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Stochastic Differential Equations with Local Growth Singular Drifts

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Abstract

In this paper, we study the weak differentiability of global strong solution of stochastic differential equations, the strong Feller property of the associated diffusion semigroups and the global stochastic flow property in which the singular drift b and the weak gradient of Sobolev diffusion σ are supposed to satisfy $\left\| |b| \cdot \mathbb{1}_{B(R)} \right\|_{p_1} \leq O((\log R)^{(p_1-d)^2/2p_1^2})$ and $\left\| \|\nabla\sigma\| \cdot \mathbb{1}_{B(R)} \right\|_{p_1} \leq O((\log(R/3))^{(p_1-d)^2/2p_1^2})$, respectively. The main tools for these results are the decomposition of global two-point motions in Fang et al. (Ann Probab 35(1):180–205, 2007), Krylov's estimate, Khasminskii's estimate, Zvonkin's transformation and the characterization for Sobolev differentiability of random fields in Xie and Zhang (Ann Probab 44(6):3661–3687, 2016).

Keywords Weak differentiability · Strong Feller property · Stochastic flow · Krylov's estimates · Zvonkin's transformation

Mathematics Subject Classification (2020) 60H10 · 60J60

1 Introduction and Main Results

In this paper, we consider the following d-dimension stochastic differential equations (SDEs, for short)

$$\begin{cases} dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \quad t \in [0, T], \\ X_0 = x \in \mathbb{R}^d. \end{cases} \tag{1.1}$$

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Here, $\{W_t\}_{t\in[0,T]}$ is a standard Wiener process in $\mathbb{R}^m$ which is defined on a complete filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{t\geq 0})$. The coefficients $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are both Borel measurable function. It is well known that stochastic differential equation defines a global stochastic homeomorphism flow if b and σ satisfy global Lipschitz conditions and linear growth conditions. In the past decades, for the non-Lipschitz coefficients SDEs there is increasing interest about their solutions and their properties (for example, the strong completeness property, the weak differentiability, stochastic homeomorphism flow property and so on).

Yamada and Ogura [22] proved the existence of global flow of homeomorphisms for one-dimensional SDEs under local Lipschitz and linear growth conditions. Li [16] proved the strong completeness property of SDEs (1.1) by studying the derivative flow equation of SDEs (1.1). Fang and Zhang [3] used the Gronwall-type estimate to study SDEs under non(local) Lipschitz conditions. Fang et al. [4] proved that Stratonovich equation defines a global stochastic homeomorphism flow if the coefficients are just locally Lipschitz and Lipschitz coefficients with mild growth. Chen and Li [1] studied Sobolev regularity of Eq. (1.1) and strong completeness property when b and σ are Sobolev coefficients.

When $\sigma = I$ and b is bounded and measurable, Veretennikov [19] first proved existence and uniqueness of the strong solution. When $\sigma = I$ and b satisfy

$$\left(\int_0^T \left(\int_{\mathbb{R}^d} |b|^p \, dx\right)^{\frac{q}{p}} dt\right)^{\frac{1}{q}} < \infty, \quad p, q \in [2, \infty), \quad \frac{2}{q} + \frac{d}{p} < 1, \qquad (1.2)$$

Krylov and Röckner [13] used the technique of PDEs to prove the existence and uniqueness of the strong solution. The similar result in time-homogeneous case was obtained by Zhang and Zhao [26], who dropped the assumption $\int_0^t |b(X_s)|^2 \, ds < \infty$, a.s.. Fedrizzi and Flandoli [5] proved the existence of a stochastic flow of α-Hölder homeomorphisms for solutions of SDEs as well as weak differentiability of solutions of SDEs under condition (1.2). Zhang [24, 25] extended the results of Krylov and Röckner [13] to the case of multiplicative noises. This extension allowed for the establishment of the well-posedness of solutions and the verification of weak differentiability in solutions. Additionally, it was proven that the solutions form a stochastic flow of homeomorphisms in $\mathbb{R}^d$. Key tools employed in this research included Krylov's estimate and Zvonkin's transformation. In [21], a characterization for Sobolev differentiability of random field was established. With the characterization, the weak differentiability of solutions was proved under local Sobolev integrability and sup-linear growth assumptions. We refer the reader to [6, 7, 20, 21, 23–25, 27] and references therein for applications of Krylov's estimate, Zvonkin's transformation and the characterization for Sobolev differentiability of random field. More recently, the critical case, i.e., $p = d$ in time-homogeneous case, $\frac{2}{q} + \frac{d}{p} = 1$ in time-inhomogeneous have been explored, see [9–12, 17, 18] and references therein.

In [4], Fang, Imkeller and Zhang obtained a global estimates by employing global decomposition of two-point motions and local estimates. In this paper, we will base on the decomposition, Krylov's estimate, Khasminskii's estimate, Zvonkin's transformation and the characterization of Sobolev differentiability of random fields to obtain the well-posedness and the weak differentiability of solutions, the strong Feller property of associated semigroups and stochastic flow property of SDEs (1.1) under the following assumptions:

($\mathbf{H^b}$) There exist two positive constants β and $\tilde{\beta}$ such that for all $R \geq 1$,

$$\left(\int_{B(R)} |b(x)|^{p_1}\, dx\right)^{\frac{1}{p_1}} \leq \beta I_b(R) + \tilde{\beta},$$

where $B(R) := \{x \in \mathbb{R}^d; |x| \leq R\}$ is a ball with center 0 and radius R, $|\cdot|$ denote the Euclidean norm, $p_1 > d$ is a constant and $I_b(R) = (\log R + 1)^{(p_1-d)^2/(2p_1^2)}$.

($\mathbf{H_1^\sigma}$) There exists a constant $\delta \in (0, 1)$ such that for all $x, \xi \in \mathbb{R}^d$,

$$\delta^{\frac{1}{2}} |\xi| \leq \left|\sigma^\top(x)\xi\right| \leq \delta^{-\frac{1}{2}} |\xi|,$$

and there exists a constant $\varpi \in (0, 1)$ such that for all $x, y \in \mathbb{R}^d$,

$$\|\sigma(x) - \sigma(y)\| \leq \delta^{-\frac{1}{2}} |x - y|^{\varpi}.$$

Here, we denote $\sigma^\top$ the transpose of matrix σ, $\|\cdot\|$ the Hilbert–Schmidt norm.

($\mathbf{H_2^\sigma}$) There exist two positive constants β and $\tilde{\beta}$ (same with ($\mathbf{H^b}$)) such that for all $R \geq 1$,

$$\left(\int_{B(R)} \|\nabla\sigma\|^{p_1}\, dx\right)^{\frac{1}{p_1}} \leq \beta I_\sigma(R) + \tilde{\beta},$$

where $\nabla\sigma := [\nabla\sigma^1, \ldots, \nabla\sigma^m]$ and $I_\sigma(R) = (\log(R/3) + 1)^{(p_1-d)^2/(2p_1^2)}$.

Our main results are given as the following theorem:

Theorem 1.1 *Under the conditions* ($\mathbf{H^b}$), ($\mathbf{H_1^\sigma}$) *and* ($\mathbf{H_2^\sigma}$)*, there exists a unique global strong solution to* (1.1)*. Moreover, we have the following conclusions:*

(A) *For all $t \in [0, T]$ and almost all ω, the mapping $x \mapsto X_t(\omega, x)$ is Sobolev differentiable and for any $p \geq 2$, there exist constants $\mathbf{C}, n > 0$ such that for Lebesgue almost all $x \in \mathbb{R}^d$,*

$$\mathbb{E}\left[\sup_{t\in[0,T]} \|\nabla X_t(x)\|^p\right] \leq \mathbf{C}(1 + |x|^n),$$

where ∇ denotes the gradient in the distributional sense.

(B) *For any $t \in [0, T]$ and any bounded measurable function f on $\mathbb{R}^d$,*

$$x \mapsto \mathbb{E}[f(X_t(x))] \text{ is continuous},$$

i.e., the semigroup $P_t f(x) := \mathbb{E}[f(X_t(x))]$ is strong Feller.

(C) *For all* $t \in [0, T]$, $x \in \mathbb{R}^d$ *and almost all* ω, *the mapping* $(t, x) \mapsto X_t(\omega, x)$ *is continuous on* $[0, T] \times \mathbb{R}^d$ *and for almost all* ω, $x \mapsto X_t(\omega, x)$ *is one-to-one on* $\mathbb{R}^d$.

These results will be proved in Sect. 6.

We would like to compare the work in [21, 24, 26] with the present paper and explain the contributions made in this paper. Following the proof of [26], we generalized [26, Theorem 3.1] to multiplicative noises (cf. Theorem 6.1). In the time-inhomogeneous case, Xie and Zhang [21] proved the weak differentiability of SDEs and the strong Feller property of the associated diffusion semigroup under local Sobolev integrability and sup-linear growth assumptions. In the present paper, we removed the sup-linear growth condition (H2) in [21] by replacing the local Sobolev integrability (H1) in [21] with stronger assumptions $(\mathbf{H^b})$, $(\mathbf{H_1^\sigma})$ and $(\mathbf{H_2^\sigma})$, proved the weak differentiability of SDEs and the strong Feller property of the associated diffusion semigroup in the time-homogeneous case. In the time-inhomogeneous case, Zhang [24] proved the solution of SDEs forms a stochastic flow of homeomorphisms under conditions:

$$|b|, \ \|\nabla\sigma\| \in L^{p_1}_{\text{loc}}(\mathbb{R}_+; L^{p_1}(\mathbb{R}^d)) \quad (p_1 > d + 2).$$

In the time-homogeneous case, the conditions will be

$$|b|, \ \|\nabla\sigma\| \in L^{p_1}(\mathbb{R}^d) \quad (p_1 > d). \tag{1.3}$$

Our main result Theorem 1.1(C) strengthens the one-to-one property of stochastic flow in [24, Theorem 1.1] by improving the conditions (1.3) with mild growth conditions $(\mathbf{H}^b)$ and $(\mathbf{H}_2^\sigma)$.

For the proof of Theorem 1.1, there are two main difficulties. The one is finer estimates depend on R is necessary for us to obtain the order of growth in $(\mathbf{H^b})$ and $(\mathbf{H_2^\sigma})$ by the decomposition of global two-point motions. By our knowledge, all existing results about Krylov's estimate and Khasminskii's estimate such as [21, 24–26] do not obviously depend on radius R.

Another difficulty is that we need an appropriate truncation for σ due to SDEs (1.1) with multiplicative noises. If we directly truncate σ by characteristic function $\mathbb{1}_{|x|\le R}$, then the truncated σ will be degenerate. Chen and Li [1] provides a truncation method which can guarantee truncated σ is not degenerate, but it seems difficult to estimate the gradient of truncated σ by $(\mathbf{H_2^\sigma})$.

We also give some remarks related to the proof of our main results and conditions posed in it.

- In Theorem 1.1, we just consider the time-homogeneous case, but by carefully tracking the proof of Theorem 1.1, our idea still work for time-inhomogeneous case.
- If the condition $(\mathbf{H_1^\sigma})$ of Theorem 1.1 is replaced by
  $(\mathbf{H_1^\sigma})_{\text{loc}}$ A constant $\delta_R \in (0, 1)$ depends on R such that for all $x \in B(R)$, $\xi \in \mathbb{R}^d$,

$$\delta_R^{\frac{1}{2}} |\xi| \le \left|\sigma^\top(x)\xi\right| \le \delta_R^{-\frac{1}{2}} |\xi|,$$

and there exist two constants $L > 0$ and $\varpi \in (0, 1)$ such that for all $x, y \in \mathbb{R}^d$,

$$\|\sigma(x) - \sigma(y)\| \le L\,|x - y|^{\varpi}\,,$$

where the growth of δ_R^{-1} is mild about R. The techniques in the proof of Theorem 1.1 still can be used. Indeed, if b and σ satisfy $\left\| |b| \cdot \mathbb{1}_{B(R)} \right\|_{p_1} \le O(\tilde{I}_b(R))$, $\left\| \|\nabla\sigma\| \cdot \mathbb{1}_{B(R)} \right\|_{p_1} \le O(\tilde{I}_b(R/3))$ and the assumption $(\mathbf{H_1^\sigma})_{\mathbf{loc}}$ holds true, then the following assumptions:

$(\mathbf{H_1^{\sigma^R}})_{\text{loc}}$ A positive constant $\tilde{\delta}_R^{-1/2} = \mathbf{C}(d, L) \cdot (\delta_R^{-1/2}) > 0$ depends on R such that for all $x, \xi \in \mathbb{R}^d$,

$$\tilde{\delta}_R^{\frac{1}{2}}\,|\xi| \le \left|(\sigma^R)^\top(x)\xi\right| \le \tilde{\delta}_R^{-\frac{1}{2}}\,|\xi|\,,$$

and for all $x, y \in \mathbb{R}^d$,

$$\left\|\sigma^R(x) - \sigma^R(y)\right\| \le \tilde{\delta}_R^{-\frac{1}{2}}\,|x - y|^{\varpi}\,.$$

