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ON THE WICK THEOREM FOR MIXTURES OF CENTERED GAUSSIAN DISTRIBUTIONS

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Abstract. We consider centered conditionally Gaussian d-dimensional vectors X with random covariance matrix Ξ having an arbitrary probability distribution law on the set of nonnegative definite symmetric $d \times d$ matrices M_d^+. The paper deals with the evaluation problem of mean values $\mathbf{E}[\prod_{i=1}^{2n}(c_i, X)]$ for $c_i \in \mathbb{R}^d$, $i = 1, \ldots, 2n$, extending the Wick theorem for a wide class of non-Gaussian distributions. We discuss in more detail the cases where the probability law $\mathcal{L}(\Xi)$ is infinitely divisible, the Wishart distribution, or the inverse Wishart distribution. An example with $\Xi = \sum_{j=1}^m Z_j \Sigma_j$, where random variables Z_j, $j = 1, \ldots, m$, are nonnegative, and $\Sigma_j \in M_d^+$, $j = 1, \ldots, m$, are fixed, includes recent results from Vignat and Bhatnagar, 2008.

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1 INTRODUCTION

Let M_d be the Euclidean space of symmetric $d \times d$ matrices A with the scalar product $\langle A_1, A_2 \rangle := \operatorname{tr}(A_1 A_2)$, $A_1, A_2 \in M_d$, $M_d^+ \subset M_d$ be the cone of nonnegative definite matrices, and $\mathcal{P}(M_d^+)$ be a class of probability distributions on M_d^+. Here $\operatorname{tr} A$ denotes the trace of a matrix A.

The probability distribution of a d-dimensional random vector X is said to be the mixture of centered Gaussian distributions with mixing distribution $U \in \mathcal{P}(M_d^+)$ (U-mixture for short) if, for all $z \in \mathbb{R}^d$,

$$\mathbf{E}\mathrm{e}^{\mathrm{i}(z,X)} = \int\limits_{M_d^+} \mathrm{e}^{-\frac{1}{2}(zA,z)}\, U(\mathrm{d}A), \tag{1.1}$$

where $(x, y) = \sum_{j=1}^d x_j y_j$ for $x, y \in \mathbb{R}^d$.

Distributional properties of infinite divisibility or self-decomposability of subclasses of such mixtures are well studied (see, e.g., [2, 3] and references therein).

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Let $c_j = (c_{j1}, \dots, c_{jd}) \in \mathbb{R}^d$, $j = 1, \dots, 2n$. The paper deals with the evaluation problem of mean values $\mathbf{E}[\prod_{j=1}^{2n}(c_j, X)]$.

Let Π_{2n} be the class of pairings σ on the set $I_{2n} = \{1, 2, \dots, 2n\}$, i.e., the partitions of I_{2n} into n disjoint pairs, implying that $\operatorname{card} \Pi_{2n} = \frac{(2n)!}{2^n n!}$. For each $\sigma \in \Pi_{2n}$, we define uniquely the subsets $I_{2n\backslash\sigma} \subset I_{2n}$ and integers $\sigma(j)$, $j \in I_{2n\backslash\sigma}$, by

$$\sigma = \{(j, \sigma(j)),\ j \in I_{2n\backslash\sigma}\}.$$

If $U = \epsilon_\Sigma$ is a Dirac measure with fixed $\Sigma \in M_d^+$, i.e., in the Gaussian case, the Wick theorem says (see, e.g., [8, 14]) that

$$\mathbf{E}\Bigg[\prod_{j=1}^{2n}(c_j, X)\Bigg] = \sum_{\sigma \in \Pi_{2n}} \prod_{j \in I_{2n\backslash\sigma}} (c_j \Sigma, c_{\sigma(j)}) := m_{2n}(c, \Sigma). \tag{1.2}$$

If $U = \mathcal{L}(Y\Sigma)$, where Y is a nonnegative random variable, $\mathbf{E}Y^n < \infty$, and $\Sigma \in M_d^+$ is fixed, then in [18] it is checked that

$$\mathbf{E}\Bigg[\prod_{j=1}^{2n}(c_j, X)\Bigg] = \mathbf{E}Y^n m_{2n}(c, \Sigma).$$

In Section 2, we shall consider an arbitrary mixing distribution and the case of infinitely divisible mixing distribution U. In Section 3, several examples are presented, including mixtures with respect to the Wishart distribution or inverse Wishart distribution.

