Lithuanian Mathematical Journal, Vol. 48, No. 3, 2008, pp. 339–356

ON THE STABILITY OF A FINITE-DIFFERENCE SCHEME FOR NONLOCAL PARABOLIC BOUNDARY-VALUE PROBLEMS

M. Sapagovas

Institute of Mathematics and Informatics, Akademijos 4, LT-08663 Vilnius, Lithuania
(e-mail: mifosap@ktl.mii.lt)

Received January 20, 2008

Abstract. We deal with the stability analysis of difference schemes for a one-dimensional parabolic equation subject to integral conditions. It is based on the spectral structure of the transition matrix of a difference scheme. The stability domain is defined by using the hyperbola which is the locus of points where the transition matrix has trivial eigenvalues. The stability conditions obtained are much more general compared with those known in the literature. We analyze three separate cases of nonlocal integral conditions and solve an example illustrating the efficiency of the technique.

Keywords: nonlocal integral conditions, parabolic equations, finite-difference schemes, stability.

1 INTRODUCTION. PROBLEM FORMULATION

We consider the one-dimensional parabolic equation subject to two integral conditions and one initial condition:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x.t), \quad 0 < x < 1,\ 0 < t \leqslant T, \tag{1.1}$$

$$u(0,t) = \int_0^1 \alpha(x) u(x,t)\,\mathrm{d}x + \mu_1(t), \tag{1.2}$$

$$u(1,t) = \int_0^1 \beta(x) u(x,t)\,\mathrm{d}x + \mu_2(t), \tag{1.3}$$

$$u(x,0) = \varphi(x). \tag{1.4}$$

To solve this problem, we apply the finite-difference method. The main objective of the article is analysis of the stability conditions for the resulting system of finite-difference equations.

The stability of finite-difference schemes in the case of various one-dimensional parabolic equations subject to integral conditions (1.2)–(1.3) has been investigated by numerous authors. In [7], a stability and

0363-1672/08/4803-0339

convergence analysis is given for both explicit and implicit difference schemes, provided that the conditions

$$\int_0^1 |\alpha(x)|\,\mathrm{d}x < 1 \quad \text{and} \quad \int_0^1 |\beta(x)|\,\mathrm{d}x < 1 \tag{1.5}$$

are satisfied. In the same article, the convergence of the Crank–Nicolson difference scheme is proved under the other, more rigorous assumption

$$\Big(\int_0^1 |\alpha(x)|^2\,\mathrm{d}x\Big)^{1/2} + \Big(\int_0^1 |\beta(x)|^2\,\mathrm{d}x\Big)^{1/2} \leqslant \sqrt{3}/2. \tag{1.6}$$

The case of conditions (1.5) is considered in [3] as well, where the stability of difference schemes is also analyzed. In [14], the stability of the difference schemes is proved under the condition

$$\int_0^1 |\alpha(x)|^2\,\mathrm{d}x + \int_0^1 |\beta(x)|^2\,\mathrm{d}x < 2. \tag{1.7}$$

In [8], a theoretical analysis of the discrete Galerkin method for the nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\Big(a(x)\frac{\partial u}{\partial x}\Big) + F(u, x, t)$$

subject to nonlocal conditions (1.2) and (1.3) is provided under the conditions

$$\int_0^1 |\alpha(x)|^2\,\mathrm{d}x < 1, \qquad \int_0^1 |\beta(x)|^2\,\mathrm{d}x < 1. \tag{1.8}$$

The main goal of our article is the stability analysis of difference schemes, based on a principle different from those discussed in the above-mentioned articles; namely, the stability is there investigated on the basis of the spectral structure of the difference operator. The peculiarity of this technique lies in that the difference operator subject to any nonlocal condition is always nonsymmetric. Moreover, the eigenvalues of this operator are not necessarily positive or even real. This technique was previously used by the author in [20] for $\alpha(x) = \alpha = \text{const}$ and $\beta(x) = \beta = \text{const}$. Analyzing the spectrum of the difference operator, there appears the opportunity to prove the stability of the difference schemes in the case of much more general conditions, compared to conditions (1.5), (1.6), (1.7), or (1.8).

The structure of the article is as follows. In Section 2, we develop the difference schemes and relate the corresponding stability problem to the eigenvalue problem for the difference operator (matrix). In Section 3, we define the concept of stability of difference schemes by using the spectrum of transition matrix and give some considerations on the definition. In Section 4, we derive an equation of a hyperbola where the difference operator has zero eigenvalue on each branch. This hyperbola is taken as a basis to determine the stability domain of the difference schemes. Sections 5.1 and 5.2 deal with a couple of specific cases of integral conditions (1.2) and (1.3) where the hyperbola degenerates to a straight line. In Section 5.3, we present the results of computer modeling in the case of variable coefficients $\alpha(x)$ and $\beta(x)$ that illustrate the efficiency of the technique. Section 6 contains some corollaries and possible extensions of the problem.

2 DEVELOPING THE DIFFERENCE SCHEMES

To study stability conditions for a difference scheme, we analyze the spectrum of the transition matrix of the corresponding system of difference equations. For this purpose, we write the integral conditions of problem

(1.1)–(1.4) in a slightly different form, introducing new parameters γ_1, γ_2. Namely, we write conditions (1.2) and (1.3) in the form

$$u(0,t) = \gamma_1 \int_0^1 \alpha(x) u(x,t)\,\mathrm{d}x + \mu_1(t), \tag{2.1}$$

$$u(1,t) = \gamma_2 \int_0^1 \beta(x) u(x,t)\,\mathrm{d}x + \mu_2(t). \tag{2.2}$$

Next, we approximate the differential problem (1.1), (2.1), (2.2), (1.4) by the following difference problem:

$$\frac{U_i^{j+1} - U_i^j}{\tau} = \sigma \Lambda U_i^{j+1} + (1-\sigma)\Lambda U_i^j + f_i^j, \tag{2.3}$$

$$U_0^{j+1} = \gamma_1(\alpha, U^{j+1}) + \mu_1^{j+1}, \tag{2.4}$$

$$U_N^{j+1} = \gamma_2(\beta, U^{j+1}) + \mu_2^{j+1}, \tag{2.5}$$

$$U_i^0 = \varphi_i, \tag{2.6}$$

where $0 \leqslant \sigma \leqslant 1;\ i = 1, 2, \ldots, N-1;\ j = 0, 1, \ldots, M-1;\ h = 1/N,\ \tau = T/M,$

$$\Lambda U_i^j = \frac{U_{i-1}^j - 2U_i^j + U_{i+1}^j}{h^2},$$

$$(\alpha, U^{j+1}) = h\Big(\frac{\alpha_0 U_0^{j+1} + \alpha_N U_N^{j+1}}{2} + \sum_{i=1}^{N-1} \alpha_i U_i^{j+1}\Big),$$

$$(\beta, U^{j+1}) = h\Big(\frac{\beta_0 U_0^{j+1} + \beta_N U_N^{j+1}}{2} + \sum_{i=1}^{N-1} \beta_i U_i^{j+1}\Big).$$