$(\mathbf{H_2^{\sigma^R}})_{\text{loc}}$ There exist constants $\mathbf{C}(d, L)$ such that for all $R \ge 1$,

$$\left(\int_{\mathbb{R}^d} \left\|\nabla\sigma^R\right\|^{p_1}\,dx\right)^{\frac{1}{p_1}} \le \mathbf{C}(d, L) \cdot \tilde{\delta}_{3R}^{-\frac{1}{2}} + O(\tilde{I}_b(R)),$$

hold true, where $O(\tilde{I}_b(R))$ means there exist two constants $C > 0$ and R_0 such that $O(\tilde{I}_b(R)) \le C\tilde{I}_b(R)\ \ \forall\, R \ge R_0$. On the other hand, by going through carefully the proof of Theorem 4.1 we can find two continuous increasing functions $G_1 : \mathbb{R}_+ \to \mathbb{R}_+$ and $G_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that C_1 and C_2 in Theorem 4.1 are equal to $G_1(\tilde{\delta}_R^{-\frac{1}{2}})$ and $G_2(\tilde{\delta}_R^{-\frac{1}{2}})$. The $C_0(\tilde{\delta}_R^{-\frac{1}{2}})$ (the key to obtain G_1) in the proof of Theorem 4.1 can be obtained by changing of coordinates to reduce $L^{\sigma^R(x_0)}$ to Δ. The $C_j(\tilde{\delta}_R^{-\frac{1}{2}})$ and $k_j(\tilde{\delta}_R^{-\frac{1}{2}})$ in (7.6) (the key to obtain G_2) can be obtained by going through carefully the proof of Page 356 to Page 378 in [15]. Finally, we can take $\tilde{\delta}_{3R}^{-\frac{1}{2}}$ satisfy $\mathbf{C}(d, L) \cdot \tilde{\delta}_{3R}^{-\frac{1}{2}} \le \mathbf{C} \cdot \tilde{I}_b(R)$ and let $\lambda^R = (2G_2(\tilde{I}_b(R))\tilde{I}_b(R))^{2p_1/(p_1-d)}$ in Lemma 4.4. Tracking the proof in Theorem 1.1, we can find a concrete $\tilde{I}_b(R)$ with enough mild growth such that the results in Theorem 1.1 still hold true.

- In [24], the well-known Bismut–Elworthy–Li's formula (cf. [2]) was proved. But even if $\sigma(x) \equiv I_{d\times d}$ (in this case, we do not need to truncate σ), it seems difficult to prove the Bismut–Elworthy–Li's formula for the solution of SDEs (1.1) under assumptions of this paper due to $\mathbb{E}[\left\|\nabla X_t^R(x)\right\|^2] \le C(R)$ and $C(R) \to \infty$ when $R \to \infty$.

- The local estimates (6.23), (6.25) and (6.24) are seemingly not enough to obtain the onto property of the map $x \mapsto X_t(\omega, x)$. In fact, if we define

$$\mathscr{X}_t(x) := \begin{cases} \left(1 + \left|X_t\left(\frac{x}{|x|^2}\right)\right|\right)^{-1}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

We just can obtain for any $k \in \mathbb{N}$, $x, y \in \{x : \frac{1}{k} \leq |x| \leq 1\} \cup \{0\}$,

$$\mathbb{E}\left[|\mathscr{X}_t(x) - \mathscr{X}_t(y)|^p\right] \leq \mathbf{C}(k)\,|x-y|^p\,.$$

Notice that, the domain $\{x : \frac{1}{k} \leq |x| \leq 1\} \cup \{0\}$ is not connected, we cannot obtain $x \mapsto \mathscr{X}_t(x)$ exist a continuous version on $\{x : |x| \leq 1\}$.
- For the critical case, i.e., $p_1 = d$, our idea will not work since Zvonkin's transformation cannot be used. On the other hand, $(\mathbf{H^b})$ and $(\mathbf{H_2^\sigma})$ seemingly indicate that the order of growth will be degenerated in the critical case.

The rest of this paper is organized as follows: In Sect. 2, we will present some preliminary knowledge. In Sect. 3, we devote to construct the cutoff functions to truncate SDEs (1.1) and verify assumptions. In Sect. 4, we provide a proof of Krylov's estimate and Khasminskii's estimate. In Sect. 5, we use Zvonkin's transformation to estimate truncated SDEs (3.1). In Sect. 6, we complete the proof of the main Theorem 1.1. Finally, we give a detailed proof of Theorem 4.1 in Appendix.

2 Preliminary

In this section, we introduce some notations, function spaces and well-known theorems which will be used in this paper.

We use := as a way of definition. Let $\mathbb{N}$ be the collection of all positive integer. For any $a, b \in \mathbb{R}$, set $a \wedge b := \min\{a, b\}$ and $a \vee b := \max\{a, b\}$. We use $a \lesssim b$ to denote there is a constant C such that $a \leq Cb$, use $a \asymp b$ to denote $a \lesssim b$ and $b \lesssim a$. For functions f and g, we use $f * g$ to denote the convolution of f and g.

Let $L^p(\mathbb{R}^d)$ be L^p-space on $\mathbb{R}^d$ with norm

$$\|f\|_p := \left(\int_{\mathbb{R}^d} |f|^p\, dx\right)^{\frac{1}{p}} < +\infty, \quad \forall f \in L^p(\mathbb{R}^d).$$

Let $W^{m,p}(\mathbb{R}^d)$ be Sobolev space on $\mathbb{R}^d$ with norm

$$\|f\|_{m,p} := \sum_{i=0}^{m} \left\|\nabla^i f\right\|_p < +\infty, \quad \forall f \in W^{m,p}(\mathbb{R}^d),$$

where ∇^i denotes the i-order gradient operator.

For $0 \le \alpha \in \mathbb{R}$ and $p \in [1, +\infty)$, the Bessel potential space $H^{\alpha,p}(\mathbb{R}^d)$ is defined by

$$H^{\alpha,p} := (I - \Delta)^{-\frac{\alpha}{2}}(L^p(\mathbb{R}^d))$$

with norm

$$\|f\|_{\alpha,p} := \left\|(I - \Delta)^{\frac{\alpha}{2}} f\right\|_p, \quad \forall f \in H^{\alpha,p}(\mathbb{R}^d).$$

Let $C^\alpha(\mathbb{R}^d)$ be Hölder space on $\mathbb{R}^d$ with norm

$$\|f\|_{C^\alpha} := \sum_{i=0}^{\lfloor\alpha\rfloor} \left\|\nabla^i f\right\|_\infty + \sup_{x\neq y} \frac{\left|\nabla^{\lfloor\alpha\rfloor} f(x) - \nabla^{\lfloor\alpha\rfloor} f(y)\right|}{|x-y|^{\alpha-\lfloor\alpha\rfloor}} < +\infty, \quad \forall f \in C^\alpha(\mathbb{R}^d),$$

where $\lfloor\alpha\rfloor$ denotes the integer part of α. Let $C_0^\infty(\mathbb{R}^d)$ be a collection of all smooth function with compact support in $\mathbb{R}^d$.

For $\alpha \in (0, 2)$ and $p \in (1, +\infty)$, we have

$$\|f\|_{\alpha,p} \asymp \left\|(I - \Delta^{\frac{\alpha}{2}})f\right\| \asymp \|f\|_p + \left\|\Delta^{\frac{\alpha}{2}} f\right\|_p, \tag{2.1}$$

where $\Delta^{\frac{\alpha}{2}} := -(-\Delta)^{\frac{\alpha}{2}}$ is the fractional Laplacian.

Let f be a locally integrable function on $\mathbb{R}^d$, $\mathcal{M}$ be the Hardy–Littlewood maximal operator defined by

$$\mathcal{M}f(x) := \sup_{0<R<+\infty} \frac{1}{|B(R)|} \int_{B(R)} f(x+y)\,dy,$$

here, with a bit of abuse of notations, $|B(R)|$ denotes the volume of ball $B(R)$.

Theorem 2.1 (Sobolev embedding theorem) *If $k > l > 0$, $p < d$ and $1 \le p < q < \infty$ satisfy $k - \frac{d}{p} = l - \frac{d}{q}$, then*

$$H^{k,p}(\mathbb{R}^d) \hookrightarrow H^{l,q}(\mathbb{R}^d).$$

If $\gamma \ge 0$ and $\gamma < \alpha - \frac{d}{p}$, then

$$H^{\alpha,p}(\mathbb{R}^d) \hookrightarrow C^\gamma(\mathbb{R}^d).$$

Theorem 2.2 (Hadamard's theorem) *If a function $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ is a k-order smooth function ($k \ge 1$) and satisfy:*

(i) $\lim_{|x|\to\infty} |\varphi(x)| = \infty$;
(ii) *for all $x \in \mathbb{R}^d$, the Jacobian matrix $\nabla\varphi(x)$ is an isomorphism of $\mathbb{R}^d$;*

Then φ is a C^k-diffeomorphism of $\mathbb{R}^d$.

Theorem 2.3 (i) *There exists a constant C_d such that for all $\varphi \in C^\infty(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$,*

$$|\varphi(x) - \varphi(y)| \leq C_d \cdot |x - y| \cdot (\mathcal{M}\,|\nabla\varphi|\,(x) + \mathcal{M}\,|\nabla\varphi|\,(y)).$$

(ii) *For any $p > 1$, there exists a constant $C_{d,p}$ such that for all $\varphi \in L^p(\mathbb{R}^d)$,*

$$\left(\int_{\mathbb{R}^d} \big(\mathcal{M}\varphi(x)\big)^p \, dx\right)^{\frac{1}{p}} \leq C_{d,p} \left(\int_{\mathbb{R}^d} |\varphi(x)|^p \, dx\right)^{\frac{1}{p}}.$$

3 Truncated SDEs

In this section, we will construct some precise cutoff functions to truncate SDEs (1.1) and verify that the truncated SDEs

$$\begin{cases} dX_t^R = b^R(X_t^R)\,dt + \sigma^R(X_t^R)d\widetilde{W}_t, \quad t \in [0, T], \\ X_0^R = x \in \mathbb{R}^d, \end{cases} \tag{3.1}$$

satisfy the following assumptions:

($\mathbf{H}^{\mathbf{b}^\mathbf{R}}$) There exist two positive constants β and $\tilde{\beta}$ such that for all $R \geq 1$,

$$\left(\int_{\mathbb{R}^d} \left|b^R(x)\right|^{p_1} \, dx\right)^{\frac{1}{p_1}} \leq \beta I_b(R) + \tilde{\beta},$$

where $p_1 > d$ is a constant.

($\mathbf{H}_\mathbf{1}^{\sigma^\mathbf{R}}$) There exists a positive constant $\tilde{\delta} \in (0, 1)$ such that for all $x, \xi \in \mathbb{R}^d$,

$$\tilde{\delta}^{\frac{1}{2}} \, |\xi| \leq \left|(\sigma^R)^\top(x)\xi\right| \leq \tilde{\delta}^{-\frac{1}{2}} \, |\xi|\,,$$

and for all $x, y \in \mathbb{R}^d$,

$$\left\|\sigma^R(x) - \sigma^R(y)\right\| \leq \tilde{\delta}^{-\frac{1}{2}} \, |x - y|^{\varpi}\,, \tag{3.2}$$

where $\tilde{\delta}$ is a constant only depend on δ and d.

($\mathbf{H}_\mathbf{2}^{\sigma^\mathbf{R}}$) There exist two positive constants β and $\tilde{\beta}$ such that for all $R \geq 1$,

$$\left(\int_{\mathbb{R}^d} \left\|\nabla\sigma^R\right\|^{p_1} \, dx\right)^{\frac{1}{p_1}} \leq \Big(C(d, \delta, p_1) + (4\beta I_\sigma(3R) + 4\tilde{\beta})\Big),$$

where $p_1 > d$ is a constant and $C(d, \delta, p_1)$ is a constant only depend on d, δ and p_1.