2 THE EXTENDED WICK THEOREM

Let

$$\phi_U(\Theta) := \int\limits_{M_d^+} \mathrm{e}^{-\operatorname{tr}(A\Theta)}\, U(\mathrm{d}A), \quad \Theta \in M_d^+. \tag{2.1}$$

Recall that, by definition, U is infinitely divisible with characteristics (Σ, V) if

$$-\log \phi_U(\Theta) = \operatorname{tr}(\Sigma\Theta) + \int\limits_{M_d^+} \big(1 - \mathrm{e}^{-\operatorname{tr}(A\Theta)}\big)\, V(\mathrm{d}A), \tag{2.2}$$

where $\Sigma \in M_d^+$, V is a measure on M_d^+ such that

$$\int\limits_{M_d^+} (\|A\| \wedge 1)\, V(\mathrm{d}A) < \infty,$$

and $\|A\| = (\operatorname{tr} A^2)^{1/2}$ (see, e.g., [6, 15]).

Theorem 1. *The following statements hold*:

1. *The probability distribution of a d-dimensional random vector X is the U-mixture of centered Gaussian distributions if and only if*

$$\mathbf{E}\mathrm{e}^{\mathrm{i}(z,X)} = \phi_U\bigg(\frac{1}{2} z^\top z\bigg), \tag{2.3}$$

where $z^\top$ is the transposed vector z.

2. *If the probability distribution of X is the U-mixture of centered Gaussian distributions and, for $j = 1, \dots, 2n$,*

$$\int_{M_d^+} (c_j A, c_j)^n \, U(\mathrm{d}A) < \infty, \tag{2.4}$$

then

$$\mathbf{E}\Bigg[\prod_{j=1}^{2n} (c_j, X)\Bigg] = \sum_{\sigma \in \Pi_{2n}} \int_{M_d^+} m_{2n}^{\sigma}(c, A) \, U(\mathrm{d}A), \tag{2.5}$$

where $m_{2n}^{\sigma}(c, A) = \prod_{j \in I_{2n \setminus \sigma}} (c_j A, c_{\sigma(j)})$.

Moreover, if U is infinitely divisible with characteristics (Σ, V) and, for $j = 1, \dots, 2n$,

$$\int_{M_d^+} (c_j A, c_j)^n \, V(\mathrm{d}A) < \infty, \tag{2.6}$$

then (2.5) *holds. In particular,*

$$\mathbf{E}(c_1, X)(c_2, X) = (c_1 \Sigma, c_2) + \int_{M_d^+} (c_1 A, c_2) \, V(\mathrm{d}A). \tag{2.7}$$

Proof. 1. The statement follows from (1.1) and (1.2), noting that, obviously,

$$\operatorname{tr}\big((z^\top z) A\big) = (zA, z).$$

2. Observe that $\operatorname{card} I_{2n \setminus \sigma} = n$ and, for all $\sigma \in \Pi_{2n}$ and $A \in M_d^+$,

$$\begin{aligned}
\prod_{j \in I_{2n \setminus \sigma}} \big|(c_j A, c_{\sigma(j)})\big|^n &\leqslant n^{-n} \bigg(\sum_{j \in I_{2n \setminus \sigma}} \big|(c_j A, c_{\sigma(j)})\big| \bigg)^n \\
&\leqslant n^{-1} \sum_{j \in I_{2n \setminus \sigma}} \big|(c_j A, c_{\sigma(j)})\big|^n \\
&\leqslant \frac{2^{n-1}}{n} \sum_{j \in I_{2n \setminus \sigma}} \big[(c_j A, c_j)^n + (c_{\sigma(j)} A, c_{\sigma(j)})^n\big] \\
&= \frac{2^{n-1}}{n} \sum_{j=1}^{2n} (c_j A, c_j)^n.
\end{aligned} \tag{2.8}$$

Using (2.8), Eq. (2.5) follows from (1.1), (1.2), and (2.4).

