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Stability, Uniqueness and Existence of Solutions to McKean–Vlasov Stochastic Differential Equations in Arbitrary Moments

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Abstract
We deduce stability and pathwise uniqueness for a McKean–Vlasov equation with random coefficients and a multidimensional Brownian motion as driver. Our analysis focuses on a non-Lipschitz continuous drift and includes moment estimates for random Itô processes that are of independent interest. For deterministic coefficients, we provide unique strong solutions even if the drift fails to be of affine growth. The theory that we develop rests on Itô's formula and leads to pth moment and pathwise exponential stability for $p \geq 2$ with explicit Lyapunov exponents.

Keywords McKean–Vlasov equation · Lyapunov stability · Moment estimate · Asymptotic behaviour · Pathwise uniqueness · Strong solution · Non-Lipschitz drift · Itô process

Mathematics Subject Classification (2020) 60H20 · 60H30 · 60F25 · 37H30 · 45M10

1 Introduction

The study of McKean–Vlasov stochastic differential equations, also called mean-field SDEs, was initiated by Kac [23], McKean [31] and Vlasov [34]. Since then, these integral equations received considerable attention in a variety of fields, such as physics, economics, finance and mathematics. A crucial reason for this is the fact

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that they can be used to model the propagation of chaos of interacting particles in a plasma, as shown in [34]. In applications of these types, questions regarding stability, uniqueness and existence of solutions arise. To give precise answers under verifiable conditions, we continue our analysis of the companion paper [25] and present new methods.

Let $d, m \in \mathbb{N}$ and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space that satisfies the usual conditions and on which there is a standard d-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion W. Then a McKean–Vlasov equation can be written in the form

$$X_t = X_0 + \int_0^t b\big(s, X_s, \mathcal{L}(X_s)\big)\,\mathrm{d}s + \int_0^t \sigma\big(s, X_s, \mathcal{L}(X_s)\big)\,\mathrm{d}W_s \quad \text{for } t \geq 0 \quad \text{a.s.} \tag{1.1}$$

Thereby, the drift and diffusion coefficients b and σ are defined on $\mathbb{R}_+ \times \mathbb{R}^m \times \mathcal{P}(\mathbb{R}^m)$ and take their values in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, respectively, and $\mathcal{P}(\mathbb{R}^m)$ stands for the convex space of all Borel probability measures on $\mathbb{R}^m$.

The theory of mean-field SDEs has undergone groundbreaking developments since the works [23, 31, 34] and proven to be an indispensable mathematical tool. For instance, another important application in physics pertains to the analysis of incompressible Navier–Stokes equations that were considered in the classical work [28] by Leray and which have a deep link with mean-field SDEs, according to Constantin and Iyer [16]. Recently, this connection was further explored by Röckner and Zhao [33].

Moreover, mean-field games, studied by Lasry and Lions [27], serve as applications in economics and may be used to explain the interaction and behaviour of agents in a vast network. For other works related to mean-field games, we refer the reader to [8–12]. Regarding applications in finance, see [13, 14, 18, 19, 26] in connection with systemic risk modelling.

Over the years, mean-field SDEs, such as (1.1), were studied from a mathematical point of view under various assumptions on the type of noise and the regularity of the coefficients. While the authors in [22] consider Lévy noise, results on unique strong solutions, based on additive Gaussian noise and a discontinuous drift, were established in [2–4] by using Malliavin calculus. Results on weak solutions can be found in [15, 17, 21, 32]. In [29], even path-dependent coefficients were treated. It is worth noting that mean-field equations of backward type were considered in [5, 6], and infinite-dimensional partial differential equations related to mean-field SDEs were derived in [7].

In the sequel, let $t_0 \geq 0$ and $\mathcal{P}$ be a separable metrisable space in $\mathcal{P}(\mathbb{R}^m)$. For $p \geq 2$ the Polish space $\mathcal{P}_p(\mathbb{R}^m)$ of all measures in $\mathcal{P}(\mathbb{R}^m)$ with a finite pth absolute moment, endowed with the pth Wasserstein metric, serves as main application. Further, assume that along with $\mathcal{P}$ the maps

$$\mathrm{B} : [t_0, \infty[\times \Omega \times \mathbb{R}^m \times \mathcal{P} \to \mathbb{R}^m \quad \text{and} \quad \Sigma : [t_0, \infty[\times \Omega \times \mathbb{R}^m \times \mathcal{P} \to \mathbb{R}^{m\times d}$$

are admissible in a suitable measurable sense, as explained in Sect. 2. We shall focus on the following *McKean–Vlasov SDE* with such *random drift* and *diffusion coefficients*:

$$\mathrm{d}X_t = \mathrm{B}_t\big(X_t, \mathcal{L}(X_t)\big)\,\mathrm{d}t + \Sigma_t\big(X_t, \mathcal{L}(X_t)\big)\,\mathrm{d}W_t \quad \text{for } t \geq t_0. \tag{1.2}$$

This comprises *controlled McKean–Vlasov SDEs*, by Example 2.2. In particular, if Σ does not depend on the measure variable $\mu \in \mathcal{P}$, then we recover the setting of the previous work [25], in which a multidimensional Yamada–Watanabe approach was developed.

The objective of this paper is to deduce stability, uniqueness and existence of solutions to (1.2), by presenting methods that handle the dependence of the diffusion coefficient with respect to the measure variable and allow for a discontinuous drift. Essentially, our contributions to the existing literature can be listed as follows:

(1) *Pathwise uniqueness* for (1.2) is shown in Corollary 3.5 if B and Σ satisfy an Osgood condition that is only partially restrictive for B. In the specific case that both coefficients are independent of $\mu \in \mathcal{P}$, which turns (1.2) into an SDE, this condition is required on compact sets only.
(2) From the *explicit L^p-comparison estimate* for (1.2) in Proposition 3.10 we obtain *(asymptotic) pth moment stability* in Corollary 3.12 under partial and complete mixed Hölder continuity conditions on B and Σ, respectively, and clear integrability conditions on the random partial Hölder coefficients relative to $(x, \mu) \in \mathbb{R}^m \times \mathcal{P}$.
(3) *Exponential pth moment stability* with explicit Lyapunov exponents follows from Corollary 3.13 if partial and complete Lipschitz conditions are valid for B and Σ, respectively, and the stability factor in (3.9), which can be viewed as functional of the partial Lipschitz coefficients, does not exceed a sum of power functions.
(4) *Pathwise exponential stability* is shown in Corollary 3.17 under the just mentioned Lipschitz conditions on B and Σ with deterministic partial Lipschitz coefficients of certain growth and the same bound involving a sum of power functions for the stability factor γ_{pq} in (3.12), where $q \geq 2$. Thereby, the pathwise Lyapunov exponent is the pqth moment Lyapunov exponent divided by pq.
(5) As the companion paper [25], this work demonstrates that for a detailed stability analysis of McKean–Vlasov equations verifiable assumptions can be given and the existence of Lyapunov functions does not have to be assumed.
(6) *Unique strong solutions* are derived in Theorem 3.24 when B and Σ are deterministic, an Osgood growth or an affine growth estimate that is only partially restrictive for B holds and partial and complete Lipschitz conditions are satisfied by B and Σ, respectively. So, B is not forced to be of affine growth relative to $(x, \mu) \in \mathbb{R}^m \times \mathcal{P}$.

We note that the contributions (1), (4) and (6) are comparable to those in [25] if Σ is independent of $\mu \in \mathcal{P}$. In this case, the pathwise uniqueness assertions of Corollary 3.7 in [25] are applicable when Σ satisfies an Osgood condition on compact sets relative to the standard basis of $\mathbb{R}^m$. Thereby, in contrast to Corollary 3.5 in this article, B is ought to satisfy a partial Osgood condition that depends on this basis.

The pathwise exponential stability statements of Corollary 3.17 in [25] require B and Σ to satisfy a partial Lipschitz condition and an $\frac{1}{2}$-Hölder condition in terms of the standard basis, respectively. Then the same bound involving a sum of power functions as in this paper is imposed on the stability factor γ_1 given by (3.9) in [25]. This entails that one half of the first moment Lyapunov exponent, studied in Corollary 3.14 there, serves as pathwise Lyapunov exponent. While γ_1 is merely influenced by the regularity of B, the relevant stability factor γ_{pq} in (3.12) for Corollary 3.17 in this work, where $q \geq 2$, is based on the partial Lipschitz coefficients stemming from partial and complete Lipschitz conditions for B and Σ, respectively.

Further, the strong existence and uniqueness result in [25, Theorem 3.25] holds if B and Σ are deterministic, B satisfies a partial affine growth and a partial Lipschitz condition relative to the standard basis, $\Sigma_s(0) = 0$ for any $s \geq t_0$ and the Osgood condition on compact sets, as mentioned before, is valid for Σ. In Theorem 3.24 of this paper, however, the partial affine growth and partial Lipschitz conditions on B are less restrictive and Σ is supposed to satisfy an affine growth and a Lipschitz condition instead.

More restrictively, the existence result in [1, Theorem 3.1] on a finite time horizon requires the deterministic drift and diffusion to be both of affine growth and Lipschitz continuous in $(x, \mu) \in \mathbb{R}^m \times \mathcal{P}$, uniformly in time, as it is based on a standard fixed-point approach. For time-independent continuous coefficients that do not need to be Lipschitz continuous, a weak solution is deduced in [17, Proposition 3.5] from an Euler–Maruyama approximation. In the case that B and Σ are integral maps, as in Example 3.26, weak existence is shown in [32, Theorem 1] under a non-degeneracy condition for Σ even if B and Σ fail to be continuous in $x \in \mathbb{R}^m$. In view of Theorem 3.24, these three existence results rely on affine growth conditions on B that do not include the specific case (3.24) of Example 3.26 with arbitrary $l \in \mathbb{N}$ and $a \in]0, \infty[^l$. This emphasises the fact that the partial affine growth condition on B in Theorem 3.24 is less restrictive.

This work is structured as follows. In Sect. 2, the setting of our paper is introduced. In this context, we recall and extend several concepts from [25] that are related to the general McKean–Vlasov equation (1.2).

The main results are formulated in Sect. 3. In detail, Sect. 3.1 gives a quantitative second moment bound for the difference of two solutions, from which pathwise uniqueness follows. In Sect. 3.2, we compare solutions in arbitrary moments, which in turn leads to standard, asymptotic and exponential stability in pth moment. Then Sect. 3.3 deals with pathwise stability and L^p-growth estimates. By combining these results, a strong existence and uniqueness result can be stated in Sect. 3.4. Thereby, all results are illustrated by a variety of examples involving integral maps.

Finally, Sect. 4 derives moment and pathwise asymptotic estimates for random Itô processes, from which our main results will be inferred in the proofs appearing in Sect. 5.

2 Preliminaries

In what follows, we use $|\cdot|$ as absolute value function, Euclidean norm or Hilbert–Schmidt norm and denote the transpose of any matrix $A \in \mathbb{R}^{m\times d}$ by A'. Moreover, for any interval I in $\mathbb{R}$ with infimum a and supremum b and every monotone function $f : I \to \mathbb{R}$, we set $f(a) := \lim_{v\downarrow a} f(v)$, if $a \notin I$, and $f(b) := \lim_{v\uparrow b} f(v)$, if $b \notin I$.

2.1 Admissible Spaces of Probability Measures and Notions of Solutions

From now on, let $\mathcal{P}$ be a separable metrisable space in $\mathcal{P}(\mathbb{R}^m)$ that is *admissible* in the sense of [25, Definition 2.1]. That is, for every metrisable space S, each probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and any continuous process $X : S \times \tilde{\Omega} \to \mathbb{R}^m$ satisfying $\mathcal{L}(X_s) \in \mathcal{P}$ for any $s \in S$, the map

$$S \to \mathcal{P}, \quad s \mapsto \mathcal{L}(X_s)$$

is Borel measurable. Sufficient conditions for a separable metrisable space in $\mathcal{P}(\mathbb{R}^m)$ to be admissible and examples of such spaces are given in [25, Section 2.1]. In particular, our main application is included.

Namely, for $p \geq 1$ the Polish space $\mathcal{P}_p(\mathbb{R}^m)$ of all $\mu \in \mathcal{P}(\mathbb{R}^m)$ admitting a finite pth absolute moment $\int_{\mathbb{R}^m} |x|^p\, \mu(\mathrm{d}x)$, endowed with the *pth Wasserstein metric* defined via

$$\vartheta_p(\mu, \nu) := \inf_{\theta\in\mathcal{P}(\mu,\nu)} \left(\int_{\mathbb{R}^m\times\mathbb{R}^m} |x-y|^p\, \mathrm{d}\theta(x, y) \right)^{\frac{1}{p}}, \tag{2.1}$$

is admissible. Here, $\mathcal{P}(\mu, \nu)$ denotes the convex space of all Borel probability measures on $\mathbb{R}^m \times \mathbb{R}^m$ with first and second marginal distributions μ and ν, respectively, for any $\mu, \nu \in \mathcal{P}(\mathbb{R}^m)$.

Next, let $\mathcal{A}$ represent the progressive σ-field, consisting of all sets A in $[t_0, \infty[\times\Omega$ for which $\mathbb{1}_A$ is progressively measurable. Then we shall call a map

$$F : [t_0, \infty[\times\Omega \times \mathbb{R}^m \times \mathcal{P} \to \mathbb{R}^{m\times d}, \quad (s, \omega, x, \mu) \mapsto F_s(x, \mu)(\omega)$$

admissible if it is $\mathcal{A} \otimes \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathcal{P})$-measurable. This property, which we also considered in [25, Section 2.2], ensures that for each $\mathbb{R}^m$-valued progressively measurable process X and every Borel measurable map $\mu : [t_0, \infty[\to \mathcal{P}$, the process

$$[t_0, \infty[\times\Omega \to \mathbb{R}^{m\times d}, \quad (s, \omega) \mapsto F_s(X_s(\omega), \mu(s))(\omega)$$

is progressively measurable. Here and subsequently, we assume that the drift B and the diffusion Σ of the McKean–Vlasov equation (1.2) are admissible, as described.

Definition 2.1 A *solution* to (1.2) is an $\mathbb{R}^m$-valued adapted continuous process X such that $\mathcal{L}(X_s) \in \mathcal{P}$ for any $s \geq t_0$, $\int_{t_0}^{\cdot} |\mathrm{B}_s(X_s, \mathcal{L}(X_s))| + |\Sigma_s(X_s, \mathcal{L}(X_s))|^2 \,\mathrm{d}s < \infty$ and

$$X = X_{t_0} + \int_{t_0}^{\cdot} \mathrm{B}_s\big(X_s, \mathcal{L}(X_s)\big)\,\mathrm{d}s + \int_{t_0}^{\cdot} \Sigma_s\big(X_s, \mathcal{L}(X_s)\big)\,\mathrm{d}W_s \quad \text{a.s.}$$

Example 2.2 Assume that there are $l \in \mathbb{N}$, an $\mathbb{R}^l$-valued progressively measurable process Y and Borel measurable maps b and σ on $[t_0, \infty[\times\mathbb{R}^m \times \mathcal{P} \times \mathbb{R}^l$ with respective values in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$ such that

$$\mathrm{B}_s(x, \mu) = b(s, x, \mu, Y_s) \quad \text{and} \quad \Sigma_s(x, \mu) = \sigma(s, x, \mu, Y_s)$$

for every $(s, x, \mu) \in [t_0, \infty[\times\mathbb{R}^m \times \mathcal{P}$. Then B and Σ are admissible and (1.2) becomes a McKean–Vlasov SDE with drift and diffusion coefficients that are *controlled* by Y.

We readily check that for B and Σ to be deterministic, it is necessary and sufficient that there are two Borel measurable maps b and σ on $[t_0, \infty[\times\mathbb{R}^m \times \mathcal{P}$ with values in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, respectively, such that

$$\mathrm{B}_s(x, \mu) = b(s, x, \mu) \quad \text{and} \quad \Sigma_s(x, \mu) = \sigma(s, x, \mu) \tag{2.2}$$

for all $(s, x, \mu) \in [t_0, \infty[\times\mathbb{R}^m \times \mathcal{P}$. In this deterministic setting, we may introduce *weak and strong solutions* and write (1.2) formally as follows:

$$\mathrm{d}X_t = b\big(t, X_t, \mathcal{L}(X_t)\big)\,\mathrm{d}t + \sigma\big(t, X_t, \mathcal{L}(X_t)\big)\,\mathrm{d}W_t \quad \text{for } t \geq t_0. \tag{2.3}$$

Namely, any $\mathbb{R}^m$-valued $\mathcal{F}_{t_0}$-measurable random vector ξ and the Brownian motion W induce a filtration by $\mathcal{E}_t^{\xi} := \sigma(\xi) \vee \sigma(W_s - W_{t_0} : s \in [t_0, t])$ for all $t \geq t_0$. Then a solution X to (2.3) with $X_{t_0} = \xi$ a.s. is *strong* if it is adapted to the right-continuous augmented filtration of $(\mathcal{E}_t^{\xi})_{t\geq t_0}$.

A *weak solution* is an $\mathbb{R}^m$-valued adapted continuous process X defined on some filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$ that satisfies the usual conditions and on which there is a standard d-dimensional $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-Brownian motion $\tilde{W}$ such that

$$\mathcal{L}(X_s) \in \mathcal{P} \quad \text{for all } s \geq t_0, \quad \int_{t_0}^{\cdot} \big|b\big(s, X_s, \mathcal{L}(X_s)\big)\big| + \big|\sigma\big(s, X_s, \mathcal{L}(X_s)\big)\big|^2 \,\mathrm{d}s < \infty$$

and $X = X_{t_0} + \int_{t_0}^{\cdot} b(s, X_s, \mathcal{L}(X_s))\,\mathrm{d}s + \int_{t_0}^{\cdot} \sigma(s, X_s, \mathcal{L}(X_s))\,\mathrm{d}\tilde{W}_s$ a.s. In such a case, we shall say that X solves (2.3) weakly relative to $\tilde{W}$.

In general, we measure the regularity of the random coefficients B and Σ with respect to the variable $\mu \in \mathcal{P}$ by means of an $\mathbb{R}_+$-valued Borel measurable functional ϑ on $\mathcal{P} \times \mathcal{P}$, and for $p \geq 1$ we assume that

$$\vartheta\big(\mathcal{L}(X), \mathcal{L}(\tilde{X})\big) \leq \mathbb{E}\big[|X - \tilde{X}|^p\big]^{\frac{1}{p}} \tag{2.4}$$

for any two $\mathbb{R}^m$-valued random vectors $X, \tilde{X}$ satisfying $\mathcal{L}(X), \mathcal{L}(\tilde{X}) \in \mathcal{P}$. For example, this condition is satisfied if ϑ is *dominated* by the pth Wasserstein metric ϑ_p in (2.1) as follows:

$$\vartheta(\mu, \nu) \leq \vartheta_p(\mu, \nu) \quad \text{for any } \mu, \nu \in \mathcal{P}. \tag{2.5}$$

Thereby, we extend the definition of $\vartheta_p(\mu, \nu)$ for all $\mu, \nu \in \mathcal{P}(\mathbb{R}^m)$ by allowing infinite values. Note that if in addition $\mathcal{P} \subseteq \mathcal{P}_p(\mathbb{R}^m)$, we have

$$\mu \circ \varphi^{-1} \in \mathcal{P} \quad \text{for all } \mu \in \mathcal{P}$$

and any bounded uniformly continuous map $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ satisfying $|\varphi| \leq |\cdot|$ and ϑ is a metric inducing the topology of $\mathcal{P}$, then $\mathcal{P}$ is always admissible, by Corollary 2.4 in [25].

Example 2.3 Suppose that $\phi : [-\infty, \infty[\to \mathbb{R}_+$ and $\varphi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ are measurable, $\rho \in C(\mathbb{R}_+)$ is increasing and vanishes at 0 and there is $c > 0$ such that

$$\varphi(x, y) \leq \rho(|x - y|) \quad \text{and} \quad \rho(v + w)/c \leq \rho(v) + \rho(w)$$

for any $x, y \in \mathbb{R}^m$ and $v, w \geq 0$. Under the condition that $\mathcal{P}$ is included in the convex space $\mathcal{P}_\rho(\mathbb{R}^m)$ of all $\mu \in \mathcal{P}(\mathbb{R}^m)$ for which $\int_{\mathbb{R}^m} \rho(|x|)\, \mu(\mathrm{d}x)$ is finite, we may take

$$\vartheta(\mu, \nu) = \phi\bigg(\inf_{\theta \in \mathcal{P}(\mu,\nu)} \int_{\mathbb{R}^m \times \mathbb{R}^m} \varphi(x, y)\, \mathrm{d}\theta(x, y) \bigg) \quad \text{for all } \mu, \nu \in \mathcal{P}$$

and the following three statements hold:

(1) If $f, g : \mathbb{R}^m \to \mathbb{R}$ are measurable and satisfy $\varphi(x, y) = f(x) - g(y) \leq \rho(|x - y|)$ for all $x, y \in \mathbb{R}^m$, then

$$\vartheta(\mu, \nu) = \phi\bigg(\int_{\mathbb{R}^m} f(x)\, \mu(\mathrm{d}x) - \int_{\mathbb{R}^m} g(y)\, \nu(\mathrm{d}y) \bigg) \quad \text{for any } \mu, \nu \in \mathcal{P}.$$

(2) For the choice $\phi(v) = v^{\frac{1}{p}}$ for any $v \geq 0$, $\varphi(x, y) = |x - y|^p$ for all $x, y \in \mathbb{R}^m$ and $\rho(v) = v^p$ for every $v \geq 0$, we get that $\mathcal{P}_\rho(\mathbb{R}^m) = \mathcal{P}_p(\mathbb{R}^m)$ and $\vartheta = \vartheta_p$.

(3) The domination condition (2.5) is valid as soon as $\phi(v) \leq (v^+)^{\frac{1}{p}}$ for all $v \in [-\infty, \infty[$ and $\rho(v) = v^p$ for any $v \geq 0$.

2.2 Concepts of Pathwise Uniqueness and Stability

First, we notice that even in the case $(\mathcal{P}, \vartheta) = (\mathcal{P}_2(\mathbb{R}^m), \vartheta_2)$ for any solution X to (1.2) the measurable function $[t_0, \infty[\to \mathbb{R}_+$, $s \mapsto \mathbb{E}[|X_s|^2]$ is not necessarily locally integrable. For instance, let for the moment the following partial and complete affine growth conditions hold:

$$x'\mathrm{B}(x, \mu) \leq |x|\big(c_0 + c_1\vartheta_2(\mu, \delta_0)\big) \quad \text{and} \quad |\Sigma(x, \mu)| \leq c_0 + c_1\vartheta_2(\mu, \delta_0)$$

for every $(x, \mu) \in \mathbb{R}^m \times \mathcal{P}_2(\mathbb{R}^m)$ and some $c_0, c_1 \geq 0$, where δ_0 is the Dirac measure in 0. Then, by recalling that $\vartheta_p(\mu, \delta_0)^p = \int_{\mathbb{R}^m} |x|^p \, \mu(\mathrm{d}x)$ for any $\mu \in \mathcal{P}_p(\mathbb{R}^m)$, Itô's formula immediately yields that

$$X_t^2 - X_{t_0}^2 - 2\int_{t_0}^t X_s' \Sigma_s\big(X_s, \mathcal{L}(X_s)\big)\,\mathrm{d}W_s = \int_{t_0}^t 2X_s' \mathrm{B}_s\big(X_s, \mathcal{L}(X_s)\big) + \big|\Sigma_s\big(X_s, \mathcal{L}(X_s)\big)\big|^2\,\mathrm{d}s$$
$$\leq \int_{t_0}^t 2|X_s|\big(c_0 + c_1\mathbb{E}\big[|X_s|^2\big]^{\frac{1}{2}}\big) + \big(c_0 + c_1\mathbb{E}\big[|X_s|^2\big]^{\frac{1}{2}}\big)^2\,\mathrm{d}s$$

for all $t \geq t_0$ a.s. Although the first appearing Lebesgue integral is finite, the second may be infinite, since the condition $\mathcal{L}(X_s) \in \mathcal{P}$ for any $s \geq t_0$ in Definition 2.1 is merely equivalent to the square-integrability of X. But if $c_1 = 0$, then Lemmas 3.20 and 3.21 show that $\mathbb{E}[|X|^2]$ is actually locally bounded.

Thus, as in [25], we shall state all uniqueness, stability and existence results under a local integrability condition, which takes *growth in the measure variable* into account. For this purpose, let Θ be an $[0, \infty]$-valued functional on $[t_0, \infty[\times\mathcal{P} \times \mathcal{P} \times \mathcal{P}(\mathbb{R}^m)$.