Let $\overline{W}$ be a d-dimensional standard Wiener process, independent of W and let

$$\widetilde{W} := \begin{bmatrix} W \\ \overline{W} \end{bmatrix}.$$

We can verify that $\widetilde{W}$ is a $(d+m)$-dimensional standard Wiener process. In SDEs (3.1), the coefficients b^R and σ^R are defined by

$$b^R(x) := b(x)\mathbb{1}_{|x|\le R}, \quad \sigma^R(x) := [\rho_R\sigma, h_R\bar{\sigma}](x),$$

where $\bar{\sigma}$ is a matrix defined by

$$\bar{\sigma}(x) \equiv \begin{pmatrix} \delta^{-\frac{1}{2}} & & \\ & \ddots & \\ & & \delta^{-\frac{1}{2}} \end{pmatrix}_{d\times d}.$$

The cutoff function h_R is defined by

$$h_R(x) = \begin{cases} 0, & |x| \le R, \\ \frac{2}{R^2}(|x|-R)^2, & R \le |x| \le \frac{3R}{2}, \\ 1-\frac{2}{R^2}(|x|-2R)^2, & \frac{3R}{2} < |x| \le 2R, \\ 1, & |x| > 2R. \end{cases}$$

It is easy to verify h_R satisfy

$$h_R(x) = \begin{cases} 0, & |x| \le R, \\ \in (0,1) & R < |x| \le 2R, \\ 1 & |x| > 2R, \end{cases} \quad |\nabla h_R|(x) = \begin{cases} 0, & |x| \le R, \\ \le \frac{2}{R} & R < |x| \le 2R, \\ 0 & |x| > 2R. \end{cases}$$

Similarly, we can construct a cutoff function ρ_R satisfy

$$\rho_R(x) = \begin{cases} 1, & |x| \le 2R, \\ \in (0,1) & 2R < |x| \le 3R, \\ 0 & |x| > 3R, \end{cases} \quad |\nabla \rho_R|(x) = \begin{cases} 0, & |x| \le 2R, \\ \le \frac{2}{R} & 2R < |x| \le 3R, \\ 0 & |x| > 3R. \end{cases}$$

Clearly, ($\mathbf{H^{b^R}}$) hold by the definition of b^R. Notice that

$$\langle \sigma^R(\sigma^R)^\top\xi, \xi\rangle = \rho_R^2\langle\sigma\sigma^\top\xi,\xi\rangle + h_R^2\langle\bar{\sigma}\bar{\sigma}^\top\xi,\xi\rangle,$$

by the definitions of $\rho_R, h_R, \bar{\sigma}$ and assumption ($\mathbf{H_1^\sigma}$), we have

$$\frac{1}{2}\delta|\xi|^2 \le \langle\sigma^R(\sigma^R)^\top\xi,\xi\rangle \le 2\delta^{-1}|\xi|^2, \quad \forall \xi \in \mathbb{R}^d. \tag{3.3}$$

On the other hand, it is easy to see for all $x, y \in B(2R)\backslash B(R)$,

$$|h_R(x) - h_R(y)| \leq \frac{2}{R}|x-y| \leq \frac{2}{R}(4R)^{1-\varpi}|x-y|^{\varpi} \leq 8|x-y|^{\varpi}, \quad \forall R \geq 1,$$

and for all $x, y \notin B(2R)\backslash B(R)$, we have $|h_R(x) - h_R(y)| \leq |x-y|^{\varpi}$, $\forall R \geq 1$. Hence, for all $x, y \in \mathbb{R}^d$, we obtain

$$|h_R(x) - h_R(y)| \leq 8|x-y|^{\varpi}, \quad \forall R \geq 1. \tag{3.4}$$

Similarly, we can obtain

$$|\rho_R(x) - \rho_R(y)| \leq 12|x-y|^{\varpi}, \quad \forall R \geq 1. \tag{3.5}$$

Therefore, we have

$$\begin{aligned}
&\left\|\sigma^R(x) - \sigma^R(y)\right\| \\
&\leq |\rho_R(x) - \rho_R(y)|\,\|\sigma(x)\| + |\rho_R(y)|\,\|\sigma(x) - \sigma(y)\| + \|\bar{\sigma}\|\,|h_R(x) - h_R(y)| \\
&\leq \left(12\delta^{-\frac{1}{2}}d^{\frac{1}{2}} + \delta^{-\frac{1}{2}} + 8\delta^{-\frac{1}{2}}d^{\frac{1}{2}}\right)|x-y|^{\varpi},
\end{aligned} \tag{3.6}$$

where the last inequality is due to (3.4) and (3.5). Combining (3.3) with (3.6), we verified the $(\mathbf{H_1^{\sigma^R}})$.

By the definition $\sigma^R = [\rho_R\sigma, h_R\bar{\sigma}]$ and direct computation, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d}\left\|\nabla\sigma^R\right\|^{p_1}dx = \int_{\mathbb{R}^d}\|\nabla[\rho_R\,\sigma, h_R\,\bar{\sigma}]\|^{p_1}\,dx \\
&= \int_{\mathbb{R}^d}\|[\nabla\rho_R(x)\,\sigma(x) + \rho_R(x)\,\nabla\sigma(x), \nabla h_R(x)\,\bar{\sigma}(x) + h_R(x)\,\nabla\bar{\sigma}(x)]\|^{p_1}\,dx \\
&\leq 4^{p_1}\left\{\int_{B(3R)\backslash B(2R)}\|\nabla\rho_R(x)\sigma(x)\|^{p_1}\,dx + \int_{B(2R)\backslash B(R)}\|\nabla h_R(x)\bar{\sigma}(x)\|^{p_1}\,dx\right. \\
&\quad\left. + \int_{B(3R)}\|\nabla\sigma\|^{p_1}\,dx\right\} \\
&:= 4^{p_1}(J_1 + J_2 + J_3).
\end{aligned}$$

Note that $|\nabla \rho_R| \leq \frac{2}{R}$ in $B(3R)\backslash B(2R)$, $|\nabla h_R| \leq \frac{2}{R}$ in $B(2R)\backslash B(R)$ and $(\mathbf{H_2^\sigma})$, there exists a constant $C(d, \delta, p_1)$ only depend on d, δ and p_1 such that for all $R \geq 1$,

$$
\begin{aligned}
J_1 &\leq \int_{B(3R)\backslash B(2R)} \left(\frac{2}{R}\delta^{-\frac{1}{2}} d^{\frac{1}{2}}\right)^{p_1} dx \leq C(d, \delta, p_1) R^{d-p_1} \leq C(d, \delta, p_1), \\
J_2 &\leq \int_{B(2R)\backslash B(R)} \left(\frac{2}{R}\delta^{-\frac{1}{2}} d^{\frac{1}{2}}\right)^{p_1} dx \leq C(d, \delta, p_1) R^{d-p_1} \leq C(d, \delta, p_1), \\
J_3 &\leq \int_{B(3R)} \|\nabla \sigma(x)\|^{p_1} dx \leq (\beta I_\sigma(3R) + \tilde{\beta})^{p_1}.
\end{aligned}
$$

Together, J_1, J_2 and J_3 imply $(\mathbf{H_2^{\sigma^R}})$.

4 Krylov's Estimate and Khasminskii's Estimate

In this section, we shall prove Krylov's estimate and Khasminskii's estimate. We need the following result about elliptic PDEs (4.1).

Theorem 4.1 *Suppose σ^R satisfies $(\mathbf{H_1^{\sigma^R}})$, $p \in (1, \infty)$, then for any $f \in L^p(\mathbb{R}^d)$, there exists a unique $u \in W^{2,p}(\mathbb{R}^d)$ such that*

$$
L^{\sigma^R(x)} u - \lambda u = f, \tag{4.1}
$$

where

$$
L^{\sigma^R(x)} u(x) := \frac{1}{2} \sum_{ijk} (\sigma^R)_{ik}(x)(\sigma^R)_{jk}(x)\partial_i \partial_j u(x)
$$

and $\lambda > C$ ($C = C(d, \varpi, \tilde{\delta}, p) \geq 2$ is a constant). Furthermore, for a $C_1 = C_1(d, \varpi, \tilde{\delta}, p) > 0$,

$$
\|u\|_{2,p} \leq C_1 \|f\|_p. \tag{4.2}
$$

Moreover, for any $\alpha \in [0, 2)$ and $p' \in [1, \infty]$ with $\frac{d}{p} < 2 - \alpha + \frac{d}{p'}$,

$$
\|u\|_{\alpha, p'} \leq C_2 \lambda^{\left(\alpha - 2 + \frac{d}{p} - \frac{d}{p'}\right)/2} \|f\|_p,
$$

where $C_1(d, \varpi, \tilde{\delta}, p)$ and $C_2(d, \varpi, \tilde{\delta}, p, \alpha, p') > 0$ are both independent of λ.

We believe that Theorem 4.1 is standard although we do not find them in any reference. In [26], authors proved Theorem 4.1 hold true when $\sigma^R \equiv I$. For convenience of the reader, we combine [26] with [25] to give a detailed proof in Appendix.

In order to prove Krylov's estimate and Khasminskii's estimate, we need to solve the following elliptic equation:

$$(L^{\sigma^R(x)} - \lambda)u^R + b^R \cdot \nabla u^R = f, \quad \lambda \geq \lambda^{b^R}, \tag{4.3}$$

where $f \in L^p(\mathbb{R}^d)$ and $\lambda^{b^R} > 1$ is a constant depend on C_2, d, p_1 and $\left\|b^R\right\|_{p_1}$.

Lemma 4.2 *If $\left\|b^R\right\|_{p_1} < \infty$ and $(\mathbf{H}_1^{\sigma^{\mathbf{R}}})$ hold, then for any $p \in (\frac{d}{2} \vee 1, p_1]$, we can find a constant*

$$\lambda^{b^R} = \left(2C_2 \left\|b^R\right\|_{p_1}\right)^{2\left(1-\frac{d}{p_1}\right)^{-1}}$$

such that for any $f \in L^p(\mathbb{R}^d)$, there exists a unique solution $u^R \in W^{2,p}(\mathbb{R}^d)$ to Eq. (4.3) and

$$\left\|u^R\right\|_{2,p} \leq 2C_1 \|f\|_p, \quad \lambda^{\left(2-\alpha+\frac{d}{p'}-\frac{d}{p}\right)/2} \left\|u^R\right\|_{\alpha,p'} \leq 2C_2 \|f\|_p \ \ (\lambda \geq \lambda^{b^R}),$$

where C_1 and C_2 are two constants in Theorem 4.1, $\alpha \in [0, 2)$ and $p' \in [1, \infty]$ with $(2 - \alpha + \frac{d}{p'} - \frac{d}{p}) > 0$.

Proof By Theorem 4.1, for any $\tilde{f} \in L^p(\mathbb{R}^d)$, we have

$$\begin{aligned} \left\|(\lambda - L^{\sigma^R(x)})^{-1}\tilde{f}\right\|_{2,p} &\leq C_1 \left\|\tilde{f}\right\|_p, \\ \lambda^{\left(2-\alpha+\frac{d}{p'}-\frac{d}{p}\right)/2} \left\|(\lambda - L^{\sigma^R(x)})^{-1}\tilde{f}\right\|_{\alpha,p'} &\leq C_2 \left\|\tilde{f}\right\|_p, \end{aligned} \tag{4.4}$$

where $\lambda > C$ $(C > 2)$, $(2 - \alpha + \frac{d}{p'} - \frac{d}{p}) > 0$ and C_1, C_2 do not depend on λ.

Since $\lambda^{b^R} = (2C_2 \left\|b^R\right\|_{p_1})^{2p_1/(p_1-d)}$, it is easy to see for any $\lambda \geq \lambda^{b^R}$,

$$C_2 \lambda^{\left(\frac{d}{p_1}-1\right)/2} \left\|b^R\right\|_{p_1} \leq \frac{1}{2}.$$

Let $u_0 = 0$ and for $n \in \mathbb{N}$ define

$$u_n^R := (L^{\sigma^R(x)} - \lambda)^{-1}(f - b^R \cdot \nabla u_{n-1}^R).$$

By (4.4) and replacing $(\Delta - \lambda)^{-1}$ with $(L^{\sigma^R(x)} - \lambda)^{-1}$ in the proof of [26, Theorem 3.3 (ii)], we completed the proof. □

Now, we provide the main result of this section.

Theorem 4.3 *If* $\left\|b^R\right\|_{p_1} < \infty$ *and* $(\mathbf{H}_1^{\sigma^{\mathbf{R}}})$ *hold and* $\{X_s^R\}_{s\in[0,T]}$ *is a solution of SDE* (3.1)*, then for any* $0 \le t_0 < t_1 \le T$, $f \in L^p(\mathbb{R}^d)$ $(p > \frac{d}{2} \vee 1)$*, we have*

$$\mathbb{E}^{\mathscr{F}_{t_0}}\left[\int_{t_0}^{t_1} f(X_s^R(x))\,ds\right] \le 4C_2\left([T\lambda^{b^R}]^{\frac{d}{2p}} + [T\lambda^{b^R}]^{\frac{d}{2p}-1}\right)(t_1-t_0)^{1-\frac{d}{2p}}\,\|f\|_p\,, \tag{4.5}$$

where C_2 *is the constant in Theorem* 4.1, $\lambda^{b^R} = (2C_2\left\|b^R\right\|_{p_1})^{2p_1/(p_1-d)}$*. Moreover, for any* $a > 0$ *we have*

$$\begin{aligned}\mathbb{E}\left[\exp\left(a\int_0^T \left|f(X_s^R(x))\right|\,ds\right)\right] \le e\\ \cdot\exp\left(T\left[\frac{4aC_2\left([T\lambda^{b^R}]^{\frac{d}{2p}} + [T\lambda^{b^R}]^{\frac{d}{2p}-1}\right)\|f\|_p}{1-e^{-1}}\right]^{\left(1-\frac{d}{2p}\right)^{-1}}\right).\end{aligned}$$

Proof The proof is divided into three steps.