We rearrange this system of difference equations (2.3)–(2.6) in a different form. To this end, we rewrite conditions (2.4), (2.5) in the form of a system of two equations with two unknowns $U_0^{j+1},\ U_N^{j+1}$:

$$\begin{cases} \Big(1 - \dfrac{\gamma_1 h \alpha_0}{2}\Big) U_0^{j+1} - \dfrac{\gamma_1 h \alpha_N}{2} U_N^{j+1} = \gamma_1 h \sum\limits_{i=1}^{N-1} \alpha_i U_i^{j+1} + \mu_1^{j+1}, \\ -\dfrac{\gamma_2 h \beta_0}{2} U_0^{j+1} + \Big(1 - \dfrac{\gamma_2 h \beta_N}{2}\Big) U_N^{j+1} = \gamma_2 h \sum\limits_{i=1}^{N-1} \beta_i U_i^{j+1} + \mu_2^{j+1}. \end{cases} \tag{2.7}$$

We solve this system for the unknowns $U_0^{j+1},\ U_N^{j+1}$ with respect to the remaining unknowns. System (2.7) has a single solution if its determinant is not equal to zero, i.e.,

$$D = \begin{vmatrix} 1 - \dfrac{\gamma_1 h \alpha_0}{2} & -\dfrac{\gamma_1 h \alpha_N}{2} \\ -\dfrac{\gamma_2 h \beta_0}{2} & 1 - \dfrac{\gamma_2 h \beta_N}{2} \end{vmatrix} \neq 0.$$

If the functions $\alpha(x)$ and $\beta(x)$ in conditions (2.1)–(2.2) are bounded in the interval $[0,1]$, i.e., if

$$|\gamma_1 \alpha(x)| \leqslant M_1 < \infty, \quad |\gamma_2 \beta(x)| \leqslant M_1 < \infty, \quad x \in [0,1],$$

and the mesh $h < 1/M_1$, then

$$D = \Big(1 - \frac{\gamma_1 h\alpha_0}{2}\Big)\Big(1 - \frac{\gamma_2 h\beta_N}{2}\Big) - \frac{\gamma_1 h\alpha_N}{2}\frac{\gamma_2 h\beta_0}{2} \geqslant \Big(1 - \frac{hM_1}{2}\Big)\Big(1 - \frac{hM_1}{2}\Big) - \frac{h^2M_1^2}{4} = 1 - hM_1 > 0.$$

Now, we can write the solution of (2.7) in the form

$$\begin{aligned} U_0^{j+1} &= \gamma_1 h \sum_{i=1}^{N-1} a_i U_i^{j+1} + \bar{\mu}_1^{j+1}, \\ U_N^{j+1} &= \gamma_2 h \sum_{i=1}^{N-1} b_i U_i^{j+1} + \bar{\mu}_2^{j+1}, \end{aligned} \tag{2.8}$$

where

$$a_i = \frac{1}{D}\Big(\alpha_i - \frac{\gamma_2 h\alpha_i\beta_N}{2} + \frac{\gamma_2 h\beta_i\alpha_N}{2}\Big),$$

$$b_i = \frac{1}{D}\Big(\beta_i - \frac{\gamma_1 h\beta_i\alpha_0}{2} + \frac{\gamma_1 h\alpha_i\beta_0}{2}\Big),$$

$$\begin{aligned} \bar{\mu}_1^{j+1} &= \frac{1}{D}\Big(\mu_1^{j+1} - \frac{\gamma_2 h\beta_N}{2}\mu_1^{j+1} + \frac{\gamma_1 h\alpha_N}{2}\mu_2^{j+1}\Big), \\ \bar{\mu}_2^{j+1} &= \frac{1}{D}\Big(\mu_2^{j+1} - \frac{\gamma_2 h\beta_0}{2}\mu_1^{j+1} - \frac{\gamma_1 h\alpha_0}{2}\mu_2^{j+1}\Big). \end{aligned}$$

Now taking $\sigma = 1$ in system (2.3) and putting the expressions (2.8) for U_0^{j+1}, U_N^{j+1} into the equations of the system for $i = 1$ and $i = N-1$, we get the expression

$$\frac{U_i^{j+1} - U_i^j}{\tau} = \frac{1}{h^2}\begin{cases} \gamma_1 h \sum\limits_{l=1}^{N-1} a_l U_l^{j+1} - 2U_1^{j+1} + U_2^{j+1} + \bar{\mu}_1^{j+1} + h^2 f_1^{j+1}, & i = 1, \\ \Big(U_{i-1}^{j+1} - 2U_i^{j+1} + U_{i+1}^{j+1}\Big) + h^2 f_i^{j+1}, & i = 2,\dots,N-2, \\ \gamma_2 h \sum\limits_{l=1}^{N-1} b_l U_l^{j+1} + U_{N-2}^{j+2} - 2U_{N-1}^{j+1} + \bar{\mu}_2^{j+1} + h^2 f_{N-1}^{j+1}, & i = N-1. \end{cases} \tag{2.9}$$

Defining the square matrix of order $(N-1)$

$$A = h^{-2}\begin{pmatrix} 2-\gamma_1 ha_1 & -1-\gamma_1 ha_2 & -\gamma_1 ha_3 & \dots & \dots & -\gamma_1 ha_{N-1} \\ -1 & 2 & -1 & \dots & \dots & 0 \\ 0 & -1 & 2 & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ & & & & 2 & -1 \\ -\gamma_2 hb_1 & -\gamma_2 hb_2 & -\gamma_2 hb_3 & \dots & -1-\gamma_2 hb_{N-2} & 2-\gamma_2 hb_{N-1} \end{pmatrix} \tag{2.10}$$

allows us to rewrite the system of difference equations (2.9) on the $(j+1)$th level in the vector form

$$\Big(E + \tau A\Big)U^{j+1} = U^j + \tau F^{j+1}, \tag{2.11}$$

where U^{j+1}, U^j, and F^{j+1} are vectors of order $N-1$, and E is the identity matrix.

Lemma 1. *The eigenvalue problem*

$$AU = \lambda U \tag{2.12}$$

for the matrix A is equivalent to the difference eigenvalue problem

$$\frac{U_{i-1} - 2U_i + U_{i+1}}{h^2} + \lambda U_i = 0, \quad i = 1, 2, \ldots, N-1, \tag{2.13}$$

$$U_0 = \gamma_1(\alpha, U), \tag{2.14}$$

$$U_N = \gamma_2(\beta, U). \tag{2.15}$$

Proof. Equation (2.12) directly follows from system (2.13)–(2.15) using the same technique as that used to get Eq. (2.11) from system (2.3)–(2.5). The inverse procedure, i.e., the derivation of system (2.3)–(2.5) from problem (2.12), is obvious if we introduce two additional variables

$$U_0 = \gamma_1 h \sum_{i=1}^{N-1} a_i U_i, \quad U_N = \gamma_2 h \sum_{i=1}^{N-1} b_i U_i.$$

The lemma is proved.

Remark 1. Difference eigenvalue problem (2.13)–(2.15) can be constructed in an ordinary way, applying the finite-difference method to the differential eigenvalue problem

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \lambda u = 0, \quad 0 < x < 1, \tag{2.16}$$

$$u(0) = \gamma_1 \int_0^1 \alpha(x) u(x)\,\mathrm{d}x, \tag{2.17}$$

$$u(1) = \gamma_2 \int_0^1 \beta(x) u(x)\,\mathrm{d}x. \tag{2.18}$$

In the case of $\sigma = 0$ (explicit scheme) instead of $\sigma = 1$ (implicit scheme) in Eq. (2.3), the system of difference equations (2.3)–(2.5) on the $(j+1)$th level can be similarly written in the form

$$U^{j+1} = \Big(E - \tau A\Big) U^j + \tau F^j, \tag{2.19}$$

where the matrix A and vectors U^{j+1}, U^j, and F^j are defined as in problem (2.11).