If a mixing distribution U is infinitely divisible with characteristics (Σ, V), (2.6) is satisfied, and a random matrix Ξ is such that $\mathcal{L}(\Xi) = U$, then, for each $c \in \mathbb{R}^d$, $\mathcal{L}((c\Xi, c))$ is infinitely divisible,

$$\mathbf{E}(c\Xi, c) = \operatorname{cum}_1(c\Xi, c) = (c\Sigma, c) + \int_{M_d^+} (cA, c) \, V(\mathrm{d}A),$$

and, for $n > 1$,

$$\mathrm{cum}_n(c\Xi, c) = \int\limits_{M_d^+} (cA, c)^n \, V(\mathrm{d}A),$$

where $\mathrm{cum}_n Z$ is the nth cumulant of Z.

Now it remains to apply the following useful statement.

Lemma 1. (*See* [16, 18].) *Let* $\mu_k = \mathbf{E}Z^k$ *and* $\kappa_k = \mathrm{cum}_k Z$. *If* $|\kappa_n| < \infty$, *then*

$$\mu_{k+1} = \sum_{j=0}^{k} \binom{k}{j} \mu_j \kappa_{k+1-j}, \quad k = 0, 1, \dots, n-1, \tag{2.9}$$

and $|\mu_n| < \infty$.

Indeed, it is well known (see, e.g., [7, 10]) that

$$\mu_k = \Gamma_k(\kappa_1, \dots, \kappa_k),$$

where the polynomials

$$\Gamma_k(x_1, \dots, x_k) = k! \sum_{m=1}^{k} \sum_{\substack{r_1, \dots, r_m \geqslant 0 \\ 1 \cdot r_1 + \dots + m r_m = k}} \frac{x_1^{r_1} \cdots x_m^{r_m}}{(1!)^{r_1} r_1! \cdots (m!)^{r_m} r_m!}$$

satisfy the recurrence formula (see [17])

$$\Gamma_{k+1}(x_1, \dots, x_{k+1}) = \sum_{j=0}^{k} \binom{k}{j} \Gamma_j(x_1, \dots, x_j) x_{k+1-j}. \qquad \square$$

Remark 1. If the probability distribution of X is the U-mixture of centered Gaussian distributions, $f : \mathbb{R}^d \to \mathbb{R}^1$ is an odd function, and $\mathbf{E}|f(X)| < \infty$, then, obviously, $\mathbf{E}f(X) = 0$.

3 EXAMPLES

Example 1 [*The extended relativistic α-stable laws*]. (Cf. [3, 4].) Let $S_d = \{A \in M_d \colon \|A\| = 1\}$ be the unit sphere of M_d, $SM_d^+ = S_d \cap M_d^+$, $\Sigma \in M_d^+$, $|\Sigma| := \det \Sigma > 0$, and $0 < \alpha < 2$. Let U be the multivariate extension of tempered α-stable law (see [3]), i.e., an infinitely divisible distribution on M_d^+ with characteristics $(0, V)$, where (in the polar representation)

$$V(\mathrm{d}A) = \frac{\mathrm{e}^{-r \ \mathrm{tr}(\Sigma A_0)}}{r^{1+\alpha/2}} \, \mathrm{d}r \, \nu(\mathrm{d}A_0), \quad r = \|A\|, \ A_0 = \frac{1}{r} A,$$

and $\nu(\mathrm{d}A_0)$ is a finite measure on SM_d^+.