Step (i) We replace $(\Delta-\lambda)^{-1}$ with $(L^{\sigma^R(x)}-\lambda)^{-1}$ in the proof of Theorem 3.4 of Zhang and Zhao [26]. Notice that

$$\lambda^{b^R} = \left(2C_2\left\|b^R\right\|_{p_1}\right)^{2\left(1-\frac{d}{p_1}\right)^{-1}}$$

is enough to ensure $C_2\lambda^{(d/p_1-1)/2}\left\|b^R\right\|_{p_1} \le \frac{1}{2}$ for all $\lambda \ge \lambda^{b^R}$. Repeating the proof of Theorem 3.4 (ii) of Zhang and Zhao [26], for all $\tilde\lambda \ge \lambda^{b^R}$, we obtain

$$\begin{aligned}\mathbb{E}^{\mathscr{F}_{t_0}}\left[\int_{t_0}^{t_1} f(X_s^R(x))\,ds\right] &\le \tilde\lambda(t_1-t_0)\left\|u^R\right\|_\infty + 2\left\|u^R\right\|_\infty\\ &\le 2C_2(t_1-t_0)\tilde\lambda^{\frac{d}{2p}}\|f\|_p + 4C_2\tilde\lambda^{\left(\frac{d}{2p}-1\right)}\|f\|_p.\end{aligned} \tag{4.6}$$

Let $\kappa = T\lambda^{b^R}$ and $\tilde\lambda = \kappa(t_1-t_0)^{-1}$. Due to $0 \le t_0 < t_1 \le T$, we have $\tilde\lambda \ge \lambda^{b^R}$. Taking $\tilde\lambda = \kappa(t_1-t_0)^{-1}$ into (4.6), we proved the Krylov's estimate (4.5).

Step (ii) Taking $0 \le t_0 < t_1 < \infty$ satisfy

$$t_1 - t_0 = \left(\frac{1-e^{-1}}{4aC_2\left(\kappa^{\frac{d}{2p}} + \kappa^{\frac{d}{2p}-1}\right)\|f\|_p}\right)^{\left(1-\frac{d}{2p}\right)^{-1}}. \tag{4.7}$$

If $t_1 - t_0 \le T$ in (4.7), by the Corollary 3.5 in Zhang and Zhao [26], we have

$$\mathbb{E}^{\mathscr{F}_{t_0}}\left[\left(\int_{t_0}^{t_1}\left|f(X_s^R(x))\right|ds\right)^n\right] \le n!\left(\frac{1-e^{-1}}{a}\right)^n.$$

Since $e^x = \sum_{n=0}^{\infty}\frac{1}{n!}x^n$, we have

$$\begin{aligned}
&\mathbb{E}^{\mathscr{F}_{t_0}}\left[\exp\left\{a\int_{t_0}^{t_1}\left|f(X_s^R(x))\right|ds\right\}\right]\\
&=\mathbb{E}^{\mathscr{F}_{t_0}}\left[\sum_{n=0}^{\infty}\frac{1}{n!}\left(a\int_{t_0}^{t_1}\left|f(X_s^R(x))\right|ds\right)^n\right]\\
&=\sum_{n=0}^{\infty}\frac{1}{n!}\mathbb{E}^{\mathscr{F}_{t_0}}\left[\left(a\int_{t_0}^{t_1}\left|f(X_s^R(x))\right|ds\right)^n\right]\\
&\le\sum_{n=0}^{\infty}(1-e^{-1})^n = e.
\end{aligned}\tag{4.8}$$

Step (iii) Finally, by virtual of the estimate (4.8), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\exp\left\{a\int_0^T\left|f(X_s^R(x))\right|ds\right\}\right]\\
&\le\mathbb{E}\left[\exp\left\{a\sum_{i=1}^{\lfloor M\rfloor+1}\int_{t_{i-1}}^{t_i}\left|f(X_s^R(x))\right|ds\right\}\right]\\
&=\mathbb{E}\left[\prod_{i=1}^{\lfloor M\rfloor+1}\exp\left\{a\int_{t_{i-1}}^{t_i}\left|f(X_s^R(x))\right|ds\right\}\right]\\
&=\mathbb{E}\left[\prod_{i=1}^{\lfloor M\rfloor}\exp\left\{a\int_{t_{i-1}}^{t_i}\left|f(X_s^R(x))\right|ds\right\}\right.\\
&\qquad\left.\times\mathbb{E}^{\mathscr{F}_{t_{\lfloor M\rfloor}}}\left[\exp\left\{a\int_{t_{\lfloor M\rfloor}}^{t_{\lfloor M\rfloor+1}}\left|f(X_s^R(x))\right|ds\right\}\right]\right]\\
&\le e\cdot\mathbb{E}\left[\prod_{i=1}^{\lfloor M\rfloor}\exp\left\{a\int_{t_{i-1}}^{t_i}\left|f(X_s^R(x))\right|ds\right\}\right]\le e^{M+1},
\end{aligned}$$

where $M = \frac{T}{t_1-t_0}$ and $0 \le t_0 < t_1 < \cdots < t_{\lfloor M\rfloor+1} = T$ satisfies $t_0 - 0 \le t_1 - t_0$, $t_i - t_{i-1} = t_1 - t_0$ $(i = 1, \ldots, \lfloor M\rfloor + 1)$.

If $t_1 - t_0 > T$ in (4.7), it is obvious that

$$\mathbb{E}\left[\int_0^T f(X_s^R(x)\,ds)\right] \le \frac{1-e^{-1}}{a},$$

by a similar argument, we have

$$\mathbb{E}\left[\exp\left\{a\int_0^T \left|f(X_s^R(x))\right|\,ds\right\}\right] \le e.$$

We completed the proof. □

In particular, in the proofs of Lemma 4.4 and Theorem 4.5, replacing λ^{b^R} with $\lambda^R = \left(4C_2^2(\beta I_b(R)+\tilde{\beta})^2\right)^{p_1/(p_1-d)}$, we can obtain the following lemma and theorem:

Lemma 4.4 *If* $(\mathbf{H}^{\mathbf{b}^{\mathbf{R}}})$ *and* $(\mathbf{H}_{\mathbf{1}}^{\sigma^{\mathbf{R}}})$ *hold, then for any* $p \in (\frac{d}{2} \vee 1, p_1]$*, we can find a constant*

$$\lambda^R = \left(4C_2^2(\beta I_b(R)+\tilde{\beta})^2\right)^{\left(1-\frac{d}{p_1}\right)^{-1}} \tag{4.9}$$

such that for any $f \in L^p(\mathbb{R}^d)$*, there exists a unique solution* $u^R \in W^{2,p}(\mathbb{R}^d)$ *to Eq. (4.3) and*

$$\left\|u^R\right\|_{2,p} \le 2C_1\,\|f\|_p\,, \quad \lambda^{(2-\alpha+\frac{d}{p'}-\frac{d}{p})/2}\left\|u^R\right\|_{\alpha,p'} \le 2C_2\,\|f\|_p \ \ (\lambda \ge \lambda^R),$$

where C_1 *and* C_2 *are two constants in Theorem 4.1,* $\alpha \in [0, 2)$ *and* $p' \in [1, \infty]$ *with* $(2-\alpha+\frac{d}{p'}-\frac{d}{p}) > 0$.

Theorem 4.5 *If* $(\mathbf{H}^{\mathbf{b}^{\mathbf{R}}})$ *and* $(\mathbf{H}_{\mathbf{1}}^{\sigma^{\mathbf{R}}})$ *hold and* $\{X_s^R\}_{s\in[0,T]}$ *is a solution of SDE (3.1), then for any* $0 \le t_0 < t_1 \le T$*,* $f \in L^p(\mathbb{R}^d)$ $(p > \frac{d}{2} \vee 1)$*, we have*

$$\mathbb{E}^{\mathscr{F}_{t_0}}\left[\int_{t_0}^{t_1} f(X_s^R(x))\,ds\right] \le 4C_2\left([T\lambda^R]^{\frac{d}{2p}} + [T\lambda^R]^{\frac{d}{2p}-1}\right)(t_1-t_0)^{1-\frac{d}{2p}}\,\|f\|_p\,, \tag{4.10}$$

where C_2 *is the constant in Theorem 4.1,* $\lambda^R = \left(4C_2^2(\beta I_b(R)+\tilde{\beta})^2\right)^{p_1/(p_1-d)}$*. Moreover, for any* $a > 0$ *we have*

$$\mathbb{E}\left[\exp\left(a\int_0^T \left|f(X_s^R(x))\right|\,ds\right)\right] \le e\cdot\exp\left(T\left[\frac{4aC_2\left([T\lambda^R]^{\frac{d}{2p}} + [T\lambda^R]^{\frac{d}{2p}-1}\right)\|f\|_p}{1-e^{-1}}\right]^{\left(1-\frac{d}{2p}\right)^{-1}}\right). \tag{4.11}$$

Corollary 4.6 (Generalized Itô's formula) *If* $(\mathbf{H}^{\mathbf{b^R}})$ *and* $(\mathbf{H}_1^{\sigma^R})$ *hold and* $\{X_s^R\}_{s\in[0,T]}$ *is a solution of SDE* (3.1)*, then for any* $f \in W^{2,p}(\mathbb{R}^d)$ *with* $p > \frac{d}{2} \vee 1$*, we have*

$$f(X_t^R) = f(x) + \int_0^t (L^{\sigma^R(x)} f + b^R \cdot \nabla f)(X_s^R)\, ds + \int_0^t \langle \nabla f(X_s^R), \sigma^R(X_s^R)\, d\widetilde{W}_s \rangle. \tag{4.12}$$

Proof We just need to consider the case $p \in (d, p_1]$ since $W^{2,p} \hookrightarrow W^{2,p_1}$ when $p > p_1$.

By Hölder's inequality and Sobolev's embedding theorem, we have

$$\left\| L^{\sigma^R(x)} f + b^R \cdot \nabla f \right\|_p \lesssim \|f\|_{2,p} + \left\| b^R \right\|_{p_1} \|\nabla f\|_{\frac{p_1 p}{p_1 - p}} \lesssim \|f\|_{2,p}. \tag{4.13}$$

Let φ be a nonnegative smooth function with compact support in the unit ball of $\mathbb{R}^d$ and $\int_{\mathbb{R}^d} \varphi(x)\, dx = 1$. Set $\varphi_n(x) := n^d \varphi(nx)$, $f_n := f * \varphi_n$ and applying Itô formula to f_n. By (4.13), we have

$$\left\| L^{\sigma^R(x)}(f - f_n) + b^R \cdot \nabla (f - f_n) \right\|_p \lesssim \|f - f_n\|_{2,p} \to 0. \tag{4.14}$$

Let $\bar{p} = \frac{dp}{2(d-p)}$, we have

$$\begin{aligned} &\mathbb{E} \left| \int_0^t \langle (\nabla f(X_s^R) - \nabla f_n(X_s^R)), \sigma^R(X_s^R)\, d\widetilde{W}_s \rangle \right|^2 \\ &\lesssim \left\| \sigma^R \right\|_\infty^2 \mathbb{E} \int_0^t \left| \nabla f(X_s^R) - \nabla f_n(X_s^R) \right|^2 ds \\ &\lesssim \left\| |\nabla f - \nabla f_n|^2 \right\|_{\bar{p}} \lesssim \|f - f_n\|_{1,2\bar{p}}^2 \\ &\lesssim \|f - f_n\|_{2,p}^2 \to 0, \end{aligned} \tag{4.15}$$

where the second inequality is due to Krylov's estimate (4.10) and the last inequality is due to Sobolev's embedding theorem. Together, (4.14) and (4.15) imply (4.12). □

5 Zvonkin's Transformation

Let u^R solve the following PDE

$$(L^{\sigma^R(x)} - \lambda) u^R + b^R \cdot \nabla u^R = -b^R.$$

By Lemma 4.4, we have

$$\left\| u^R \right\|_{2,p_1} \le 2C_1 \left\| b^R \right\|_{p_1}, \quad \lambda^{\left(1 - \frac{d}{p_1}\right)/2} \left\| u^R \right\|_{1,\infty} \le 2C_2 \left\| b^R \right\|_{p_1} \ (\lambda \ge \lambda^R). \tag{5.1}$$

Let $\lambda_H^R = \gamma\lambda^R$ and $\gamma^{(\frac{d}{2p_1}-\frac{1}{2})} = \frac{1}{2}$, it is easy to check

$$\left\|\nabla u^R\right\|_\infty \leq \left\|u^R\right\|_{1,\infty} \leq \gamma^{\left(\frac{d}{2p_1}-\frac{1}{2}\right)} = \frac{1}{2}. \tag{5.2}$$

Define

$$\Phi_R(x) := x + u^R(x),$$

then

$$L^{\sigma^R(x)}\Phi_R + b^R \cdot \nabla\Phi_R = \lambda u^R.$$

By (5.2), for all $\lambda \geq \lambda_H^R$, we have

$$\left\|u^R\right\|_\infty \leq \frac{1}{2}, \quad \left\|\nabla u^R\right\|_\infty \leq \frac{1}{2}. \tag{5.3}$$

By the definition of $\Phi_R(x)$ and (5.3), we have

$$\lim_{|x|\to\infty} |\Phi_R(x)| = \infty, \quad \frac{1}{2}|x-y| \leq |\Phi_R(x) - \Phi_R(y)| \leq 2|x-y|.$$

Therefore, by Theorem 2.2, we obtain $\Phi_R : \mathbb{R}^d \to \mathbb{R}^d$ is a C^1-diffeomorphism and

$$\|\nabla\Phi_R\|_\infty \leq 2, \quad \left\|\nabla\Phi_R^{-1}\right\|_\infty \leq 2. \tag{5.4}$$

Theorem 5.1 *Let $Y_t^R := \Phi_R(X_t^R)$, then X_t^R solve equation (3.1) if and only if Y_t^R solves*

$$\begin{cases} dY_t^R = \tilde{b}^R(Y_t^R)\,dt + \tilde{\sigma}^R(Y_t^R)\,d\widetilde{W}_t, \quad t \in [0,T], \\ Y_0^R = \Phi_R(x), \end{cases} \tag{5.5}$$

where $\tilde{b}^R(y) := \lambda u^R \circ \Phi_R^{-1}(y)$ and $\tilde{\sigma}^R(y) := (\nabla\Phi_R(\cdot)\sigma^R(\cdot)) \circ \Phi_R^{-1}(y)$.