Definition 2.4 *Pathwise uniqueness* holds for (1.2) (with respect to Θ) if every two solutions X and $\tilde{X}$ with $X_{t_0} = \tilde{X}_{t_0}$ a.s. (and for which the function

$$[t_0, \infty[\to [0, \infty], \quad s \mapsto \Theta\big(s, \mathcal{L}(X_s), \mathcal{L}(\tilde{X}_s), \mathcal{L}(X_s - \tilde{X}_s)\big) \tag{2.6}$$

is measurable and locally integrable) are indistinguishable.

Example 2.5 Let $\rho, \varrho \in C(\mathbb{R}_+)$ vanish at 0, ρ be concave and $\eta, \lambda : [t_0, \infty[\to \mathbb{R}_+$ be measurable and locally integrable such that

$$\Theta(s, \mu, \tilde{\mu}, \nu) = \lambda(s)\varrho\big(\vartheta(\mu, \tilde{\mu})\big) + \eta(s)\int_{\mathbb{R}^m} \rho(|x|^p)\,\nu(\mathrm{d}x)$$

for all $s \geq t_0$, $\mu, \tilde{\mu} \in \mathcal{P}$ and $\nu \in \mathcal{P}(\mathbb{R}^m)$. Then $\Theta(s, \mu, \tilde{\mu}, \nu) < \infty$ as soon as $\nu \in \mathcal{P}_p(\mathbb{R}^m)$, and for any two continuous processes $X, \tilde{X}$ satisfying $\mathcal{L}(X_s), \mathcal{L}(\tilde{X}_s) \in \mathcal{P}$ for all $s \geq t_0$ it holds that

$$\Theta\big(\cdot, \mathcal{L}(X_s), \mathcal{L}(\tilde{X}_s), \mathcal{L}(X_s - \tilde{X}_s)\big) = \lambda\varrho\big(\vartheta(\mathcal{L}(X), \mathcal{L}(\tilde{X}))\big) + \eta\mathbb{E}\big[\rho(|X - \tilde{X}|^p)\big].$$

So, the measurable function (2.6) is locally integrable if $\mathbb{E}[|X - \tilde{X}|^p]$ is locally bounded, for instance.

Based on the stability concepts of [25, Definition 2.12] that involve first moments, we formulate *generalised notions of stability* for (1.2) in a *global meaning*. In this regard, shifts of the stochastic drift and diffusion coefficients are not required, as the explicit argumentation preceding Definition 2.12 in [25] explains.

Moreover, we stress the fact that the stability definitions below and Definition 2.4 of pathwise uniqueness apply to (2.3) in the case (2.2) of deterministic coefficients by considering weak solutions on common filtered probability spaces instead of solutions on the underlying space, as noted in the end of Section 2.2 in [25].

Definition 2.6 Let $\alpha > 0$.

(i) Equation (1.2) is *stable in pth moment* (relative to Θ) if for any two solutions X and $\tilde{X}$ (for which (2.6) is measurable and locally integrable) it holds that

$$\sup_{t \geq t_0} \mathbb{E}\big[|X_t - \tilde{X}_t|^p\big] < \infty,$$

provided $\mathbb{E}[|X_{t_0} - \tilde{X}_{t_0}|^p] < \infty$. If additionally $\lim_{t\uparrow\infty} \mathbb{E}[|X_t - \tilde{X}_t|^p] = 0$, then we refer to *asymptotic stability in pth moment*.

(ii) We say that (1.2) is *α-exponentially stable in pth moment* (relative to Θ) if there exist $\lambda < 0$ and $c \geq 0$ such that any two solutions X and $\tilde{X}$ satisfy

$$\mathbb{E}\big[|X_t - \tilde{X}_t|^p\big] \leq ce^{\lambda(t-t_0)^\alpha}\mathbb{E}\big[|X_{t_0} - \tilde{X}_{t_0}|^p\big] \tag{2.7}$$

for each $t \geq t_0$ (as soon as (2.6) is measurable and locally integrable). In this case, λ is a *pth moment α-Lyapunov exponent* for (1.2).

(iii) We call (1.2) *pathwise α-exponentially stable* (relative to an initial absolute pth moment and Θ) if there is $\lambda < 0$ such that

$$\limsup_{t\uparrow\infty} \frac{1}{t^\alpha} \log\big(|X_t - \tilde{X}_t|\big) \leq \lambda \quad \text{a.s.}$$

(provided $\mathbb{E}[|X_{t_0} - \tilde{X}_{t_0}|^p] < \infty$ and (2.6) is measurable and locally integrable). In such a case, λ is a *pathwise α-Lyapunov exponent* for (1.2).

Remark 2.7 Suppose that $\mathbb{E}[|X - \tilde{X}|^p]$ is locally bounded. Then from (2.7) we directly infer that

$$\limsup_{t\uparrow\infty} \frac{1}{t^\alpha} \log\big(\mathbb{E}\big[|X_t - \tilde{X}_t|^p\big]\big) \leq \lambda.$$

Conversely, the latter bound yields the former for $\lambda + \varepsilon$ instead of λ for any $\varepsilon > 0$.

3 Main Results

3.1 A Quantitative Second Moment Bound and Pathwise Uniqueness

By comparing solutions to (1.2) with possibly different drift and diffusion coefficients, we obtain pathwise uniqueness. In this regard, let the two maps

$$\tilde{\mathrm{B}} : [t_0, \infty[\times\Omega \times \mathbb{R}^m \times \mathcal{P} \to \mathbb{R}^m \quad \text{and} \quad \tilde{\Sigma} : [t_0, \infty[\times\Omega \times \mathbb{R}^m \times \mathcal{P} \to \mathbb{R}^{m\times d}$$

be admissible. We introduce two sublinear functionals $[\cdot]_p$ and $[\cdot]_\infty$ with values in $[0, \infty]$ and $]-\infty, \infty]$, respectively, on the linear space of all random variables by

$$\big[X\big]_p := \mathbb{E}\big[\big(X^+\big)^p\big]^{\frac{1}{p}} \quad \text{and} \quad \big[X\big]_\infty := \operatorname{ess\,sup} X.$$

Then the inequalities of Hölder and Young show that for any $\beta, \gamma \in [0, p]$ with $\beta \leq \gamma$, each $x \geq 0$ and any two random variables X and Y with $Y \geq 0$, $[X]_{\frac{p}{\gamma}} < \infty$ and $\mathbb{E}[Y^p] < \infty$, the product $XY^{p-\gamma}$ is quasi-integrable and

$$\begin{aligned} px\mathbb{E}\big[XY^{p-\gamma}\big]\mathbb{E}\big[Y^p\big]^{\frac{\beta}{p}} &\leq px\big[X\big]_{\frac{p}{\gamma}}\mathbb{E}\big[Y^p\big]^{1-\frac{\gamma-\beta}{p}} \\ &\leq \big[X\big]_{\frac{p}{\gamma}}\big((\gamma-\beta)x^{\frac{p}{\gamma-\beta}} + (p+\beta-\gamma)\mathbb{E}\big[Y^p\big]\big), \end{aligned} \tag{3.1}$$

provided $x = 1$ whenever $\beta = \gamma$, since $1^\infty = \lim_{q\uparrow\infty} 1^q = 1$. By means of these bounds we derive the quantitative moment estimates of Proposition 4.4 and Theorem 4.6, on which our main results rely.

First, let us introduce an *uniform error and continuity condition* for the coefficients (B, Σ) and $(\tilde{\mathrm{B}}, \tilde{\Sigma})$ that is only partially restrictive for the drift coefficients B and $\tilde{\mathrm{B}}$:

(C.1) There are $\rho, \varrho \in C(\mathbb{R}_+)$ that are positive on $]0, \infty[$ and vanish at 0 and $\mathbb{R}_+$-valued progressively measurable processes $\varepsilon, \eta, \lambda$ with locally integrable paths so that

$$\begin{aligned} 2(x-\tilde{x})'\big(\mathrm{B}(x,\mu) - \tilde{\mathrm{B}}(\tilde{x},\tilde{\mu})\big) &+ |\Sigma(x,\mu) - \tilde{\Sigma}(\tilde{x},\tilde{\mu})|^2 \\ &\leq \varepsilon + \eta\rho(|x-\tilde{x}|^2) + \lambda\varrho\big(\vartheta(\mu,\tilde{\mu})^2\big) \end{aligned}$$

for any $x, \tilde{x} \in \mathbb{R}^m$ and $\mu, \tilde{\mu} \in \mathcal{P}$ a.s. In addition, $\rho^{\frac{1}{\alpha}}$ is concave for some $\alpha \in]0, 1]$, ϱ is increasing and $\mathbb{E}[\varepsilon]$, $[\eta]_{\frac{1}{1-\alpha}}$, $\mathbb{E}[\lambda]$ are locally integrable.

Remark 3.1 If $(\mathrm{B}, \Sigma) = (\tilde{\mathrm{B}}, \tilde{\Sigma})$ and $\varepsilon = 0$, then (C.1) is simply a *partial uniform continuity condition* for B and Σ. Moreover, we may interpret ε as *error bound* for the differences $\mathrm{B} - \tilde{\mathrm{B}}$ and $\Sigma - \tilde{\Sigma}$.

For the succeeding L^2-estimate based on Bihari's inequality, we recall that for any $\rho \in C(\mathbb{R}_+)$ that is positive on $]0, \infty[$ and vanishes at 0, the function $\Phi_\rho \in C^1(]0, \infty[)$ given by

$$\Phi_\rho(w) := \int_1^w \frac{1}{\rho(v)}\,\mathrm{d}v \tag{3.2}$$

is a strictly increasing C^1-diffeomorphism onto the interval $]\Phi_\rho(0), \Phi_\rho(\infty)[$. Let D_ρ be the set of all $(v, w) \in \mathbb{R}_+^2$ with $\Phi_\rho(v) + w < \Phi_\rho(\infty)$ and note that $\Psi_\rho : D_\rho \to \mathbb{R}_+$ given by

$$\Psi_\rho(v, w) := \Phi_\rho^{-1}\big(\Phi_\rho(v) + w\big) \tag{3.3}$$

is a continuous extension of a locally Lipschitz continuous function. Moreover, Ψ_ρ is increasing in each coordinate.

Hence, under (C.1), we use for fixed $\beta \in]0, 1]$ the two measurable locally integrable functions

$$\gamma := \alpha[\eta]_{\frac{1}{1-\alpha}} + \beta\mathbb{E}[\lambda] \quad \text{and} \quad \delta := (1-\alpha)[\eta]_{\frac{1}{1-\alpha}} + (1-\beta)\mathbb{E}[\lambda]$$

to give a *quantitative* L^2*-bound*. Thereby, we require until the end of this section that the L^p-norm bound (2.4) is valid for $p = 2$.

Proposition 3.2 *Let (C.1) hold, X and $\tilde{X}$ be two solutions to (1.2) with respective coefficients* (B, Σ) *and* $(\tilde{\mathrm{B}}, \tilde{\Sigma})$ *such that*

$$\mathbb{E}\big[|Y_{t_0}|^2\big] < \infty \quad \textit{for } Y := X - \tilde{X}$$

and $\mathbb{E}[\lambda]\varrho(\vartheta(\mathcal{L}(X), \mathcal{L}(\tilde{X}))^2)$ *be locally integrable. Define* $\varrho_0 \in C(\mathbb{R}_+)$ *via*

$$\varrho_0(v) := \rho(v)^{\frac{1}{\alpha}} \vee \varrho(v)^{\frac{1}{\beta}}$$

and assume that $\Phi_{\rho^{\frac{1}{\alpha}}}(\infty) = \infty$ *or* $\mathbb{E}[\eta\rho(|Y|^2)]$ *is locally integrable. Then* $\mathbb{E}[|Y|^2]$ *is locally bounded and*

$$\sup_{s\in[t_0,t]} \mathbb{E}\big[|Y_s|^2\big] \le \Psi_{\varrho_0}\left(\mathbb{E}\big[|Y_{t_0}|^2\big] + \int_{t_0}^t \mathbb{E}[\varepsilon_s] + \delta(s)\,\mathrm{d}s, \int_{t_0}^t \gamma(s)\,\mathrm{d}s\right)$$

for each $t \in [t_0, t_0^+[$*, where* $t_0^+ > t_0$ *denotes the supremum over all* $t \ge t_0$ *for which*

$$\left(\mathbb{E}\big[|Y_{t_0}|^2\big] + \int_{t_0}^t \mathbb{E}[\varepsilon_s] + \delta(s)\,\mathrm{d}s, \int_{t_0}^t \gamma(s)\,\mathrm{d}s\right) \in D_{\varrho_0}.$$

Remark 3.3 From $\Phi_{\varrho_0}(\infty) = \infty$ it follows that $\Phi_{\rho^{\frac{1}{\alpha}}}(\infty) = \infty$ and $D_{\varrho_0} = \mathbb{R}_+^2$. Thus, $\mathbb{E}[|Y|^2]$ is bounded in this case if $\mathbb{E}[\varepsilon]$, γ and δ are integrable. Moreover, the conditions

$$\Phi_{\varrho_0}(0) = -\infty, \quad Y_{t_0} = 0 \;\text{ a.s. and }\; \mathbb{E}[\varepsilon] = \delta = 0 \;\text{ a.e.}$$

imply $t_0^+ = \infty$ and $Y = 0$ a.s. This fact will be used to derive pathwise uniqueness.

Example 3.4 Suppose that $\alpha = \beta = 1$ and $\rho(v) = \varrho(v) = cv(|\log(v)| + 1)$ for any $v \ge 0$ and some $c \in]0, 1]$. Then we have $\Phi_{\varrho_0}(0) = -\infty$ and $\Phi_{\varrho_0}(\infty) = \infty$. Further,

$$\log\big(\Psi_{\varrho_0}(v, w)\big) = \begin{cases} (1 + \log(v))e^{cw} - 1, & \text{if } v \ge 1, \\ \frac{e^{cw}}{1-\log(v)} - 1, & \text{if } 1 > v \ge \exp(1 - e^{cw}), \\ 1 - (1 - \log(v))e^{-cw}, & \text{if } v < \exp(1 - e^{cw}), \end{cases}$$

for any $v, w \ge 0$, which leads to an explicit L^2-estimate in Proposition 3.2.

To deduce pathwise uniqueness from the comparison, we restrict (C.1) to the case when $(\mathrm{B}, \Sigma) = (\tilde{\mathrm{B}}, \tilde{\Sigma})$, $\epsilon = 0$, $\alpha = 1$ and η is deterministic. Further, if B and Σ are independent of $\mu \in \mathcal{P}$, then this condition will be imposed on compact sets only:

(C.2) There are $\rho, \varrho \in C(\mathbb{R}_+)$ that are positive on $]0,\infty[$ and vanish at 0, a measurable locally integrable function $\eta : [t_0,\infty[\to \mathbb{R}_+$ and an $\mathbb{R}_+$-valued progressively measurable process λ with locally integrable paths such that

$$\begin{aligned}2(x-\tilde{x})'\big(\mathrm{B}(x,\mu)-\mathrm{B}(\tilde{x},\tilde{\mu})\big) &+ |\Sigma(x,\mu)-\Sigma(\tilde{x},\tilde{\mu})|^2\\ &\leq \eta\rho(|x-\tilde{x}|^2)+\lambda\varrho\big(\vartheta(\mu,\tilde{\mu})^2\big)\end{aligned}$$

for any $x,\tilde{x}\in\mathbb{R}^m$ and $\mu,\tilde{\mu}\in\mathcal{P}$ a.s. Additionally, ρ is concave, ϱ is increasing and $\mathbb{E}[\lambda]$ is locally integrable.

(C.3) B and Σ are independent of $\mu\in\mathcal{P}$, and for each $n\in\mathbb{N}$ there are a concave $\rho_n\in C(\mathbb{R}_+)$ that is positive on $]0,\infty[$ and vanishes at 0 and a measurable locally integrable function $\eta_n : [t_0,\infty[\to\mathbb{R}_+$ so that

$$2(x-\tilde{x})'\big(\hat{\mathrm{B}}(x)-\hat{\mathrm{B}}(\tilde{x})\big)+|\hat{\Sigma}(x)-\hat{\Sigma}(\tilde{x})|^2\leq \eta_n\rho_n(|x-\tilde{x}|^2)$$

for all $x,\tilde{x}\in\mathbb{R}^m$ with $|x|\vee|\tilde{x}|\leq n$ a.s., where $\hat{\mathrm{B}} := \mathrm{B}(\cdot,\hat{\mu})$ and $\hat{\Sigma} := \Sigma(\cdot,\hat{\mu})$ for fixed $\hat{\mu}\in\mathcal{P}$.

Under (C.2), pathwise uniqueness for (1.2) follows with respect to the Borel measurable functional $\Theta : [t_0,\infty[\times\mathcal{P}\times\mathcal{P}\times\mathcal{P}(\mathbb{R}^m)\to[0,\infty]$ defined via

$$\Theta(s,\mu,\tilde{\mu},\nu) := \mathbb{E}\big[\lambda_s\big]\varrho\big(\vartheta(\mu,\tilde{\mu})^2\big)+\mathbb{1}_{]0,\infty[}\big(\Phi_\rho(\infty)\big)\eta(s)\int_{\mathbb{R}^m}\rho(|y|^2)\,\nu(\mathrm{d}y).$$

Corollary 3.5 *The following two assertions hold:*

(i) If (C.2) is satisfied and $\int_0^1 \frac{1}{\rho(v)\vee\varrho(v)}\,\mathrm{d}v=\infty$, *then pathwise uniqueness for (1.2) relative to* Θ *is valid.*
(ii) Assume that (C.3) holds and $\int_0^1\frac{1}{\rho_n(v)}\,\mathrm{d}v=\infty$ *for any* $n\in\mathbb{N}$. *Then we have pathwise uniqueness for the SDE (1.2).*

Remark 3.6 If B and Σ are deterministic, in which case (2.2) holds, then pathwise uniqueness for (2.3) in the standard sense follows from the corollary if the assumptions are restricted as follows:

(1) The uniform continuity condition (C.2) is stated when λ is independent of $\omega\in\Omega$.
(2) The domination condition (2.5) instead of the L^p-bound (2.4) is used for $p=2$.

As application, we consider the case that B and Σ are *integral maps*. Thereby, an $\mathbb{R}^{m\times d}$-valued map on $[t_0,\infty[\times\Omega\times\mathbb{R}^m\times\mathbb{R}^m$ will be called admissible, as introduced in Sect. 2.1, by viewing it as a map on $[t_0,\infty[\times\Omega\times\mathbb{R}^{2m}\times\mathcal{P}$ that is independent of $\mu\in\mathcal{P}$.

Example 3.7 Let $\mathrm{B}^{(0)}$ and $\Sigma^{(0)}$ be admissible maps on $[t_0,\infty[\times\Omega\times\mathbb{R}^m\times\mathbb{R}^m$ with values in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, respectively, such that $\mathrm{B}^{(0)}(x,\cdot)$ and $\Sigma^{(0)}(x,\cdot)$ are μ-integrable,

$$\mathrm{B}(x,\mu)=\int_{\mathbb{R}^m}\mathrm{B}^{(0)}(x,y)\,\mu(\mathrm{d}y)\quad\text{and}\quad\Sigma(x,\mu)=\int_{\mathbb{R}^m}\Sigma^{(0)}(x,y)\,\mu(\mathrm{d}y)$$

for all $(x,\mu)\in[t_0,\infty[\times\mathcal{P}$. Then the following two assertions hold:

(1) Suppose that there are concave $\rho, \varrho \in C(\mathbb{R}_+)$ that are positive on $]0, \infty[$ and vanish at 0, a measurable locally integrable function $\eta : [t_0, \infty[\to \mathbb{R}_+$ and an $\mathbb{R}_+$-valued progressively measurable process λ with locally integrable paths such that

$$2(x - \tilde{x})'\big(\mathrm{B}^{(0)}(x, y) - \mathrm{B}^{(0)}(\tilde{x}, \tilde{y})\big) + |\Sigma^{(0)}(x, y) - \Sigma^{(0)}(\tilde{x}, \tilde{y})|^2 \\ \leq \eta\rho(|x - \tilde{x}|^2) + \lambda\varrho(|y - \tilde{y}|^2)$$

for all $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}^m$. If in addition ϱ is increasing and $\mathbb{E}[\lambda]$ is locally integrable, then (C.2) is valid when $\mathcal{P} \subseteq \mathcal{P}_2(\mathbb{R}^m)$ and $\vartheta(\mu, \nu) = \vartheta_2(\mu, \nu)$ for all $\mu, \nu \in \mathcal{P}$.

(2) Assume that there are a measurable locally integrable map $\eta : [t_0, \infty[\to \mathbb{R}_+^2$, a measurable locally square-integrable function $\hat{\eta}_1 : [t_0, \infty[\to \mathbb{R}_+$ and an $\mathbb{R}_+$-valued progressively measurable process $\hat{\eta}^{(2)}$ with locally square-integrable paths so that

$$(x - \tilde{x})'\big(\mathrm{B}^{(0)}(x, y) - \mathrm{B}^{(0)}(\tilde{x}, \tilde{y})\big) \leq |x - \tilde{x}|\big(\eta_1|x - \tilde{x}| + \eta_2|y - \tilde{y}|\big) \\ \text{and} \quad |\Sigma^{(0)}(x, y) - \Sigma^{(0)}(\tilde{x}, \tilde{y})| \leq \hat{\eta}_1|x - \tilde{x}| + \hat{\eta}^{(2)}|y - \tilde{y}|$$

for any $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}^m$ and $[\hat{\eta}^{(2)}]_2^2$ is locally integrable. Then (C.2) follows when $\mathcal{P} \subseteq \mathcal{P}_1(\mathbb{R}^m)$, $\vartheta(\mu, \nu) \geq \vartheta_1(\mu, \nu)$ for any $\mu, \nu \in \mathcal{P}$ and $\rho(v) = \varrho(v) = v$ for each $v \geq 0$. In this case, we may set

$$\Theta(\cdot, \mu, \tilde{\mu}) := \frac{1}{2}\big(\eta_2 + 3\big[\hat{\eta}^{(2)}\big]_2^2\big)\vartheta(\mu, \tilde{\mu})^2$$

for all $\mu, \tilde{\mu} \in \mathcal{P}$ and pathwise uniqueness for (1.2) with respect to Θ holds.

3.2 An Explicit Moment Estimate and Moment Stability

In this section, we compare two solutions with varying drift and diffusion coefficients in the L^p-norm for $p \geq 2$. The resulting estimate implies standard, asymptotic and exponential stability in pth moment.

To this end, we require a *uniform error and mixed Hölder continuity condition* for $(\mathrm{B}, \tilde{\mathrm{B}})$ and $(\Sigma, \tilde{\Sigma})$ that is only *partially restrictive for the drift coefficients*:

(C.4) There are $l \in \mathbb{N}$, $\alpha, \beta \in [0, 1]^l$, measurable maps $\zeta, \hat{\zeta} : [t_0, \infty[\to \mathbb{R}_+^l$ and progressively measurable processes η and $\hat{\eta}$ with respective values in $\mathbb{R}^l$ and $\mathbb{R}_+^l$ such that

$$(x - \tilde{x})'(\mathrm{B}(x, \mu) - \tilde{\mathrm{B}}(\tilde{x}, \tilde{\mu})) \leq \sum_{k=1}^{l} \zeta_k \eta^{(k)} |x - \tilde{x}|^{1+\alpha_k} \vartheta(\mu, \tilde{\mu})^{\beta_k} \\ \text{and} \quad |\Sigma(x, \mu) - \tilde{\Sigma}(\tilde{x}, \tilde{\mu})| \leq \sum_{k=1}^{l} \hat{\zeta}_k \hat{\eta}^{(k)} |x - \tilde{x}|^{\alpha_k} \vartheta(\mu, \tilde{\mu})^{\beta_k} \tag{3.4}$$

for all $x, \tilde{x} \in \mathbb{R}^m$ and $\mu, \tilde{\mu} \in \mathcal{P}$ a.s. Further, $\alpha_k + \beta_k \leq 1$, $\zeta_k \eta^{(k)}$ and $(\hat{\zeta}_k \hat{\eta}^{(k)})^2$ admit locally integrable paths, we have $\zeta_k = \hat{\zeta}_k = 1$, if $\alpha_k + \beta_k = 1$, and

$$\big(1 + \zeta_k^{\frac{p}{1-\alpha_k-\beta_k}}\big)\big[\eta^{(k)}\big]_{\frac{p}{1-\alpha_k}}, \quad \big(1 + \hat{\zeta}_k^{\frac{2p}{1-\alpha_k-\beta_k}}\big)\big[\hat{\eta}^{(k)}\big]^2_{\frac{p}{1-\alpha_k}}$$

are locally integrable for each $k \in \{1, \ldots, l\}$.