We find that, for all $n = 1, 2, \ldots$ and $j = 1, \ldots, 2n$,

$$\int\limits_{M_d^+} (c_j A, c_j)^n \, V(\mathrm{d}A) = \int\limits_{SM_d^+} \int\limits_0^\infty r^n (c_j A_0, c_j)^n \frac{\mathrm{e}^{-r \, \mathrm{tr}(\Sigma A_0)}}{r^{1+\alpha/2}} \, \mathrm{d}r \, \nu(\mathrm{d}A_0)$$

$$= \Gamma\left(n - \frac{\alpha}{2}\right) \int\limits_{SM_d^+} \frac{(c_j A_0, c_j)^n}{(\mathrm{tr}(\Sigma A_0))^{n-\alpha/2}} \, \nu(\mathrm{d}A_0) < \infty.$$

Thus, assumption (2.6) is always satisfied, and, if the law of X is the U-mixture of centered Gaussian distributions, then, for all $n \geqslant 1$, formula (2.5) holds, and

$$\mathbf{E}(c_1, X)(c_2, X) = \Gamma\left(1 - \frac{\alpha}{2}\right) \int\limits_{SM_d^+} \frac{(c_1 A_0, c_2)}{(\mathrm{tr}(\Sigma A_0))^{1-\alpha/2}} \, \nu(\mathrm{d}A_0).$$

Using (2.3), for $z \in \mathbb{R}^d$, we have that

$$\begin{aligned}
\mathbf{E}\mathrm{e}^{\mathrm{i}(z,X)} &= \exp\biggl\{ - \int\limits_{M_d^+} \bigl(1 - \mathrm{e}^{-\frac{1}{2}(zA,z)}\bigr) \, V(\mathrm{d}A) \biggr\} \\
&= \exp\biggl\{ - \int\limits_{SM_d^+} \int\limits_0^\infty \bigl(1 - \mathrm{e}^{-\frac{r}{2}(zA_0,z)}\bigr) \frac{\mathrm{e}^{-r \, \mathrm{tr}(\Sigma A_0)}}{r^{1+\alpha/2}} \, \mathrm{d}r \, \nu(\mathrm{d}A_0) \biggr\} \\
&= \exp\biggl\{ - \left| \Gamma\left(-\frac{\alpha}{2}\right) \right| \int\limits_{SM_d^+} \left(\left[\mathrm{tr}(\Sigma A_0) + \frac{1}{2}(zA_0, z) \right]^{\alpha/2} - \bigl(\mathrm{tr}(\Sigma A_0)\bigr)^{\alpha/2} \right) \nu(\mathrm{d}A_0) \biggr\} \\
&=: \hat{\mu}_{\alpha,\Sigma}(z).
\end{aligned}$$

Remark 2. Let $X_{\alpha,\Sigma} := \{X_{\alpha,\Sigma}(t), \ t \geqslant 0\}$ be a Lévy process with

$$\mathbf{E} \exp\bigl\{\mathrm{i}\bigl(z, X_{\alpha,\Sigma}(1)\bigr)\bigr\} = \hat{\mu}_{\alpha,\Sigma}(z), \quad z \in \mathbb{R}^d, \ \Sigma \in M_d^+; \qquad X_{\alpha,\Sigma,h} := \bigl\{h^{-1/\alpha} X_{\alpha,\Sigma}(ht), \ t \geqslant 0\bigr\};$$

and

$$X'_{\alpha,\Sigma,h} := \bigl\{h^{-1/2} X_{\alpha,\Sigma}(ht), \ t \geqslant 0\bigr\}, \quad h > 0.$$

Similarly to [4], $X_{\alpha,\Sigma,h} \Rightarrow X_{\alpha,0}$ as $h \downarrow 0$, and, for $|\Sigma| > 0$, $X'_{\alpha,\Sigma,h} \Rightarrow G_{\alpha,\Sigma}$ as $h \to \infty$ in the space $D_{[0,\infty)}(\mathbb{R}^d)$ of càdlàg functions $\omega : [0, \infty) \to \mathbb{R}^d$ equipped with the $\mathcal{J}_1$-topology of Skorokhod, where "$\Rightarrow$" means the weak convergence of stochastic processes, and $G_{\alpha,\Sigma}$ is a centered Gaussian Lévy process with the covariance matrix

$$\Sigma_\alpha := \Gamma\left(1 - \frac{\alpha}{2}\right) \int\limits_{SM_d^+} \bigl(\mathrm{tr}(\Sigma A_0)\bigr)^{\alpha/2-1} A_0 \, \nu(\mathrm{d}A_0).$$