3 STABILITY OF DIFFERENCE SCHEMES

A sufficient stability condition for difference scheme of the form

$$U^{j+1} = SU^j + \bar{f}^j, \tag{3.1}$$

can be written in the form [17]

$$\|S\| \leqslant 1 + c_0 \tau, \tag{3.2}$$

where c_0 is a constant independent of both τ and h. In the case of symmetric matrix S, we can define

$$\|S\| = \rho(S) = \max_{1\leqslant i\leqslant N-1} |\lambda_i(S)|,$$

where $\lambda_i(S)$ are the eigenvalues of S, and $\rho(S)$ is the spectral radius of S. Thus, the stability of difference scheme (3.1) is confirmed by the condition $\rho(S) \leqslant 1$.

In the case of nonsymmetric matrix S, which is typical for the difference schemes with nonlocal conditions, the sufficient stability condition (3.2) is usually replaced by the necessary von Neumann condition

$$|\lambda_i(S)| \leqslant 1 + c_1\tau, \tag{3.3}$$

where c_1 is a constant independent of both τ and h (see, e.g., [9], [16]).

In the case where S is a nonsymmetric matrix, the inequality $\rho(S) < 1$ is a necessary and sufficient condition to define a norm $\|S\|_*$ of the matrix S such that $||S||_* < 1$ (see [1]). As is noted in ([9], Section 25.2), if the necessary von Neumann condition (3.3) does not hold, then it is practically impossible to define the norms of vectors or matrices so that the difference scheme is stable. And vice versa, if condition (3.3) is true, then one can always succeed in defining norms so that the difference scheme is stable.

Monograph [18] (Section 2.2.3) gives a method to define a norm $||S||_*$, under the condition $\rho(S) < 1$, such that inequality $||S||_* \leqslant 1 + \varepsilon$ holds for every sufficiently small positive number $\varepsilon > 0$ given in advance. We observe that if S is a simple-structured matrix, i.e., the number of linearly independent eigenvectors is equal to the order of the matrix, then one can define the norm

$$||S||_* = \rho(S).$$

In this case, the vector norm, compatible to the matrix norm $||S||_*$, is defined by

$$||U||_* = ||P^{-1}U||_3,$$

where P is a matrix with columns that are linearly independent eigenvectors of the matrix S, and

$$||v||_3 = \Big(h\sum_{i=1}^{N-1} v_i^2\Big)^{1/2}.$$

This definition of the norm of a nonsymmetric matrix, $||S||_* = \rho(S)$, was used in [19] to investigate the iterative methods for the systems of difference equations with nonlocal conditions.

In the following sections, we will use the stability condition $\rho(S) < 1$ of the difference scheme. It is argued in [4] that this condition ensures the step stability of the difference scheme defined by the inequality

$$|U_i^j| \leqslant C, \quad j = 1, 2, \dots,$$

where the constant C depends on τ and h.

4 ANALYSIS OF THE SPECTRUM OF MATRIX A

Here we investigate the spectrum of the matrix A given by (2.10). Simply speaking, we will find, according to Lemma 1, all the eigenvalues of the difference problem (2.13)–(2.15). For this purpose, we will apply the same technique as in [2], [6], [15], [21], [22], where the spectrum of both differential and difference operators was analyzed.

Dealing with the stability of the system of difference equations (2.3)–(2.6), it is important to clearify the conditions under which the eigenvalue of the matrix A equals zero. First of all, we will answer this question.

Lemma 2. *The necessary and sufficient condition for the eigenvalue of the difference problem* (2.13)–(2.15) *to be zero, $\lambda = 0$, is as follows:*

$$\gamma_1\gamma_2[(\alpha, x)(\beta, 1) - (\beta, x)(\alpha, 1)] + \gamma_1(\alpha, 1 - x) + \gamma_2(\beta, x) - 1 = 0. \tag{4.1}$$

Proof. For $\lambda = 0$, the general solution of the difference equations (2.13) is

$$U_1 = c_1 ih + c_2, \quad i = 0, 1, \ldots, N, \tag{4.2}$$

for any free constants c_1 and c_2. We choose the values of the constants so that solution (4.2) also satisfies nonlocal conditions (2.14) and (2.15). Therefore, we put solution (4.2) into conditions (2.14), (2.15):

$$\begin{cases} c_2 = \gamma_1 c_1(\alpha, x) + \gamma_1 c_2(\alpha, 1), \\ c_1 + c_2 = \gamma_2 c_1(\beta, x) + \gamma_2 c_2(\beta, 1). \end{cases} \tag{4.3}$$

For $\lambda = 0$ to be an eigenvalue of the problem (2.13)–(2.15), it is necessary and sufficient that solution (4.2) be not identically equal to zero, i.e., system (4.3) has a nontrivial solution (c_1, c_2). Therefore, the determinant of the system must be zero:

$$D = \begin{vmatrix} -\gamma_1(\alpha, x) & 1 - \gamma_1(\alpha, 1) \\ 1 - \gamma_2(\beta, x) & 1 - \gamma_2(\beta, 1) \end{vmatrix} = 0.$$

This yields (4.1). The lemma is proved.

Equation (4.1) in the coordinate system $(\gamma_1,\ \gamma_2)$ represents a hyperbola. Depending on the expressions of $\alpha(x)$ and $\beta(x)$, the hyperbola (4.1) can degenerate to a straight line. This happens when the coefficient $A = (\alpha, x)(\beta, 1) - (\beta, x)(\alpha, 1)$ at the term $\gamma_1\gamma_2$ equals zero. We point out the main cases where the hyperbola degenerates to a straight line, i.e., $A = 0$.

Lemma 3. *If at least one of the conditions listed below is true, then* (4.1) *defines a straight line in the coordinate system* (γ_1, γ_2) :

1. $\alpha(x) = \alpha = const$ and $\beta(x) = \beta = const$;
2. $\alpha(x) = c\beta(x), \quad c = const$;
3. $(\alpha, 1) = 0, \quad (\beta, 1) = 0$;
4. $\alpha(x) = 0$ or $\beta(x) = 0$.

Now we explain the technique for the stability analysis of difference schemes. Two branches of hyperbola divide the plane $(\gamma_1,\ \gamma_2)$ into three unbounded regions. If hyperbola degenerates to a straight line, then it divides the plane only into two unbounded parts. One of these regions contains the point $(\gamma_1 = 0,\ \gamma_2 = 0)$. It never belongs to the hyperbola or the straight line and can be located either somewhere between the branches of the hyperbola or on one side of both branches. The point $(\gamma_1 = 0,\ \gamma_2 = 0)$ is significant because the case $\gamma_1 = 0,\ \gamma_2 = 0$ corresponds to the parabolic equation (1.1) with classical boundary-value conditions (Dirichlet conditions in this case)

$$u(0) = \mu_1(t), \quad u(1) = \mu_2(t).$$

In the case $\gamma_1 = 0, \gamma_2 = 0$, the following two propositions are important:

1. The matrix A is symmetric, and all of its eigenvalues are distinct and positive:

$$\lambda_k = \frac{4}{h^2}\sin^2\frac{k\pi h}{2}, \quad k = 1, 2, \dots, N-1,$$

and the eigenvectors are linearly independent (and orthogonal):

$$U^k = \{U_i^k\} = \{\sin k\pi i h\}, \quad k = 1, 2, \dots, N-1.$$

2. Implicit difference scheme (2.11) is unconditionally stable for all values of $h > 0$ and $\tau > 0$, and explicit difference scheme (2.19) is conditionally stable for $\tau/h^2 \leqslant 1/2$.