Remark 3.8 All the coefficients ζ_k, $\eta^{(k)}$, $\hat{\zeta}_k$, $\hat{\eta}^{(k)}$ appearing in (C.4), where $k \in \{1, \ldots, l\}$ satisfies $\alpha_k = \beta_k = 0$, serve as *error terms* for $\mathrm{B} - \tilde{\mathrm{B}}$ and $\Sigma - \tilde{\Sigma}$, respectively. Further, it is feasible to take

$$\zeta_k = \hat{\zeta}_k = 1 \quad \text{for all } k \in \{1, \ldots, l\}$$

whenever there are a measurable locally integrable map $\kappa : [t_0, \infty[\to \mathbb{R}^l$ and a measurable locally square-integrable map $\hat{\kappa} : [t_0, \infty[\to \mathbb{R}^l_+$ such that the inequalities (3.4) hold for

$$\eta = \kappa \quad \text{and} \quad \hat{\eta} = \hat{\kappa}.$$

However, the possibility to choose ζ and $\hat{\zeta}$ appropriately leads to the error estimate (3.22) in Theorem 3.24, the announced strong existence result.

Example 3.9 For $\alpha, \beta \in]0, 1]$ let $\zeta_1, \hat{\zeta}_1 : [t_0, \infty[\to \mathbb{R}_+$ be measurable and η and $\hat{\eta}$ be two progressively measurable processes with values in $\mathbb{R}^3$ and $\mathbb{R}^3_+$, respectively, such that

$$\begin{aligned} (x - \tilde{x})'(\mathrm{B}(x, \mu) - \tilde{\mathrm{B}}(\tilde{x}, \tilde{\mu})) &\leq |x - \tilde{x}|(\zeta_1 \eta^{(1)} + \eta^{(2)}|x - \tilde{x}|^\alpha + \eta^{(3)} \vartheta(\mu, \tilde{\mu})^\beta) \\ \text{and} \quad |\Sigma(x, \mu) - \tilde{\Sigma}(\tilde{x}, \tilde{\mu})| &\leq \hat{\zeta}_1 \hat{\eta}^{(1)} + \hat{\eta}^{(2)}|x - \tilde{x}|^\alpha + \hat{\eta}^{(3)} \vartheta(\mu, \tilde{\mu})^\beta \end{aligned}$$

for any $x, \tilde{x} \in \mathbb{R}^m$ and $\mu, \tilde{\mu} \in \mathcal{P}$ a.s. Then (C.4) holds if $\zeta_1 \eta^{(1)}$, $\eta^{(k)}$, $(\hat{\zeta}_1 \hat{\eta}^{(1)})^2$, $(\hat{\eta}^{(k)})^2$ have locally integrable paths for $k \in \{2, 3\}$ and

$$\big(1 + \zeta_1^p\big)\big[\eta^{(1)}\big]_p, \quad \big[\eta^{(2)}\big]_{\frac{p}{1-\alpha}}, \quad \big[\eta^{(3)}\big]_p, \quad \big(1 + \hat{\zeta}_1^{2p}\big)\big[\hat{\eta}^{(1)}\big]^2_p, \quad \big[\hat{\eta}^{(2)}\big]^2_{\frac{p}{1-\alpha}}, \quad \big[\hat{\eta}^{(3)}\big]^2_p$$

are locally integrable.

Under (C.4), we introduce two measurable locally integrable functions γ_p and $\hat{\delta}_p$ on $[t_0, \infty[$ with respective values in $]-\infty, \infty]$ and $[0, \infty]$ by

$$\begin{aligned} \gamma_p(s) := &\sum_{k=1}^{l} (p - 1 + \alpha_k + \beta_k)\big[\eta_s^{(k)}\big]_{\frac{p}{1-\alpha_k}} \\ &+ \frac{p-1}{2} \sum_{j,k=1}^{l} (p - 2 + \alpha_j + \beta_j + \alpha_k + \beta_k)\big[\hat{\eta}_s^{(j)} \hat{\eta}_s^{(k)}\big]_{\frac{p}{2-\alpha_j-\alpha_k}} \end{aligned} \tag{3.5}$$

and

$$\hat{\delta}_p(s) := \sum_{k=1}^{l}(1-\alpha_k-\beta_k)\zeta_k(s)^{\frac{p}{1-\alpha_k-\beta_k}}\big[\eta_s^{(k)}\big]_{\frac{p}{1-\alpha_k}} + \frac{p-1}{2}\sum_{j,k=1}^{l}(2-\alpha_j-\beta_j-\alpha_k-\beta_k)(\hat{\zeta}_j\hat{\zeta}_k)(s)^{\frac{p}{2-\alpha_j-\beta_j-\alpha_k-\beta_k}}\big[\hat{\eta}_s^{(j)}\hat{\eta}_s^{(k)}\big]_{\frac{p}{2-\alpha_j-\alpha_k}}. \tag{3.6}$$

Thereby, we observe that the term $[\hat{\eta}^{(j)}\hat{\eta}^{(k)}]_{\frac{p}{2-\alpha_j-\alpha_k}}$ is indeed locally integrable for any $j,k \in \{1,\ldots,l\}$, since Hölder's inequality gives

$$\big[\hat{\eta}^{(j)}\hat{\eta}^{(k)}\big]_{\frac{p}{2-\alpha_j-\alpha_k}} \leq \big[\hat{\eta}^{(j)}\big]_{\frac{p}{1-\alpha_j}}\big[\hat{\eta}^{(k)}\big]_{\frac{p}{1-\alpha_k}}.$$

Moreover, since $\zeta_k = \hat{\zeta}_k = 1$ for all $k \in \{1,\ldots,l\}$ with $\alpha_k + \beta_k = 1$ and $1^\infty = \lim_{q\uparrow\infty} 1^q = 1$, Young's inequality yields that

$$(2-\alpha_j-\beta_j-\alpha_k-\beta_k)(\hat{\zeta}_j\hat{\zeta}_k)^{\frac{p}{2-\alpha_j-\beta_j-\alpha_k-\beta_k}} \leq (1-\alpha_j-\beta_j)\hat{\zeta}_j^{\frac{p}{1-\alpha_j-\beta_j}} + (1-\alpha_k-\beta_k)\hat{\zeta}_k^{\frac{p}{1-\alpha_k-\beta_k}}$$

for all $j,k \in \{1,\ldots,l\}$. This clarifies the local integrability of the expressions appearing within the second sum in (3.6).

By means of the coefficients γ_p and $\hat{\delta}_p$ we get an *explicit L^p-comparison estimate* under a local integrability condition involving the $[0,\infty]$-valued Borel measurable functional Θ on $[t_0,\infty[\times\mathcal{P}\times\mathcal{P}$ given by

$$\Theta(\cdot,\mu,\tilde{\mu}) := \sum_{\substack{k=1,\\ \beta_k>0}}^{l} \zeta_k\big[\eta^{(k)}\big]_{\frac{p}{1-\alpha_k}}\vartheta(\mu,\tilde{\mu})^{\beta_k} + \hat{\zeta}_k^2\big[\hat{\eta}^{(k)}\big]^2_{\frac{p}{1-\alpha_k}}\vartheta(\mu,\tilde{\mu})^{2\beta_k}. \tag{3.7}$$

For instance, let $\mu,\tilde{\mu} : [t_0,\infty[\to \mathcal{P}$ be two Borel measurable maps for which the function $[t_0,\infty[\to \mathbb{R}_+$, $s \mapsto \vartheta(\mu,\tilde{\mu})(s)$ is locally bounded. Then $\Theta(\cdot,\mu,\tilde{\mu})$ is locally integrable, because

$$\zeta_k\big[\eta^{(k)}\big]_{\frac{p}{1-\alpha_k}} \quad \text{and} \quad \hat{\zeta}_k^2\big[\hat{\eta}^{(k)}\big]^2_{\frac{p}{1-\alpha_k}}$$

possess this property for each $k \in \{1,\ldots,l\}$, by Young's inequality. In particular, if $\beta = 0$, then there is no dependence on the measure variable to consider and $\Theta = 0$.

Proposition 3.10 *Let (C.4) hold and X and $\tilde{X}$ be two solutions to (1.2) with respective coefficients* (B,Σ) *and* $(\tilde{\mathrm{B}},\tilde{\Sigma})$ *such that*

$$\mathbb{E}\big[|Y_{t_0}|^p\big] < \infty \quad \textit{for } Y := X - \tilde{X}$$

and $\Theta(\cdot, \mathcal{L}(X), \mathcal{L}(\tilde{X}))$ *is locally integrable. Then*

$$\mathbb{E}\big[|Y_t|^p\big] \leq e^{\int_{t_0}^t \gamma_p(s)\,\mathrm{d}s}\mathbb{E}\big[|Y_{t_0}|^p\big] + \int_{t_0}^t e^{\int_s^t \gamma_p(\tilde{s})\,\mathrm{d}\tilde{s}}\hat{\delta}_p(s)\,\mathrm{d}s \tag{3.8}$$

for all $t \geq t_0$. *In particular, if* γ_p^+ *and* $\hat{\delta}_p$ *are integrable, then* $\mathbb{E}[|Y|^p]$ *is bounded. If in addition* γ_p^- *fails to be integrable, then*

$$\lim_{t\uparrow\infty} \mathbb{E}\big[|Y_t|^p\big] = 0.$$

Remark 3.11 The term $\hat{\delta}_p$ contains both coefficients ζ and $\hat{\zeta}$ and is based on all Hölder exponents in $]0, 1[$ appearing in (C.4) in the following sense: $\hat{\delta}_p(s) = 0$ for fixed $s \geq t_0$ if and only if for every $k \in \{1, \ldots, l\}$ with $\alpha_k + \beta_k < 1$ we have

$$\zeta_k(s) \wedge \eta_s^{(k)} \leq 0 \quad \text{and} \quad \hat{\zeta}_k(s) \wedge \hat{\eta}_s^{(k)} = 0 \quad \text{a.s.}$$

Until the end of this section, let $(\mathrm{B}, \Sigma) = (\tilde{\mathrm{B}}, \tilde{\Sigma})$. Then (C.4) turns into a *mixed Hölder continuity condition* if all the error terms disappear, that is, $\alpha_k + \beta_k > 0$ for any $k \in \{1, \ldots, l\}$. Noteworthy, even if these expressions are in place, stability still follows.

Corollary 3.12 *Let (C.4) be valid. Then (1.2) is (asymptotically) stable in pth moment with respect to* Θ *if* γ_p^+ *and* $\hat{\delta}_p$ *are integrable (and* $\int_{t_0}^\infty \gamma_p^-(s)\,\mathrm{d}s = \infty$*).*

To analyse the L^p*-boundedness* and the *rate of* L^p*-convergence for solutions* in the succeeding Corollary 3.13, we strengthen (C.4) to a *partial Lipschitz condition on* B and a *complete Lipschitz condition on* Σ:

(C.5) There are a measurable locally integrable function $\eta_1 : [t_0, \infty[\to \mathbb{R}$, an $\mathbb{R}_+$-valued progressively measurable process $\eta^{(2)}$ and an $\mathbb{R}_+^2$-valued progressively measurable process $\hat{\eta}$ such that

$$(x - \tilde{x})'\big(\mathrm{B}(x, \mu) - \mathrm{B}(\tilde{x}, \tilde{\mu})\big) \leq |x - \tilde{x}|\big(\eta_1|x - \tilde{x}| + \eta^{(2)}\vartheta(\mu, \tilde{\mu})\big)$$
$$\text{and} \quad |\Sigma(x, \mu) - \Sigma(\tilde{x}, \tilde{\mu})| \leq \hat{\eta}^{(1)}|x - \tilde{x}| + \hat{\eta}^{(2)}\vartheta(\mu, \tilde{\mu})$$

for any $x, \tilde{x} \in \mathbb{R}^m$ and $\mu, \tilde{\mu} \in \mathcal{P}$ a.s. In addition, $\eta^{(2)}$ and $|\hat{\eta}|^2$ have locally integrable paths and $[\eta^{(2)}]_p$, $[\hat{\eta}^{(1)}]_\infty^2$ and $[\hat{\eta}^{(2)}]_p^2$ are locally integrable.

Let us suppose that (C.5) holds, in which case (C.4) follows for $l = 2$, $\alpha = (1, 0)$ and $\beta = (0, 1)$, as Example 3.9 shows. Thus, the function $\hat{\delta}_p$ in (3.6) is identically zero and the formula for the *stability coefficient* γ_p in (3.5) reduces to

$$\frac{\gamma_p}{p} = \eta_1 + \big[\eta^{(2)}\big]_p + \frac{p-1}{2}\Big(\big[\hat{\eta}^{(1)}\big]_\infty^2 + 2\big[\hat{\eta}^{(1)}\hat{\eta}^{(2)}\big]_p + \big[\hat{\eta}^{(2)}\big]_p^2\Big). \tag{3.9}$$

Moreover, the functional (3.7) is of the form

$$\Theta(\cdot,\mu,\tilde{\mu}) = \big[\eta^{(2)}\big]_p \vartheta(\mu,\tilde{\mu}) + \big[\hat{\eta}^{(2)}\big]_p^2 \vartheta(\mu,\tilde{\mu})^2 \tag{3.10}$$

for all $\mu,\tilde{\mu} \in \mathcal{P}$. Hence, by using the following upper bound on γ_p that involves sums of power functions, we derive *exponential moment stability*.

(C.6) Condition (C.5) is satisfied and there are $l \in \mathbb{N}$, $\alpha \in]0,\infty[^l$ and $\lambda, s \in \mathbb{R}^l$ such that $\alpha_1 < \cdots < \alpha_l$, $\lambda_l < 0$ and

$$\gamma_p(s) \le \lambda_1\alpha_1(s-s_1)^{\alpha_1-1} + \cdots + \lambda_l\alpha_l(s-s_l)^{\alpha_l-1} \quad \text{for a.e. } \ s \ge t_1$$

for some $t_1 \ge t_0$ with $\max_{k=1,\ldots,l} s_k \le t_1$.

Based on the fact that the preceding condition implies the existence of some $\hat{t}_1 \ge t_0$ such that $\gamma_p < 0$ a.e. on $[\hat{t}_1,\infty[$, we state the subsequent stability properties.

Corollary 3.13 *The following two assertions hold:*

(i) Suppose that (C.5) is valid and γ_p^+ is integrable. Then for the difference Y of any two solutions X and $\tilde{X}$ to (1.2), we have

$$\sup_{t\ge t_0} e^{\int_{t_0}^t \gamma_p^-(s)\,\mathrm{d}s}\mathbb{E}\big[|Y_t|^p\big] < \infty,$$

assuming that $\mathbb{E}[|Y_{t_0}|^p] < \infty$ and $\Theta(\cdot,\mathcal{L}(X),\mathcal{L}(\tilde{X}))$ is locally integrable. Moreover, if in addition $\int_{t_0}^\infty \gamma_p^-(s)\,\mathrm{d}s = \infty$, then

$$\lim_{t\uparrow\infty} e^{a\int_{t_0}^t \gamma_p^-(s)\,\mathrm{d}s}\mathbb{E}\big[|Y_t|^p\big] = 0 \ \ \textit{for all } a \in [0,1[.$$

(ii) Let (C.6) be satisfied. Then (1.2) is α_l-exponentially stable in pth moment with respect to Θ with any pth moment α_l-Lyapunov exponent in $]\lambda_l,0[$, and λ_l is a Lyapunov exponent if

$$\max_{k=1,\ldots,l} \lambda_k \le 0 \quad \textit{and} \quad s_l \le t_0.$$

To illustrate the preceding results, let us consider affine and integral maps.

Example 3.14 For $l \in \mathbb{N}$ let κ, ζ and η be progressively measurable processes with values in $\mathbb{R}^m$, $\mathbb{R}^{m\times m}$ and $\mathbb{R}_+^l$, respectively, and locally integrable paths. Further, let

$$f_1,\ldots,f_l : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$$

be measurable and $\Sigma^{(0)}$ be an $\mathbb{R}^{m\times d}$-valued admissible map on $[t_0,\infty[\times\Omega\times\mathbb{R}^m\times\mathbb{R}^m$ such that $f_1(x,\cdot),\ldots,f_l(x,\cdot)$, $\Sigma^{(0)}(x,\cdot)$ are μ-integrable,

$$\begin{aligned} \mathrm{B}(x,\mu) &= \kappa+\zeta x \\ &\quad + \eta^{(1)}\int_{\mathbb{R}^m} f_1(x,y)\,\mu(\mathrm{d}y)+\cdots+\eta^{(l)}\int_{\mathbb{R}^m} f_l(x,y)\,\mu(\mathrm{d}y) \\ \text{and}\quad \Sigma(x,\mu) &= \int_{\mathbb{R}^m}\Sigma^{(0)}(x,y)\,\mu(\mathrm{d}y) \end{aligned} \tag{3.11}$$

for any $(x,\mu)\in\mathbb{R}^m\times\mathcal{P}$. Then (C.4) is satisfied under the following three conditions:

(1) $\mathcal{P}\subseteq\mathcal{P}_1(\mathbb{R}^m)$, $\vartheta(\mu,\nu)\geq\vartheta_1(\mu,\nu)$ for all $\mu,\nu\in\mathcal{P}$ and there is a measurable locally integrable function $\overline{\zeta}:[t_0,\infty[\to\mathbb{R}$ satisfying $x'\zeta x\leq\overline{\zeta}|x|^2$ for all $x\in\mathbb{R}^m$.

(2) There exist $\alpha,\beta\in]0,1]$ and $\tilde{\eta}\in\mathbb{R}^{l\times 2}$ such that $\tilde{\eta}_{1,2},\ldots,\tilde{\eta}_{l,2}\geq 0$ and

$$(x-\tilde{x})'\big(f_k(x,y)-f_k(\tilde{x},\tilde{y})\big)\leq|x-\tilde{x}|\big(\tilde{\eta}_{k,1}|x-\tilde{x}|^{\alpha}+\tilde{\eta}_{k,2}|y-\tilde{y}|^{\beta}\big)$$

for all $x,\tilde{x},y,\tilde{y}\in\mathbb{R}^m$ and $k\in\{1,\ldots,l\}$. Furthermore, $\overline{\eta}^{(1)}:=\sum_{k=1}^{l}\eta^{(k)}\tilde{\eta}_{k,1}$ and $\overline{\eta}^{(2)}:=\sum_{k=1}^{l}\eta^{(k)}\tilde{\eta}_{k,2}$ are such that $[\overline{\eta}^{(1)}]_{\frac{p}{1-\alpha}}$ and $[\overline{\eta}^{(2)}]_p$ are locally integrable.

(3) There are $\hat{\alpha},\hat{\beta}\in]0,1]$ and an $\mathbb{R}_+^2$-valued progressively measurable process $\hat{\eta}$ with locally square-integrable paths such that

$$|\Sigma^{(0)}(x,y)-\Sigma^{(0)}(\tilde{x},\tilde{y})|\leq\hat{\eta}^{(1)}|x-\tilde{x}|^{\hat{\alpha}}+\hat{\eta}^{(2)}|y-\tilde{y}|^{\hat{\beta}}$$

for any $x,\tilde{x},y,\tilde{y}\in\mathbb{R}^m$. Moreover, $[\hat{\eta}^{(1)}]^2_{\frac{p}{1-\hat{\alpha}}}$ and $[\hat{\eta}^{(2)}]^2_p$ are locally integrable.

Thus, under these requirements, Proposition 3.10 and Corollaries 3.12 and 3.13 entail the following assertions:

(4) The bound (3.8) holds for the difference Y of any two solutions X and $\tilde{X}$ to (1.2) for which $\mathbb{E}[|Y_{t_0}|^p]<\infty$ and

$$\big[\overline{\eta}^{(2)}\big]_p\vartheta\big(\mathcal{L}(X),\mathcal{L}(\tilde{X})\big)^{\beta}+\big[\hat{\eta}^{(2)}\big]_p^2\vartheta\big(\mathcal{L}(X),\mathcal{L}(\tilde{X})\big)^{2\hat{\beta}}$$

is locally integrable. Thereby, the formulas (3.5) and (3.6) reduce to

$$\begin{aligned} \gamma_p &= p\overline{\zeta}+(p-1+\alpha)\big[\overline{\eta}^{(1)}\big]_{\frac{p}{1-\alpha}}+(p-1+\beta)\big[\overline{\eta}^{(2)}\big]_p \\ &\quad+\frac{p-1}{2}\Big(\big(p-2+2\hat{\alpha}\big)\big[\hat{\eta}^{(1)}\big]^2_{\frac{p}{1-\hat{\alpha}}}+\big(p-2+2\hat{\beta}\big)\big[\hat{\eta}^{(2)}\big]^2_p\Big) \\ &\quad+(p-1)(p-2+\hat{\alpha}+\hat{\beta})\big[\hat{\eta}^{(1)}\hat{\eta}^{(2)}\big]_{\frac{p}{2-\hat{\alpha}}} \end{aligned}$$

and

$$\hat{\delta}_p = (1-\alpha)\big[\overline{\eta}^{(1)}\big]_{\frac{p}{1-\alpha}} + (1-\beta)\big[\overline{\eta}^{(2)}\big]_p + (p-1)\Big((1-\hat{\alpha})\big[\hat{\eta}^{(1)}\big]^2_{\frac{p}{1-\hat{\alpha}}} + (2-\hat{\alpha}-\hat{\beta})\big[\hat{\eta}^{(1)}\hat{\eta}^{(2)}\big]_{\frac{p}{2-\hat{\alpha}}} + (1-\hat{\beta})\big[\hat{\eta}^{(2)}\big]^2_p\Big).$$

(5) Equation (1.2) is (asymptotically) stable in pth moment with respect to the Borel measurable functional $\Theta : [t_0,\infty[\times\mathcal{P}\times\mathcal{P}\to[0,\infty]$ defined via

$$\Theta(\cdot,\mu,\tilde{\mu}) := \big[\overline{\eta}^{(2)}\big]_p\vartheta(\mu,\tilde{\mu})^{\beta} + \big[\hat{\eta}^{(2)}\big]^2_p\vartheta(\mu,\tilde{\mu})^{2\hat{\beta}}$$

if the coefficients $(\overline{\zeta} + [\overline{\eta}^{(1)}]_{\frac{p}{1-\alpha}}\mathbb{1}_{\{1\}}(\alpha))^+$, $[\overline{\eta}^{(1)}]_{\frac{p}{1-\alpha}}\mathbb{1}_{]0,1[}(\alpha)$,

$$\big[\overline{\eta}^{(2)}\big]_p, \quad \big[\hat{\eta}^{(1)}\big]^2_{\frac{p}{1-\hat{\alpha}}} \quad \text{and} \quad \big[\hat{\eta}^{(2)}\big]^2_p$$

are integrable (and $(\overline{\zeta} + [\overline{\eta}^{(1)}]_{\frac{p}{1-\alpha}}\mathbb{1}_{\{1\}}(\alpha))^-$ fails to be integrable).

(6) Suppose that $\alpha=\beta=\hat{\alpha}=\hat{\beta}=1$, in which case (C.5) is satisfied. Consequently, if there exist $\lambda<0$ and $a>0$ such that

$$\overline{\zeta} + \big[\overline{\eta}^{(1)}\big]_\infty + \big[\overline{\eta}^{(2)}\big]_p + \frac{p-1}{2}\Big(\big[\hat{\eta}^{(1)}\big]^2_\infty + 2\big[\hat{\eta}^{(1)}\hat{\eta}^{(2)}\big]_p + \big[\hat{\eta}^{(2)}\big]^2_p\Big) \leq \frac{\lambda}{p}a(s-t_0)^{a-1} \quad \text{for a.e. } s\geq t_0,$$

then (1.2) is a-exponentially stable in pth moment relative to Θ with Lyapunov exponent λ.

