \alpha\gamma(x_2 - x_1) \neq 0$ and

$$G(x,s) = \frac{1}{2k}\begin{vmatrix} \alpha & \gamma & 0 & 1 \\ \alpha x_1 - \beta & \gamma x_2 + \delta & 0 & x \\ \alpha x_1^2 - 2\beta x_1 & \gamma x_2^2 + 2\delta x_2 & 2 & x^2 \\ 0 & \theta(x_2 - s)(\gamma\frac{(s-x_2)^2}{2} - \delta(s - x_2)) & 1 & \frac{\theta(x-s)(s-x)^2}{2} \end{vmatrix}.$$

Expanding this determinant, we get the following expression for Green's function (see [2, Theorem 1.2]):

$$G(x,s) = \begin{cases} s \in [x_1, x_2]\colon & \begin{cases} -u_1(x,s) & x \leqslant s, \\ -v_1(x,s) & x \geqslant s, \end{cases} \\ s \in [x_2, x_3]\colon & \begin{cases} -u_2(x,s) & x \leqslant s, \\ -v_2(x,s) & x \geqslant s, \end{cases} \end{cases}$$

where

$$u_1(x,s) := \frac{1}{k}(s - x_1)\big[\alpha(x - x_1) + \beta\big]\left[\delta + \frac{\gamma}{2}(2x_2 - x_1 - s)\right] - \frac{1}{2}(x - x_1)^2,$$

$$v_1(x,s) := u_1(x,s) + \frac{1}{2}(x - s)^2 = \frac{1}{2k}(s - x_1)\big[\alpha(s - x_1) + 2\beta\big]\big[\gamma(x_2 - x) + \delta\big],$$

$$u_2(x,s) := \frac{1}{k}\big[\alpha(x - x_1) + \beta\big]\left[\delta(x_2 - x_1) + \frac{\gamma}{2}(x_2 - x_1)^2\right] - \frac{1}{2}(x - x_1)^2,$$

$$v_2(x,s) := u_2(x,s) + \frac{1}{2}(x - s)^2.$$

We get the same expression for Green's function as in [2]. In this example, we express Green's function by formula (4.3) directly. We can write Green's function for the "classical" case $\alpha = \delta = 1$, $\beta = \gamma = 0$:

$$\begin{aligned} G^{\mathrm{cl}}(x,s) &= \begin{vmatrix} 1 & 0 & 0 & 1 \\ x_1 & 1 & 0 & x \\ \frac{x_1^2}{2} & x_2 & 1 & \frac{x^2}{2} \\ 0 & \theta(x_2 - s)(x_2 - s) & 1 & \frac{\theta(x-s)(s-x)^2}{2} \end{vmatrix} \\ &= \theta(x - s)\frac{(s - x)^2}{2} - \frac{x^2}{2} + \big(x_2 - \theta(x_2 - s)(x_2 - s)\big)x + \frac{x_1^2}{2} \\ &\quad - x_1x_2 + x_1\theta(x_2 - s)(x_2 - s). \end{aligned}$$

Then Green's function for general problem (4.16) can be expressed via Green's function G^{cl}:

$$\begin{aligned} G(x,s) &= G^{\mathrm{cl}}(x,s) + \beta\frac{\mathrm{d}}{\mathrm{d}x}G^{\mathrm{cl}}(x,s)\bigg|_{x=x_1} \cdot \frac{(x_2 - x)\gamma + \delta}{\alpha\delta + \beta\gamma + \alpha\gamma(x_2 - x_1)} \\ &\quad + \gamma G^{\mathrm{cl}}(x_2,s) \cdot \frac{\alpha(x_1 - x - \delta) - \beta}{\alpha\delta + \beta\gamma + \alpha\gamma(x_2 - x_1)}. \end{aligned}$$

5 CONCLUSIONS

The main result of this paper is that Green's function for ODE with additional conditions is related to Green's function of a similar problem, and this relation is expressed by formulas (4.6). When $D(\boldsymbol{L})[\boldsymbol{u}] \neq 0$, the functionals L_1, L_2, L_3 are linearly independent. This condition is necessary and sufficient for the existence of Green's function for the problem with three functional conditions. We present a few examples, but formulas (4.6) can be applied to a very wide class of problems with nonconstant coefficients and various boundary conditions as well as NBCs.

We can easily generalize all the results of this paper for an nth-order differential equation with n additional functional conditions.

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