Indeed, because, for all $z \in \mathbb{R}^d$, we easily find that

$$\mathbf{E} \exp\bigl\{\mathrm{i}\bigl(z, X_{\alpha,\Sigma,h}(1)\bigr)\bigr\} = \bigl[\hat{\mu}_{\alpha,\Sigma}\bigl(h^{1/\alpha} z\bigr)\bigr]^h \to \hat{\mu}_{\alpha,0}(z) \quad \text{as } h \downarrow 0$$

and

$$\mathbf{E}\exp\{\mathrm{i}(z, X'_{\alpha,\Sigma,h}(1))\} = \left[\hat{\mu}_{\alpha,\Sigma}(h^{1/2}z)\right]^h \to \exp\left\{-\frac{1}{2}(z\Sigma_\alpha, z)\right\} \quad \text{as } h\to\infty,$$

it suffices to apply the well-known Skorokhod theorem on weak convergence of Lévy processes.

Example 2 [*The Wishart mixtures*]. Let $Y_1,\dots,Y_k$ be i.i.d. d-dimensional centered Gaussian vectors with covariance matrix Σ, $|\Sigma|>0$, $k\geqslant d$, and $U=\mathcal{L}(W_k) := W_d(\Sigma,k)$, where $W_k=\sum_{j=1}^k Y_j^\top Y_j$. It is known (see, e.g., [1, 12]) that $W_d(\Sigma,k)$, called the Wishart distribution, is a multivariate analogue of χ_k^2-distribution and enjoys the following properties:

$$\int_{M_d^+} \exp\{-\operatorname{tr}(A\Theta)\}\, W_d(\Sigma,k\mid \mathrm{d}A) = \frac{|\Sigma^{-1}|^{k/2}}{|\Sigma^{-1}+2\Theta|^{k/2}}, \quad \Theta\in M_d^+,$$

$$\mathcal{L}\big((cW_k,c)\big) = \mathcal{L}\big((c\Sigma,c)\chi_k^2\big), \quad c\in\mathbb{R}^d,$$

and

$$W_d(\Sigma,k\mid \mathrm{d}A) = w_d(\Sigma,k\mid A)\,\mathrm{d}A,$$

where, for $A\in M_d^+$,

$$w_d(\Sigma,k\mid A) = \begin{cases} \frac{|A|^{(k-d-1)/2}\exp\{-\frac{1}{2}\operatorname{tr}(\Sigma^{-1}A)\}}{(2^d|\Sigma|)^{k/2}\pi^{d(d-1)/4}\prod_{j=1}^d\Gamma(\frac{k-j+1}{2})} & \text{if } |A|>0,\\ 0 & \text{otherwise.}\end{cases}$$

If $d\geqslant 2$, the Wishart distribution is not infinitely divisible (see [9, 13]).

We have that, for all $c\in\mathbb{R}^d$ and $n=1,2,\dots,$

$$\int_{M_d^+} (cA,c)^n\, W_d(\Sigma,k\mid \mathrm{d}A) = (c\Sigma,c)^n\mathbf{E}(\chi_k^2)^n = \frac{2^n\Gamma(n+\frac{k}{2})}{\Gamma(\frac{k}{2})}(c\Sigma,c)^n<\infty.$$