Next, we prove two propositions on relating stability of difference schemes to spectral properties of the nonsymmetric matrix A defined by (2.10).

Theorem 1. *If all eigenvalues of the matrix A are real and positive, then difference scheme* (2.11) *is unconditionally stable and explicit difference scheme* (2.19) *is stable if* $\tau < 2/\rho(A)$.

Proof. For $\sigma = 1$, we directly derive

$$|\lambda_k(S)| = |\lambda_k((E+\tau A)^{-1})| = \frac{1}{1+\tau\lambda_k(A)},$$

which shows that $\rho(S) < 1$ for all $\tau > 0$. For $\sigma = 0$,

$$|\lambda_k(S)| = |\lambda_k(E-\tau A)| = |1-\tau\lambda_k(A)|.$$

Thus, $\rho(S) < 1$ if

$$\tau < \frac{2}{\rho(A)}.$$

The condition $\rho(S) < 1$ ensures the stability of difference schemes (2.11) and (2.19).

A more general proposition concerning the complex eigenvalues of matrix A also holds. Denote

$$\lambda_k(A) = \mathrm{Re}\,\lambda_k(A) \pm i\mathrm{Im}\,\lambda_k(A).$$

Theorem 2. *If* $\mathrm{Re}\,\lambda_k(A) > 0$, *then difference scheme* (2.11) *is unconditionally stable and difference scheme* (2.19) *is stable if* τ *is sufficiently small, i.e.,*

$$\tau < \frac{2\mathrm{Re}\,\lambda_k(A)}{[\mathrm{Re}\,\lambda_k(A)]^2 + [\mathrm{Im}\,\lambda_k(A)]^2}. \tag{4.4}$$

Proof. If $\lambda_k(A)$ is a complex number, then (2.11) implies

$$|\lambda_k(S)| = \frac{1}{|1+\tau\lambda_k(A)|} = \frac{1}{|1+\tau\mathrm{Re}\,\lambda_k(A) + i\tau\mathrm{Im}\,\lambda_k(A)|}$$
$$= \Big([1+\tau\mathrm{Re}\,\lambda_k(A)]^2 + [\tau\mathrm{Im}\,\lambda_k(A)]^2\Big)^{-1/2}.$$

Therefore, $\rho(S) < 1$ for any $\tau > 0$. Similarly, Eq. (2.19) yields

$$|\lambda_k(S)| = |1 - \tau\lambda_k(A)|^2 = [1 - \tau \mathrm{Re}\,\lambda_k(A)]^2 + [\tau \mathrm{Im}\,\lambda_k(A)]^2$$
$$= \tau^2\Big((\mathrm{Re}\,\lambda_k(A))^2 + (\mathrm{Im}\,\lambda_k(A))^2\Big) - 2\tau \mathrm{Re}\,\lambda_k(A) + 1.$$

Therefore, the equality $\rho(S) < 1$ holds if (4.4) is true.

5 EXAMPLES OF STABILITY ANALYSIS OF DIFFERENCE SCHEMES

5.1 Problem 1: $\alpha(x) = \mathbf{const}$, $\beta(x) = \mathbf{const}$.

In this case, one can take $\alpha(x) = 1$ and $\beta(x) = 1$ and express the stability condition of the difference scheme (2.3)–(2.6) in terms of parameters γ_1, γ_2. By Lemma 3, the hyperbola degenerates to the straight line $\gamma_1 + \gamma_2 = 2$. This problem is investigated in [20]. When γ_1 and γ_2 satisfy the condition

$$\gamma_1 + \gamma_2 < 2, \tag{5.1}$$

then all eigenvalues of the matrix A are real and positive. Thus, by Theorem 1 we have the stability of the difference schemes (2.11) and (2.19), and the stability domain is the half-plane $\gamma_1 + \gamma_2 < 2$.

5.2 Problem 2: $\alpha(x) = 0$, $\beta(x) = x$.

By Lemma 3, in this case, hyperbola (4.1) also degenerates to the straight line $\gamma_2 = \mathrm{const}$. We investigate this problem in detail. In order to understand fully the spectral structure of the matrix A, we investigate both the difference eigenvalue problem (2.13)–(2.15) and differential eigenvalue problem (2.16)–(2.18).

Lemma 4. *If $\alpha(x) = 0$ and $\beta(x) = x$, then the number $\lambda = 0$ is an eigenvalue of the differential eigenvalue problem* (2.16)–(2.18) *if and only if $\gamma_2 = 3$. The corresponding eigenvector is $u(x) = c_1 x$, where c_1 is a free constant.*

Proof. When $\lambda = 0$, the solution of differential equation (2.16) satisfying condition (2.17) with $\alpha(x) = 0$ is $u(x) = c_1 x$. Putting it into nonlocal condition (2.18), we obtain

$$c_1 = c_1\gamma_2 \int_0^1 x^2\,\mathrm{d}x$$

or

$$c_1\Big(1 - \frac{\gamma_2}{3}\Big) = 0.$$

Hence, the condition $u(x) \not\equiv 0$ is equivalent to the condition $\gamma_2 = 3$. The lemma is proved.

The following lemma is proved similarly.

Lemma 5. *If $\alpha(x) = 0$ and $\beta(x) = x$, then the number $\lambda = 0$ is an eigenvalue of the difference eigenvalue problem* (2.13)–(2.15) *if and only if*

$$\gamma_2 = 3 - \frac{3h^2}{2 + h^2}. \tag{5.2}$$

The corresponding eigenvector is $U_i = Cih$, $i = 0, 1, \ldots, N$.

We observe that the proposition of Lemma 5 can be obtained directly by using the equation of hyperbola (4.1). If $\alpha(x) = 0$ and $\beta(x) = x$, then Eq. (4.1) becomes

$$\gamma_2(x, x) = 1$$

or

$$\gamma_2 h\Big(\frac{1}{2} + \sum_{i=1}^{N-1}(ih)^2\Big) = 1,$$

which implies condition (5.2).

Lemma 6. *If $\alpha(x) = 0$ and $\beta(x) = x$, then the negative eigenvalues of the differential eigenvalue problem* (2.16)–(2.18)*, provided that they exist, are given by $\lambda_k = -\beta_k^2$, where $\beta_k > 0$ are the solutions of the equation*

$$\tanh\beta = \frac{\gamma_2\beta}{\beta^2 + \gamma_2}. \tag{5.3}$$

Proof. When $\lambda < 0$, the solution of (2.16) satisfying condition (2.17) with $\alpha(x) = 0$ is given by

$$u(x) = c_2 \sinh\beta x, \quad \beta = \sqrt{-\lambda} > 0.$$

Putting this expression for $u(x)$ into integral condition (2.18), we obtain

$$c_2 \sinh\beta = c_2\gamma_2 \int_0^1 x \sinh\beta x \,\mathrm{d}x.$$

Elementary transformations give

$$c_2 \sinh\beta = c_2\gamma_2\Big(\frac{\cosh\beta}{\beta} - \frac{\sinh\beta}{\beta^2}\Big).$$

Hence, $c_2 \neq 0$ if

$$\sinh\beta = \gamma_2\Big(\frac{\cosh\beta}{\beta} - \frac{\sinh\beta}{\beta^2}\Big).$$

This yields Eq. (5.3).