3.3 Pathwise Stability and Moment Growth Bounds

In the first part of this section, we establish pathwise exponential stability for (1.2). In this regard, we restrict the partial Lipschitz condition (C.5) to the case that all the regularity coefficients are deterministic and a certain integral estimate holds:

(C.7) There are a measurable locally integrable map $\eta : [t_0,\infty[\to\mathbb{R}^2$ and a measurable locally square-integrable map $\hat{\eta} : [t_0,\infty[\to\mathbb{R}^2_+$ such that $\eta_2\geq 0$ and

$$(x-\tilde{x})'\big(\mathrm{B}(x,\mu)-\mathrm{B}(\tilde{x},\tilde{\mu})\big) \leq |x-\tilde{x}|\big(\eta_1|x-\tilde{x}| + \eta_2\vartheta(\mu,\tilde{\mu})\big)$$
$$\text{and} \quad |\Sigma(x,\mu)-\Sigma(\tilde{x},\tilde{\mu})| \leq \hat{\eta}_1|x-\tilde{x}| + \hat{\eta}_2\vartheta(\mu,\tilde{\mu})$$

for all $x,\tilde{x}\in\mathbb{R}^m$ and $\mu,\tilde{\mu}\in\mathcal{P}$ a.s. Additionally, there exists some $\hat{\delta}>0$ such that $\sup_{t\geq t_0}\int_t^{t+\hat{\delta}} f(s)\,\mathrm{d}s < \infty$ for $f\in\{\eta_2,\hat{\eta}_1^2,\hat{\eta}_2^2\}$.

Remark 3.15 The preceding integral estimate is always satisfied if η_2 and $\hat{\eta}$ are in fact locally bounded.

Given $q \geq 2$, we shall just for the next two results assume that (2.4) holds when p is replaced by pq but may fail to be valid for p. Thus, under (C.7), the stability coefficient γ_{pq} and the functional Θ, which we considered in (3.9) and (3.10) for p instead of pq, can be written in the form

$$\frac{\gamma_{pq}}{pq} = \eta_1 + \eta_2 + \frac{pq-1}{2}\big(\hat{\eta}_1 + \hat{\eta}_2\big)^2 \tag{3.12}$$

and $\Theta(\cdot, \mu, \tilde{\mu}) = \eta_2 \vartheta(\mu, \tilde{\mu}) + \hat{\eta}_2^2 \vartheta(\mu, \tilde{\mu})^2$ for all $\mu, \tilde{\mu} \in \mathcal{P}$. In addition, we impose the following abstract condition on γ_{pq} to deduce a *general pathwise stability bound* from Theorem 4.13, a pathwise result for random Itô processes.

(C.8) Condition (C.7) is valid and there are $\hat{\varepsilon} \in]0, 1[$ and a strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ in $[t_0, \infty[$ such that $\gamma_{pq} \leq 0$ a.e. on $[t_1, \infty[$,

$$\sup_{n\in\mathbb{N}}(t_{n+1} - t_n) < \hat{\delta}, \quad \lim_{n\uparrow\infty} t_n = \infty$$

and $\sum_{n=1}^{\infty} \exp(\frac{\varepsilon}{pq}\int_{t_1}^{t_n} \gamma_{pq}(s)\,\mathrm{d}s) < \infty$ for each $\varepsilon \in]0, \hat{\varepsilon}[$.

Proposition 3.16 *Let (C.8) be valid and X and $\tilde{X}$ be two solutions to (1.2) for which $\Theta(\cdot, \mathcal{L}(X), \mathcal{L}(\tilde{X}))$ is locally integrable. Then for $Y := X - \tilde{X}$ we have*

$$\limsup_{t\uparrow\infty} \frac{1}{\varphi(t)} \log\big(|Y_t|\big) \leq \frac{1}{pq} \limsup_{n\uparrow\infty} \frac{1}{\varphi(t_n)} \int_{t_1}^{t_n} \gamma_{pq}(s)\,\mathrm{d}s \quad a.s.$$

for each increasing continuous function $\varphi : [t_1, \infty[\to \mathbb{R}_+$ that is positive on $]t_1, \infty[$ as soon as $\mathbb{E}[|Y_{t_0}|^{pq}] < \infty$ or $\eta_2 = \hat{\eta}_2 = 0$.

Since the following condition, which involves the same sum of power functions as in (C.6), implies (C.8), we obtain pathwise exponential stability.

(C.9) Condition (C.7) is satisfied and there exist $l \in \mathbb{N}$, $\alpha \in]0, \infty[^l$, $\lambda, s \in \mathbb{R}^l$ and $t_1 \geq t_0$ such that $\alpha_1 < \cdots < \alpha_l$, $\lambda_l < 0$, $\max_{k=1,\ldots,l} s_k \leq t_1$ and

$$\gamma_{pq}(s) \leq \lambda_1\alpha_1(s - s_1)^{\alpha_1 - 1} + \cdots + \lambda_l\alpha_l(s - s_l)^{\alpha_l - 1} \quad \text{for a.e. } s \geq t_1.$$

Corollary 3.17 *Under (C.9), the following two statements hold:*

(i) The McKean–Vlasov SDE (1.2) is pathwise α_l-exponentially stable with Lyapunov exponent $\frac{\lambda_l}{pq}$ relative to an initial absolute pqth moment and Θ.
(ii) If B and Σ are actually independent of $\mu \in \mathcal{P}$, then the SDE (1.2) is pathwise α_l-exponentially stable with Lyapunov exponent $\frac{\lambda_l}{pq}$.

Example 3.18 Assume that B and Σ admit the representations in (3.11). Then (C.7) follows from condition (1) and the subsequent sharpened versions of conditions (2) and (3), respectively, in Example 3.14:

(4) There is $\tilde{\eta} \in \mathbb{R}^{l \times 2}$ such that $\tilde{\eta}_{1,2}, \ldots, \tilde{\eta}_{l,2} \geq 0$ and

$$(x - \tilde{x})'\big(f_k(x, y) - f_k(\tilde{x}, \tilde{y})\big) \leq |x - \tilde{x}|\big(\tilde{\eta}_{k,1}|x - \tilde{x}| + \tilde{\eta}_{k,2}|y - \tilde{y}|\big)$$

for all $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}^m$ and $k \in \{1, \ldots, l\}$. In addition, there exists a measurable map $\overline{\eta} : [t_0, \infty[\to \mathbb{R}^2$ such that $\overline{\eta}_1$ is locally integrable, $\overline{\eta}_2$ is locally bounded and $\sum_{k=1}^{l} \eta^{(k)} \tilde{\eta}_{k,j} \leq \overline{\eta}_j$ for $j \in \{1, 2\}$.

(5) There is a measurable locally bounded map $\hat{\eta} : [t_0, \infty[\to \mathbb{R}_+^2$ satisfying

$$|\Sigma^{(0)}(x, y) - \Sigma^{(0)}(\tilde{x}, \tilde{y})| \leq \hat{\eta}_1|x - \tilde{x}| + \hat{\eta}_2|y - \tilde{y}|$$

for any $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}^m$.

Based on these assumptions, our considerations in Example 3.14 show that for the stability coefficient γ_{pq} in (3.12) and the functional Θ we have

$$\frac{\gamma_{pq}}{pq} = \overline{\zeta} + \overline{\eta}_1 + \overline{\eta}_2 + \frac{pq - 1}{2}\big(\hat{\eta}_1 + \hat{\eta}_2\big)^2$$

and $\Theta(\cdot, \mu, \tilde{\mu}) = \overline{\eta}_2 \vartheta(\mu, \tilde{\mu}) + \hat{\eta}_2^2 \vartheta(\mu, \tilde{\mu})^2$ for all $\mu, \tilde{\mu} \in \mathcal{P}$. Thus, if there are $\lambda < 0$ and $a > 0$ such that

$$\gamma_{pq}(s) \leq \lambda a (s - t_0)^{a-1} \quad \text{for a.e. } s \geq t_0,$$

then Corollary 3.17 yields the following two assertions:

(4) Equation (1.2) is pathwise a-exponentially stable with Lyapunov exponent $\frac{\lambda}{pq}$ with respect to an initial absolute pqth moment and Θ.

(5) If $f_1, \ldots, f_l$ are independent of the second variable $y \in \mathbb{R}^m$ and $\hat{\eta}_2 = 0$, then $\overline{\eta}_2 = 0$ is feasible and the SDE (1.2) is pathwise a-exponentially stable with Lyapunov exponent $\frac{\lambda}{pq}$.

Now we deduce a second and a pth moment estimate for solutions to (1.2). As the first bound implies that their second moment functions are locally bounded, local integrability in terms of the functional Θ in Corollary 3.5 is always satisfied.

Similarly, the second bound ensures that the absolute pth moment function of any solution is locally bounded, in which case all local integrability requirements with respect to Θ in Corollaries 3.12, 3.13 and 3.17 hold.

So, let us give two *growth conditions* on (B, Σ) that are only *partially restrictive* for B. The first is required for the second moment estimate and includes different classes of growth behaviour. The second yields the pth moment estimate and is of affine nature:

(C.10) There are $\phi, \varphi \in C(\mathbb{R}_+)$ that are positive on $]0, \infty[$ and vanish at 0 and $\mathbb{R}_+$-valued progressively measurable processes κ, υ, χ with locally integrable paths so that

$$2x'\mathrm{B}(x, \mu) + |\Sigma(x, \mu)|^2 \leq \kappa + \upsilon\phi(|x|^2) + \chi\varphi(\vartheta(\mu, \delta_0)^2)$$

for all $(x,\mu) \in \mathbb{R}^m \times \mathcal{P}$ a.s. Furthermore, $\phi^{\frac{1}{\alpha}}$ is concave for some $\alpha \in]0,1]$, φ is increasing and $\mathbb{E}[\kappa]$, $[\upsilon]_{\frac{1}{1-\alpha}}$, $\mathbb{E}[\chi]$ are locally integrable.

(C.11) There are $l \in \mathbb{N}$, $\alpha, \beta \in [0,1]^l$, measurable maps $\kappa, \hat{\kappa} : [t_0, \infty[\to \mathbb{R}_+^l$ and progressively measurable processes υ and $\hat{\upsilon}$ with values in $\mathbb{R}^l$ and $\mathbb{R}_+^l$, respectively, such that

$$x'\mathrm{B}(x,\mu) \le \sum_{k=1}^{l} \kappa_k \upsilon^{(k)} |x|^{1+\alpha_k} \vartheta(\mu,\delta_0)^{\beta_k}$$
$$\text{and} \quad |\Sigma(x,\mu)| \le \sum_{k=1}^{l} \hat{\kappa}_k \hat{\upsilon}^{(k)} |x|^{\alpha_k} \vartheta(\mu,\delta_0)^{\beta_k} \tag{3.13}$$

for any $(x,\mu) \in \mathbb{R}^m \times \mathcal{P}$ a.s. Moreover, $\alpha_k + \beta_k \le 1$, $\kappa_k \upsilon^{(k)}$ and $(\hat{\kappa}_k \hat{\upsilon}^{(k)})^2$ have locally integrable paths, it holds that $\kappa_k = \hat{\kappa}_k = 1$, if $\alpha_k + \beta_k = 1$, and

$$\big(1 + \kappa_k^{\frac{p}{1-\alpha_k-\beta_k}}\big)[\upsilon^{(k)}]_{\frac{p}{1-\alpha_k}}, \quad \big(1 + \hat{\kappa}_k^{\frac{2p}{1-\alpha_k-\beta_k}}\big)[\hat{\upsilon}^{(k)}]^2_{\frac{p}{1-\alpha_k}}$$

are locally integrable for any $k \in \{1,\ldots,l\}$.

Remark 3.19 As in (C.4), we could have

$$\kappa_k = \hat{\kappa}_k = 1 \quad \text{for all } k \in \{1,\ldots,l\}$$

as soon as there are a measurable locally integrable map $\zeta : [t_0,\infty[\to \mathbb{R}^l$ and a measurable locally square-integrable map $\hat{\zeta} : [t_0,\infty[\to \mathbb{R}_+^l$ such that the estimates (3.13) hold for

$$\upsilon = \zeta \quad \text{and} \quad \hat{\upsilon} = \hat{\zeta}.$$

By admitting more general choices of κ and $\hat{\kappa}$, the growth estimate (3.23) in Theorem 3.24 can be derived.

Provided (C.10) is valid, we define for $\beta \in]0,1]$ two measurable locally integrable functions by

$$g := \alpha\big[\upsilon\big]_{\frac{1}{1-\alpha}} + \beta\mathbb{E}\big[\chi\big] \quad \text{and} \quad h := (1-\alpha)\big[\upsilon\big]_{\frac{1}{1-\alpha}} + (1-\beta)\mathbb{E}\big[\chi\big]$$

and apply Proposition 4.4 to state the following *quantitative L^2-estimate*, which becomes explicit in the setting of Example 3.4. To this end, let (2.4) be valid for $p = 2$.

Lemma 3.20 *Let (C.10) hold and X be a solution to (1.2) such that $\mathbb{E}[|X_{t_0}|^2] < \infty$ and $\mathbb{E}[\chi]\varphi(\vartheta(\mathcal{L}(X),\delta_0)^2)$ is locally integrable. Define $\varphi_0 \in C(\mathbb{R}_+)$ via*

$$\varphi_0(v) := \phi(v)^{\frac{1}{\alpha}} \vee \varphi(v)^{\frac{1}{\beta}}$$

and suppose that $\Phi_{\varphi_0}(\infty) = \infty$. *Then* $\mathbb{E}[|X|^2]$ *is locally bounded and*

$$\sup_{s \in [t_0,t]} \mathbb{E}\big[|X_s|^2\big] \leq \Psi_{\varphi_0}\left(\mathbb{E}\big[|X_{t_0}|^2\big] + \int_{t_0}^t \mathbb{E}\big[\kappa_s\big] + h(s)\,\mathrm{d}s, \int_{t_0}^t g(s)\,\mathrm{d}s\right)$$

for any $t \geq t_0$. *In particular, if* $\mathbb{E}[\kappa]$, g *and* h *are integrable, then* $\mathbb{E}[|X|^2]$ *is bounded.*

Next, let again (2.4) hold for $p \geq 2$. If the growth estimate (C.11) is satisfied, then we may introduce two measurable locally integrable functions by

$$\begin{aligned} g_p := &\sum_{k=1}^{l}(p-1+\alpha_k+\beta_k)\big[\upsilon^{(k)}\big]_{\frac{p}{1-\alpha_k}} \\ &+ \frac{p-1}{2}\sum_{j,k=1}^{l}(p-2+\alpha_j+\beta_j+\alpha_k+\beta_k)\big[\hat{\upsilon}^{(j)}\hat{\upsilon}^{(k)}\big]_{\frac{p}{2-\alpha_j-\alpha_k}} \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} h_p := &\sum_{k=1}^{l}(1-\alpha_k-\beta_k)\kappa_k^{\frac{p}{1-\alpha_k-\beta_k}}\big[\upsilon^{(k)}\big]_{\frac{p}{1-\alpha_k}} \\ &+ \frac{p-1}{2}\sum_{j,k=1}^{l}(2-\alpha_j-\beta_j-\alpha_k-\beta_k)(\hat{\kappa}_j\hat{\kappa}_k)^{\frac{p}{2-\alpha_j-\beta_j-\alpha_k-\beta_k}}\big[\hat{\upsilon}^{(j)}\hat{\upsilon}^{(k)}\big]_{\frac{p}{2-\alpha_j-\alpha_k}}. \end{aligned} \tag{3.15}$$

Further, we define an $[0,\infty]$-valued Borel measurable functional Θ on $[t_0,\infty[\times\mathcal{P}\times\mathcal{P}$ by

$$\Theta(\cdot,\mu,\tilde{\mu}) := \sum_{\substack{k=1,\\ \beta_k>0}}^{l} \kappa_k\big[\upsilon^{(k)}\big]_{\frac{p}{1-\alpha_k}}\vartheta(\mu,\tilde{\mu})^{\beta_k} + \hat{\kappa}_k^2\big[\hat{\upsilon}^{(k)}\big]^2_{\frac{p}{1-\alpha_k}}\vartheta(\mu,\tilde{\mu})^{2\beta_k}.$$

These formulas are in essence the same as those for the stability coefficients in (3.5) and (3.6) and the functional in (3.7), since the following *explicit* L^p*-growth estimate* and the preceding L^p-comparison estimate in Proposition 3.10 are implied by Theorem 4.6.

Lemma 3.21 *Let (C.11) hold and* X *be a solution to (1.2) such that* $\mathbb{E}[|X_{t_0}|^p] < \infty$ *and* $\Theta(\cdot,\mathcal{L}(X),\delta_0)$ *is locally integrable. Then*

$$\mathbb{E}\big[|X_t|^p\big] \leq e^{\int_{t_0}^t g_p(s)\,\mathrm{d}s}\mathbb{E}\big[|X_{t_0}|^p\big] + \int_{t_0}^t e^{\int_s^t g_p(\tilde{s})\,\mathrm{d}\tilde{s}}h_p(s)\,\mathrm{d}s \tag{3.16}$$

for all $t \geq t_0$. *In particular, if* g_p^+ *and* h_p *are integrable, then* $\mathbb{E}[|X|^p]$ *is bounded and from* $\int_{t_0}^\infty g_p^-(s)\,\mathrm{d}s = \infty$ *it follows that* $\lim_{t\uparrow\infty}\mathbb{E}[|X_t|^p] = 0$.

Example 3.22 Let B and Σ be of the form (3.11). Then (C.11) is ensured by condition (1) in Example 3.14 and the following two conditions:

(2) There exist $\alpha, \beta \in]0, 1]$ and $\tilde{\upsilon} \in \mathbb{R}^{l\times 3}$ such that $\tilde{\upsilon}_{1,3}, \ldots, \tilde{\upsilon}_{l,3} \geq 0$ and

$$x' f_k(x, y) \leq |x|\big(\tilde{\upsilon}_{k,1} + \tilde{\upsilon}_{k,2}|x|^{\alpha} + \tilde{\upsilon}_{k,3}|y|^{\beta}\big)$$

for any $x, y \in \mathbb{R}^m$ and $k \in \{1, \ldots, l\}$. In addition, for $\overline{\upsilon}^{(j)} := \sum_{k=1}^{l} \eta^{(k)}\tilde{\upsilon}_{k,j}$, where $j \in \{1, 2, 3\}$, the local integrability of $[|\kappa| + \overline{\upsilon}^{(1)}]_p$, $[\overline{\upsilon}^{(2)}]_{\frac{p}{1-\alpha}}$ and $[\overline{\upsilon}^{(3)}]_p$ holds.

(3) There are $\hat{\alpha}, \hat{\beta} \in]0, 1]$ and an $\mathbb{R}^3_+$-valued progressively measurable process $\hat{\upsilon}$ with locally square-integrable paths satisfying

$$|\Sigma^{(0)}(x, y)| \leq \hat{\upsilon}^{(1)} + \hat{\upsilon}^{(2)}|x|^{\hat{\alpha}} + \hat{\upsilon}^{(3)}|y|^{\hat{\beta}}$$

for all $x, y \in \mathbb{R}^m$. Further, $[\hat{\upsilon}^{(1)}]_p$, $[\hat{\upsilon}^{(2)}]_{\frac{p}{1-\hat{\alpha}}}$ and $[\hat{\upsilon}^{(3)}]_p$ are locally square-integrable.

If these three requirements are met, then Lemma 3.21 yields the following two assertions:

(4) For any solution X to (1.2) such that $\mathbb{E}[|X_{t_0}|^p] < \infty$ the estimate (3.16) is valid if the local integrability of

$$\big[\overline{\upsilon}^{(3)}\big]_p \vartheta\big(\mathcal{L}(X), \delta_0\big)^{\beta} + \big[\hat{\upsilon}^{(3)}\big]_p^2 \vartheta\big(\mathcal{L}(X), \delta_0\big)^{2\hat{\beta}}$$

holds, where the coefficients (3.14) and (3.15) become

$$\begin{aligned} g_p = {} & (p-1)\big[|\kappa| + \overline{\upsilon}^{(1)}\big]_p + p\overline{\zeta} + (p-1+\alpha)\big[\overline{\upsilon}^{(2)}\big]_{\frac{p}{1-\alpha}} + (p-1+\beta)\big[\overline{\upsilon}^{(3)}\big]_p \\ & + \frac{p-1}{2}\Big((p-2)\big[\hat{\upsilon}^{(1)}\big]_p^2 + (p-2+2\hat{\alpha})\big[\hat{\upsilon}^{(2)}\big]^2_{\frac{p}{1-\hat{\alpha}}} + (p-2+2\hat{\beta})\big[\hat{\upsilon}^{(3)}\big]_p^2\Big) \\ & + (p-1)\Big((p-2+\hat{\alpha})\big[\hat{\upsilon}^{(1)}\hat{\upsilon}^{(2)}\big]_{\frac{p}{2-\hat{\alpha}}} + (p-2+\hat{\beta})\big[\hat{\upsilon}^{(1)}\hat{\upsilon}^{(3)}\big]_{\frac{p}{2}}\Big) \\ & + (p-1)(p-2+\hat{\alpha}+\hat{\beta})\big[\hat{\upsilon}^{(2)}\hat{\upsilon}^{(3)}\big]_{\frac{p}{2-\hat{\alpha}}} \end{aligned}$$

and

$$\begin{aligned} h_p = {} & \big[|\kappa| + \overline{\upsilon}^{(1)}\big]_p + (1-\alpha)\big[\overline{\upsilon}^{(2)}\big]_{\frac{p}{1-\alpha}} + (1-\beta)\big[\overline{\upsilon}^{(3)}\big]_p \\ & + (p-1)\Big(\big[\hat{\upsilon}^{(1)}\big]_p^2 + (1-\hat{\alpha})\big[\hat{\upsilon}^{(2)}\big]^2_{\frac{p}{1-\hat{\alpha}}} + (1-\hat{\beta})\big[\hat{\upsilon}^{(3)}\big]_p^2 + (2-\hat{\alpha})\big[\hat{\upsilon}^{(1)}\hat{\upsilon}^{(2)}\big]_{\frac{p}{2-\hat{\alpha}}}\Big) \\ & + (p-1)\Big((2-\hat{\beta})\big[\hat{\upsilon}^{(1)}\hat{\upsilon}^{(3)}\big]_{\frac{p}{2}} + (2-\hat{\alpha}-\hat{\beta})\big[\hat{\upsilon}^{(2)}\hat{\upsilon}^{(3)}\big]_{\frac{p}{2-\hat{\alpha}}}\Big). \end{aligned}$$

(5) Assume that $[|\kappa| + \overline{\upsilon}^{(1)}]_p$, $(\overline{\zeta} + [\overline{\upsilon}^{(2)}]_{\frac{p}{1-\alpha}}\mathbb{1}_{\{1\}}(\alpha))^+$, $[\overline{\upsilon}^{(2)}]_{\frac{p}{1-\alpha}}\mathbb{1}_{]0,1[}(\alpha)$,

$$\big[\overline{\upsilon}^{(3)}\big]_p, \quad \big[\hat{\upsilon}^{(1)}\big]_p^2, \quad \big[\hat{\upsilon}^{(2)}\big]^2_{\frac{p}{1-\hat{\alpha}}} \quad \text{and} \quad \big[\hat{\upsilon}^{(3)}\big]_p^2$$

are integrable. Then $\mathbb{E}[|X|^p]$ is bounded, and we have $\lim_{t\uparrow\infty}\mathbb{E}[|X_t|^p]=0$ as soon as $(\overline{\zeta}+[\overline{\upsilon}^{(2)}]_{\frac{p}{1-\alpha}}\mathbb{1}_{\{1\}}(\alpha))^-$ fails to be integrable.

3.4 Strong Solutions with Locally Bounded Absolute Moment Functions

Now we suppose that b and σ are two Borel measurable maps on $[t_0,\infty[\times\mathbb{R}^m\times\mathcal{P}$ with values in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, respectively, and $\xi:\Omega\to\mathbb{R}^m$ is $\mathcal{F}_{t_0}$-measurable. The aim of this section is to deduce a strong solution X to (2.3) such that $X_{t_0}=\xi$ a.s. and the measurable absolute pth moment function

$$[t_0,\infty[\to[0,\infty],\quad t\mapsto\mathbb{E}\big[|X_t|^p\big]$$

is finite and locally bounded for $p\geq 2$. Namely, we use the preceding comparison and growth results to construct the law of the solution as local uniform limit of a Picard iteration. In this setting, $b(s,\cdot,\mu)$ is not required to be Lipschitz continuous or of affine growth for any $(s,\mu)\in[t_0,\infty[\times\mathcal{P}$.