Thus, if the law of X is the $W_d(\Sigma,k)$-mixture of centered Gaussian distributions, then, for all $n\geqslant 1$, the formula (2.5) holds,

$$\mathbf{E}(c_1,X)(c_2,X) = \frac{2\Gamma(1+\frac{k}{2})}{\Gamma(\frac{k}{2})}(c_1,\Sigma,c_2) = k(c_1\Sigma,c_2),$$

and, by (2.3), for $z\in\mathbb{R}^d$,

$$\mathbf{E}\mathrm{e}^{\mathrm{i}(z,X)} = \frac{|\Sigma^{-1}|^{k/2}}{|\Sigma^{-1}+z^\top z|^{k/2}}.$$

Example 3 [*The multivariate t-distributions*]. Under the assumptions of Example 2, the matrix W_k is invertible with probability 1. In this case, taking $U=\mathcal{L}(kW_k^{-1})$, it is known that, for all $c\in\mathbb{R}^d$ (see, e.g., [12]),

$$\mathcal{L}\big(k(cW_k^{-1},c)\big) = \mathcal{L}\bigg(k(c\Sigma^{-1},c)\frac{1}{\chi_{k-d+1}^2}\bigg),$$

implying that, for $c \neq 0$,

$$\int\limits_{M_d^+} (cA, c)^n \, U(\mathrm{d}A) = \begin{cases} k^n (c\Sigma^{-1}, c)^n \Gamma\left(\frac{k-d+1}{2} - n\right) & \text{if } n < \frac{k-d+1}{2}, \\ \infty & \text{if } n \geqslant \frac{k-d+1}{2}. \end{cases}$$

So, if the law of X is the U-mixture of centered Gaussian distributions, then formula (2.5) holds for $n < \frac{k-d+1}{2}$. If $k > d+1$, then

$$\mathbf{E}(c_1, X)(c_2, X) = k(c_1\Sigma^{-1}, c_2)\Gamma\left(\frac{k-d-1}{2}\right).$$

It is also known (see, e.g., [11]) that the law $\mathcal{L}(X)$ has the density f_X of the multivariate t-distribution:

$$f_X(x) = \frac{\Gamma(\frac{k+d}{2})}{(\pi k)^{d/2}\Gamma(k/2)|\Sigma|^{1/2}} \left(1 + \frac{(x\Sigma^{-1}, x)}{k}\right)^{-\frac{k+d}{2}}, \quad x \in \mathbb{R}^d.$$

From (3.3) and [5] we find that

$$\mathbf{E}\mathrm{e}^{\mathrm{i}(z,X)} = \frac{[k(z\Sigma, z)]^{k/4}}{2^{k/2-1}\Gamma(k/2)} K_{k/2}\big(\sqrt{k(z\Sigma, z)}\big), \quad z \in \mathbb{R}^d, \tag{3.1}$$

where K_ν is a modified Bessel function of the third kind, i.e.,

$$K_\nu(x) = \frac{1}{2}\int\limits_0^\infty u^{-\nu-1} \exp\left\{-\frac{1}{2}x\left(u + \frac{1}{u}\right)\right\} \mathrm{d}u, \quad x > 0, \ \nu \in \mathbb{R}^1.$$

From (2.3) and (3.1) we derive that, for $\Theta = \frac{1}{2}z^\top z \in M_d^+$, $z \in \mathbb{R}^d$,

$$\phi_U(\Theta) = \mathbf{E}\mathrm{e}^{-k\,\mathrm{tr}(W_k^{-1}\Theta)} = \frac{[2k\,\mathrm{tr}(\Sigma\Theta)]^{k/4}}{2^{k/2-1}\Gamma(\frac{k}{2})} K_{k/2}\big(\sqrt{2k\,\mathrm{tr}(\Sigma\Theta)}\big) \tag{3.2}$$

and conjecture that (3.2) holds true for all $\Theta \in M_d^+$.

Example 4. Let $\mathbb{R}_+^m = \{x = (x_1, \dots, x_m)\colon x_j \geqslant 0, \ j = 1, \dots, m\}$. Suppose that a random vector $Z = (Z_1, \dots, Z_m)$ takes its values in $\mathbb{R}_+^m$, $\Xi = \sum_{j=1}^m Z_j\Sigma_j$, where the matrices $\Sigma_j \in M_d^+$, $j = 1, \dots, m$, are fixed, and $U = \mathcal{L}(\Xi)$.