Proof Applying Itô formula (4.12) to $\Phi_R(X_t^R)$, we obtain

$$\Phi_R(X_t^R) = \Phi_R(x) + \lambda\int_0^t u^R(X_s^R)\,ds + \int_0^t \nabla\Phi_R(X_s^R)\sigma^R(X_s^R)\,d\widetilde{W}_s.$$

Noticing that $Y_t^R = \Phi_R(X_t^R)$, we obtain Y_t^R solves (5.5). Similarly, applying Itô formula (4.12) to $\Phi_R^{-1}(Y_t^R)$, we completed the proof. □

6 The Proof of Theorem 1.1

Proof In this section, the letters $\mathbf{C}$ and $\widetilde{\mathbf{C}}$ will denote some unimportant constant whose value is independent of R and may change in different places. Whose dependence on parameters can be traced from the context. We also use $\mathbf{C}(T)$ and $\mathbf{C}(N)$ to emphasize the constant $\mathbf{C}$ depend on T and N, respectively.

Firstly, we prove SDE (3.1) exists a unique strong solution.

Theorem 6.1 *Under* $(\mathbf{H}_{\mathbf{1}}^{\mathbf{b}^{\mathbf{R}}})$, $(\mathbf{H}_{\mathbf{1}}^{\sigma^{\mathbf{R}}})$ *and* $(\mathbf{H}_{\mathbf{2}}^{\sigma^{\mathbf{R}}})$, *for all* $x \in \mathbb{R}^d$, *the SDE* (3.1) *exists a unique strong solution.*

Proof By Theorem 5.1, we only need to prove SDE (5.5) exists a unique strong solution. By the definition of $\tilde{b}^R$, $\tilde{\sigma}^R$ and Lemma 4.4, for all $\lambda \geq \lambda_H^R$, we have

$$\begin{aligned}
&\left\|\tilde{b}^R\right\|_\infty \leq \frac{1}{2}\lambda, \quad \left\|\nabla\tilde{b}^R\right\|_\infty \leq \lambda, \\
&\left\|\tilde{\sigma}^R\right\|_\infty \leq 2\left\|\sigma^R\right\|_\infty, \quad \left\|\nabla\tilde{\sigma}^R\right\|_{p_1} \leq C\left(\left\|b^R\right\|_{p_1} + \left\|\nabla\sigma^R\right\|_{p_1}\right)
\end{aligned} \tag{6.1}$$

Note that $\tilde{b}^R$ and $\tilde{\sigma}^R$ are both continuous and bounded. By Yamada–Watanabe's theorem, we only need to show the pathwise uniqueness. Performing the same procedure in [26, Theorem 3.1], we completed the proof. □

Lemma 6.2 *Under* $(\mathbf{H}^{\mathbf{b}^{\mathbf{R}}})$, $(\mathbf{H}_{\mathbf{1}}^{\sigma^{\mathbf{R}}})$ *and* $(\mathbf{H}_{\mathbf{2}}^{\sigma^{\mathbf{R}}})$, *let* $\{X_s^R(x)\}_{s\in[0,T]}$ *and* $\{X_s^R(y)\}_{s\in[0,T]}$ *be two solutions of SDE* (3.1) *with initial conditions* $X_0^R(x) = x$ *and* $X_0^R(y) = y$, *respectively, then for any* $\alpha \in \mathbb{R}$, *we have*

$$\mathbb{E}\left[\left|X_t^R(x) - X_t^R(y)\right|^\alpha\right] \leq \widetilde{\mathbf{C}}\left(\exp\left(\widetilde{\mathbf{C}}\,(\lambda^R)^{\frac{p_1}{p_1-d}}\right)\right)|x-y|^\alpha, \tag{6.2}$$

$$\mathbb{E}\left[\left(1 + \left|X_t^R(x)\right|^2\right)^\alpha\right] \leq \widetilde{\mathbf{C}}\left(\exp\left(\widetilde{\mathbf{C}}\,\lambda^R\right)\right)\left(1 + |x|^2\right)^\alpha, \tag{6.3}$$

and for all $p \geq 2$,

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\left|X_s^R(x)\right|^p\right] \leq \widetilde{\mathbf{C}}\,(1 + |x|^p + (\lambda^R)^p), \tag{6.4}$$

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\left|X_s^R(x) - X_s^R(y)\right|^p\right] \leq \widetilde{\mathbf{C}}\left(\exp\left(\widetilde{\mathbf{C}}\,(\lambda^R)^{\frac{p_1}{p_1-d}}\right)\right)|x-y|^p, \tag{6.5}$$

where $\widetilde{\mathbf{C}}$ *is independent of* β, $\tilde{\beta}$ *and* R.

Proof For $\Phi_R(x) \neq \Phi_R(y)$, take $0 < \varepsilon < |\Phi_R(x) - \Phi_R(y)|$ and set

$$\tau_\varepsilon := \inf\left\{\left|Y_t^R(\Phi_R(x)) - Y_t^R(\Phi_R(y))\right| \leq \epsilon\right\}.$$

For convenience, we define $Z_t^R := Y_t^R(\Phi_R(x)) - Y_t^R(\Phi_R(y))$ where $\{Y_s^R(\Phi_R(x))\}_{s\in[0,T]}$ and $\{Y_s^R(\Phi_R(y))\}_{s\in[0,T]}$ are the solutions of SDE (5.5) with initial conditions $Y_0^R(\Phi_R(x)) = \Phi_R(x)$ and $Y_0^R(\Phi_R(y)) = \Phi_R(y)$, respectively.

By Itô formula, we have

$$\begin{aligned}\left|Z_{t\wedge\tau_\varepsilon}^R\right|^\alpha &= |\Phi_R(x) - \Phi_R(y)|^\alpha \\ &\quad + \int_0^{t\wedge\tau_\varepsilon} \alpha \left|Z_s^R\right|^{\alpha-2} \langle Z_s^R, (\tilde\sigma^R(Y_s^R(x)) - \tilde\sigma^R(Y_s^R(y)))\, d\widetilde{W}_s\rangle \\ &\quad + \int_0^{t\wedge\tau_\varepsilon} \alpha \left|Z_s^R\right|^{\alpha-2} \langle Z_s^R, (\tilde b^R(Y_s^R(x)) - \tilde b^R(Y_s^R(y)))\rangle\, ds \\ &\quad + \int_0^{t\wedge\tau_\varepsilon} \frac{\alpha}{2} \left|Z_s^R\right|^{\alpha-2} \left\|\tilde\sigma^R(Y_s^R(x)) - \tilde\sigma^R(Y_s^R(y))\right\|^2 ds \\ &\quad + \int_0^{t\wedge\tau_\varepsilon} \frac{\alpha(\alpha-2)}{2} \left|Z_s^R\right|^{\alpha-4} \left|(\tilde\sigma^R(Y_s^R(x)) - \tilde\sigma^R(Y_s^R(y)))^\top Z_s^R\right|^2 ds.\end{aligned} \tag{6.6}$$

Set

$$\mathbf{B}_s := \frac{\alpha\big(\tilde\sigma^R(Y_s^R(x)) - \tilde\sigma^R(Y_s^R(y))\big)^\top Z_s^R}{\left|Z_s^R\right|^2} \tag{6.7}$$

and

$$\begin{aligned}\mathbf{A}_s &:= \frac{\alpha\langle Z_s^R, (\tilde b^R(Y_s^R(x)) - \tilde b^R(Y_s^R(y)))\rangle}{\left|Z_s^R\right|^2} + \frac{\frac{\alpha}{2}\left\|\tilde\sigma^R(Y_s^R(x)) - \tilde\sigma^R(Y_s^R(y))\right\|^2}{\left|Z_s^R\right|^2} \\ &\quad + \frac{\frac{\alpha(\alpha-2)}{2}\left|\tilde\sigma^R(Y_s^R(x)) - \tilde\sigma^R(Y_s^R(y)))^\top Z_s^R\right|^2}{\left|Z_s^R\right|^4}.\end{aligned} \tag{6.8}$$

By (6.6), we have

$$\left|Z_{t\wedge\tau_\varepsilon}^R\right|^\alpha = |\Phi_R(x) - \Phi_R(y)|^\alpha + \int_0^{t\wedge\tau_\varepsilon} \left|Z_{s\wedge\tau_\varepsilon}^R\right|^\alpha \left(\mathbf{A}_s\, ds + \mathbf{B}_s\, d\widetilde{W}_s\right).$$

By the Doléans–Dade's exponential, we obtain

$$\begin{aligned}\left|Z_{t\wedge\tau_\varepsilon}^R\right|^\alpha &= |\Phi_R(x) - \Phi_R(y)|^\alpha \exp\left(\int_0^{t\wedge\tau_\varepsilon} \mathbf{B}_s\, d\widetilde{W}_s\right. \\ &\qquad \left. -\frac{1}{2}\int_0^{t\wedge\tau_\varepsilon} |\mathbf{B}_s|^2\, ds + \int_0^{t\wedge\tau_\varepsilon} \mathbf{A}_s\, ds\right).\end{aligned} \tag{6.9}$$

By the definitions of $\tilde{b}^R$ and $\tilde{\sigma}^R$ in Theorem 5.1 and Lemma 2.3(i), it is easy to see

$$\begin{aligned}\left|\tilde{\sigma}^R(x)-\tilde{\sigma}^R(y)\right| &\le C_d\,|x-y|\left(\mathcal{M}\left|\nabla\sigma^R\right|(\Phi_R^{-1}(x))+\mathcal{M}\left|\nabla\sigma^R\right|(\Phi_R^{-1}(y))\right)\\ &\quad + C_d\,|x-y|\left(\mathcal{M}\left|\nabla^2u^R\right|(\Phi_R^{-1}(x))+\mathcal{M}\left|\nabla^2u^R\right|(\Phi_R^{-1}(y))\right),\end{aligned} \tag{6.10}$$

and

$$\begin{aligned}\left|\tilde{b}^R(x)-\tilde{b}^R(y)\right| &= \left|\lambda u^R\circ\Phi_R^{-1}(x)-\lambda u^R\circ\Phi_R^{-1}(y)\right|\\ &\le \lambda C_d\left|\Phi_R^{-1}(x)-\Phi_R^{-1}(y)\right|\\ &\quad\times\left(\mathcal{M}\left|\nabla u^R\right|(\Phi_R^{-1}(x))+\mathcal{M}\left|\nabla u^R\right|(\Phi_R^{-1}(y))\right)\\ &\le \lambda C_d\,|x-y|\left(\mathcal{M}\left|\nabla u^R\right|(\Phi_R^{-1}(x))+\mathcal{M}\left|\nabla u^R\right|(\Phi_R^{-1}(y))\right).\end{aligned} \tag{6.11}$$

Firstly, we shall prove that for any $\mu > 0$,

$$\mathbb{E}\left[\exp\left(\mu\int_0^{T\wedge\tau_\varepsilon}|\mathbf{B}_s|^2\,ds\right)\right]\le C(e)\cdot\exp\left(\widetilde{\mathbf{C}}\,[\lambda^R]^{\left(1-\frac{d}{p_1}\right)^{-1}}\right),$$

and

$$\mathbb{E}\left[\exp\left(\mu\int_0^{T\wedge\tau_\varepsilon}|\mathbf{A}_s|\,ds\right)\right]\le C(e)\cdot\exp\left(\widetilde{\mathbf{C}}\,[\lambda^R]^{\left(1-\frac{d}{p_1}\right)^{-1}}\right).$$

Combine the definitions of (6.8), (6.7) with (6.10), (6.11), we only need to estimate

$$\begin{aligned}M_1 &:= \mathbb{E}\left[\exp\left(\int_0^{T\wedge\tau_\varepsilon}\mathcal{M}\left|\nabla^2u^R\right|^2(X_s^R(x))\,ds\right)\right],\\ M_2 &:= \mathbb{E}\left[\exp\left(\int_0^{T\wedge\tau_\varepsilon}\mathcal{M}\left\|\nabla\sigma^R\right\|^2(X_s^R(x))\,ds\right)\right],\end{aligned}$$

and

$$M_3 := \mathbb{E}\left[\exp\left(\int_0^{T\wedge\tau_\varepsilon}\lambda\mathcal{M}\left|\nabla u^R\right|(X_s^R(x))\,ds\right)\right].$$

Take $f = \mathcal{M}\left|\nabla^2 u^R\right|^2$ and $p = \frac{p_1}{2}$ in (4.11), then we have

$$M_1 \le e \cdot \exp\left(T \left[\frac{p_1(p_1-2)C_2\left((T\lambda^R)^{\frac{d}{p_1}} + (T\lambda^R)^{\frac{d}{p_1}-1}\right)\left\|\mathcal{M}\left|\nabla^2 u^R\right|^2\right\|_{\frac{p_1}{2}}}{1-e^{-1}} \right]^{\left(1-\frac{d}{p_1}\right)^{-1}} \right).$$

We can take $T\lambda^R > 1$, then $(T\lambda^R)^{\frac{d}{p_1}-1} < (T\lambda^R)^{\frac{d}{p_1}}$. By Theorem 2.3 (ii) and (5.1), we have

$$\left\|\mathcal{M}\left|\nabla^2 u^R\right|^2\right\|_{\frac{p_1}{2}} \lesssim \left\|\nabla^2 u^R\right\|_{p_1}^2 \lesssim \left\|b^R\right\|_{p_1}^2.$$

Therefore,

$$\begin{aligned} M_1 &\le e \cdot \exp\left(\widetilde{\mathbf{C}}\left[(\lambda^R)^{\frac{d}{p_1}}\left\|b^R\right\|_{p_1}^2\right]^{\left(1-\frac{d}{p_1}\right)^{-1}} \right) \\ &\le e \cdot \exp\left(\widetilde{\mathbf{C}}\,[\lambda^R]^{\left(1-\frac{d}{p_1}\right)^{-1}} \right), \end{aligned}$$

where the second inequality is due to ($\mathbf{H}^{\mathbf{b}^{\mathbf{R}}}$) and (4.9).