Lemma 7. *Since $\alpha(x) = 0$ and $\beta(x) = x$, the differential eigenvalue problem* (2.16)–(2.18) *has a single negative eigenvalue if and only if $\gamma_2 > 3$.*

Proof. The functions $f_1(\beta) = \tanh\beta$ and $f_2(\beta) = \gamma_2\beta/(\beta^2 + \gamma_2)$ both are equal to zero at the point $\beta = 0$. Moreover, $f_1(\beta)$ increases over the interval $[0, \infty)$ from 0 to 1.

If $\gamma_2 < 0$, then the function $f_2(\beta)$ has an infinite discontinuity at the point $\beta_0 = \sqrt{-\gamma_2}$. Next, $f_1(\beta) < f_2(\beta)$ for $\beta \in (0, \beta_0)$ and $f_2(\beta) < 0$ for $\beta \in (\beta_0, \infty)$. Thus, Eq. (5.3) has no solutions in $(0, \infty)$.

If $0 < \gamma_2 < 3$, then $f_1(\beta) < f_2(\beta)$ in the interval $(0, \infty)$, i.e., Eq. (5.3) has no solutions in $(0, \infty)$ as well.

If $\gamma_2 > 3$, then $f_1(\beta) > 0$ and is continuous in $(0, \infty)$, and there exists a unique β^* such that $f_1(\beta) < f_2(\beta)$ for $\beta \in (0, \beta^*)$ and $f_1(\beta) > f_2(\beta)$ for $\beta \in (\beta^*, \infty)$. In this case, Eq. (5.3) has a unique solution $\beta^* > 0$.

Lemma 8. *If $\alpha(x) = 0$ and $\beta(x) = x$, then the differential eigenvalue problem* (2.16)–(2.18) *has infinitely many positive eigenvalues for all values of γ_2.*

Proof. When $\lambda > 0$, the solution of (2.16) satisfying condition (2.17) with $\alpha(x) = 0$ is given by

$$u(x) = c_2 \sin \alpha x, \quad \alpha = \sqrt{\lambda} > 0.$$

Putting this expression for $u(x)$ into condition (2.18), we obtain

$$c_2 \sin \alpha = c_2 \gamma_2 \int_0^1 x \sin \alpha x \, \mathrm{d}x$$

or

$$c_2 \sin \alpha = c_2 \gamma_2 \Big(-\frac{\cos \alpha}{\alpha} + \frac{\sin \alpha}{\alpha^2} \Big).$$

To have $u(x) \not\equiv 0$, it is necessary that

$$\sin \alpha = \gamma_2 \Big(-\frac{\cos \alpha}{\alpha} + \frac{\sin \alpha}{\alpha^2} \Big) \tag{5.4}$$

or

$$\tan \alpha = \frac{\gamma_2 \alpha}{\gamma_2 - \alpha^2}.$$

This equation has infinitely many solutions $\alpha > 0$. For each α_k, the value $\lambda_k = \alpha_k^2$ is an eigenvalue of differential problem (2.16)–(2.18).

We observe that the question on the existence or nonexistence of complex eigenvalues of problem (2.16)–(2.18) remains unanswered.

Theorem 3. *If $\alpha(x) = 0$ and $\beta(x) = x$, then, depending on the value of γ_2, the difference eigenvalue problem* (2.13)–(2.15) *has $N-1$ or $N-2$ positive eigenvalues.*

Proof. We write differential equation (2.13) in the form

$$U_{i-1} - 2\Big(1 - \frac{\lambda h^2}{2}\Big) U_i + U_{i+1} = 0.$$

We look for the positive eigenvalues $\lambda > 0$; therefore,

$$1 - \frac{\lambda h^2}{2} < 1.$$

Let us begin with the eigenvalues $\lambda > 0$ such that

$$\Big| 1 - \frac{\lambda h^2}{2} \Big| < 1.$$

Denote

$$\cos \alpha h = 1 - \frac{\lambda h^2}{2}. \tag{5.5}$$

Then

$$\lambda = \frac{4}{h^2}\sin^2\frac{\alpha h}{2}. \tag{5.6}$$

One can directly check that the equation

$$U_{i-1} - 2\cos\alpha h U_i + U_{i+1} = 0$$

has a solution satisfying condition (2.14) with $\alpha(x) = 0$, which is given by

$$U_i = c_2 \sin i\alpha h, \quad i = 0, 1, \ldots, N. \tag{5.7}$$

Putting this expression into nonlocal condition (2.15) yields

$$c_2 \sin\alpha = c_2\gamma_2 h\Big(\frac{\sin\alpha}{2} + \sum_{i=1}^{N-1} ih\sin i\alpha h\Big). \tag{5.8}$$

As usual, we assume that $c_2 \neq 0$. Rearranging the right-hand side of Eq. (5.8), we obtain

$$\begin{aligned}
\sin\alpha &= \gamma_2 h\left(\frac{\sin\alpha}{2} + h\left(\frac{\sin\alpha}{4\sin^2\frac{\alpha h}{2}} - \frac{N\cos\frac{2N-1}{2}\alpha h}{2\sin\frac{\alpha h}{2}}\right)\right)\\
&= \gamma_2 h\left(\frac{\sin\alpha}{\alpha} + h\left(\frac{\sin\alpha}{4\sin^2\frac{\alpha h}{2}} - \frac{N(\cos\alpha\cos\frac{\alpha h}{2} + \sin\alpha\sin\frac{\alpha h}{2})}{2\sin\frac{\alpha h}{2}}\right)\right)\\
&= \gamma_2 h\left(\frac{h\sin\alpha}{4\sin^2\frac{\alpha h}{2}} - \frac{\cos\alpha}{2\tan\frac{\alpha h}{2}}\right).
\end{aligned}$$

Denoting

$$A(\alpha h) = \frac{4}{\alpha^2h^2}\sin^2\frac{\alpha h}{2}, \quad B(\alpha h) = \frac{2}{\alpha h}\tan\frac{\alpha h}{2},$$

we get

$$\sin\alpha = \gamma_2\Big(A(\alpha h)\frac{\sin\alpha}{\alpha^2} - \frac{1}{B(\alpha h)}\frac{\cos\alpha}{\alpha}\Big). \tag{5.9}$$

When αh is a sufficiently small positive number, $A(\alpha h) \approx 1$, $B(\alpha h) \approx 1$. Thus, when h is a sufficiently small positive number, the investigation of solutions of Eq. (5.9) does not differ essentially from that for Eg. (5.4) given in the proof of Lemma 8.

Writing (5.9) in the form

$$\tan\alpha = \frac{\frac{A(\alpha h)}{B(\alpha h)}\gamma_2\alpha}{\gamma_2 - A(\alpha h)\alpha^2} \tag{5.10}$$

and taking into account Eq. (5.6), we notice that the roots λ_k are all distinct as $\alpha_k h/2$ varies from 0 to $\pi/2$, and, outside this range, the eigenvalues repeat themselves. Therefore, we seek the solutions of (5.10) in the interval $\alpha \in (0, N\pi)$ rather that in an infinite one. We denote

$$f_1(\alpha) = \tan\alpha, \quad f_2(\alpha) = \frac{A(\alpha h)}{B(\alpha h)}\gamma_2 h\Big/(\gamma_2 - A(\alpha h)\alpha^2)$$

and investigate separate cases, depending on the value of γ_2.

Case 1. $\gamma_2 < 0$. In this case, $f_2(\alpha)$ is a positive continuous function over the interval $\alpha \in (0, N\pi)$. Taking into account the properties of the periodic function $f_1(\alpha)$, we obtain that (5.10) has exactly one root α_k in each interval $(k\pi,\ k\pi + \pi/2),\ k = 1, 2, \dots, N-1$.