If $\phi_Z(t) := \mathbf{E}\mathrm{e}^{-(Z,t)}$, $t \in \mathbb{R}_+^m$, then

$$\begin{aligned} \phi_U(\Theta) &= \mathbf{E}\mathrm{e}^{-\mathrm{tr}(\Xi\Theta)} = \mathbf{E}\mathrm{e}^{-\sum_{j=1}^m Z_j \mathrm{tr}(\Sigma_j\Theta)} \\ &= \phi_Z\big(\mathrm{tr}(\Sigma_1\Theta), \dots, \mathrm{tr}(\Sigma_m\Theta)\big). \end{aligned} \tag{3.3}$$

Obviously, for each $c \in \mathbb{R}^d$,

$$\mathbf{E}(c\Xi, c)^n = \mathbf{E}\left(\sum_{j=1}^m Z_j(c\Sigma_j, c)\right)^n < \infty$$

if and only if

$$\mathbf{E}Z_j^n < \infty, \quad j = 1, \dots, m. \tag{3.4}$$

If $\mathcal{L}(X)$ is the U-mixture of centered Gaussian distributions and if (3.4) is satisfied, then

$$\mathbf{E}\Biggl[\prod_{j=1}^{2n}(c_j, X)\Biggr] = \sum_{\sigma\in\Pi_{2n}} \int\limits_{\mathbb{R}_+^m} m_{2n}^{\sigma}\Biggl(c, \sum_{j=1}^{m} x_j \Sigma_j\Biggr) P(Z \in \mathrm{d}x). \tag{3.5}$$

In particular, if $\mathbf{E}Z_j < \infty$, $j = 1, \dots, m$, then

$$\mathbf{E}(c_1, X)(c_2, X) = \sum_{j=1}^{m} \mathbf{E}Z_j (c_1\Sigma_j, c_2). \tag{3.6}$$

Applying (2.3) and (3.3), we find that

$$\mathbf{E}\mathrm{e}^{\mathrm{i}(z,X)} = \phi_Z\biggl(\frac{1}{2}(z\Sigma_1, z), \dots, \frac{1}{2}(z\Sigma_m, z)\biggr). \tag{3.7}$$

If $\mathcal{L}(Z)$ is infinitely divisible distribution on $\mathbb{R}_+^m$ with characteristics (x^0, ν), i.e.,

$$\phi_Z(t) = \exp\biggl\{-(x^0, t) - \int\limits_{\mathbb{R}_+^m} (1 - \mathrm{e}^{-(x,t)})\, \nu(\mathrm{d}x)\biggr\}, \tag{3.8}$$

where $x^0 \in \mathbb{R}_+^m$, $\nu(\{0\}) = 0$, and

$$\int\limits_{\mathbb{R}_+^m} |x| \wedge 1\, \nu(\mathrm{d}x) < \infty,$$

then $\mathcal{L}(U)$ is infinitely divisible distribution on M_d^+ with

$$\phi_U(\Theta) = \exp\Biggl\{-\sum_{j=1}^{m} x_j^0 \operatorname{tr}(\Sigma_j\Theta) - \int\limits_{\mathbb{R}_+^m} \Biggl(1 - \exp\Biggl\{-\sum_{j=1}^{m} x_j \operatorname{tr}(\Sigma_j\Theta)\Biggr\}\Biggr) \nu(\mathrm{d}x)\Biggr\}, \quad \Theta \in M_d^+.$$