Similarly, taking $f = \mathcal{M}\left\|\nabla\sigma^R\right\|^2$ and $p = \frac{p_1}{2}$ in (4.11), we obtain

$$\begin{aligned} M_2 &\le e \cdot \exp\left(\widetilde{\mathbf{C}}\left[(\lambda^R)^{\frac{d}{p_1}}\left\|\nabla\sigma^R\right\|_{p_1}^2\right]^{\left(1-\frac{d}{p_1}\right)^{-1}} \right) \\ &\le e \cdot \exp\left(\widetilde{\mathbf{C}}\left[\lambda^R + (\lambda^R)^{\frac{d}{p_1}}\right]^{\left(1-\frac{d}{p_1}\right)^{-1}} \right) \\ &\le e \cdot \exp\left(\widetilde{\mathbf{C}}\,[\lambda^R]^{\left(1-\frac{d}{p_1}\right)^{-1}} \right). \end{aligned}$$

Taking $f = \lambda_H^R \cdot \mathcal{M}\left|\nabla u^R\right|$ and $p = \infty$, we obtain

$$M_3 \le e \cdot \exp\left(\widetilde{\mathbf{C}} \cdot \lambda^R\right) \le e \cdot \exp\left(\widetilde{\mathbf{C}}\,[\lambda^R]^{\left(1-\frac{d}{p_1}\right)^{-1}} \right).$$

By Novikov's criterion, the process

$$t \mapsto \exp\left(2\int_0^{t\wedge\tau_\varepsilon} \mathbf{B}_s\, d\widetilde{W}_s - 2\int_0^{t\wedge\tau_\varepsilon} |\mathbf{B}_s|^2\, ds\right) =: M_t^\varepsilon$$

is a continuous exponential martingale. By Hölder's inequality, we obtain

$$\mathbb{E}\left|Z^R_{t\wedge\tau_\varepsilon}\right|^\alpha \le 2^\alpha\,|x-y|^\alpha\left(\mathbb{E}M^\varepsilon_t\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\exp\left(\int_0^{t\wedge\tau_\varepsilon}|\mathbf{B}_s|^2\,ds+2\int_0^{t\wedge\tau_\varepsilon}|\mathbf{A}_s|\,ds\right)\right]\right)^{\frac{1}{2}}$$
$$\le C(\alpha,e)\exp\left(\widetilde{\mathbf{C}}\,[\lambda^R]^{\left(1-\frac{d}{p_1}\right)^{-1}}\right)|x-y|^\alpha.$$

Let $\varepsilon\downarrow 0$, we have

$$\mathbb{E}\left[\left|Y^R_t(\Phi_R(x))-Y^R_t(\Phi_R(y))\right|^\alpha\right]\le C(\alpha,e)\exp\left(\widetilde{\mathbf{C}}\,[\lambda^R]^{\left(1-\frac{d}{p_1}\right)^{-1}}\right)|x-y|^\alpha.$$

Moreover, if $\alpha>0$, then

$$\mathbb{E}\left[\left|X^R_t(x)-X^R_t(y)\right|^\alpha\right]=\mathbb{E}\left[\left|\Phi^{-1}_R(Y^R_t(\Phi_R(x)))-\Phi^{-1}_R(Y^R_t(\Phi_R(y)))\right|^\alpha\right]$$
$$\le\left\|\nabla\Phi^{-1}_R\right\|^\alpha_\infty\mathbb{E}\left[\left|Z^R_t\right|^\alpha\right]$$
$$\le C(\alpha,e)\exp\left(\widetilde{\mathbf{C}}\,[\lambda^R]^{\left(1-\frac{d}{p_1}\right)^{-1}}\right)|x-y|^\alpha. \tag{6.12}$$

Notice that

$$\left|Y^R_t(\Phi_R(x))-Y^R_t(\Phi_R(y))\right|=\left|\Phi_R(X^R_t(x))-\Phi_R(X^R_t(y))\right|$$
$$\le 2\left|X^R_t(x)-X^R_t(y)\right|,$$

if $\alpha<0$, then

$$\left|X^R_t(x)-X^R_t(y)\right|^\alpha\le 2^{-\alpha}\left|Y^R_t(\Phi_R(x))-Y^R_t(\Phi_R(y))\right|^\alpha$$
$$\le C(\alpha,e)\exp\left(\widetilde{\mathbf{C}}\,[\lambda^R]^{\left(1-\frac{d}{p_1}\right)^{-1}}\right)|x-y|^\alpha. \tag{6.13}$$

Together, (6.12) and (6.13) imply (6.2).

Notice that

$$\Phi_R(\Phi^{-1}_R(x))=x,\quad \Phi_R(x)=x+u^R(x),$$

we have

$$\Phi^{-1}_R(x)+u^R(\Phi^{-1}_R(x))=x.$$

Therefore,

$$|\Phi_R(x)| \vee \left|\Phi_R^{-1}(x)\right| \le |x| + \left\|u^R\right\|_\infty \le |x| + \frac{1}{2}. \tag{6.14}$$

By $X_s^R(x) = \Phi_R^{-1}(Y_s^R(\Phi_R(x)))$, (5.4) and (6.14), we have

$$\frac{1}{2}\left(1 + \left|Y_s^R(\Phi_R(x))\right|\right) \le 1 + \left|X_s^R(x)\right| \le 2\left(1 + \left|Y_s^R(\Phi_R(x))\right|\right).$$

Combining the inequality

$$\frac{1}{2}(1+|x|)^2 \le (1+|x|^2) \le (1+|x|)^2,$$

we can obtain

$$\left(1 + \left|X_s^R(x)\right|^2\right)^\alpha \le C(\alpha)\left(1 + \left|Y_s^R(\Phi_R(x))\right|^2\right)^\alpha,$$

where $C(\alpha) = 8^\alpha \vee 8^{-\alpha}$. Therefore, we just need to consider the estimate of $\mathbb{E}\left[\left(1 + \left|Y_s^R(\Phi_R(x))\right|^2\right)^\alpha\right]$.

Applying Itô formula to $\left(1 + \left|Y_s^R(\Phi_R(x))\right|^2\right)^\alpha$, we have

$$\begin{aligned}\left(1 + \left|Y_t^R\right|^2\right)^\alpha &= (1 + |\Phi_R(x)|^2)^\alpha + 2\alpha \int_0^t \left(1 + \left|Y_s^R\right|^2\right)^{\alpha-1} \langle Y_s^R, \tilde{\sigma}^R(Y_s^R) d\widetilde{W}_s\rangle \\ &\quad + 2\alpha \int_0^t \left(1 + \left|Y_s^R\right|^2\right)^{\alpha-1} \langle \tilde{b}(Y_s^R), Y_s^R\rangle \, ds \\ &\quad + \alpha \int_0^t \left(1 + \left|Y_s^R\right|^2\right)^{\alpha-1} \left\|\sigma(Y_s^R)\right\|^2 ds \\ &\quad + 2\alpha(\alpha-1) \int_0^t \left(1 + \left|Y_s^R\right|^2\right)^{\alpha-2} \left|\tilde{\sigma}^R(Y_s^R) Y_s^R\right|^2 ds.\end{aligned}$$

By (6.1) and (6.15), we obtain

$$\mathbb{E}\left[\left(1 + \left|Y_t^R\right|^2\right)^\alpha\right] \le \tilde{\mathbf{C}}(1+|x|^2)^\alpha + (\tilde{\mathbf{C}}\lambda^R + \tilde{\mathbf{C}}) \int_0^t \mathbb{E}\left[\left(1 + \left|Y_s^R\right|^2\right)^\alpha\right] ds.$$

Using Gronwall's inequality, we proved (6.3).

It is easy to see

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le s\le t}\left|X_s^R(x)\right|^p\right]\\
&\le \mathbb{E}\left[\sup_{0\le s\le t}\left|\Phi_R^{-1}(Y_s^R(\Phi_R(x)))\right|^p\right]\\
&\le \mathbb{E}\left[\sup_{0\le s\le t}\left|\Phi_R^{-1}(Y_s^R(\Phi_R(x)))-\Phi_R^{-1}(0)+\Phi_R^{-1}(0)\right|^p\right]\\
&\le C(p)\mathbb{E}\left[\sup_{0\le s\le t}\left|Y_s^R(\Phi_R(x))\right|^p\right]+C(p)\left|\Phi_R^{-1}(0)\right|^p\\
&\le C(p)\mathbb{E}\left[\sup_{0\le s\le t}\left|Y_s^R(\Phi_R(x))\right|^p\right]+C(p),
\end{aligned}
$$

where the last inequality is due to $\left\|\nabla\Phi_R^{-1}\right\|_\infty \le 2$ and $\Phi_R^{-1}(0)\le 1/2$. So, we only need to estimate $\mathbb{E}\left[\sup_{0\le s\le t}\left|Y_s^R(\Phi_R(x))\right|^p\right],\ p\ge 2.$

By Eq. (5.5), we have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le s\le t}\left|Y_s^R\right|^p\right]\\
&\le C(p)\mathbb{E}\left[|\Phi_R(x)|^p+\sup_{0\le s\le t}\left|\int_0^s \tilde{b}^R(Y_r^R)\,dr\right|^p+\sup_{0\le s\le t}\left|\int_0^s \tilde{\sigma}^R(Y_r^R)\,d\widetilde{W}_r\right|^p\right]\\
&:= C(p)(I_1+I_2+I_3).
\end{aligned}
\tag{6.15}
$$

It is not hard to see

$$
\begin{aligned}
I_1 &\le \left(x+\left\|u^R\right\|_\infty\right)^p \le C(p)(1+|x|^p),\\
I_2 &\le \mathbb{E}\left[t^{p-1}\int_0^t\left|\tilde{b}^R(Y_r^R)\right|^p\,dr\right]\le t^p\left\|\tilde{b}^R\right\|_\infty^p \le \frac{1}{2^p}t^p\lambda^p,\\
I_3 &\le \mathbb{E}\left[\left(\int_0^t\left\|\tilde{\sigma}^R(Y_r^R)\right\|^2\,dr\right)^{\frac{p}{2}}\right]\le t^{\frac{p}{2}}\left\|\tilde{\sigma}^R\right\|_\infty^p \le t^{\frac{p}{2}}2^p\left\|\sigma^R\right\|_\infty^p.
\end{aligned}
$$

So, we obtained (6.4).