Consequently, since $\gamma_2 < 0$ for the problem (2.13)–(2.15), there exist exactly $N-1$ positive eigenvalues:

$$\lambda_k = \frac{4}{h^2}\sin^2\frac{\alpha_k h}{2}, \quad \alpha_k \in (k\pi, k\pi + \frac{\pi}{2}).$$

Case 2. $0 < \gamma_2 < 3 - 3h^2/(2+h^2)$. In this case, the function $f_2(\alpha)$ has a discontinuity at $\bar{\alpha} \approx \sqrt{\gamma_2}$. It increases over the interval $(0, \bar{\alpha})$ from 0 to ∞ and increases over the interval $(\bar{\alpha}, N\pi)$ from ∞ to some negative value. Equation (5.10) has $N-1$ solutions, one of which lies in the interval $(0, \pi)$, while the remaining belong to the intervals $(k\pi - \pi/2, k\pi),\ k = 2, \dots, N-1$, one in each of them. Consequently, Eq. (5.9) has $N-1$ solutions in the interval $(0, N\pi)$; therefore, there exist $N-1$ positive eigenvalues of problem (2.13)–(2.15).

Case 3. $\gamma_2 \geqslant 3 - 3h^2/(2+h^2)$. In this case, the function $f_2(\alpha)$ has a discontinuity at $\bar{\alpha} \approx \sqrt{\gamma_2}$ as well. Depending on the value of γ_2, Eq. (5.10) has a solution in each of $N-2$ intervals:

$$(\pi, \pi + \frac{\pi}{2}), \dots, \Big((k-1)\pi, (k-1)\pi + \frac{\pi}{2}\Big); \Big((k+1)\pi - \frac{\pi}{2}, (k+1)\pi\Big) \dots, \Big((N-1)\pi - \frac{\pi}{2}, (N-1)\pi\Big);$$

where k is an integer such that $\gamma \in (k\pi - \pi/2, k\pi + \pi/2]$. Therefore, the total number of solutions of (5.10) in the interval $(0, N\pi)$ is $N-2$, and problem (2.13)–(2.15) has $N-2$ positive eigenvalues. The theorem is proved.

Remark 2. The proof of the theorem is based upon the assumption that the number λ satisfies the inequality $|1 - \lambda h^2/2| < 1$, i.e., $\lambda \in (0, 4/h^2)$. Other eigenvalues, $\lambda_k \geqslant 4/h^2$ do not exist because we found all of the $N-1$ eigenvalues of matrix A. For the values of γ that yield $N-2$ positive eigenvalues, there exist single eigenvalues that are equal to zero or are negative.

Corollary 1. The difference eigenvalue problem (2.13)–(2.15) has no complex eigenvalues.

It is noteworthy that, for the differential problem, we cannot make a similar conclusion without a deeper investigation.

Lemmas 4–8 and Theorem 3 yield the main result of this section stated as Theorem 4.

Theorem 4. *If $\alpha(x) = 0,\ \beta(x) = x$, and $\gamma_2 < 3 + \mathrm{O}(h^2)$, then difference scheme* (2.11) *is unconditionally stable, while difference scheme* (2.19) *is stable under the additional condition $\tau/h^2 < 1/2$.*

Proof. Since all the eigenvalues of the matrix A are real, taking into account the condition that $\gamma_2 < 3 - 3h^2/(2+h^2)$ is positive, the stability follows from Theorem 1. Along with the Remark for Theorem 3, we have $0 < \alpha_k < 4/h^2$. Hence, since $\sigma = 0$, the additional stability condition $\tau < 2/\rho(A)$ of difference scheme (2.19) can be replaced by $\tau < h^2/2$ or $\tau/h^2 < 1/2$.

5.3 Problem 3: $\alpha(x) = 1 + x,\ \beta(x) = 1 - x$.

In this case, Eq. (4.1) defines a hyperbola as the locus of points where difference problem (2.13)–(2.15) has an eigenvalue equal to zero, which does not degenerate to a straight line.

Lemma 9. *If $\alpha(x) = 1 + x$ and $\beta(x) = 1 - x$, then the number $\lambda = 0$ is the eigenvalue of the difference eigenvalue problem* (2.13)–(2.15) *if and only if*

$$\gamma_1\gamma_2(1 + 2h^2) + \gamma_1(4 - h^2) + \gamma_2(1 - h^2) - 6 = 0. \tag{5.11}$$

The proof of the lemma is similar to that of Lemma 4. If we drop terms of order $O(h^2)$ in (5.11), then we get the equation of the hyperbola

$$\gamma_1\gamma_2 + 4\gamma_1 - \gamma_2 - 6 = 0. \tag{5.12}$$

Equation (5.12) can be considered as a necessary and sufficient condition on parameters γ_1, γ_2 for $\lambda = 0$ to be an eigenvalue of differential problem (2.16)–(2.18).

Lemma 10. *If $\alpha(x) = 1 + x$ and $\beta(x) = 1 - x$, then the positive eigenvalues λ_k of the difference problem* (2.13)–(2.15) *are given by*

$$\lambda_k = \frac{4}{h^2}\sin^2\frac{\alpha_k h}{2},$$

where α_k are the solutions of the equation

$$\gamma_1\gamma_2\Big(\frac{4AB(\cos\alpha - 1)}{\alpha^3} + \frac{2A^2\sin\alpha}{\alpha^2} - \frac{2AC\sin\alpha}{\alpha}\Big) + \gamma_1\Big(\frac{A(-2+\cos\alpha)}{\alpha} + \frac{B\sin\alpha}{\alpha^2} + C\cos\alpha\Big)$$
$$- \gamma_2\Big(\frac{A}{\alpha} - \frac{B\sin\alpha}{\alpha^2} - C\Big) + \sin\alpha = 0 \tag{5.13}$$

in the interval $\alpha \in (0, N\pi)$, where

$$A = \frac{\alpha h}{2}\Big/\tan\frac{\alpha h}{2}, \quad B = \frac{\alpha^2 h^2}{4}\Big/\sin^2\frac{\alpha h}{2}, \quad C = \frac{h(\sin\alpha - \sin^2\alpha)}{2}.$$

We see that

$$A \approx 1, \quad B \approx 1, \quad C \approx 0$$

when αh is a sufficiently small number.

Equation (5.13) can be derived similarly to (5.9). If, in (5.13), we drop the terms of order $O(h)$ and $O(h^2)$, we get the following equation with the solutions defining the eigenvalues $\lambda_k = \alpha_k^2$ of differential problem (2.16)–(2.18):

$$\gamma_1\gamma_2\Big(\frac{4(\cos\alpha - 1)}{\alpha^3} + \frac{2\sin\alpha}{\alpha^2}\Big) + \gamma_1\Big(\frac{-2+\cos\alpha}{\alpha} + \frac{\sin\alpha}{\alpha^2}\Big) - \gamma_2\Big(\frac{1}{\alpha} - \frac{\sin\alpha}{\alpha^2}\Big) + \sin\alpha = 0. \tag{5.14}$$

For the differential problem, Eq. (5.14) has infinitely many solutions in the interval $(0, \infty)$, i.e., problem (2.16)–(2.18) has an infinite set of eigenvalues. However, for the difference problem, Eq. (5.13) has only a finite number of eigenvalues over the interval $(0, N\pi)$, which depends on the values of γ_1, γ_2 and is not greater than $N - 1$. Contrary to Problems 1–2 analyzed in Sections 5.1 and 5.2, the present problem possesses complex eigenvalues for some values of γ_1, γ_2. Therefore, when investigating the stability of difference schemes (2.11) and (2.19) in the case of complex eigenvalues, one has to apply Theorem 2, i.e., take into account condition $\mathrm{Re}\,\lambda_k(A) > 0$.