For a Borel measurable map $\mu:[t_0,\infty[\to\mathcal{P}$ we define two measurable maps b_μ and σ_μ on $[t_0,\infty[\times\mathbb{R}^m$ with values in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, respectively, by $b_\mu(t,x):=b(t,x,\mu(t))$ and $\sigma_\mu(t,x):=\sigma(t,x,\mu(t))$. These two coefficients induce the SDE

$$\mathrm{d}X_t=b_\mu(t,X_t)\,\mathrm{d}t+\sigma_\mu(t,X_t)\,\mathrm{d}W_t\quad\text{for } t\geq t_0. \tag{3.17}$$

To obtain strong solutions for this equation, we introduce a growth as well as a continuity and boundedness condition and a spatial Osgood condition on compact sets on (b,σ):

(D.1) There exist $\phi,\varphi\in C(\mathbb{R}_+)$ that are positive on $]0,\infty[$ and vanish at 0 and measurable locally integrable functions $\kappa,\upsilon,\chi:[t_0,\infty[\to\mathbb{R}_+$ such that

$$2x'b(\cdot,x,\mu)+|\sigma(\cdot,x,\mu)|^2\leq\kappa+\upsilon\phi(|x|^2)+\chi\varphi(\vartheta(\mu,\delta_0)^2)$$

for all $(x,\mu)\in\mathbb{R}^m\times\mathcal{P}$. Moreover, ϕ is concave, φ is increasing and $\int_1^\infty\frac{1}{\phi(v)}\,\mathrm{d}v=\infty$.

(D.2) $b(s,\cdot,\mu)$ and $\sigma(s,\cdot,\mu)$ are continuous for any $(s,\mu)\in[t_0,\infty[\times\mathcal{P}$, and for each $n\in\mathbb{N}$ there is $c_n\geq 0$ such that

$$|b(s,x,\mu)|\vee|\sigma(s,x,\mu)|\leq c_n$$

for every $(s,x,\mu)\in[t_0,t_0+n]\times\mathbb{R}^m\times\mathcal{P}$ with $|x|\leq n$ and $\vartheta(\mu,\delta_0)\leq n$.

(D.3) For every $n\in\mathbb{N}$ there are a concave $\rho_n\in C(\mathbb{R}_+)$ that is positive on $]0,\infty[$ and a measurable locally integrable function $\eta_n:[t_0,\infty[\to\mathbb{R}_+$ such that

$$\begin{aligned}2(x-\tilde{x})'\big(b(\cdot,x,\mu)-b(\cdot,\tilde{x},\mu)\big)+|\sigma(\cdot,x,\mu)-\sigma(\cdot,\tilde{x},\mu)|^2\\ \leq\eta_n\rho_n(|x-\tilde{x}|^2)\end{aligned}$$

for all $x,\tilde{x}\in\mathbb{R}^m$ with $|x|\vee|\tilde{x}|\leq n$ and $\mu\in\mathcal{P}$. In addition, $\int_0^1\frac{1}{\rho_n(v)}\,\mathrm{d}v=\infty$.

We write $B_{b,loc}(\mathcal{P})$ for the set of all Borel measurable maps $\mu : [t_0, \infty[\to \mathcal{P}$ for which $\vartheta(\mu, \delta_0)$ is locally bounded. Then a local weak existence result from [20], Corollary 3.5 and Lemma 3.20 allow for a concise analysis of the SDE (3.17).

Proposition 3.23 *For $\mu \in B_{b,loc}(\mathcal{P})$ the following three assertions hold:*

(i) Under (D.3), we have pathwise uniqueness for (3.17).
(ii) Let (D.1) and (D.2) be satisfied. Then there is a weak solution X to (3.17) with $\mathcal{L}(X_{t_0}) = \mathcal{L}(\xi)$. Further, if $\mathbb{E}[|\xi|^2] < \infty$, then $\vartheta_2(\mathcal{L}(X), \delta_0)$ is locally bounded.
(iii) Assume that (D.1)-(D.3) are valid. Then (3.17) admits a unique strong solution $X^{\xi,\mu}$ such that $X^{\xi,\mu}_{t_0} = \xi$ a.s.

Next, consider the convex space $B_{b,loc}(\mathcal{P}_p(\mathbb{R}^m))$ of all $\mathcal{P}_p(\mathbb{R}^m)$-valued Borel measurable maps μ on $[t_0, \infty[$ such that $\int_{\mathbb{R}^m} |x|^p \, \mu(\mathrm{d}x)$ is locally bounded, endowed with the topology of local uniform convergence.

Then $B_{b,loc}(\mathcal{P}_p(\mathbb{R}^m))$ is completely metrisable, as ϑ_p is complete, and a sequence $(\mu_n)_{n\in\mathbb{N}}$ in this space converges locally uniformly to some $\mu \in B_{b,loc}(\mathcal{P}_p(\mathbb{R}^m))$ if and only if $\lim_{n\uparrow\infty} \sup_{s\in[t_0,t]} \vartheta_p(\mu_n, \mu)(s) = 0$ for all $t \geq t_0$.

To deduce a strong solution to (2.3) as local uniform limit of a Picard iteration in $B_{b,loc}(\mathcal{P}_p(\mathbb{R}^m))$, we replace the spatial Osgood condition (D.3) on compacts sets by a Lipschitz condition, which implies the former:

(D.4) There are a measurable locally integrable map $\eta : [t_0, \infty[\to \mathbb{R}^2$ and a measurable locally square-integrable map $\hat{\eta} : [t_0, \infty[\to \mathbb{R}^2_+$ such that $\eta_2 \geq 0$ and

$$(x - \tilde{x})'\big(b(\cdot, x, \mu) - b(\cdot, \tilde{x}, \tilde{\mu})\big) \leq |x - \tilde{x}|\big(\eta_1|x - \tilde{x}| + \eta_2\vartheta(\mu, \tilde{\mu})\big)$$
$$\text{and} \quad |\sigma(\cdot, x, \mu) - \sigma(\cdot, \tilde{x}, \tilde{\mu})| \leq \hat{\eta}_1|x - \tilde{x}| + \hat{\eta}_2\vartheta(\mu, \tilde{\mu})$$

for all $x, \tilde{x} \in \mathbb{R}^m$ and $\mu, \tilde{\mu} \in \mathcal{P}$.

Assuming that (D.4) is satisfied, from which (C.5) follows for $(\mathrm{B}, \Sigma) = (b, \sigma)$, we define two measurable locally integrable functions by

$$\begin{aligned} \gamma_{p,0} &:= p\eta_1 + (p-1)\eta_2 \\ &\quad + \frac{p-1}{2}\big(p\hat{\eta}_1^2 + 2(p-1)\hat{\eta}_1\hat{\eta}_2 + (p-2)\hat{\eta}_2^2\big) \\ \text{and} \quad \hat{\delta}_0 &:= \eta_2 + (p-1)\big(\hat{\eta}_1\hat{\eta}_2 + \hat{\eta}_2^2\big). \end{aligned} \tag{3.18}$$

So, $\gamma_{p,0}$ and $\hat{\delta}_0$ are defined according to the formulas in (3.5) and (3.6), respectively, for the particular choice

$$l = 2, \quad \alpha = (1, 0), \quad \beta = (0, 0) \quad \text{and} \quad \zeta_2 = \hat{\zeta}_2 = 1,$$

because we will use the L^p-estimate of Proposition 3.10 when $(\mathrm{B}, \Sigma) = (b_\mu, \sigma_\mu)$ and $(\tilde{\mathrm{B}}, \tilde{\Sigma}) = (b_{\tilde{\mu}}, \sigma_{\tilde{\mu}})$ for $\mu, \tilde{\mu} \in B_{b,loc}(\mathcal{P})$.

If in addition the following affine growth condition for (b, σ) is valid, from which (D.1) follows, then an estimate in $B_{b,loc}(\mathcal{P}_p(\mathbb{R}^m))$ for the Picard iteration holds.

(D.5) There are $l \in \mathbb{N}$, $\alpha, \beta \in [0,1]^l$, a measurable locally integrable map $\upsilon : [t_0, \infty[\to \mathbb{R}^l$ and a measurable locally square-integrable map $\hat{\upsilon} : [t_0, \infty[\to \mathbb{R}^l_+$ such that

$$x'b(\cdot, x, \mu) \leq \sum_{k=1}^{l} \upsilon_k |x|^{1+\alpha_k} \vartheta(\mu, \delta_0)^{\beta_k}$$

$$\text{and} \quad |\sigma(\cdot, x, \mu)| \leq \sum_{k=1}^{l} \hat{\upsilon}_k |x|^{\alpha_k} \vartheta(\mu, \delta_0)^{\beta_k}$$

for every $(x, \mu) \in \mathbb{R}^m \times \mathcal{P}$ and $\alpha + \beta \in [0,1]^l$.

As (D.5) ensures that (C.11) is satisfied for $(\mathrm{B}, \Sigma) = (b, \sigma)$, we may use the coefficients g_p and h_p given by (3.14) and (3.15) when $\kappa_k = \hat{\kappa}_k = 1$ for all $k \in \{1, \ldots, l\}$. That is,

$$\begin{aligned} g_p = &\sum_{k=1}^{l} (p - 1 + \alpha_k + \beta_k)\big(\upsilon_k^+ - \upsilon_k^- \mathbb{1}_{\{1\}}(\alpha_k)\big) \\ &+ \frac{p-1}{2} \sum_{j,k=1}^{l} (p - 2 + \alpha_j + \beta_j + \alpha_k + \beta_k)\hat{\upsilon}_j \hat{\upsilon}_k \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} h_p = &\sum_{k=1}^{l} (1 - \alpha_k - \beta_k)\upsilon_k^+ \\ &+ \frac{p-1}{2} \sum_{j,k=1}^{l} (2 - \alpha_j - \beta_j - \alpha_k - \beta_k)\hat{\upsilon}_j \hat{\upsilon}_k. \end{aligned} \tag{3.20}$$

As a result, we obtain a *unique strong solution* to (2.3) with initial value condition ξ together with a *semi-explicit error estimate* and an *explicit growth estimate*.

Theorem 3.24 *Assume that (D.1) and $p = 2$ or (D.5) holds and let (D.2), (D.4) be valid. Further, let $\mathcal{P}_p(\mathbb{R}^m) \subseteq \mathcal{P}$, $\mu_0 \in B_{b,loc}(\mathcal{P})$, $\mathbb{E}[|\xi|^p] < \infty$ and $\Theta : [t_0, \infty[\times \mathcal{P} \times \mathcal{P} \to \mathbb{R}_+$ be given by*

$$\Theta(\cdot, \mu, \tilde{\mu}) := \eta_2 \vartheta(\mu, \tilde{\mu}) + \hat{\eta}_2^2 \vartheta(\mu, \tilde{\mu})^2. \tag{3.21}$$

(i) Then we have pathwise uniqueness for (2.3) with respect to Θ and there is a unique strong solution X^ξ such that $X^\xi_{t_0} = \xi$ a.s. and $\mathbb{E}[|X^\xi|^p]$ is locally bounded.

(ii) The map $[t_0, \infty[\to \mathcal{P}_p(\mathbb{R}^m)$, $t \mapsto \mathcal{L}(X^\xi_t)$ is the local uniform limit of the sequence $(\mu_n)_{n\in\mathbb{N}}$ in $B_{b,loc}(\mathcal{P}_p(\mathbb{R}^m))$ recursively defined via $\mu_n := \mathcal{L}(X^{\xi,\mu_{n-1}})$

and

$$\sup_{s\in[t_0,t]} \vartheta_p(\mu_n(s), \mathcal{L}(X_s^\xi)) \leq \Delta(t) \sum_{i=n}^{\infty} \left(\frac{1}{i!}\right)^{\frac{1}{p}} \left(\int_{t_0}^{t} e^{\int_s^t \gamma_{p,0}^+(\tilde{s})\,\mathrm{d}\tilde{s}} \hat{\delta}_0(s)\,\mathrm{d}s\right)^{\frac{i}{p}} \tag{3.22}$$

for any $t \geq t_0$ with $\Delta(t) := \sup_{s\in[t_0,t]} \vartheta(\mathcal{L}(X_s^{\xi,\mu_0}), \mu_0(s))$.

(iii) *Suppose that (D.5) is satisfied. If μ_0 lies in the closed and convex space M_p of all $\mu \in B_{b,loc}(\mathcal{P}_p(\mathbb{R}^m))$ such that*

$$\vartheta_p(\mu(t), \delta_0)^p \leq e^{\int_{t_0}^t g_p(s)\,\mathrm{d}s}\mathbb{E}\big[|\xi|^p\big] + \int_{t_0}^{t} e^{\int_s^t g_p(\tilde{s})\,\mathrm{d}\tilde{s}} h_p(s)\,\mathrm{d}s \tag{3.23}$$

for any $t \geq t_0$, then so does μ_n for each $n \in \mathbb{N}$.

Remark 3.25 While the choice $\mu_0 = \delta_0$ yields $\Delta(t) \leq \sup_{s\in[t_0,t]} \mathbb{E}[|X_s^{\xi,\delta_0}|^p]^{\frac{1}{p}}$ for any $t \geq t_0$, we have $\mu_n = \mu_0$ for all $n \in \mathbb{N}$ whenever $\mu_0 = \mathcal{L}(X^\xi)$.

Example 3.26 Let b_0 and σ_0 be measurable maps on $[t_0, \infty[\times\mathbb{R}^m \times \mathbb{R}^m$ with values in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, respectively, such that $b_0(s, x, \cdot)$ and $\sigma_0(s, x, \cdot)$ are μ-integrable,

$$b(s, x, \mu) = \int_{\mathbb{R}^m} b_0(s, x, y)\,\mu(\mathrm{d}y) \quad \text{and} \quad \sigma(s, x, \mu) = \int_{\mathbb{R}^m} \sigma_0(s, x, y)\,\mu(\mathrm{d}y)$$

for any $(s, x, \mu) \in [t_0, \infty[\times\mathbb{R}^m \times \mathcal{P}$. Further, let $\mathcal{P} \subseteq \mathcal{P}_1(\mathbb{R}^m)$ and $\vartheta(\mu, \nu) \geq \vartheta_1(\mu, \nu)$ for all $\mu, \nu \in \mathcal{P}$. Then the following three assertions hold:

(1) Assume that there are a measurable locally integrable map $\eta : [t_0, \infty[\to \mathbb{R}^2$ and a measurable locally square-integrable map $\hat{\eta} : [t_0, \infty[\to \mathbb{R}_+^2$ such that $\eta_2 \geq 0$ and

$$(x - \tilde{x})'\big(b_0(\cdot, x, y) - b_0(\cdot, \tilde{x}, \tilde{y})\big) \leq |x - \tilde{x}|\big(\eta_1|x - \tilde{x}| + \eta_2|y - \tilde{y}|\big)$$
$$\text{and} \quad |\sigma_0(\cdot, x, y) - \sigma_0(\cdot, \tilde{x}, \tilde{y})| \leq \hat{\eta}_1|x - \tilde{x}| + \hat{\eta}_2|y - \tilde{y}|$$

for any $x, \tilde{x}, y, \tilde{y} \in \mathbb{R}^m$. Then the Lipschitz condition (D.4) for (b, σ) is satisfied.

(2) If there are $l \in \mathbb{N}$, $\alpha, \beta \in [0, 1]^l$, a measurable locally integrable map $\upsilon : [t_0, \infty[\to \mathbb{R}_+^l$ and a measurable locally square-integrable map $\hat{\upsilon} : [t_0, \infty[\to \mathbb{R}_+^l$ such that

$$x'b_0(\cdot, x, y) \leq \sum_{k=1}^{l} \upsilon_k|x|^{1+\alpha_k}|y|^{\beta_k} \quad \text{and} \quad |\sigma_0(\cdot, x, y)| \leq \sum_{k=1}^{l} \hat{\upsilon}_k|x|^{\alpha_k}|y|^{\beta_k}$$

for any $x, y \in \mathbb{R}^m$ and $\alpha + \beta \in [0, 1]^l$, then $\sigma_0(s, x, \cdot)$ is always μ-integrable for all $(s, x, \mu) \in [t_0, \infty[\times\mathbb{R}^m \times \mathcal{P}$ and the affine growth condition (D.5) for (b, σ) follows.

(3) Let $b_0(s,\cdot,y)$ and $\sigma_0(s,\cdot,y)$ be continuous for all $(s,y) \in [t_0,\infty[\times\mathbb{R}^m$ and assume that for each $n \in \mathbb{N}$ there is $c_n \geq 0$ such that

$$|b_0(s,x,y)| \leq c_n(1+|y|)$$

for any $s \in [t_0, t_0+n]$ and $x, y \in \mathbb{R}^m$ with $|x| \leq n$. Further, let the estimate for σ_0 in (2) be valid such that $\hat{\upsilon}$ is locally bounded. Then the continuity and boundedness condition (D.2) for (b,σ) is valid.

For instance, let $l \in \mathbb{N}$, $a \in]0,\infty[^l$ and $f_1,\ldots,f_l : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$ be Lipschitz continuous. Further, let $\hat{b}$ and c be measurable locally integrable maps on $[t_0,\infty[$ with respective values in $\mathbb{R}^l_+$ and $\mathbb{R}^l$ such that

$$\begin{aligned} b_0(\cdot,x,y) = &-x\left(\hat{b}_1|x|^{a_1-1} + \cdots + \hat{b}_l|x|^{a_l-1}\right) \\ &+ c_1 f_1(x,y) + \cdots + c_l f_l(x,y) \end{aligned} \tag{3.24}$$

for any $x, y \in \mathbb{R}^m$ with $x \neq 0$ and $b_0(\cdot,0,\cdot) = c_1 f_1(0,\cdot) + \cdots + c_l f_l(0,\cdot)$. Then all the conditions for b_0 in (1) and (2) are satisfied, and the conditions in (3) are valid for b_0 if

$$\hat{b} \text{ and } c \text{ are in fact locally bounded.}$$

In the general case, each statement of Theorem 3.24 holds as soon as $\mathcal{P}_p(\mathbb{R}^m) \subseteq \mathcal{P}$, $\mathbb{E}[|\xi|^p] < \infty$ and the requirements in (1)–(3) are met. Thereby, the coefficients

$$\gamma_{p,0}, \quad \hat{\delta}_0, \quad g_p, \quad h_p \quad \text{and} \quad \Theta$$

remain exactly as specified in the formulas (3.18)–(3.21).

4 Moment and Pathwise Asymptotic Estimations for Random Itô Processes

4.1 Auxiliary Moment Bounds

From now on, let B and Σ be two progressively measurable processes with respective values in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$ satisfying

$$\int_{t_0}^{\cdot} |\mathrm{B}_s| + |\Sigma_s|^2 \, \mathrm{d}s < \infty$$

and Y be a *random Itô process* with drift B and diffusion Σ, as introduced and analysed in [25, Section 4.1]. That is, Y is an $\mathbb{R}^m$-valued adapted continuous process such that

$$Y = Y_{t_0} + \int_{t_0}^{\cdot} \mathrm{B}_s \, \mathrm{d}s + \int_{t_0}^{\cdot} \Sigma_s \, \mathrm{d}W_s \quad \text{a.s.}$$

Our aim is to give *quantitative L^p-estimates* for Y when $p \geq 2$. For this purpose, we consider the following representation.

Lemma 4.1 *Let $l \in \mathbb{N}$, V be an $\mathbb{R}^{l\times m}$-valued adapted locally absolutely continuous process and $u : [t_0, \infty[\to \mathbb{R}$ be continuous and locally of bounded variation. Then*

$$\begin{aligned} u|VY|^p = u(t_0)|V_{t_0}Y_{t_0}|^p + \int_{t_0}^{\cdot} |V_sY_s|^p \,\mathrm{d}u(s) \\ + p\int_{t_0}^{\cdot} u(s)|Y_sV_s|^{p-2}(V_sY_s)'\Big(\big(\dot{V}_sY_s + V_s\mathrm{B}_s\big)\,\mathrm{d}s + V_s\Sigma_s\,\mathrm{d}W_s\Big) \\ + \frac{p}{2}\int_{t_0}^{\cdot} u(s)|V_sY_s|^{p-2}\Big(|V_s\Sigma_s|^2 + (p-2)|(V_s\Sigma_s)'\psi_m(V_sY_s)|^2\Big)\,\mathrm{d}s \quad \textit{a.s.,} \end{aligned}$$

where $\psi_m : \mathbb{R}^m \to \mathbb{R}^m$ is given by $\psi_m(x) := \frac{x}{|x|}$, if $x \neq 0$, and $\psi_m(0) := 0$, if $x = 0$.

Proof The function $\varphi : (\mathbb{R}^l)^m \times \mathbb{R}^m \to \mathbb{R}_+$ defined by $\varphi(a_1, \ldots, a_m, x) := |\sum_{j=1}^m a_jx_j|^p$ is twice continuously differentiable with first-order derivatives with respect to the jth and the last variable given by

$$D_{a_j}\varphi(a_1, \ldots, a_m, x) = p|Ax|^{p-2}(Ax)'x_j \quad \text{and} \quad D_x\varphi(a_1, \ldots, a_m, x) = p|Ax|^{p-2}(Ax)'A$$

for all $j \in \{1, \ldots, m\}$, $a_1, \ldots, a_m \in \mathbb{R}^l$ and $x \in \mathbb{R}^m$, where $A \in \mathbb{R}^{l\times m}$ is of the form $A = (a_1, \ldots, a_m)$. Its second-order derivative relative to the last variable equals

$$D_x^2\varphi(a_1, \ldots, a_m, x) = p|Ax|^{p-2}\big(A'A + (p-2)A'\psi_m(Ax)\psi_m(Ax)'A\big),$$

which in turn gives us that

$$\mathrm{tr}(D_x^2\varphi(a_1, \ldots, a_m, x)BB') = p|Ax|^{p-2}\big(|AB|^2 + (p-2)|(AB)'\psi_m(Ax)|^2\big)$$

for any $B \in \mathbb{R}^{m\times d}$. Moreover, from Itô's formula we know that

$$\begin{aligned} |VY|^p - |V_{t_0}Y_{t_0}|^p = \sum_{j=1}^m \int_{t_0}^{\cdot} D_{a_j}\varphi\big(V_s^{(1)}, \ldots, V_s^{(m)}, Y_s\big)\,\mathrm{d}V_s^{(j)} \\ + \int_{t_0}^{\cdot} D_x\varphi\big(V_s^{(1)}, \ldots, V_s^{(m)}, Y_s\big)\,\mathrm{d}Y_s \\ + \frac{1}{2}\int_{t_0}^{\cdot} \mathrm{tr}\big(D_x^2\varphi\big(V_s^{(1)}, \ldots, V_s^{(m)}, Y_s\big)\Sigma_s\Sigma_s'\big)\,\mathrm{d}s \quad \text{a.s.,} \end{aligned}$$

where $V^{(j)}$ stands for the jth column of the process V for all $j \in \{1, \ldots, m\}$. Hence, Itô's product rule completes the verification. □

We readily employ the just considered identity to get an auxiliary L^p-estimate.

Lemma 4.2 *Let $\mathbb{E}[|Y_{t_0}|^p] < \infty$ and assume that there are a progressively measurable process Z with locally integrable paths and a stopping time τ such that*

$$Y_s'\mathrm{B}_s + \frac{p-1}{2}|\Sigma_s|^2 \leq Z_s \quad \text{for all } s \in [t_0, \tau[\text{ a.s.}$$

If $u : [t_0, \infty[\to \mathbb{R}_+$ is locally absolutely continuous, then

$$\mathbb{E}\big[u(t \wedge \tau)|Y_t^\tau|^p\big] \leq u(t_0)\mathbb{E}\big[|Y_{t_0}|^p\big] + \mathbb{E}\bigg[\int_{t_0}^{t\wedge\tau} |Y_s|^{p-2}\big(\dot{u}(s)|Y_s|^2 + u(s)pZ_s\big)\,\mathrm{d}s\bigg]$$

for every $t \geq t_0$ for which $\int_{t_0}^{t\wedge\tau} |Y_s|^{p-2}(\dot{u}(s)|Y_s|^2 + u(s)pZ_s)^+\,\mathrm{d}s$ is integrable.