If $\mathcal{L}(X)$ is the U-mixture of centered Gaussian distributions and

$$\int\limits_{\mathbb{R}_+^m} x_j^n\, \nu(\mathrm{d}x) < \infty, \quad j = 1, \dots, m, \tag{3.9}$$

then formula (3.5) holds, and from (3.3), (3.7), and (3.8) it follows that $\mathcal{L}(X)$ is an infinitely divisible distribution on $\mathbb{R}^d$ with

$$\mathbf{E}\mathrm{e}^{\mathrm{i}(z,X)} = \exp\Biggl\{-\frac{1}{2}\sum_{j=1}^{m} x_j^0 (z\Sigma_j, z) - \int\limits_{\mathbb{R}_+^m} \Biggl(1 - \exp\Biggl\{-\frac{1}{2}\sum_{j=1}^{m} x_j (z\Sigma_j, z)\Biggr\}\Biggr) \nu(\mathrm{d}x)\Biggr\}.$$

In particular, if (3.9) is satisfied with $n = 1$, then formula (3.6) holds.

REFERENCES

1. T.W. Anderson, *An Introduction to Multivariate Statistical Analysis,* 3rd edition, Wiley Ser. Probab. Stat., Wiley-Interscience [John Wiley & Sons], Hoboken, NJ, 2003.
2. O.E. Barndorff-Nielsen, J. Pedersen, and Ken-iti Sato, Multivariate subordination, self-decomposability and stability, *Adv. Appl. Probab.*, **33**(1):160–187, 2001.
3. O.E. Barndorff-Nielsen and V. Pérez-Abreu, Extensions of type G and marginal infinite divisibility, *Teor. Veroyatn. Primen.*, **47**(2):301–319, 2002.
4. B. Grigelionis, On stochastic processes associated with relativistic stable distributions, *Lith. Math. J.*, **48**(1):61–69, 2008. Letter to Editor, *Lith. Math. J.*, **48**(2):235, 2008.
5. C.C. Heyde and N.N. Leonenko, Student processes, *Adv. Appl. Probab.*, **37**(2):342–365, 2005.
6. J. Jonasson, Infinite divisibility of random objects in locally compact positive convex cones, *J. Multivariate Anal.*, **65**(2):129–138, 1998.
7. M. Kendall and A. Stuart, *The Advanced Theory of Statistics, Vol. 1: Distribution Theory,* 4th edition, Macmillan Publishing Co., Inc., New York, 1977.
8. V.P. Leonov and A.N. Shiryaev, On a method of calculation of semi-invariants, *Theory Probab. Appl.*, **4**:319–329, 1959.
9. P. Lévy, The arithmetic character of the Wishart distribution, *Proc. Cambridge Philos. Soc.*, **44**:295–297, 1948.
10. P. McCullagh, *Tensor Methods in Statistics*, Monogr. Stat. Appl. Probab., Chapman & Hall, London, 1987.
11. V.H. de la Peña, T. Lai, and Q.-M. Shao, *Self-Normalized Processes: Limit Theory and Statistical Applications*, Probab. Appl., Springer, Berlin, 2009.
12. C. Radhakrishna Rao, *Linear Statistical Inference and its Applications*, John Wiley & Sons Inc., New York, 1965.
13. D.N. Shanbhag, The Davidson–Kendall problem and related results on the structure of the Wishart distribution, *Aust. J. Stat.*, **30A**:272–280, 1988.
14. B. Simon, *The* $P(\phi)_2$ *Euclidean (Quantum) Field Theory*, Princeton Ser. Phys., Princeton Univ. Press, Princeton, NJ, 1974.
15. A.V. Skorohod, *Random Processes with Independent Increments*, Math. Appl. Sov. Ser., Vol. 47, Kluwer Academic Publishers Group, Dordrecht, 1991. Translated from the 2nd Russian edition by P.V. Malyshev.
16. J.L. Solé and F. Utzet, Time-space harmonic polynomials relative to a Lévy process, *Bernoulli*, **14**(1):1–13, 2008.
17. R.P. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge Stud. Adv. Math., Vol. 62, Cambridge Univ. Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and Appendix 1 by Sergei Fomin.
18. C. Vignat and S. Bhatnagar, An extension of Wick's theorem, *Stat. Probab. Lett.*, **78**(15):2400–2403, 2008.