We give a brief explanation of when and how the complex roots do appear. The hyperbola defined by (5.11) divides the coordinate plane (γ_1, γ_2) into three unbounded regions S_1, S_2, S_3. The region S_1 is located between both branches of the hyperbola, the region S_2 is above both branches in the northeast direction from the origin of coordinates, and the region S_3 is below both branches of the hyperbola in the southwest direction from the origin of coordinates. We define the regions in the following way. Denote

$$l(\gamma_1) = \frac{6 - \gamma_1(4 - h^2)}{\gamma_1(1 + 2h^2) + 1 - h^2}, \quad \tilde{\gamma}_1 = \frac{-1 + h^2}{1 + 2h^2}.$$

Then

$$S_1 = \{\gamma_1 \leqslant \tilde{\gamma}_1,\ \ \gamma_2 \geqslant l(\gamma_1)\} \cup \{\gamma_1 \geqslant \tilde{\gamma}_1,\ \ \gamma_2 \leqslant l(\gamma_1)\},$$
$$S_2 = \{\gamma_1 > \tilde{\gamma}_1,\ \ \gamma_2 > l(\gamma_1)\},$$
$$S_3 = \{\gamma_1 < \tilde{\gamma}_1,\ \ \gamma_2 < l(\gamma_1)\}.$$

The origin $\gamma_1 = 0,\ \gamma_2 = 0$ of the coordinate system belongs to the region S_1 (for different expressions of $\alpha(x)$ and $\beta(x)$, it can be in any other region). The region S_1 is separated from the regions S_2 and S_3 by the hyperbola, at the points of which problem (2.13)–(2.15) has the zero eigenvalue $\lambda = 0$. For the values $(\gamma_1, \gamma_2) \in S_l,\ l = 2, 3$, problem (2.13)–(2.15) has a single negative eigenvalue. Therefore, for the values $(\gamma_1, \gamma_2) \in S_l,\ l = 2, 3$, the difference scheme does not satisfy the stability condition $\rho(S) < 1$. Since all eigenvalues of the matrix A are positive in the origin $(\gamma_1 = 0,\ \gamma_2 = 0)$, i.e., $\rho(S) < 1$, the stability region of difference scheme (2.13)–(2.15) is located in the vicinity of the origin of coordinates and can be a part of the region S_1 or even fully coincide with it. The stability region coincides with S_1 in the case where problem (2.13)–(2.15) has no complex eigenvalues in the region S_1 possessing the property $\operatorname{Re}\lambda_k(A) < 0$.

Computer-aided modeling gives us a lot of complex solutions of Eq. (5.13) in the region S_1. The complex root of (5.13) appears as the parameters γ_1, γ_2 vary so that the point γ_1, γ_2 moves away from the origin $(0, 0)$, and two distinct real roots of (5.13) merge into one real multiple root.

As the parameters γ_1, γ_2 vary further on, the multiple root of Eq. (5.13) can turn into two complex conjugate roots $\operatorname{Re}\lambda_k(A) \pm i\operatorname{Im}\lambda_k(A)$. At the beginning, those two roots have the property $\operatorname{Re}\lambda_k(A) > 0$. As the parameters γ_1, γ_2 vary further on, this property can vanish.

Using computer aided modeling, it is easy to make sure that there are no complex roots in the rectangle $\Omega = \{-2,64 \leqslant \gamma_1 \leqslant 1; -3,97 \leqslant \gamma_2 \leqslant 1\}$. If $(\gamma_1, \gamma_2) \in \Omega$, then all eigenvalues of the matrix A are real and positive, except at one point $(\gamma_1 = 1,\ \gamma_2 = 1)$. At this point, which belongs to hyperbola (5.11), the matrix A has exactly $N - 2$ positive eigenvalues and one zero eigenvalue $\lambda = 0$.

A particular problem was solved applying difference schemes (2.11) and (2.19). When $\alpha(x) = 1 + x$ and $\beta(x) = 1 - x$, the functions $f(x,t),\ \mu_1(t),\ \mu_2(t)$, and $\varphi(x)$ in problem (1.1), (2.1), (2.2), (1.4) were chosen so that

$$u(x,t) = e^{x+2t}$$

is a solution of the differential problem. Tables 1 and 2 give the results of numerical solution according to difference scheme (2.19) with $h = 0.00625\ (N = 160),\ \tau = h^2/2,\ T = 2$. Also the tables show the maximal absolute value of the approximation error

$$\varepsilon_i^j = \max_{1 \leqslant i \leqslant N-1} \Bigl(u(x_i, t_j) - U_i^j\Bigr).$$

We note that

$$\max_{0 \leqslant x \leqslant 1} u(x,2) = u(1,2) \approx 148.41,$$
$$\min_{0 \leqslant x \leqslant 1} u(x,2) = u(0,2) \approx 54.60.$$

The numerical results show that the stability region for difference scheme (2.19) approximating problem (1.1), (2.1), (2.2), (1.4) covers a sufficiently large range of values of parameters γ_1, γ_2. A similar situation was observed in [5], where the difference scheme in the case of $\alpha(x) = x$ and $\beta(x) = x^2$ is stable for values of γ_1, γ_2 significantly greater than 1.

Table 1. Values of ε_i^j for $\gamma_1 \leqslant -2$

γ_2	γ_1		
	-10	-5	-2
1.5	-0.180	$-0.253 \cdot 10^{-1}$	$-0.258 \cdot 10^{-1}$
1.2	$-0.768 \cdot 10^{-2}$	$-0.858 \cdot 10^{-2}$	$-0.108 \cdot 10^{-1}$
1.1	$-0.604 \cdot 10^{-2}$	$-0.662 \cdot 10^{-2}$	$-0.769 \cdot 10^{-2}$
1	$-0.481 \cdot 10^{-2}$	$-0.517 \cdot 10^{-2}$	$-0.574 \cdot 10^{-2}$
0	$-0.101 \cdot 10^{-2}$	$-0.969 \cdot 10^{-3}$	$-0.916 \cdot 10^{-3}$
-2	$0.201 \cdot 10^{-2}$	$0.185 \cdot 10^{-2}$	$0.172 \cdot 10^{-2}$
-5	$0.256 \cdot 10^{-2}$	$0.268 \cdot 10^{-2}$	$0.226 \cdot 10^{-2}$
-10	unstable	unstable	$0.243 \cdot 10^{-2}$
-20	unstable	unstable	$0.227 \cdot 10^{-2}$

Table 2. Values of ε_i^j for $\gamma_1 \geqslant -0.5$

γ_2	γ_1		
	-0.5	0	3
1.5	-0.243	$-0.646 \cdot 10^{-1}$	unstable
1.2	$-0.157 \cdot 10^{-1}$	$-0.206 \cdot 10^{-1}$	unstable
1.1	$-0.964 \cdot 10^{-2}$	$-0.115 \cdot 10^{-1}$	unstable
1	$-0.667 \cdot 10^{-2}$	$-0.740 \cdot 10^{-2}$	$-0.921 \cdot 10^{-1}$
0	$-0.847 \cdot 10^{-3}$	$-0.793 \cdot 10^{-3}$	$0.837 \cdot 10^{-2}$
-5	$0.201 \cdot 10^{-2}$	$0.160 \cdot 10^{-2}$	$0.449 \cdot 10^{-2}$
-10	$0.206 \cdot 10^{-2}$	$0.162 \cdot 10^{-2}$	$0.439 \cdot 10^{-2}$
-20	$0.246 \cdot 10^{-2}$	$0.176 \cdot 10^{-2}$	$0.436 \cdot 10^{-2}$
-40	$0.237 \cdot 10^{-2}$	$0.185 \cdot 10^{-2}$	$0.432 \cdot 10^{-2}$
-100	$0.201 \cdot 10^{-2}$	$0.188 \cdot 10^{-2}$	$0.430 \cdot 10^{-2}$