Proof By the preceding lemma, the claimed inequality holds when τ is replaced for each $k \in \mathbb{N}$ by the stopping time

$$\tau_k := \inf\bigg\{t \geq t_0 \,\bigg|\, |Y_t| \geq k \text{ or } \int_{t_0}^{t} |Z_s| + |\Sigma_s|^2\,\mathrm{d}s \geq k\bigg\} \wedge \tau, \tag{4.1}$$

since $\int_{t_0}^{\cdot\wedge\tau_k} u(s)|Y_s|^{p-2}Y_s'\Sigma_s\,\mathrm{d}W_s$ is a square-integrable martingale. Hence, Fatou's lemma and dominated and monotone convergence give the asserted bound, as $\sup_{k\in\mathbb{N}} \tau_k = \tau$. □

Now we apply a Burkholder–Davis–Gundy inequality for stochastic integrals driven by W from [30, Theorem 7.3]. For $q \geq 2$ set $\overline{w}_q := (q^{q+1}/(2(q-1)^{q-1}))^{q/2}$, if $q > 2$, and $\overline{w}_q := 4$, if $q = 2$. Then

$$\mathbb{E}\bigg[\sup_{\tilde{s}\in[t_0,t]}\bigg|\int_{t_0}^{\tilde{s}} X_s\,\mathrm{d}W_s\bigg|^q\bigg] \leq \overline{w}_q\mathbb{E}\bigg[\bigg(\int_{t_0}^{t} |X_s|^2\,\mathrm{d}s\bigg)^{\frac{q}{2}}\bigg] \tag{4.2}$$

for every $\mathbb{R}^{m\times d}$-valued progressively measurable process X and each $t \geq t_0$ satisfying $\int_{t_0}^{t} |X_s|^2\,\mathrm{d}s < \infty$. The next result is an auxiliary moment estimate in the supremum norm.

Proposition 4.3 *Let $q \geq 1$, Z be a progressively measurable process with locally integrable paths and τ be a stopping time such that*

$$Y_s'\mathrm{B}_s \leq Z_s \quad \text{for all } s \in [t_0, \tau[\text{ a.s.}$$

Then $\hat{Z} := Z + \frac{p-1}{2}|\Sigma|^2$ *and any locally absolutely continuous function* $u : [t_0, \infty[\to \mathbb{R}_+$ *satisfy*

$$\mathbb{E}\bigg[\bigg(\sup_{s\in[t_0,t]} u(s\wedge\tau)|Y_s^\tau|^p - u(t_0)|Y_{t_0}|^p\bigg)^q\bigg]^{\frac{1}{q}} \le \mathbb{E}\bigg[\bigg(\int_{t_0}^{t\wedge\tau} |Y_s|^{p-2}\big(\dot{u}(s)|Y_s|^2 + u(s)p\hat{Z}_s\big)^+\,\mathrm{d}s\bigg)^q\bigg]^{\frac{1}{q}} + p\mathbb{E}\bigg[\overline{w}_{q_0}\bigg(\int_{t_0}^{t\wedge\tau} u(s)^2|Y_s|^{2p-2}|\Sigma_s|^2\,\mathrm{d}s\bigg)^{\frac{q_0}{2}}\bigg]^{\frac{1}{q_0}}$$

for each $t \ge t_0$ *with* $q_0 := q \vee 2$.

Proof Because $\sup_{\tilde{s}\in[t_0,t]}\int_{t_0}^{\tilde{s}}\kappa(s)\,\mathrm{d}s \le \int_{t_0}^t \kappa^+(s)\,\mathrm{d}s$ for each measurable locally integrable function $\kappa : [t_0, \infty[\to \mathbb{R}$, we infer from Lemma 4.1 that the stopping time (4.1) satisfies

$$\sup_{s\in[t_0,t]} u(s\wedge\tau_k)|Y_s^{\tau_k}|^p \le u(t_0)|Y_{t_0}|^p + \int_{t_0}^{t\wedge\tau_k} |Y_s|^{p-2}\big(\dot{u}(s)|Y_s|^2 + u(s)p\hat{Z}_s\big)^+\,\mathrm{d}s + \sup_{s\in[t_0,t]} I_s^{\tau_k} \quad \text{a.s.} \tag{4.3}$$

for any fixed $k \in \mathbb{N}$ and $t \ge t_0$, where I denotes a continuous local martingale with $I_{t_0} = 0$ that is indistinguishable from the stochastic integral

$$p\int_{t_0}^{\cdot} u(s)|Y_s|^{p-2}Y_s'\Sigma_s\,\mathrm{d}W_s.$$

Thus, Hölder's inequality, (4.2) and the estimate $|\Sigma'Y| \le |\Sigma||Y|$ yield that

$$\overline{w}_{q_0}^{-1}\mathbb{E}\bigg[\sup_{s\in[t_0,t]}|I_s^{\tau_k}|^q\bigg]^{\frac{q_0}{q}} \le \mathbb{E}\bigg[\bigg(\int_{t_0}^{t\wedge\tau_k} p^2u(s)^2|Y_s|^{2p-2}|\Sigma_s|^2\,\mathrm{d}s\bigg)^{\frac{q_0}{2}}\bigg]. \tag{4.4}$$

For this reason, the claimed inequality follows when τ is replaced by τ_k from (4.3), (4.4) and the triangle inequality in the L^q-norm. Since $\sup_{k\in\mathbb{N}}\tau_k = \tau$, monotone convergence completes the proof. □

4.2 Quantitative Moment Estimates

First, we derive an L^2-estimate from Sect. 4.1 and Bihari's inequality. For this purpose, let $l \in \mathbb{N}$ and consider the following assumption on the random Itô process Y:

(A.1) There are $\alpha \in]0,1]^l$, $\rho_1, \ldots, \rho_l, \varrho_1, \ldots, \varrho_l \in C(\mathbb{R}_+)$ that are positive on $]0, \infty[$ and vanish at 0, a measurable map $\theta : [t_0, \infty[\to \mathbb{R}_+^l$ and

an $\mathbb{R}_+$-valued process κ and two $\mathbb{R}_+^l$-valued processes η, λ

that are all progressively measurable and have locally integrable paths such that

$$2Y'\mathrm{B} + |\Sigma|^2 \leq \kappa + \sum_{k=1}^{l} \eta^{(k)}\rho_k(|Y|^2) + \lambda^{(k)}\varrho_k \circ \theta_k^2 \quad \text{a.s.}$$

In addition, $\rho_k^{\frac{1}{\alpha_k}}$ is concave, ϱ_k is increasing, $\theta_k(s) \leq \mathbb{E}[|Y_s|^2]^{\frac{1}{2}}$ for all $s \geq t_0$ with $\mathbb{E}[\lambda_s^{(k)}] > 0$ and

$$\mathbb{E}\big[\kappa\big], \quad \big[\eta^{(k)}\big]_{\frac{1}{1-\alpha_k}}, \quad \mathbb{E}\big[\lambda^{(k)}\big]$$

are locally integrable for every $k \in \{1, \ldots, l\}$.

Under (A.1), we define for $\beta \in]0, 1]^l$ two measurable locally integrable functions by

$$\gamma := \sum_{k=1}^{l} \alpha_k \big[\eta^{(k)}\big]_{\frac{1}{1-\alpha_k}} + \beta_k \mathbb{E}\big[\lambda^{(k)}\big] \quad \text{and} \quad \delta := \sum_{k=1}^{l} (1-\alpha_k)\big[\eta^{(k)}\big]_{\frac{1}{1-\alpha_k}} + (1-\beta_k)\mathbb{E}\big[\lambda^{(k)}\big].$$

Based on definitions (3.2) and (3.3), this allows for a general bound.

Proposition 4.4 *Let (A.1) hold,* $\mathbb{E}[|Y_{t_0}|^2] < \infty$, $\sum_{k=1}^{l} \mathbb{E}[\lambda^{(k)}]\varrho_k \circ \theta_k^2$ *be locally integrable and* $\rho_0, \varrho_0 \in C(\mathbb{R}_+)$ *be given by*

$$\rho_0(v) := \max_{k=1,\ldots,l} \rho_k(v)^{\frac{1}{\alpha_k}} \quad \text{and} \quad \varrho_0(v) := \rho_0(v) \vee \max_{k=1,\ldots,l} \varrho_k(v)^{\frac{1}{\beta_k}}.$$

If $\Phi_{\rho_0}(\infty) = \infty$ *or* $\sum_{k=1}^{l} \mathbb{E}[\eta^{(k)}\rho_k(|Y|^2)]$ *is locally integrable, then* $\mathbb{E}[|Y|^2]$ *is locally bounded and*

$$\sup_{s\in[t_0,t]} \mathbb{E}\big[|Y_s|^2\big] \leq \Psi_{\varrho_0}\bigg(\mathbb{E}\big[|Y_{t_0}|^2\big] + \int_{t_0}^{t} \mathbb{E}[\kappa_s] + \delta(s)\,\mathrm{d}s, \int_{t_0}^{t} \gamma(s)\,\mathrm{d}s\bigg)$$

for all $t \in [t_0, t_0^+[$, *where* $t_0^+ > t_0$ *stands for the supremum over all* $t \geq t_0$ *for which*

$$\bigg(\mathbb{E}\big[|Y_{t_0}|^2\big] + \int_{t_0}^{t} \mathbb{E}[\kappa_s] + \delta(s)\,\mathrm{d}s, \int_{t_0}^{t} \gamma(s)\,\mathrm{d}s\bigg) \in D_{\varrho_0}.$$

Proof We take the stopping time $\tau_n := \inf\{t \geq t_0 \,|\, |Y_t| \geq n\}$ for given $n \in \mathbb{N}$, define $\hat{\kappa} := \mathbb{E}[\kappa] + \sum_{k=1}^{l} \mathbb{E}[\lambda^{(k)}]\varrho_k \circ \theta_k^2$ and observe that

$$\mathbb{E}\big[|Y_t^{\tau_n}|^2\big] \leq \mathbb{E}\big[|Y_{t_0}|^2\big] + \int_{t_0}^{t} \hat{\kappa}(s) + \sum_{k=1}^{l} \mathbb{E}\big[\eta_s^{(k)}\rho_k(|Y_s|^2)\mathbb{1}_{\{\tau_n > s\}}\big]\,\mathrm{d}s \tag{4.5}$$

for fixed $t \geq t_0$, according to Lemma 4.2. Moreover, for any stopping time τ for which $\mathbb{E}[|Y^\tau|^2] < \infty$ we infer from (3.1) and the concavity of $\rho^{\frac{1}{\alpha_k}}$ that

$$\mathbb{E}\big[\eta_s^{(k)} \rho_k(|Y_s|^2) \mathbb{1}_{\{\tau > s\}}\big] \leq \big[\eta_s^{(k)}\big]_{\frac{1}{1-\alpha_k}} \big(1 - \alpha_k + \alpha_k \rho_k \big(\mathbb{E}\big[|Y_s^\tau|^2\big]\big)^{\frac{1}{\alpha_k}}\big) \tag{4.6}$$

for any $s \in [t_0, t]$ and $k \in \{1, \ldots, l\}$. So, if $\Phi_{\rho_0}(\infty) = \infty$, then an application of Bihari's inequality to (4.5) and Fatou's lemma show that $\mathbb{E}[|Y|^2]$ is locally bounded.

In this case, we may take $\tau = \infty$ in (4.6) to see that $\sum_{k=1}^l \mathbb{E}[\eta^{(k)} \rho_k(|Y|^2)]$ is locally integrable. For this reason, we merely suppose that the latter holds. Then

$$\mathbb{E}\big[|Y_t|^2\big] \leq \mathbb{E}\big[|Y_{t_0}|^2\big] + \int_{t_0}^t (\hat{\kappa} + \hat{\delta})(s) + \sum_{k=1}^l \alpha_k \big[\eta_s^{(k)}\big]_{\frac{1}{1-\alpha_k}} \rho_k \big(\mathbb{E}\big[|Y_s|^2\big]\big)^{\frac{1}{\alpha_k}} \, \mathrm{d}s$$

for $\hat{\delta} := \sum_{k=1}^l (1-\alpha_k)[\eta^{(k)}]_{\frac{1}{1-\alpha_k}}$, by (4.5) and Fatou's lemma. Thus, as in the previous case, $\mathbb{E}[|Y|^2]$ is locally bounded. Lastly, since Young's inequality gives

$$\mathbb{E}\big[\lambda^{(k)}\big] \varrho_k \circ \theta_k^2 \leq \mathbb{E}\big[\lambda^{(k)}\big] \big(1 - \beta_k + \beta_k \varrho_k \big(\mathbb{E}\big[|Y|^2\big]\big)^{\frac{1}{\beta_k}}\big)$$

on $[t_0, t]$ for all $k \in \{1, \ldots, l\}$, the asserted estimate follows from Bihari's inequality. □

Next, we seek to give an explicit L^p-estimate for $p \geq 2$ and impose an abstract mixed power condition on Y:

(A.2) There are $\alpha, \beta \in [0, 2]^l$ with $\alpha + \beta \in [0, 2]^l$, measurable maps $\zeta, \theta : [t_0, \infty[\to \mathbb{R}_+^l$ and a progressively measurable process η such that

$$Y' \mathrm{B} + \frac{p-1}{2} |\Sigma|^2 \leq \zeta_1 \eta^{(1)} |Y|^{\alpha_1} \theta_1^{\beta_1} + \cdots + \zeta_l \eta^{(l)} |Y|^{\alpha_l} \theta_l^{\beta_l} \quad \text{a.s.}$$

Further, $\zeta_k \eta^{(k)}$ has locally integrable paths, $\theta_k(s) \leq \mathbb{E}[|Y_s|^p]^{\frac{1}{p}}$ for all $s \geq t_0$ with $\zeta_k(s)[\eta_s^{(k)}]_{\frac{p}{2-\alpha_k}} \mathbb{1}_{]0,2]}(\beta_k) > 0$, we have $\zeta_k = 1$, if $\alpha_k + \beta_k = 2$, and

$$\big(1 + \zeta_k^{\frac{p}{2-\alpha_k-\beta_k}}\big) \big[\eta^{(k)}\big]_{\frac{p}{2-\alpha_k}}$$

is locally integrable for any $k \in \{1, \ldots, l\}$.

Remark 4.5 The preceding condition implies that $\zeta_k[\eta^{(k)}]_{\frac{p}{2-\alpha_k}}$ is locally integrable, since Young's inequality entails that

$$p\zeta_k \leq p - 2 + \alpha_k + \beta_k + (2 - \alpha_k - \beta_k) \zeta_k^{\frac{p}{2-\alpha_k-\beta_k}} \quad \text{for all } k \in \{1, \ldots, l\}.$$

If (A.2) is satisfied, then we may define two measurable locally integrable functions $\gamma_{p,2} : [t_0, \infty[\to]-\infty, \infty]$ and $\hat{\delta}_{p,2} : [t_0, \infty[\to [0, \infty]$ by

$$\gamma_{p,2}(s) := \sum_{k=1}^{l} (p - 2 + \alpha_k + \beta_k)\big[\eta_s^{(k)}\big]_{\frac{p}{2-\alpha_k}} \tag{4.7}$$

and

$$\hat{\delta}_{p,2}(s) := \sum_{k=1}^{l} (2 - \alpha_k - \beta_k)\zeta_k(s)^{\frac{p}{2-\alpha_k-\beta_k}} \big[\eta_s^{(k)}\big]_{\frac{p}{2-\alpha_k}}, \tag{4.8}$$

which yield an explicit pth moment estimate. As a direct consequence, we can provide sufficient conditions for boundedness and convergence in $L^p(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem 4.6 *Let (A.2) hold, $\mathbb{E}[|Y_{t_0}|^p] < \infty$ and $\sum_{k=1,\, \beta_k>0}^{l} \zeta_k[\eta^{(k)}]_{\frac{p}{2-\alpha_k}} \theta_k^{\beta_k}$ be locally integrable. Then*

$$\mathbb{E}\big[|Y_t|^p\big] \le e^{\int_{t_0}^t \gamma_{p,2}(s)\, ds} \mathbb{E}\big[|Y_{t_0}|^p\big] + \int_{t_0}^t e^{\int_s^t \gamma_{p,2}(\tilde{s})\, d\tilde{s}} \hat{\delta}_{p,2}(s)\, ds$$

for all $t \ge t_0$. In particular, if $\gamma_{p,2}^+$ and $\hat{\delta}_{p,2}$ are integrable, then $\mathbb{E}[|Y|^p]$ is bounded. If additionally $\int_{t_0}^{\infty} \gamma_{p,2}^-(s)\, ds = \infty$, then $\lim_{t\uparrow\infty} \mathbb{E}[|Y_t|^p] = 0$.

Proof According to Lemma 4.2 and (A.2), the process $\hat{Z} := Y'\mathrm{B} + \frac{p-1}{2}|\Sigma|^2$ and any stopping time τ for which $\mathbb{E}[|Y^\tau|^p]$ is locally bounded satisfy

$$\begin{aligned}\mathbb{E}\big[u(t \wedge \tau)|Y_t^\tau|^p\big] \le \mathbb{E}\big[|Y_{t_0}|^p\big] \\ + \int_{t_0}^t \mathbb{E}\big[|Y_s|^{p-2}\big(\dot{u}(s)|Y_s|^2 + u(s)p\hat{Z}_s\big)\mathbb{1}_{\{\tau>s\}}\big]\, ds\end{aligned} \tag{4.9}$$

for any $t \ge t_0$ and each locally absolutely continuous function $u : [t_0, \infty[\to \mathbb{R}_+$. Thereby, we immediately infer from (3.1) that

$$p\mathbb{E}\big[|Y_s|^{p-2}\hat{Z}_s\mathbb{1}_{\{\tau>s\}}\big] \le \hat{\delta}_{p,1}(s) + \gamma_{p,1}(s)\mathbb{E}\big[|Y_s|^p\mathbb{1}_{\{\tau>s\}}\big]$$

for all $s \ge t_0$ with the two measurable locally integrable functions

$$\gamma_{p,1} := \sum_{k=1}^{l} (p - 2 + \alpha_k)\zeta_k\big[\eta^{(k)}\big]_{\frac{p}{2-\alpha_k}} \theta_k^{\beta_k} \quad \text{and} \quad \hat{\delta}_{p,1} := \sum_{k=1}^{l} (2 - \alpha_k)\zeta_k\big[\eta^{(k)}\big]_{\frac{p}{2-\alpha_k}} \theta_k^{\beta_k}.$$

Thus, if we choose the function $u(t) = \exp(-\int_{t_0}^t \gamma_{p,1}(s)\, ds)$ for all $t \ge t_0$ and the stopping time $\tau = \inf\{t \ge t_0 \mid |Y_t| \ge n\}$ in (4.9) for any $n \in \mathbb{N}$, then

$$e^{-\int_{t_0}^t \gamma_{p,1}(s)\, ds}\mathbb{E}\big[|Y_t|^p\big] \le \mathbb{E}\big[|Y_{t_0}|^p\big] + \int_{t_0}^t e^{-\int_{t_0}^s \gamma_{p,1}(s_0)\, ds_0} \hat{\delta}_{p,1}(s)\, ds$$

for each $t \geq t_0$, by an application of Fatou's lemma. In particular, $\mathbb{E}[|Y|^p]$ is locally bounded. Thus, a second estimation by means of (3.1) shows that

$$p\mathbb{E}\big[|Y_s|^{p-2}\hat{Z}_s\big] \leq \hat{\delta}_{p,2}(s) + \gamma_{p,2}(s)\mathbb{E}\big[|Y_s|^p\big]$$

for any $s \geq t_0$, since $[\eta^{(k)}]_{\frac{p}{2-\alpha_k}} \geq 0$ whenever $\beta_k > 0$ for all $k \in \{1, \ldots, l\}$. Now we take $u(t) = \exp(-\int_{t_0}^t \gamma_{p,2}(s)\,\mathrm{d}s)$ for any $t \geq t_0$ and $\tau = \infty$ in (4.9) to obtain the asserted estimate after dividing by $u(t)$. □

Remark 4.7 Assume that $\hat{\delta}_{p,2} = 0$ a.e., which is the case if $\alpha_k + \beta_k = 2$ for every $k \in \{1, \ldots, l\}$. If $\gamma_{p,2}^+$ is integrable, then Theorem 4.6 gives

$$\sup_{t \geq t_0} e^{\int_{t_0}^t \gamma_{p,2}^-(s)\,\mathrm{d}s}\mathbb{E}\big[|Y_t|^p\big] < \infty,$$

because $a\gamma_{p,2}^- + \gamma_{p,2} = \gamma_{p,2}^+ - (1-a)\gamma_{p,2}^-$ for each $a \in [0, 1]$. Thus, if in addition $\gamma_{p,2}^-$ fails to be integrable, then

$$\lim_{t\uparrow\infty} e^{a\int_{t_0}^t \gamma_{p,2}^-(s)\,\mathrm{d}s}\mathbb{E}\big[|Y_t|^p\big] = 0 \quad \text{for any } a \in [0, 1[.$$

This describes the rate of convergence more accurately.

4.3 Moment Bounds in the Supremum Norm

This section provides general methods to obtain L^{pq}-moment estimates in the supremum norm for $p, q \geq 2$ under the following abstract mixed power condition for $l \in \mathbb{N}$, which implies (A.2) when p is replaced by pq and $\zeta_1 = \cdots = \zeta_l = 1$ holds there:

(A.3) There are $\alpha, \hat{\alpha}, \beta, \hat{\beta} \in [0, 2]^l$, a measurable map $\theta : [t_0, \infty[\to \mathbb{R}_+^l$ and progressively measurable processes η and $\hat{\eta}$ with values in $\mathbb{R}^l$ and $\mathbb{R}_+^l$, respectively, such that

$$Y'\mathrm{B} + \frac{pq-1}{2}|\Sigma|^2 \leq \eta^{(1)}|Y|^{\alpha_1}\theta^{\beta_1} + \cdots + \eta^{(l)}|Y|^{\alpha_l}\theta^{\beta_l}$$
$$\text{and} \quad |\Sigma|^2 \leq \hat{\eta}^{(1)}|Y|^{\hat{\alpha}_1}\theta_1^{\hat{\beta}_1} + \cdots + \hat{\eta}^{(l)}|Y|^{\hat{\alpha}_l}\theta_l^{\hat{\beta}_l} \quad \text{a.s.}$$

Moreover, $\alpha + \beta, \hat{\alpha} + \hat{\beta} \in [0, 2]^l$, η and $\hat{\eta}$ have locally integrable paths and the following three conditions hold for each $k \in \{1, \ldots, l\}$:

(1) $\alpha_k = 2 \Leftrightarrow k = l$, $[\eta^{(k)}]_{\frac{pq}{2-\alpha_k}}$ is locally integrable and $\theta_k(s) \leq \mathbb{E}[|Y_s|^{pq}]^{\frac{1}{pq}}$ for all $s \geq t_0$ with

$$\big[\eta_s^{(k)}\big]_{\frac{pq}{2-\alpha_k}}\mathbb{1}_{]0,2]}(\beta_k) > 0 \quad \text{or} \quad \big[\hat{\eta}_s^{(k)}\big]_{\frac{pq}{2-\hat{\alpha}_k}}\mathbb{1}_{]0,2]}(\hat{\beta}_k) > 0.$$

(2) If $\alpha_k > 0$, then there are a real-valued measurable locally integrable function $\eta_{k,1}$ on $[t_0, \infty[$ and an $\mathbb{R}_+$-valued progressively measurable process $\eta^{(k,2)}$ so that

$$\eta^{(k)} = \eta_{k,1}\eta^{(k,2)} \quad \text{and} \quad \int_{t_0}^{\cdot} \eta_{k,1}^{+}(s)\big[\eta_s^{(k,2)}\big]^{q}_{\frac{pq}{2-\alpha_k}}\, \mathrm{d}s < \infty.$$

(3) There are a measurable locally integrable function $\hat{\eta}_{k,1} : [t_0, \infty[\to \mathbb{R}_+$ and an $\mathbb{R}_+$-valued progressively measurable process $\hat{\eta}^{(k,2)}$ satisfying

$$\hat{\eta}^{(k)} = \hat{\eta}_{k,1}\hat{\eta}^{(k,2)} \quad \text{and} \quad \int_{t_0}^{\cdot} \hat{\eta}_{k,1}(s)[\hat{\eta}_s^{(k,2)}]^{\frac{q}{2}}_{\frac{pq}{2-\hat{\alpha}_k}}\, \mathrm{d}s < \infty.$$

Remark 4.8 If the two processes η and $\hat{\eta}$ are in fact deterministic, then the conditions (2) and (3) are redundant.