Notice that

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\Phi_R^{-1}(Y_t^R(\Phi_R(x)))-\Phi_R^{-1}(Y_t^R(\Phi_R(y)))\right|^p\right]$$
$$\le 2^p\mathbb{E}\left[\sup_{0\le t\le T}\left|Y_t^R(\Phi_R(x))-Y_t^R(\Phi_R(y))\right|^p\right],$$

we only need to estimate $\mathbb{E}[\sup_{0\le t\le T}\left|Z_t^R\right|^p]$. By (6.9), we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left|Z_t^R\right|^p\right]\\
&\le |\Phi_R(x)-\Phi_R(y)|^p\left(\mathbb{E}\sup_{0\le t\le T}M_1^2(t)\right)^{\frac{1}{2}}\left(\exp\left(2\int_0^T|\mathbf{A}_s|\,ds\right)\right)^{\frac{1}{2}}\\
&\le |\Phi_R(x)-\Phi_R(y)|^p\left(\mathbb{E}M_1^2(T)\right)^{\frac{1}{2}}\left(\exp\left(2\int_0^T|\mathbf{A}_s|\,ds\right)\right)^{\frac{1}{2}}\\
&\le |\Phi_R(x)-\Phi_R(y)|^p\left(\mathbb{E}M_4(T)\right)^{\frac{1}{4}}\left(\exp\left(6\int_0^T|\mathbf{B}_s|^2\,ds\right)\right)^{\frac{1}{4}}\\
&\quad\times\left(\exp\left(2\int_0^T|\mathbf{A}_s|\,ds\right)\right)^{\frac{1}{2}}\\
&\le \widetilde{\mathbf{C}}\left(\exp\left(\widetilde{\mathbf{C}}\,(\lambda^R)^{\frac{p_1}{p_1-d}}\right)\right)|x-y|^p,
\end{aligned}$$

where

$$M_k(t):=\exp\left(k\int_0^t\mathbf{B}_s\,d\widetilde{W}_s-\frac{k^2}{2}\int_0^t|\mathbf{B}_s|^2\,ds\right).$$

We proved (6.5). □

Let $D_t(x):=\sup_{0\le s\le t}|X_s(x)|$, $\tau_R(x):=\inf\{t\ge 0,|X_t(x)|>R\}$ and similarly, let $D_t^R(x):=\sup_{0\le s\le t}\left|X_s^R(x)\right|$, $\tau_R^R(x):=\inf\{t\ge 0,\left|X_t^R(x)\right|>R\}$. It is easy to see

$$\{D_t(x)\ge R\}=\{\tau_R\le t\},\{D_t^R(x)\ge R\}=\{\tau_R^R\le t\}.$$

By the definitions of b^R and σ^R, it is not hard to obtain

$$\{\tau_R\le t\}\subset\{\tau_R^R\le t\}.$$

For all $x \in B(N)$, we have

$$\begin{aligned}\mathbb{P}(\tau_R \le t) \le \mathbb{P}(\tau_R^R \le t) &= \mathbb{P}(D_t^R(x) \ge R)\\ &\le \frac{\mathbb{E}[\left|D_t^R(x)\right|^n]}{R^n}\\ &\le \frac{\widetilde{\mathbf{C}}(1+|x|^n+(\lambda^R)^n)}{R^n},\end{aligned}$$

where the second inequality is due to Markov's inequality, the last inequality is due to Lemma 6.2. By the definition of λ^R in (4.9), we can obtain $(\lambda^R)^n/R^n \to 0$ when $R \to \infty$. Hence, we have $\tau_R \to \infty$ when $R \to \infty$. On the other hand, by the definitions of b^R and σ^R, we observe that if $D_t(x) < R$, then $X_t(x) = X_t^R(x)$, i.e., $X_t(x) = X_t^R(x)$ for all $t < \tau_R$. By Theorem 6.1, SDE (3.1) exists a unique strong solution. We can define $X_t(x) = X_t^R(x)$ for $t < \tau_R$. It is clear that $\{X_t(x)\}_{t\in[0,T]}$ is the unique strong solution of SDE (1.1).

By (6.4) and definition of λ^R, for all $x \in B(N)$, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|X_t(x)|^p\right] &\le \sum_{R=1}^{\infty}\mathbb{E}\left[\left|D_T^R(x)\right|^p \mathbb{1}_{\{R-1\le D_T(x)<R\}}\right]\\ &\le \sum_{R=2}^{\infty}\mathbb{E}\left[\left|D_T^R(x)\right|^p \mathbb{1}_{\{R-1\le D_T(x)<R\}}\right] + \mathbf{C}(N)\\ &\le \sum_{R=2}^{\infty}\mathbb{E}\left[\left|D_T^R(x)\right|^{2p}\right]^{\frac{1}{2}}\left[\mathbb{P}(D_T^{R-1}(x)\ge R-1)\right]^{\frac{1}{2}} + \mathbf{C}(N)\\ &\le \sum_{R=2}^{\infty}\mathbb{E}\left[\left|D_T^R(x)\right|^{2p}\right]^{\frac{1}{2}}\cdot\frac{\mathbb{E}[(D_t^{R-1}(x))^{2p}]^{\frac{1}{2}}}{(R-1)^p} + \mathbf{C}(N)\\ &\le \sum_{R=2}^{\infty}\frac{\mathbb{E}[(D_T^R(x))^{2p}]^{\frac{1}{2}}\cdot\mathbb{E}[(D_T^{R-1}(x))^{2p}]^{\frac{1}{2}}}{(R-1)^p} + \mathbf{C}(N)\\ &\le \mathbf{C}(N).\end{aligned} \tag{6.16}$$

where the last inequality is due to (6.4) and the definition of λ^R.

For all $x, y \in B(N)$, we consider the following estimate

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T}|X_t(x)-X_t(y)|^p\right]\\ &= \sum_{R=1}^{\infty}\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^R(x)-X_t^R(y)\right|^p \mathbb{1}_{\{R-1\le D_T(x)\vee D_T(y)<R\}}\right]\\ &\le \sum_{R=1}^{\infty}\left(\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^R(x)-X_t^R(y)\right|^{2p}\right]\right)^{\frac{1}{2}}\mathbb{P}\Big(D_T(x)\vee D_T(y)\ge R-1\Big)^{\frac{1}{2}}\end{aligned}$$

$$\begin{aligned}
&\le \sum_{R=1}^{\infty}\left(\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^R(x)-X_t^R(y)\right|^{2p}\right]\right)^{\frac{1}{2}}\\
&\quad\times\Big(\mathbb{P}(D_T(x)\ge R-1)+\mathbb{P}(D_T(y)\ge R-1)\Big)^{\frac{1}{2}}.\\
&\le \sum_{R=1}^{\infty}\left(\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^R(x)-X_t^R(y)\right|^{2p}\right]\right)^{\frac{1}{2}}\\
&\quad\times\Big(\mathbb{P}(D_T^{R-1}(x)\ge R-1)+\mathbb{P}(D_T^{R-1}(y)\ge R-1)\Big)^{\frac{1}{2}}\\
&\le \sum_{R=2}^{\infty}\left(\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^R(x)-X_t^R(y)\right|^{2p}\right]\right)^{\frac{1}{2}}\\
&\quad\times\left(\frac{\mathbb{E}[(D_T^{R-1}(x))^{2n}]}{(R-1)^{2n}}+\frac{\mathbb{E}[(D_T^{R-1}(y))^{2n}]}{(R-1)^{2n}}\right)^{\frac{1}{2}}+\mathbf{C}\,|x-y|^p\\
&\le \sum_{R=2}^{\infty}\widetilde{\mathbf{C}}\,|x-y|^p\left(\exp\left(\widetilde{\mathbf{C}}\,(\lambda^R)^{\frac{p_1}{p_1-d}}\right)\right)\frac{(1+|x|^n)}{(R-1)^n}\\
&\quad+\sum_{R=2}^{\infty}\widetilde{\mathbf{C}}\,|x-y|^p\left(\exp\left(\widetilde{\mathbf{C}}\,(\lambda^R)^{\frac{p_1}{p_1-d}}\right)\right)\frac{(\lambda^R)^n}{(R-1)^n}\\
&\quad+\sum_{R=2}^{\infty}\widetilde{\mathbf{C}}\,|x-y|^p\left(\exp\left(\widetilde{\mathbf{C}}\,(\lambda^R)^{\frac{p_1}{p_1-d}}\right)\right)\frac{(1+|y|^n)}{(R-1)^n}+\mathbf{C}\,|x-y|^p\\
&\le \sum_{R=2}^{\infty}\widetilde{\mathbf{C}}\,|x-y|^p\left(\exp\left(2\widetilde{\mathbf{C}}\,(\lambda^R)^{\frac{p_1}{p_1-d}}\right)\right)\frac{(2+|x|^n)}{(R-1)^n}+\mathbf{C}\,|x-y|^p\\
&\quad+\sum_{R=2}^{\infty}\widetilde{\mathbf{C}}\,|x-y|^p\left(\exp\left(2\widetilde{\mathbf{C}}\,(\lambda^R)^{\frac{p_1}{p_1-d}}\right)\right)\frac{(2+|y|^n)}{(R-1)^n},
\end{aligned}\tag{6.17}$$

where the last inequality we used the fact that we can find a constant $C(\widetilde{\mathbf{C}}, p_1, d, n(\beta))$ such that for all $\lambda^R \ge C(\widetilde{\mathbf{C}}, p_1, d, n(\beta))$,

$$(\lambda^R)^n \le \exp\left(\widetilde{\mathbf{C}}\,(\lambda^R)^{\frac{p_1}{p_1-d}}\right).\tag{6.18}$$

In fact, if let $\tilde{\beta}$ satisfy $(2C_2\tilde{\beta})^{2(1-\frac{d}{p_1})^{-1}} = C(\widetilde{\mathbf{C}}, p_1, d, n(\beta))$, then for all $R \ge 1$, λ^R satisfy (6.18), where $n(\beta)$ be decided by (6.19).

On the other hand, by the definitions of λ^R and $I_b(R)$, we have

$$\mathbb{E}\left[\sup_{0\le t\le T}|X_t(x)-X_t(y)|^p\right]\le\sum_{R=2}^{\infty}\mathbf{C}(\beta,\tilde{\beta})R^{\mathbf{C}(\beta)}\frac{(2+|x|^n)}{(R-1)^n}+\sum_{R=2}^{\infty}\mathbf{C}(\beta,\tilde{\beta})R^{\mathbf{C}(\beta)}\frac{(2+|y|^n)}{(R-1)^n}+\mathbf{C}\,|x-y|^p\,.$$

Therefore, take n satisfy

$$\mathbf{C}(\beta)+1<n, \tag{6.19}$$

we obtain

$$\mathbb{E}\left[\sup_{0\le t\le T}|X_t(x)-X_t(y)|^p\right]\le\mathbf{C}\Big((1+|x|^n)+(1+|y|^n)\Big)\,|x-y|^p\,. \tag{6.20}$$

By Lemma 2.1 in [21], (6.16) and (6.20), we proved Theorem 1.1(A).

Following the proof of Zhang [24], it is not hard to prove for any bounded measurable function f and $t\in[0,T]$,

$$x\mapsto\mathbb{E}[f(X_t^R(x))]\text{ is continuous.} \tag{6.21}$$

For any $x, y\in B(N)$, we have

$$\begin{aligned}
&|\mathbb{E}\,[f(X_t(x)-f(X_t(y)))]|\\
&\quad\le\left|\mathbb{E}\left[(f(X_t(x)-f(X_t(y))))\mathbb{1}_{\{t\le\tau_R\}}\right]\right|+2\,\|f\|_\infty\,\mathbb{P}(t>\tau_R)\\
&\quad\le\left|\mathbb{E}\left[(f(X_t^R(x)-f(X_t^R(y))))\mathbb{1}_{\{t\le\tau_R\}}\right]\right|+2\,\|f\|_\infty\,\mathbb{P}(t>\tau_R)\\
&\quad\le\left|\mathbb{E}\left[(f(X_t^R(x)-f(X_t^R(y))))\right]\right|+4\,\|f\|_\infty\,\mathbb{P}(t>\tau_R).
\end{aligned} \tag{6.22}$$

Together, (6.22), (6.21) and $\tau_R\to\infty$ when $R\to\infty$ imply Theorem 1.1(B).

Lemma 6.3 *Under* $(\mathbf{H^b})$, $(\mathbf{H_1^\sigma})$ *and* $(\mathbf{H_2^\sigma})$, *let* $\{X_t(x)\}_{t\in[0,T]}$ *and* $\{X_t(y)\}_{t\in[0,T]}$ *are two solutions of SDE* (1.1) *with initial conditions* $X_0(x)=x$ *and* $X_0(y)=y$, *respectively, then for all* $0\le t\le T$, $\alpha\in\mathbb{R}$ *and* $x, y\in B(N)$, *we have*

$$\mathbb{E}[|X_t(x)-X_t(y)|^\alpha]\le\mathbf{C}(N)\,|x-y|^\alpha\,, \tag{6.23}$$

$$\mathbb{E}\left[\left(1+|X_t(x)|^2\right)^\alpha\right]\le\mathbf{C}(N)\left(1+|x|^2\right)^\alpha, \tag{6.24}$$

and for all $p\ge2$,

$$\mathbb{E}[|X_t(x)-X_s(x)|^p]\le\mathbf{C}(N)\,|t-s|^{\frac{p}{2}}\,. \tag{6.25}$$