6 CONCLUSIONS AND EXTENSIONS

We propose a new technique for the stability analysis of difference schemes for parabolic equations subject to integral conditions, which requires investigation of the spectrum of the transition matrix of the system of difference equations. The characteristic feature of nonlocal problems is that the transition matrix is always nonsymmetric. Depending on the values of parameters γ_1, γ_2 included into the integral conditions, the spectrum of the matrix can be sufficiently intricate: the matrix can have one negative or trivial eigenvalue, and they can be multiple or complex. The proposed technique for stability investigation has one important feature: for numerous problems, the stability conditions it yields are much more general compared to those in [3], [7], [8], [14].

The hyperbola defined so that, at every point of this hyperbola, the transition matrix has the trivial eigenvalue $\lambda = 0$ plays an important role in the proposed stability analysis method. The coefficients of the equation of the hyperbola are calculated in terms of the functionals of the weight coefficients $\alpha(x)$ and $\beta(x)$ of the integral conditions. Having the equation of the hyperbola, one can determine the values of parameters γ_1, γ_2 such that the difference scheme is unstable and such that it can be stable.

Numerical experiments showed that this technique of stability analysis of difference schemes is quite efficient in practice. The values of parameters γ_1, γ_2 for which the difference scheme is stable can be sufficiently distant from the origin $\gamma_1 = 0$, $\gamma_2 = 0$.

In the case where the transition matrix of the difference scheme is nonsymmetric, the idea to use the stability criterion $\rho(S) < 1$ related to the vector norm

$$||u||_* = ||P^{-1}U||_3$$

(see Section 3), where P is the matrix such that its columns are linearly independent eigenvectors of the initial matrix, is not new. In [10], [11], [12], [13], quite a similar way of defining the vector norm using a matrix constructed of eigenvectors and adjoint vectors is exploited. Such a situation is typical for parabolic equations subject to nonlocal conditions only at the endpoints of the interval.

The stability analysis technique proposed in this article can naturally be extended to some two-dimensional parabolic equations subject to integral conditions. As an example, we present the following problem:

$$\begin{aligned}
&\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + f(x,y,t), \quad (x,y) \in D = \{0 < x < 1, \ \ 0 < y < 1\},\\
&u(x,0,t) = \mu_1(x,t), \quad u(x,1,t) = \mu_2(x,t),\\
&u(0,y,t) = \gamma_1 \int_0^1 \alpha(x) u(x,y,t)\,\mathrm{d}x + \mu_3(y,t),\\
&u(1,y,t) = \gamma_2 \int_0^1 \beta(x) u(x,y,t)\,\mathrm{d}t + \mu_4(y,t),\\
&u(x,y,0) = \varphi(x,y).
\end{aligned}$$

REFERENCES

1. K.E. Atkinson, *Introduction to Numerical Analysis*, John Wiley & Sons, New York, 1978.
2. B.I. Bandyrskii, I. Lazurchak, V.L. Makarov, and M. Sapagovas, Eigenvalue problem for the second order differential equation with nonlocal conditions, *Nonlin. Anal. Model. Control*, **11**(1):13–32, 2006.
3. N. Borovykh, Stability in the numerical solution of the heat equation with nonlocal boundary conditions, *Appl. Numer. Math.*, **42**:17–27, 2002.
4. B. Cahlon, D.M. Kulkarni, and P. Shi, Stepwise stability for the heat equation with a nonlocal constrain, *SIAM J. Numer. Anal.*, **32**(2):571–593, 1995.
5. R. Čiegis, A. Štikonas, O. Štikonienė, and O. Subač, A monotonic finite-diference scheme for a parabolic problem with nonlocal conditions, *Differ. Equ.*, **38**(7):1027–1037, 2002.
6. R. Čiupaila, Ž. Jesevičiūtė, and M. Sapagovas, On the eigenvalue problem for one-dimensional differential operator with nonlocal integral condition, *Nonlinear Anal. Model. Control*, **9**(2):109–116, 2004.
7. G. Ekolin, Finite difference methods for a nonlocal boundary value problem for heat equation, *BIT*, **31**:245–261, 1991.
8. G. Fairweather and J.C. Lopez-Marcos, Galerkin methods for a semilinear parabolic problem with nonlocal conditions, *Adv. Comput. Math.*, **6**:243–262, 1996.
9. S.K. Godunov and V.S. Ryabenjkii, *Difference Schemes Introduction to the Theory*, Moscow, Nauka, 1977 (in Russian).
10. A.V. Gulin, N.I. Ionkin, and V.A. Morozova, Stability of a nonlocal two-dimensional finite-difference problem, *Differ. Equ.*, **37**(7):970–978, 2001.
11. A.V. Gulin, N.I. Ionkin, and V.A. Morozova, Stability criterion of difference schemes for the heat conduction equation with nonlocal boundary conditions, *Comput. Methods Appl. Math.*, **6**(1):31–55, 2006.
12. A.V. Gulin, N.I. Ionkin, and V.A. Morozova, Study of the norm in stability problems for nonlocal difference schemes, *Differ. Equ.*, **42**(7):974–984, 2006.
13. A.V. Gulin and V.A. Morozova, On the stability of nonlocal finite-difference boundary value problem, *Differ. Equ.*, **39**(7):962–967, 2003.
14. Y. Liu, Numerical solution of the heat equation with nonlocal boundary conditions, *J. Comput. Appl. Math.*, **110**(1):115–127, 1999.

15. S. Pečiulytė, O. Štikonienė, and A. Štikonas, Sturm–Liouville problem for stationary differential operator with nonlocal integral boundary conditions, *Nonlin. Anal. Model. Control*, **11**(1):47–78, 2006.

16. R.D. Richtmyer and K.W. Marton, *Difference Methods for Initial-Value Problems*, Second Edition, John Wiley & Sons, 1967.

17. A.A. Samarskii, *The Theory of Difference Schemes*, Moscow, Nauka, 1977 (in Russian); Marcel Dekker, Inc., New York and Basel, 2001.

18. A.A. Samarskii and A.V. Gulin, *Numerical Methods*, Moscow, Nauka, 1989 (in Russian).

19. M.P. Sapagovas, The eigenvalue of some problems with a nonlocal condition, *Differ. Equ.*, **38**(7):1020–1026, 2002.

20. M.P. Sapagovas, On stability of finite-difference schemes for one-dimensional parabolic equations subject to integral condition, *Zh. Obchysl. Prykl. Mat.*, **92**:70–90, 2005.

21. M.P. Sapagovas and A.D. Štikonas, On the structure of the spectrum of a differential operator with a nonlocal condition, *Differ. Equ.*, **41**(7):1010–1018, 2005.

22. A. Štikonas, The Sturm–Liouville problem with a nonlocal boundary condition, *Lith. Math. J.*, **47**(3):336–351, 2007.