Given (A.3) is satisfied, we introduce $\alpha_p, \hat{\alpha}_p \in [0, 1]^l$ coordinatewise by

$$\alpha_{p,k} := \frac{p - 2 + \alpha_k}{p} \quad \text{and} \quad \hat{\alpha}_{p,k} := \frac{2p - 2 + \hat{\alpha}_k}{2p}.$$

For any $j \in \{1, \ldots, l\}$ with $\alpha_j > 0$ and each $k \in \{1, \ldots, l\}$, let the two $\mathbb{R}_+$-valued continuous functions $c_{j,q}$ and $\hat{c}_{k,q}$ on the set of all $(t_1, t) \in [t_0, \infty[^2$ with $t_1 \leq t$ be given by

$$c_{j,q}(t_1, t) := \bigg(\int_{t_1}^{t} \eta_{j,1}^{+}(s)\, \mathrm{d}s\bigg)^{1-\frac{1}{q}} \quad \text{and} \quad \hat{c}_{k,q}(t_1, t) := \bigg(\int_{t_1}^{t} \hat{\eta}_{k,1}(s)\, \mathrm{d}s\bigg)^{\frac{1}{2}-\frac{1}{q}}.$$

Further, we define three $[0, \infty]$-valued measurable functions $f_{p,q}$, $g_{p,q}$ and $h_{p,q}$ on the set of all $(t_1, t) \in [t_0, \infty[^2$ with $t_1 \leq t$ by

$$\begin{aligned} f_{p,q}(t_1, t) &:= p\mathbb{E}\bigg[\bigg(\int_{t_1}^{t} |Y_s|^{p-2} \sum_{k=1,\, \alpha_k=0}^{l} (\eta_s^{(k)})^{+}\theta_k^{\beta_k}(s)\, \mathrm{d}s\bigg)^{q}\bigg]^{\frac{1}{q}}, \\ g_{p,q}(t_1, t) &:= f_{p,q}(t_1, t) \\ &\quad + p \sum_{k=1,\, \alpha_k>0}^{l} c_{k,q}(t_1, t)\bigg(\int_{t_1}^{t} \eta_{k,1}^{+}(s)\mathbb{E}\big[(\eta_s^{(k,2)})^{q}|Y_{t_1}|^{\alpha_{p,k}pq}\big]\theta_k(s)^{\beta_k q}\, \mathrm{d}s\bigg)^{\frac{1}{q}} \\ &\quad + p \sum_{k=1}^{l} \hat{c}_{k,q}(t_1, t)\bigg(\overline{w}_q \int_{t_1}^{t} \hat{\eta}_{k,1}(s)\mathbb{E}\big[(\hat{\eta}_s^{(k,2)})^{\frac{q}{2}}|Y_{t_1}|^{\hat{\alpha}_{p,k}pq}\big]\theta_k(s)^{\hat{\beta}_k\frac{q}{2}}\, \mathrm{d}s\bigg)^{\frac{1}{q}} \end{aligned}$$

and

$$h_{p,q}(t_1,t) := p^q \sum_{k=1,\,\alpha_k>0}^{l} c_{k,q}(t_1,t)^q \int_{t_1}^{t} \eta_{k,1}^{+}(s)\big[\eta_s^{(k,2)}\big]_{\frac{pq}{2-\alpha_k}}^{q}\,\theta_k(s)^{\beta_k q}\,\mathrm{d}s$$
$$+ p^q \sum_{k=1}^{l} \hat{c}_{k,q}(t_1,t)^q\,\overline{w}_q \int_{t_1}^{t} \hat{\eta}_{k,1}(s)\big[\hat{\eta}_s^{(k,2)}\big]_{\frac{pq}{2-\hat{\alpha}_k}}^{\frac{q}{2}}\,\theta_k(s)^{\hat{\beta}_k\frac{q}{2}}\,\mathrm{d}s,$$

which are finite if $\mathbb{E}[|Y|^{pq}]$ is locally bounded. This fact follows from Hölder's inequality, applied to any two $[0,\infty]$-valued random variables ζ and X as follows:

$$\mathbb{E}\big[\zeta^q X^{\alpha_{p,k}pq}\big] \le [\zeta]_{\frac{pq}{2-\alpha_k}}^{q}\,\mathbb{E}\big[X^{pq}\big]^{\alpha_{p,k}}$$
$$\text{and}\quad \mathbb{E}\big[\zeta^{\frac{q}{2}} X^{\hat{\alpha}_{p,k}pq}\big] \le [\zeta]_{\frac{pq}{2-\hat{\alpha}_k}}^{\frac{q}{2}}\,\mathbb{E}\big[X^{pq}\big]^{\hat{\alpha}_{p,k}} \tag{4.10}$$

for each $k \in \{1,\ldots,l\}$. Finally, let us introduce $\underline{\alpha} := \min_{k=1,\ldots,l}(\alpha_{p,k} \wedge \hat{\alpha}_{p,k})$ and $\overline{\alpha} := \max_{k=1,\ldots,l}(\alpha_{p,k} \vee \hat{\alpha}_{p,k})$.

Proposition 4.9 *Let (A.3) be valid, $\sum_{k=1,\,\beta_k>0}^{l}[\eta^{(k)}]_{\frac{pq}{2-\alpha_k}}\theta_k^{\beta_k}$ be locally integrable and $\rho_0 \in C(\mathbb{R}_+)$ be given by $\rho_0(v) := v^{\underline{\alpha}}\mathbb{1}_{[0,1]}(v) + v^{\overline{\alpha}}_{]1,\infty[}(v)$. If*

$$\mathbb{E}\big[|Y_{t_0}|^{pq}\big] \quad \textit{and} \quad \mathbb{E}\bigg[\bigg(\int_{t_0}^{t} |Y_s|^{p-2} \sum_{k=1,\,\alpha_k=0}^{l} (\eta_s^{(k)})^{+}\,\mathrm{d}s\bigg)^q\bigg] \tag{4.11}$$

are finite, then $\sup_{s\in[t_0,t]}|Y_s|$ is pq-fold integrable and

$$\mathbb{E}\bigg[\bigg(\sup_{s\in[t_1,t]}|Y_s|^p - |Y_{t_1}|^p\bigg)^q\bigg] \le \Psi_{\rho_0}\Big((2l+1)^{q-1}g_{p,q}(t_1,t)^q,\,(2l+1)^{q-1}h_{p,q}(t_1,t)\Big)$$

for all $t_1, t \ge t_0$ with $t_1 \le t$. In particular, $\mathbb{E}[|Y|^{pq}]$ is continuous.

Proof Under the integrability assertion, dominated convergence gives us that

$$\lim_{n\uparrow\infty} \mathbb{E}\big[|Y_{t_n}|^{pq}\big] = \mathbb{E}\big[|Y_t|^{pq}\big]$$

for each sequence $(t_n)_{n\in\mathbb{N}}$ in $[t_0,\infty[$ converging to some $t \ge t_0$. Hence, we merely need to show the first two claims.

As (A.3) implies (A.2) when p is replaced by pq in the latter assumption, we know from Theorem 4.6 that $\mathbb{E}[|Y|^{pq}]$ is locally bounded. Thus, Proposition 4.3, the triangle inequality in the L^q-norm and Jensen's inequality yield that

$$\mathbb{E}\bigg[\bigg(\sup_{s\in[t_1,t]} |Y_s^{\tau_n}|^p - |Y_{t_1}|^p\bigg)^q\bigg]^{\frac{1}{q}} \le f_{p,q}(t_1,t)$$
$$+ p\sum_{k=1,\,\alpha_k>0}^{l} c_{k,q}(t_1,t)\bigg(\int_{t_1}^t \eta_{k,1}^+(s)\mathbb{E}\big[(\eta_s^{(k,2)})^q|Y_s^{\tau_n}|^{\alpha_{p,k}pq}\big]\theta_k(s)^{\beta_k q}\,\mathrm{d}s\bigg)^{\frac{1}{q}}$$
$$+ p\sum_{k=1}^{l}\hat{c}_{k,q}(t_1,t)\bigg(\overline{w}_q\int_{t_1}^t \hat{\eta}_{k,1}(s)\mathbb{E}\big[(\hat{\eta}_s^{(k,2)})^{\frac{q}{2}}|Y_s^{\tau_n}|^{\hat{\alpha}_{p,k}pq}\big]\theta_k(s)^{\hat{\beta}_k\frac{q}{2}}\,\mathrm{d}s\bigg)^{\frac{1}{q}}$$

for any fixed $t_1, t \ge t_0$ with $t_1 \le t$ and $n \in \mathbb{N}$, where $\tau_n := \inf\{t \ge t_1 \,|\, |Y_t| \ge n\}$. Hence, if $\mathbb{E}[|Y_{t_1}|^{pq}] < \infty$, then Minkowski's inequality, (4.10) and Bihari's inequality give the claimed estimate for

$$\mathbb{E}\bigg[\bigg(\sup_{s\in[t_1,t]} |Y_s^{\tau_n}|^p - |Y_{t_1}|^p\bigg)^q\bigg]^{\frac{1}{q}}.$$

Afterwards, Fatou's lemma implies the asserted bound. Moreover, if we take $t_1 = t_0$, then the pq-fold integrability of $|Y_{t_0}|$ implies that of $\sup_{s\in[t_0,t]}|Y_s|$. For this reason, the proposition is proven. □

Remark 4.10 The second expectation in (4.11) is finite if for each $k \in \{1,\ldots,l\}$ with $\alpha_k = 0$ there exist a measurable locally integrable function $\eta_{k,1} : [t_0,\infty[\to \mathbb{R}$ and an $\mathbb{R}_+$-valued progressively measurable process $\eta^{(k,2)}$ so that

$$\eta^{(k)} = \eta_{k,1}\eta^{(k,2)} \quad \text{and} \quad \eta_{k,1}^+\big[\eta^{(k,2)}\big]_{\frac{pq}{2}}^q$$

is locally integrable, by the inequalities of Jensen and Hölder.

If (A.3) is satisfied and $\gamma : [t_0,\infty[\to]-\infty,\infty]$ is measurable and locally integrable, then we define an $[0,\infty]$-valued measurable function $h_{\gamma,p,q}$ on the set of all $(t_1,t) \in [t_0,\infty[^2$ with $t_1 \le t$ via

$$h_{\gamma,p,q}(t_1,t) := p\mathbb{E}\bigg[\bigg(\int_{t_1}^t e^{-\int_{t_1}^s \gamma(s_0)\,\mathrm{d}s_0}|Y_s|^{p-2}\sum_{k=1,\,\alpha_k=0}^{l}(\eta_s^{(k)})^+\theta_k^{\beta_k}(s)\,\mathrm{d}s\bigg)^q\bigg]^{\frac{1}{q}}$$
$$+ p\sum_{k=1,\,\alpha_k\in]0,2[}^{l} c_{k,q}(t_1,t)\bigg(\int_{t_1}^t \eta_{k,1}^+(s)\big[\eta_s^{(k,2)}\big]_{\frac{pq}{2-\alpha_k}}^q e^{-q\int_{t_1}^s\gamma(s_0)\,\mathrm{d}s_0}\mathbb{E}\big[|Y_s|^{pq}\big]^{\alpha_{p,k}}\theta(s)^{\beta_k q}\,\mathrm{d}s\bigg)^{\frac{1}{q}}$$
$$+ p\sum_{k=1}^{l}\hat{c}_{k,q}(t_1,t)\bigg(\overline{w}_q\int_{t_0}^t \hat{\eta}_{k,1}(s)\big[\hat{\eta}_s^{(k,2)}\big]_{\frac{pq}{2-\hat{\alpha}_k}}^{\frac{q}{2}} e^{-q\int_{t_1}^s\gamma(s_0)\,\mathrm{d}s_0}\mathbb{E}\big[|Y_s|^{pq}\big]^{\hat{\alpha}_{p,k}}\theta_k(s)^{\hat{\beta}_k\frac{q}{2}}\,\mathrm{d}s\bigg)^{\frac{1}{q}},$$

then an auxiliary pqth moment stability estimate in the supremum norm follows.

Lemma 4.11 *Let (A.3) be valid and* $\sum_{k=1,\,\beta_k>0}^{l}[\eta^{(k)}]_{\frac{pq}{2-\alpha_k}}\theta_k^{\beta_k}$ *be locally integrable. If the expectations in (4.11) are finite, then*

$$\mathbb{E}\Big[\Big(\sup_{s\in[t_1,t]} e^{-\int_{t_1}^{s}\gamma(s_0)\,\mathrm{d}s_0}|Y_s|^p - |Y_{t_1}|^p\Big)^q\Big]^{\frac{1}{q}} \le h_{\gamma,p,q}(t_1,t)$$
$$+\Big(\int_{t_1}^{t}\big(p[\eta_s^{(l)}]_\infty - \gamma(s)\big)^+\,\mathrm{d}s\Big)^{1-\frac{1}{q}}\Big(\int_{t_1}^{t}\big(p[\eta_s^{(l)}]_\infty - \gamma(s)\big)^+ e^{-q\int_{t_1}^{s}\gamma(s_0)\,\mathrm{d}s_0}\mathbb{E}\big[|Y_s|^{pq}\big]\,\mathrm{d}s\Big)^{\frac{1}{q}}$$

for any measurable locally integrable function $\gamma : [t_0,\infty[\to\mathbb{R}$ *and all* $t_1, t \ge t_0$ *with* $t_1 \le t$.

Proof From Theorem 4.6 we deduce that $h_{\gamma,p,q}$ is finite, Proposition 4.9 gives the pq-fold integrability of $\sup_{s\in[t_0,t]}|Y_s|$ and we readily observe that

$$-\gamma|Y|^2 + p\sum_{k=1}^{l}\eta^{(k)}|Y|^{\alpha_k}\theta_k^{\beta_k} = p\Big(\sum_{k=1}^{l-1}\eta^{(k)}|Y|^{\alpha_k}\theta_k^{\beta_k}\Big) + (p\eta^{(l)}-\gamma)|Y|^2.$$

Hence, we may infer the assertion from Proposition 4.3 by using the triangle inequality in the L^q-norm, Jensen's inequality and (4.10). □

At last, we introduce a condition that forces the function $\hat{\delta}_{pq,2}$ in (4.8) when p is replaced by pq and $\zeta_1 = \cdots = \zeta_l = 1$ to vanish a.e. on $[t_1,\infty[$ for some $t_1 \ge t_0$.

(A.4) Assumption (A.3) holds and for any $k \in \{1,\ldots,l\}$ with $\alpha_k = 0$ there are a measurable locally integrable function $\eta_{k,1} : [t_0,\infty[\to\mathbb{R}_+$ and an $\mathbb{R}_+$-valued progressively measurable process $\eta^{(k,2)}$ such that

$$\eta^{(k)} = \eta_{k,1}\eta^{(k,2)} \quad\text{and}\quad \int_{t_0}^{\cdot}\eta_{k,1}^+(s)\big[\eta_s^{(k,2)}\big]_{\frac{pq}{2}}^q\,\mathrm{d}s < \infty.$$

Moreover, there are $t_1 \ge t_0$, $\hat{\delta} > 0$ and $\overline{c}_0 \ge 0$ such that $\eta^{(k)}$ (resp. $\hat{\eta}^{(k)}$) vanishes on $[t_1,\infty[$ for any $k \in \{1,\ldots,l\}$ with $\alpha_k + \beta_k < 2$ (resp. $\hat{\alpha}_k + \hat{\beta}_k < 2$) and

$$\int_t^{t+\hat{\delta}}\eta_{j,1}^+(s)\max\big\{1,\big[\eta_s^{(j,2)}\big]_{\frac{pq}{2-\alpha_j}}\big\}^q\,\mathrm{d}s \vee \int_t^{t+\hat{\delta}}\hat{\eta}_{k,1}(s)\max\big\{1,\big[\hat{\eta}_s^{(k,2)}\big]_{\frac{pq}{2-\hat{\alpha}_k}}\big\}^{\frac{q}{2}}\,\mathrm{d}s$$

is bounded by $\overline{c}_0$ for any $t \ge t_1$ and $j,k \in \{1,\ldots,l\}$ with $j \le l-1$.

Then the pathwise asymptotic behaviour of Y in the next section can be handled with the subsequent pqth moment estimate in the supremum norm by using the function $\gamma_{pq,2}$ in (4.7).

Proposition 4.12 *Let (A.4) hold,* $\mathbb{E}[|Y_{t_0}|^{pq}] < \infty$ *and* $\sum_{k=1,\,\beta_k>0}^{l}[\eta^{(k)}]_{\frac{pq}{2-\alpha_k}}\theta_k^{\beta_k}$ *be locally integrable and assume that there are a measurable locally integrable function* $\gamma : [t_0,\infty[\to\mathbb{R}$ *and* $\overline{c}_{\gamma,-1}, \overline{c}_{\gamma,0}, \overline{c}_{\gamma,q}, \hat{c}_{\gamma,0} \ge 0$ *so that*

$$\int_{t_2}^{t}\big(p[\eta_s^{(l)}]_\infty - \gamma(s)\big)^+\,\mathrm{d}s \le \overline{c}_{\gamma,-1},\quad \int_{t_2}^{t}(\gamma_{pq,2}-q_0\gamma)(s)\,\mathrm{d}s \le \overline{c}_{\gamma,q_0},\quad \int_{t_2}^{t}\gamma(s)\,\mathrm{d}s \le \hat{c}_{\gamma,0}$$

for any $t_2, t \geq t_1$ with $t_2 \leq t < \hat{\delta}$ and $q_0 \in \{0, q\}$. Then there is $\overline{c} > 0$ such that

$$\mathbb{E}\Big[\sup_{s\in[t_2,t]} |Y_s|^{pq}\Big]^{\frac{1}{q}} \leq \overline{c}\Big(\mathbb{E}\big[|Y_{t_0}|^{pq}\big] + \int_{t_0}^{t_1} \hat{\delta}_{pq,2}(s)\,\mathrm{d}s\Big)^{\frac{1}{q}} e^{\frac{1}{q}\int_{t_1}^{t_2} \gamma_{pq,2}(s)\,\mathrm{d}s} \tag{4.12}$$

for every $t_2, t \geq t_1$ with $t_2 \leq t < \hat{\delta}$.

Proof As $\eta^{(k)} = 0$ (resp. $\hat{\eta}^{(k)} = 0$) on $[t_1, \infty[$ for any $k \in \{1, \ldots, l\}$ with $\alpha_k + \beta_k < 2$ (resp. $\hat{\alpha}_k + \hat{\beta}_k < 2$), it follows from Lemma 4.11, (4.10) and Jensen's inequality that

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{s\in[t_2,t]} e^{-q\int_{t_2}^{s}\gamma(s_0)\,\mathrm{d}s_0}|Y_s|^{pq}\Big]^{\frac{1}{q}} \leq \mathbb{E}\big[|Y_{t_2}|^{pq}\big]^{\frac{1}{q}}\\
&+ p\overline{c}_0^{1-\frac{1}{q}}\sum_{k=1}^{l-1}\Big(\int_{t_2}^{t}\eta_{k,1}^{+}(s)\big[\eta_s^{(k,2)}\big]_{\frac{pq}{2-\alpha_k}}^{q} e^{-q\int_{t_2}^{s}\gamma(s_0)\,\mathrm{d}s_0}\mathbb{E}\big[|Y_s|^{pq}\big]\,\mathrm{d}s\Big)^{\frac{1}{q}}\\
&+ \overline{c}_{\gamma,-1}^{1-\frac{1}{q}}\Big(\int_{t_2}^{t}\big(p\big[\eta_s^{(l)}\big]_\infty - \gamma(s)\big)^{+} e^{-q\int_{t_2}^{s}\gamma(s_0)\,\mathrm{d}s_0}\mathbb{E}\big[|Y_s|^{pq}\big]\,\mathrm{d}s\Big)^{\frac{1}{q}}\\
&+ p\overline{c}_0^{\frac{1}{2}-\frac{1}{q}}\sum_{k=1}^{l}\Big(\overline{w}_q\int_{t_2}^{t}\hat{\eta}_{k,1}(s)\big[\hat{\eta}_s^{(k,2)}\big]_{\frac{pq}{2-\hat{\alpha}_k}}^{\frac{q}{2}} e^{-q\int_{t_2}^{s}\gamma(s_0)\,\mathrm{d}s_0}\mathbb{E}\big[|Y_s|^{pq}\big]\,\mathrm{d}s\Big)^{\frac{1}{q}}.
\end{aligned} \tag{4.13}$$

Moreover, as $\hat{\delta}_{pq,2} = 0$ a.e. on $[t_1, \infty[$, the moment stability estimate of Theorem 4.6 gives us that

$$e^{-q_0\int_{t_2}^{s}\gamma(s_0)\,\mathrm{d}s_0 - \overline{c}_{\gamma,q_0}}\mathbb{E}\big[|Y_s|^{pq}\big] \leq e^{\int_{t_0}^{t_2}\gamma_{pq,2}(s_0)\,\mathrm{d}s_0}\mathbb{E}\big[|Y_{t_0}|^{pq}\big] + \int_{t_0}^{t_1} e^{\int_{s_0}^{t_2}\gamma_{pq,2}(s_1)\,\mathrm{d}s_1}\hat{\delta}_{pq,2}(s_0)\,\mathrm{d}s_0$$

for all $s \in [t_2, t]$ and $q_0 \in \{0, q\}$. Thus, the sum of the four right-hand terms in (4.13) is bounded by the right-hand expression in (4.12) when $\overline{c}$ is replaced by the constant

$$\overline{c}_1 := e^{\frac{1}{q}\int_{t_0}^{t_1}\gamma_{pq,2}^{+}(s)\,\mathrm{d}s}\Big(e^{\frac{1}{q}\overline{c}_{\gamma,0}} + e^{\frac{1}{q}\overline{c}_{\gamma,q}}\Big(p\overline{c}_0(l-1) + \overline{c}_{\gamma,-1} + p\overline{c}_0^{\frac{1}{2}}\overline{w}_q^{\frac{1}{q}}l\Big)\Big).$$

Because $\exp(-\int_{t_2}^{s}\gamma(s_0)\,\mathrm{d}s_0) \geq \exp(-\hat{c}_{\gamma,0})$ for any $s \in [t_2, t]$, the claimed bound holds for $\overline{c} := \exp(\hat{c}_{\gamma,0})\overline{c}_1$. □

4.4 Pathwise Asymptotic Behaviour

Finally, we derive a limiting bound for Y from the moment estimate of Proposition 4.12. To this end, we use an application of the Borel–Cantelli lemma in [25, Lemma 4.11].

Namely, let $A \in \mathcal{F}$ and X be an $\mathbb{R}_+$-valued right-continuous process for which there are a strictly increasing sequence $(t_n)_{n\in\mathbb{N}}$ in $[t_0, \infty[$ with $\lim_{n\uparrow\infty} t_n = \infty$, a sequence $(c_n)_{n\in\mathbb{N}}$ in $]0,\infty[$, $\hat{c}>0$ and $\hat{\varepsilon}\in]0,1[$ such that

$$\mathbb{E}\Big[\sup_{s\in]t_n,t_{n+1}]} X_s\mathbb{1}_A\Big] \leq \hat{c}c_n \quad \text{for all } n\in\mathbb{N} \quad \text{and} \quad \sum_{n=1}^{\infty} c_n^{\varepsilon} < \infty$$

for each $\varepsilon\in]0,\hat{\varepsilon}[$. Then any $]0,\infty[$-valued lower semicontinuous function φ on $]t_1,\infty[$ satisfies

$$\limsup_{t\uparrow\infty}\frac{\log(X_t)}{\varphi(t)} \leq \limsup_{n\uparrow\infty}\frac{\log(c_n)}{\inf_{s\in]t_n,t_{n+1}]}\varphi(s)} \quad \text{a.s. on } A. \tag{4.14}$$

Theorem 4.13 *Let (A.4) be satisfied and $\sum_{k=1,\,\beta_k>0}^{l}[\eta^{(k)}]_{\frac{pq}{2-\alpha_k}}\theta_k^{\beta_k}$ be locally integrable. Assume that $\gamma_{pq,2}\leq 0$ a.e. on $[t_1,\infty[$ and there is an increasing sequence $(t_n)_{n\in\mathbb{N}\setminus\{1\}}$ in $[t_1,\infty[$ such that*

$$\sup_{n\in\mathbb{N}}(t_{n+1}-t_n) < \hat{\delta}, \quad \lim_{n\uparrow\infty} t_n = \infty$$

and $\sum_{n=1}^{\infty}\exp(\frac{\varepsilon}{pq}\int_{t_1}^{t_n}\gamma_{pq,2}(s)\,\mathrm{d}s) < \infty$ for all $\varepsilon\in]0,\hat{\varepsilon}[$ and some $\hat{\varepsilon}\in]0,1[$. If $\mathbb{E}[|Y_{t_0}|^{pq}]$ is finite or $\beta=\hat{\beta}=0$, then

$$\limsup_{t\uparrow\infty}\frac{1}{\varphi(t)}\log\big(|Y_t|\big) \leq \frac{1}{pq}\limsup_{n\uparrow\infty}\frac{1}{\varphi(t_n)}\int_{t_1}^{t_n}\gamma_{pq,2}(s)\,\mathrm{d}s \quad \textit{a.s.}$$

for each increasing continuous function $\varphi:[t_1,\infty[\to\mathbb{R}_+$ that is positive on $]t_1,\infty[$.

Proof Since $\gamma_{pq,2} = pq([\eta^{(l)}]_\infty + \sum_{k=1}^{l-1}[\eta^{(k)}]_{\frac{pq}{2-\alpha_k}})$ on $[t_1,\infty[$, it follows that $[\eta^{(l)}]_\infty \leq 0$ a.e. on the same interval. Consequently, if $\mathbb{E}[|Y_{t_0}|^{pq}]<\infty$, then

$$\mathbb{E}\Big[\sup_{s\in[t_n,t_{n+1}]}|Y_s|^{pq}\Big] \leq \hat{c}e^{\int_{t_1}^{t_n}\gamma_{pq,2}(s)\,\mathrm{d}s}$$

for each $n\in\mathbb{N}$ and some $\hat{c}>0$ by taking $\gamma = p[\eta^{(l)}]_\infty$ a.e. in Proposition 4.12. Indeed, from Jensen's inequality and (A.4) we immediately obtain that

$$\int_{t_2}^{t}\big(\gamma_{pq,2}(s) - pq\big[\eta_s^{(l)}\big]_\infty\big)\,\mathrm{d}s \leq pq(l-1)\overline{c}_0$$

for any $t_2, t\geq t_1$ satisfying $t_2\leq t<\hat{\delta}$. Next, we suppose that $\beta=\hat{\beta}=0$ and set $A_k := \{|Y_{t_0}|\leq k\}$ for given $k\in\mathbb{N}$. Then Proposition 4.12 yields $\hat{c}_k>0$ such that

$$\mathbb{E}\Big[\sup_{s\in[t_n,t_{n+1}]}|Y_s\mathbb{1}_{A_k}|^{pq}\Big] \leq \hat{c}_k e^{\int_{t_1}^{t_n}\gamma_{pq,2}(s)\,\mathrm{d}s}$$

for all $n \in \mathbb{N}$, as the random Itô process $Y\mathbb{1}_{A_k}$ with drift $\mathrm{B}\mathbb{1}_{A_k}$ and diffusion $\Sigma\mathbb{1}_{A_k}$ satisfies (A.3) and (A.4). Hence, in both cases the claimed pathwise inequality follows from the result recalled at (4.14) and the fact that $\bigcup_{k\in\mathbb{N}} A_k = \Omega$. □

5 Proofs of the Main Results

5.1 Proofs of the Moment Estimates, Uniqueness and Moment Stability

Proof of Proposition 3.2 Let the two progressively measurable processes $\hat{\mathrm{B}}$ and $\hat{\Sigma}$ with values in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$, respectively, be defined via

$$\begin{aligned} \hat{\mathrm{B}}_s &:= \mathrm{B}_s\big(X_s, \mathcal{L}(X_s)\big) - \tilde{\mathrm{B}}_s\big(\tilde{X}_s, \mathcal{L}(\tilde{X}_s)\big) \quad \text{and} \\ \hat{\Sigma}_s &:= \Sigma_s\big(X_s, \mathcal{L}(X_s)\big) - \tilde{\Sigma}_s\big(\tilde{X}_s, \mathcal{L}(\tilde{X}_s)\big). \end{aligned} \tag{5.1}$$

Then Y is a random Itô process with drift $\hat{\mathrm{B}}$ and diffusion $\hat{\Sigma}$ satisfying

$$2Y'\hat{\mathrm{B}} + |\hat{\Sigma}|^2 \leq \varepsilon + \eta\rho(|Y|^2) + \lambda\varrho\circ\theta^2 \quad \text{a.s.}$$

with the measurable function $\theta := \vartheta(\mathcal{L}(X), \mathcal{L}(\tilde{X}))$. For this reason, the proposition is a special case of Proposition 4.4. □

Proof of Corollary 3.5 For both claims in (i) and (ii), let X and $\tilde{X}$ be two solutions to (1.2) with $X_{t_0} = \tilde{X}_{t_0}$ a.s. Suppose first that (C.2) holds and the expression

$$\mathbb{E}[\lambda]\varrho\big(\vartheta(\mathcal{L}(X), \mathcal{L}(\tilde{X}))^2\big) + \mathbb{1}_{]0,\infty[}\big(\Phi_\rho(\infty)\big)\eta\mathbb{E}\big[\rho(|X - \tilde{X}|^2)\big]$$

is locally integrable. Then Proposition 3.2 gives $\mathbb{E}[|X_t - \tilde{X}_t|^2] = 0$ for any $t \geq t_0$, as $\varrho_0 := \rho \vee \varrho$ satisfies $(0, w) \in D_{\varrho_0}$ and $\Psi_{\varrho_0}(0, w) = 0$ for all $w \geq 0$. Hence, path continuity implies that X and $\tilde{X}$ are indistinguishable.

Now let (C.3) hold and set $\tau_n := \inf\{t \geq t_0 \,|\, |X_t| \geq n \text{ or } |\tilde{X}_t| \geq n\}$ for fixed $n \in \mathbb{N}$. Then $Y := X^{\tau_n} - \tilde{X}^{\tau_n}$ is a random Itô process with drift $\tilde{\mathrm{B}}$ and diffusion $\tilde{\Sigma}$ given by

$$\tilde{\mathrm{B}}_s := \big(\hat{\mathrm{B}}_s(X_s) - \hat{\mathrm{B}}_s(\tilde{X}_s)\big)\mathbb{1}_{\{\tau_n > s\}} \quad \text{and} \quad \tilde{\Sigma}_s := \big(\hat{\Sigma}_s(X_s) - \hat{\Sigma}_s(\tilde{X}_s)\big)\mathbb{1}_{\{\tau_n > s\}},$$

and we have $2Y'\tilde{\mathrm{B}} + |\tilde{\Sigma}|^2 \leq \eta_n\rho_n(|Y|^2)$ a.s. So, Proposition 4.4 yields that $Y = 0$ a.s. This in turn implies that $X = \tilde{X}$ a.s., because $\sup_{n\in\mathbb{N}} \tau_n = \infty$. □

Proof of Proposition 3.10 We define two processes $\hat{\mathrm{B}}$ and $\hat{\Sigma}$ with respective values in $\mathbb{R}^m$ and $\mathbb{R}^{m\times d}$ by (5.1) and observe that

$$\begin{aligned} Y'\hat{\mathrm{B}} &\leq \sum_{k=1}^{l} \zeta_k \eta^{(k)} |Y|^{1+\alpha_k}\theta^{\beta_k} \\ \text{and} \quad |\hat{\Sigma}|^2 &\leq \sum_{j,k=1}^{l} \hat{\zeta}_j\hat{\zeta}_k\hat{\eta}^{(j)}\hat{\eta}^{(k)}|Y|^{\alpha_j+\alpha_k}\theta^{\beta_j+\beta_k} \end{aligned} \tag{5.2}$$

a.s., where $\theta := \vartheta(\mathcal{L}(X), \mathcal{L}(\tilde{X}))$. So, all claims follow from Theorem 4.6, as our considerations succeeding definitions (3.5) and (3.6) explain. In this context, the required local integrability of

$$\hat{\zeta}_j \hat{\zeta}_k \big[\hat{\eta}^{(j)} \hat{\eta}^{(k)}\big]_{\frac{p}{2-\alpha_j-\alpha_k}} \theta^{\beta_j+\beta_k}$$

for all $j, k \in \{1, \ldots, l\}$ follows from that of $\Theta(\cdot, \mathcal{L}(X), \mathcal{L}(\tilde{X}))$, by Young's inequality. □

Proof of Corollary 3.12 Definition 2.6 clarifies that both stability claims are immediate consequences of Proposition 3.10. □

Proof of Corollary 3.13 (i) From Remark 4.7 we infer the first two assertions by replacing $\gamma_{p,2}$ by γ_p there, since Proposition 3.10 is a special case of Theorem 4.6.

(ii) Let X and $\tilde{X}$ be two solutions to (1.2) such that $\mathbb{E}[|X_{t_0} - \tilde{X}_{t_0}|^p] < \infty$ and $\Theta(\cdot, \mathcal{L}(X), \mathcal{L}(\tilde{X}))$ is locally integrable. Then

$$\mathbb{E}\big[|X_t - \tilde{X}_t|^p\big] \leq e^{\int_{t_0}^t \gamma_p(s)\,\mathrm{d}s} \mathbb{E}\big[|X_{t_0} - \tilde{X}_{t_0}|^p\big]$$

for each $t \geq t_0$, according to Proposition 3.10. Thus, let us directly exclude the case that $X_{t_0} = \tilde{X}_{t_0}$ a.s. Then (C.6) implies that

$$\limsup_{t\uparrow\infty} \frac{1}{t^{\alpha_l}} \log\big(\mathbb{E}\big[|X_t - \tilde{X}_t|^p\big]\big) \leq \limsup_{t\uparrow\infty} \sum_{k=1}^{l} \lambda_k \frac{(t-s_k)^{\alpha_k} - (t_1 - s_k)^{\alpha_k}}{t^{\alpha_l}} = \lambda_l$$

and we obtain the first assertion from Remark 2.7. To prove the second claim, we may suppose that $l = 1$ and see that $\hat{c}_0 := \max_{t\in[t_0,t_1]} \exp(\int_{t_0}^t \gamma_p(s)\,\mathrm{d}s - \lambda_1(t-t_0)^{\alpha_1})$ satisfies

$$\mathbb{E}\big[|X_t - \tilde{X}_t|^p\big] \leq \hat{c}_0 e^{\lambda_1(t-t_0)^{\alpha_1}} \mathbb{E}\big[|X_{t_0} - \tilde{X}_{t_0}|^p\big] \tag{5.3}$$

for every $t \in [t_0, t_1]$. Hence, to ensure that (5.3) is satisfied for all $t \geq t_0$, we take $\hat{c} := \hat{c}_0 \vee \exp(\int_{t_0}^{t_1} \gamma_p(s)\,\mathrm{d}s - \lambda_1(t_1 - s_1)^{\alpha_1})$ instead of $\hat{c}_0$. □

5.2 Proofs for Pathwise Stability and the Moment Growth Bounds

Proof of Proposition 3.16 Since Y is random Itô process with drift $\hat{\mathrm{B}}$ and diffusion $\hat{\Sigma}$ defined via (5.1) for the choice $(\tilde{\mathrm{B}}, \tilde{\Sigma}) = (\mathrm{B}, \Sigma)$, the assertion follows from Theorem 4.13. □

Proof of Corollary 3.17 To show both claims simultaneously, we argue as in the proof of Corollary 3.17 in [25]. Namely, we take $\hat{t}_1 \geq t_1$ and $\tilde{\delta} > 0$ such that $\gamma_{pq} \leq 0$ a.e. on $[\hat{t}_1, \infty[$ and set $t_n := \hat{t}_1 + \tilde{\delta}(n-1)$ for all $n \in \mathbb{N}$ with $n \geq 2$. Then

$$\int_0^\infty \exp\left(\frac{\varepsilon}{pq} \int_{\hat{t}_1}^{\hat{t}_1+\tilde{\delta}t} \gamma_{pq}(s)\,\mathrm{d}s\right)\mathrm{d}t \leq \int_0^\infty \exp\left(\frac{\varepsilon}{pq} \sum_{k=1}^{l} \lambda_k \int_{\hat{t}_1}^{\hat{t}_1+\tilde{\delta}t} \alpha_k (s-s_k)^{\alpha_{k-1}}\,\mathrm{d}s\right)\mathrm{d}t$$

for any given $\varepsilon > 0$. Since $\lambda_l < 0$, the integral on the right-hand side is finite, see Lemma 5.1 in [25], for instance. Thus, the integral test for the convergence of series implies that $\sum_{n=1}^{\infty} \exp(\frac{\varepsilon}{pq} \int_{\hat{t}_1}^{t_n} \gamma_{pq}(s)\,\mathrm{d}s) < \infty$.

Consequently, (C.8) follows as soon as $\tilde{\delta} < \hat{\delta}$, and Proposition 3.16 entails that the difference Y of any two solutions X and $\tilde{X}$ to (1.2) for which $\Theta(\cdot, \mathcal{L}(X), \mathcal{L}(\tilde{X}))$ is locally integrable satisfies

$$\limsup_{t\uparrow\infty} \frac{\log(|Y_t|)}{t^{\alpha_l}} \leq \frac{1}{pq} \limsup_{n\uparrow\infty} \sum_{k=1}^{l} \lambda_k \frac{(t_n - s_k)^{\alpha_k} - (\hat{t}_1 - s_k)^{\alpha_k}}{t_n^{\alpha_l}} = \frac{\lambda_l}{pq} \quad \text{a.s.},$$

provided $\mathbb{E}[|Y_{t_0}|^{pq}] < \infty$ or both B and Σ are independent of $\mu \in \mathcal{P}$. □

Proof of Lemma 3.20 By hypothesis, X is a random Itô process with drift and diffusion given by $\hat{\mathrm{B}} := \mathrm{B}(X, \mathcal{L}(X))$ and $\hat{\Sigma} := \Sigma(X, \mathcal{L}(X))$, respectively, so that

$$2X'\hat{\mathrm{B}} + |\hat{\Sigma}|^2 \leq \kappa + \upsilon\phi(|X|^2) + \chi\varphi(\theta^2) \quad \text{a.s.}$$

with the measurable function $\theta := \vartheta(\mathcal{L}(X), \delta_0)$. For this reason, the lemma is implied by Proposition 4.4. □

Proof of Lemma 3.21 As in the proof of Lemma 3.20, let us set $\hat{\mathrm{B}} := \mathrm{B}(X, \mathcal{L}(X))$ and $\hat{\Sigma} := \Sigma(X, \mathcal{L}(X))$ and $\theta := \vartheta(\mathcal{L}(X), \delta_0)$. Then we readily see that

$$X'\hat{\mathrm{B}} \leq \sum_{k=1}^{l} \kappa_k \upsilon^{(k)} |X|^{1+\alpha_k} \theta^{\beta_k} \quad \text{and} \quad |\hat{\Sigma}|^2 \leq \sum_{j,k=1}^{l} \hat{\kappa}_j \hat{\kappa}_k \hat{\upsilon}^{(j)} \hat{\upsilon}^{(k)} |X|^{\alpha_k} \theta^{\beta_k} \quad \text{a.s.}$$

Hence, Theorem 4.6 yields all assertions. □

5.3 Proofs for Unique Strong Solutions

Proof of Proposition 3.23 (i) Because the uniform continuity condition (C.3) is satisfied when $(\mathrm{B}, \Sigma) = (b_\mu, \sigma_\mu)$, pathwise uniqueness for (3.17) follows from Corollary 3.5.

(ii) In essence, we may proceed as in the proof of Proposition 3.24 in [25]. First, let ξ be essentially bounded. Mainly, Theorem 2.3 in [20, Chapter IV] yields a local weak solution $\tilde{X}$ to (3.17).

That is, based on the one-point compactification, $\tilde{X}$ is an $\mathbb{R}^m \cup \{\infty\}$-valued adapted continuous process on a filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$ on which there is an $(\tilde{\mathcal{F}}_t)_{t\geq 0}$-Brownian motion $\tilde{W}$ such that the usual and the following conditions hold:

(1) If $(s, \omega) \in [t_0, \infty[\times\tilde{\Omega}$ satisfies $\tilde{X}_s(\omega) = \infty$, then $\tilde{X}_t(\omega) = \infty$ for any $t \geq s$.
(2) $\mathcal{L}(\tilde{X}_{t_0}) = \mathcal{L}(\xi)$ and for the supremum τ of the sequence $(\tau_n)_{n\in\mathbb{N}}$ of stopping times defined by $\tau_n := \inf\{t \geq t_0 \,|\, |\tilde{X}_t| \geq n\}$ we have $\tau > t_0$ a.s.
(3) $\tilde{X}^{\tau_n}$ solves (1.2) relative to $\tilde{W}$ when B and Σ are replaced by the admissible maps $b_\mu \mathbb{1}_{\{\tau_n > \cdot\}}$ and $\sigma_\mu \mathbb{1}_{\{\tau_n > \cdot\}}$, respectively, for every $n \in \mathbb{N}$.

Clearly, (D.1) ensures that (C.10) holds for $(\mathrm{B}, \Sigma) = (b_\mu, \sigma_\mu)$, as $\kappa_\mu := \kappa + \chi\varphi(\vartheta(\mu, \delta_0)^2)$ is locally integrable. Consequently, Lemma 3.20 and Fatou's lemma assert that

$$\tilde{\mathbb{E}}\big[|\tilde{X}_t^\tau|^2\big] \leq \liminf_{n\uparrow\infty} \tilde{\mathbb{E}}\big[|\tilde{X}_t^{\tau_n}|^2\big] \leq \Psi_\phi\bigg(\mathbb{E}\big[|\xi|^2\big] + \int_{t_0}^t \kappa_\mu(s)\,\mathrm{d}s, \int_{t_0}^t \upsilon(s)\,\mathrm{d}s\bigg) \quad (5.4)$$

for each $t \geq t_0$, which implies that $\tau = \infty$ and $\tilde{X} \in \mathbb{R}^m$ $\tilde{\mathbb{P}}$-a.s. Thus, $X := \tilde{X}\mathbb{1}_{\{\tau=\infty\}}$ serves as weak solution to (2.3) in the standard sense and $\tilde{\mathbb{E}}[|X|^2]$ is locally bounded. In particular, this derivation applies to the case when ξ is deterministic.

Therefore, Remark 2.1 in [20, Chapter IV] entails that there is a weak solution X to (2.3) with $X_{t_0} = \xi$ a.s., regardless of whether ξ is essentially bounded. If, however, $\mathbb{E}[|\xi|^2] < \infty$, then the second moment function of X is bounded by the right-hand term in (5.4), according to Lemma 3.20.

(iii) By what we have just shown, pathwise uniqueness for (3.17) holds and there exists a weak solution for any $\mathbb{R}^m$-valued $\mathcal{F}_{t_0}$-measurable random vector used as initial condition. Hence, Theorem 1.1 in [20, Chapter IV] entails the assertion. □

Proof of Theorem 3.24 (i) and (ii) Pathwise uniqueness with respect to Θ follows from Proposition 3.10, as the underlying filtered probability space and Brownian motion were arbitrarily chosen. In particular, there exists at most a unique solution X to (2.3) such that $X_{t_0} = \xi$ a.s. and $\mathbb{E}[|X|^p]$ is locally bounded.

Next, for any $\mu \in B_{b,loc}(\mathcal{P})$ Proposition 3.23 gives a unique strong solution $X^{\xi,\mu}$ to (3.17) with $X_{t_0}^{\xi,\mu} = \xi$ a.s., and $\mathbb{E}[|X^{\xi,\mu}|^p]$ is locally bounded, by Lemmas 3.20 and 3.21. We note that this process is also a strong solution to (2.3) if μ is a fixed-point of the operator

$$\Psi : B_{b,loc}(\mathcal{P}) \to B_{b,loc}(\mathcal{P}_p(\mathbb{R}^m)), \quad \Psi(\nu)(t) := \mathcal{L}(X_t^{\xi,\nu}).$$

For given $\mu, \tilde{\mu} \in B_{b,loc}(\mathcal{P})$, we directly check that (C.4) is valid when (B, Σ) and $(\tilde{\mathrm{B}}, \tilde{\Sigma})$ are replaced by (b_μ, σ_μ) and $(b_{\tilde{\mu}}, \sigma_{\tilde{\mu}})$, respectively. Hence, Proposition 3.10 implies that

$$\begin{aligned} \vartheta_p(\Psi(\mu), \Psi(\tilde{\mu}))(t)^p &\leq \mathbb{E}\big[|X_t^{\xi,\mu} - X_t^{\xi,\tilde{\mu}}|^p\big] \\ &\leq \int_{t_0}^t e^{\int_s^t \gamma_{p,0}(\tilde{s})\,\mathrm{d}\tilde{s}}\hat{\delta}_0(s)\vartheta(\mu, \tilde{\mu})(s)^p\,\mathrm{d}s \end{aligned} \quad (5.5)$$

for every $t \geq t_0$. In particular, this shows that there is at most a unique fixed-point of Ψ, due to Gronwall's inequality.

Further, because $B_{b,loc}(\mathcal{P}_p(\mathbb{R}^m))$ is completely metrisable, the fixed-point theorem for time evolution operators in [24] yields the existence of a fixed-point and the error estimate (3.22). Namely, it follows inductively that

$$\sup_{s\in[t_0,t]} \vartheta_p(\mu_m, \mu_n)(s) \leq \Delta(t) \sum_{i=n}^{m-1} \left(\frac{1}{i!}\right)^{\frac{1}{p}} \left(\int_{t_0}^t e^{\int_s^t \gamma_{p,0}^+(\tilde{s})\,\mathrm{d}\tilde{s}}\hat{\delta}_0(s)\,\mathrm{d}s\right)^{\frac{i}{p}} \quad (5.6)$$

for all $m, n \in \mathbb{N}$ with $m > n$ and $t \geq t_0$. Hence, $(\mu_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $B_{b,loc}(\mathcal{P}_p(\mathbb{R}^m))$, and from (5.5) we infer that its limit μ must be a fixed-point of Ψ. Now we may take the limit $m \uparrow \infty$ in (5.6) to get the desired bound (3.22).

(iii) The set M_p is closed and convex, since the estimate in (3.23) does not depend on $\mu \in B_{b,loc}(\mathcal{P}_p(\mathbb{R}^m))$. An application of Lemma 3.21 entails that

$$\vartheta_p(\Psi(\mu)(t), \delta_0)^p \leq e^{\int_{t_0}^t g_{p,0}(s)\,\mathrm{d}s}\mathbb{E}\big[|\xi|^p\big] + \int_{t_0}^t e^{\int_s^t g_{p,0}(\tilde{s})\,\mathrm{d}\tilde{s}} h_{p,\mu}(s)\,\mathrm{d}s \tag{5.7}$$

for every $\mu \in B_{b,loc}(\mathcal{P})$ and $t \geq t_0$ with the two measurable locally integrable functions

$$g_{p,0} := \sum_{k=1}^{l}(p-1+\alpha_k)\big(\upsilon_k^+ - \upsilon_k^- \mathbb{1}_{\{1\}}(\alpha_k)\big) + \frac{p-1}{2}\sum_{j,k=1}^{l}(p-2+\alpha_j+\alpha_k)\hat{\upsilon}_j\hat{\upsilon}_k$$

and

$$\begin{aligned} h_{p,\mu} := &\sum_{\substack{k=1,\\ \alpha_k<1}}^{l}(1-\alpha_k)\vartheta_p(\mu,\delta_0)^{\frac{\beta_k}{1-\alpha_k}p}\upsilon_k^+ \\ &+ \frac{p-1}{2}\sum_{\substack{j,k=1,\\ \alpha_j<1 \text{ or } \alpha_k<1}}^{l}(2-\alpha_j-\alpha_k)\vartheta_p(\mu,\delta_0)^{\frac{\beta_j+\beta_k}{2-\alpha_j-\alpha_k}p}\hat{\upsilon}_j\hat{\upsilon}_k. \end{aligned}$$

Thereby, we used the fact that (b_μ, σ_μ) satisfies (C.11) for the choice $\beta = 0$. Next, Young's inequality gives us that

$$(1-\alpha_k)\vartheta_p(\mu,\delta_0)^{\frac{\beta_k}{1-\alpha_k}p} \leq 1-\alpha_k-\beta_k+\beta_k\vartheta_p(\mu,\delta_0)^p$$

for every $k \in \{1,\ldots,l\}$ with $\alpha_k < 1$ and

$$(2-\alpha_j-\alpha_k)\vartheta_p(\mu,\delta_0)^{\frac{\beta_j+\beta_k}{2-\alpha_j-\alpha_k}p} \leq 2-\alpha_j-\beta_j-\alpha_k-\beta_k+(\beta_j+\beta_k)\vartheta_p(\mu,\delta_0)^p$$

for any $j, k \in \{1,\ldots,l\}$ with $\alpha_j < 1$ or $\alpha_k < 1$. From these estimates we infer that $g_{p,1} := g_p - g_{p,0}$ satisfies

$$h_{p,\mu} \leq h_p + g_{p,1}\vartheta_p(\mu,\delta_0)^p.$$

So, the inequality (5.7), the fundamental theorem of calculus for Lebesgue–Stieltjes integrals and Fubini's theorem show that Ψ maps M_p into itself, which implies the claim. □

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