Proof Set $D_t(x) := \sup_{0\le s\le t} |X_t(x)|$ and $D_t(y) := \sup_{0\le s\le t} |X_t(y)|$. It is easy to see if $D_t(x) < R$ and $D_t(y) < R$, then $X_t(x) = X_t^R(x)$, $X_t(y) = X_t^R(y)$. Moreover, by Lemma 6.2, similar to (6.17), for all $t \in [0, T]$ and $x, y \in B(N)$, we have

$$\begin{aligned}
&\mathbb{E}[|X_t(x) - X_t(y)|^\alpha] \\
&\quad = \sum_{R=1}^{\infty} \mathbb{E}\left[\left|X_t^R(x) - X_t^R(y)\right|^\alpha \mathbb{1}_{\{R-1\le D_T(x)\vee D_T(y)<R\}}\right] \\
&\quad \le \sum_{R=1}^{\infty} \left(\mathbb{E}\left[\left|X_t^R(x) - X_t^R(y)\right|^{2\alpha}\right]\right)^{\frac{1}{2}} \mathbb{P}\Big(D_T(x)\vee D_T(y) \ge R-1\Big)^{\frac{1}{2}} \\
&\quad \le \sum_{R=1}^{\infty} \left(\mathbb{E}\left[\left|X_t^R(x) - X_t^R(y)\right|^{2\alpha}\right]\right)^{\frac{1}{2}} \Big(\mathbb{P}(D_T(x) \ge R-1) + \mathbb{P}(D_T(y) \ge R-1)\Big)^{\frac{1}{2}} \\
&\quad \le \sum_{R=2}^{\infty} \left(\mathbb{E}\left[\left|X_t^R(x) - X_t^R(y)\right|^{2\alpha}\right]\right)^{\frac{1}{2}} \left(\frac{\mathbb{E}[(D_T^{R-1}(x))^{2n}]}{(R-1)^{2n}} + \frac{\mathbb{E}[(D_T^{R-1}(y))^{2n}]}{(R-1)^{2n}}\right)^{\frac{1}{2}} \\
&\qquad + \mathbf{C}\,|x-y|^\alpha \\
&\quad \le \mathbf{C}\,(1 + |x|^n + |y|^n)\,|x-y|^\alpha \\
&\quad \le \mathbf{C}(N)\,|x-y|^\alpha\,,
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\left(1 + |X_t(x)|^2\right)^\alpha\right] \\
&\quad = \sum_{R=1}^{\infty} \mathbb{E}\left[\left(1 + \left|X_t^R(x)\right|^2\right)^\alpha \mathbb{1}_{\{R-1\le D_T(x)<R\}}\right] \\
&\quad \le \sum_{R=2}^{\infty} \left(\mathbb{E}\left[\left(1 + \left|X_t^R(x)\right|^2\right)^{2\alpha}\right]\right)^{\frac{1}{2}} \left(\frac{\mathbb{E}[(D_T^{R-1}(x))^{2n}]}{(R-1)^{2n}}\right)^{\frac{1}{2}} + \mathbf{C}(1+|x|^2)^\alpha \\
&\quad \le \mathbf{C}\left(1 + |x|^n\right)\left(1 + |x|^2\right)^\alpha \\
&\quad \le \mathbf{C}(N)(1+|x|^2)^\alpha.
\end{aligned}$$

On the other hand, it is not hard to obtain

$$\begin{aligned}
&\mathbb{E}\left[\left|X_t^R(x) - X_s^R(x)\right|^p\right] \\
&\quad \le C(p)\mathbb{E}\left[\left|Y_t^R(\Phi_R(x)) - Y_s^R(\Phi_R(x))\right|^p\right] \\
&\quad \le \mathbf{C}(T)\Big(1 + (\lambda^R)^p\Big)\,|t-s|^{\frac{p}{2}}\,,
\end{aligned}$$

where the last inequality is due to

$$\mathbb{E}\left[\left|\int_s^t \tilde{b}^R(Y_r^R)\,dr\right|^p\right] \le ||\tilde{b}^R||_\infty^p\,|t-s|^p\,,$$

and

$$\mathbb{E}\left[\left|\int_s^t \tilde{\sigma}^R(Y_r^R)\,d\widetilde{W}_r\right|^p\right] \le ||\tilde{\sigma}^R||_\infty^p\,|t-s|^{\frac{p}{2}}\,.$$

Moreover, for all $t, s \in [0, T]$ and $x \in B(N)$, we have

$$\begin{aligned}
&\mathbb{E}[|X_t(x) - X_s(x)|^p] \\
&= \sum_{R=1}^{\infty} \mathbb{E}\left[\left|X_t^R(x) - X_s^R(x)\right|^p \mathbb{1}_{\{R-1\le D_T(x)<R\}}\right] \\
&\le \sum_{R=2}^{\infty} \left(\mathbb{E}\left[\left|X_t^R(x) - X_s^R(x)\right|\right]^{2p}\right)^{\frac{1}{2}} \left(\frac{\mathbb{E}[(D_T^{R-1}(x))^{2p}]}{(R-1)^{2p}}\right)^{\frac{1}{2}} + \mathbf{C}\,|t-s|^{\frac{p}{2}} \\
&\le \sum_{R=2}^{\infty} \mathbf{C}(T)\frac{\left(1+|x|^p+(\lambda^R)^p\right)^2}{(R-1)^p}\,|t-s|^{\frac{p}{2}} + \mathbf{C}\,|t-s|^{\frac{p}{2}} \\
&\le \mathbf{C}(1+|x|^{2p})\,|t-s|^{\frac{p}{2}} \\
&\le \mathbf{C}(N)\,|t-s|^{\frac{p}{2}}\,.
\end{aligned}$$

We completed the proof. □

By Lemma 6.3, for all $p \ge 2$, $t, s \in [0, T]$ and $x, y \in B(N)$, we have

$$\mathbb{E}\left[|X_t(x) - X_s(y)|^p\right] \le \mathbf{C}(N)\left(|x-y|^p + |t-s|^{\frac{p}{2}}\right).$$

By Kolmogorov's lemma, we can obtain for any $N \in \mathbb{N}$, there exists a $\mathbb{P}$-null set Ξ_N such that for any $\omega \notin \Xi_N$, $X_\cdot(\omega, \cdot) : [0, T] \times B(N) \to \mathbb{R}^d$ is continuous. If we set $\Xi := \cup_{N=1}^{\infty} \Xi_N$, then $\mathbb{P}(\Xi) = 0$ and

$$X_\cdot(\omega, \cdot) : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \text{ is continuous}, \quad \forall \omega \notin \Xi.$$

Similar to the standard argument (cf. [14]), the proof for any $t \in [0, T]$, almost all ω, the maps $x \mapsto X_t(\omega, x)$ are one-to-one due to (6.23) and (6.25). For the reader's convenience, we give the details of one-to-one property.

For $x \neq y \in \mathbb{R}^d$, set

$$\mathscr{R}(t, x, y) := \frac{1}{|X_t(x) - X_t(y)|},$$

then

$$
\begin{aligned}
&\left|\mathscr{R}(t,x,y)-\mathscr{R}(s,x',y')\right|\\
&\quad\le \frac{\left|X_t(x)-X_t(y)-X_s(x')+X_s(y')\right|}{|X_t(x)-X_t(y)|\,|X_s(x')-X_s(y')|}\\
&\quad\le \frac{\left|X_t(x)-X_t(x')\right|+\left|X_t(x')-X_s(x')\right|+\left|X_t(y)-X_t(y')\right|+\left|X_t(y')-X_s(y')\right|}{|X_t(x)-X_t(y)|\,|X_s(x')-X_s(y')|}.
\end{aligned}
$$

By Hölder inequality, we have

$$
\begin{aligned}
\mathbb{E}\left|\mathscr{R}(t,x,y)-\mathscr{R}(s,x',y')\right|^p \le \mathbf{C}\cdot\mathbb{E}\Big[&\left|X_t(x)-X_t(x')\right|^{2p}+\left|X_t(x')-X_s(x')\right|^{2p}\\
&+\left|X_t(y)-X_t(y')\right|^{2p}+\left|X_t(y')-X_s(y')\right|^{2p}\Big]^{\frac{1}{2}}\\
&\cdot\mathbb{E}\left[|X_t(x)-X_t(y)|^{-4p}\right]^{\frac{1}{4}}\\
&\cdot\mathbb{E}\left[\left|X_s(x')-X_s(y')\right|^{-4p}\right]^{\frac{1}{4}}.
\end{aligned}
$$

Moreover, for all $x, y, x', y' \in B(N)$ and $|x-y| \wedge \left|x'-y'\right| > \varepsilon$, we obtain

$$
\begin{aligned}
&\mathbb{E}\left|\mathscr{R}(t,x,y)-\mathscr{R}(s,x',y')\right|^p\\
&\quad\le \mathbf{C}(N)\left(\left|x-x'\right|^p+|t-s|^{\frac{p}{2}}+\left|y-y'\right|^p+|t-s|^{\frac{p}{2}}\right)\varepsilon^{-2p}.
\end{aligned}
$$

Choose $p > 4(d+1)$, by Kolmogorov's lemma, there exists a $\mathbb{P}$-null set $\Xi_{k,N}$ such that for all $\omega \notin \Xi_{k,N}$, the mapping $(t,x,y) \mapsto \mathscr{R}(t,x,y)$ is continuous on

$$
\left\{(t,x,y)\in[0,T]\times B(N)\times B(N) : |x-y| > \frac{1}{k}\right\} \quad \forall k \in \mathbb{N}_+.
$$

Set $\Xi := \cup_{k,N=1}^{\infty}, \Xi_{k,N}$, then for any $\omega \notin \Xi$, the mapping $(t,x,y) \mapsto \mathscr{R}(t,x,y)$ is continuous on

$$
\{(t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d : x \neq y\}.
$$

We proved one-to-one property. □

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Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The author has no conflicts of interest to declare that are relevant to the content of this article.

7 Appendix

The Proof of Theorem 4.1: **Step (i)** Suppose $\sigma^R(x)$ does not depend on x, Krylov proved the estimate (4.2) in [8, Page 109]. Therefore, If $\sigma^R(x) \equiv \sigma^R(x_0)$, then

$$\left\| (\lambda - L^{\sigma^R(x_0)})^{-1} f \right\|_{2,p} \leq C_0 \left\| f \right\|_p .$$

Step (ii) Suppose for some $x_0 \in \mathbb{R}^d$

$$\left\| \sigma^R(x) - \sigma^R(x_0) \right\| \leq \frac{1}{2\tilde{\delta}^{-\frac{1}{2}} C_0}, \tag{7.1}$$

we consider the following equation

$$L^{\sigma^R(x_0)} u - \lambda u + g = 0,$$

where $g := L^{\sigma^R(x)} - L^{\sigma^R(x_0)} + f$. By (7.1) and the definition of $L^{\sigma^R(x)}$, we obtain

$$\|g\|_p \leq \frac{1}{2C_0} \|u_{xx}\|_p + \|f\|_p .$$

Hence, by **Step (i)**, we have

$$\|u_{xx}\|_p \leq C_0 \|g\|_p \leq \frac{1}{2} \|u_{xx}\|_p + C_0 \|f\|_p ,$$

i.e.,

$$\|u_{xx}\|_p \leq 2C_0 \|f\|_p .$$

Step (iii) Define a smooth cutoff function as follows:

$$\zeta(x) = \begin{cases} 1, & |x| \leq 1, \\ \in [0, 1], & 1 < x < 2, \\ 0 & |x| \geq 2. \end{cases}$$

Fix a small constant ε which will be determined below.

For fixed $z \in \mathbb{R}^d$, let

$$\zeta_z^\varepsilon(x) := \zeta\left(\frac{x - z}{\varepsilon}\right).$$

It is easy to check that

$$\int_{\mathbb{R}^d} \left|\nabla_x^j \zeta_z^\varepsilon(x)\right|^p dz = \varepsilon^{d-jp} \int_{\mathbb{R}^d} \left|\nabla^j \zeta(z)\right|^p dz > 0, \quad j = 0, 1, 2. \tag{7.2}$$

Multiply both side of (4.1) by $\zeta_z^\varepsilon(x)$, we have

$$L^{\sigma^R(x)}(u\zeta_z^\varepsilon) - \lambda(u\zeta_z^\varepsilon) + g_z^\varepsilon = 0,$$

where $g_z^\varepsilon := (L^{\sigma^R(x)}u)\zeta_z^\varepsilon - L^{\sigma^R(x)}(u\zeta_z^\varepsilon) - f\zeta_z^\varepsilon$.

Let

$$\hat{\sigma}^R(x) := \sigma^R((x-z)\zeta_z^{2\varepsilon}(x) + z).$$

It is easy to obtain

$$L^{\sigma^R(x)}(u\zeta_z^\varepsilon) = L^{\hat{\sigma}^R(x)}(u\zeta_z^\varepsilon),$$

since $\zeta_z^{2\varepsilon}(x) = 1$ for $|x - z| \le 2\varepsilon$ and $\zeta_z^\varepsilon(x) = 0$ for $|x - z| > 2\varepsilon$.

By (3.2) and the definition of g_z^ε, we have

$$\left\|\hat{\sigma}^R(x) - \hat{\sigma}^R(z)\right\| \le \tilde{\delta}^{-\frac{1}{2}} \left|(x-z)\zeta_z^{2\varepsilon}\right|^\varpi \le \tilde{\delta}^{-\frac{1}{2}} |4\varepsilon|^\varpi,$$

and

$$\left\|g_z^\varepsilon\right\|_p \le \left\|f\zeta_z^\varepsilon\right\|_p + \tilde{\delta}^{-1} \left\||u_x| \left|(\zeta_z^\varepsilon)_x\right|\right\|_p + \tilde{\delta}^{-1} \left\||u| \left|(\zeta_z^\varepsilon)_{xx}\right|\right\|_p.$$

By **Step (ii)**, if

$$L^{\sigma^R(x)}u - \lambda u + f = 0, \quad \left\|\sigma^R(x) - \sigma^R(x_0)\right\| \le \frac{1}{2\tilde{\delta}^{-\frac{1}{2}} C_0},$$

then

$$\|u_{xx}\|_p \le 2C_0 \|f\|_p.$$

Now, we consider the following equation:

$$L^{\hat{\sigma}^R(x)}(u\zeta_z^\varepsilon) - \lambda(u\zeta_z^\varepsilon) = g_z^\varepsilon$$

and take ε to be small enough so that

$$\left\|\hat{\sigma}^R(x) - \hat{\sigma}^R(z)\right\| \le \tilde{\delta}^{-\frac{1}{2}} |4\varepsilon|^\varpi \le \frac{1}{2\tilde{\delta}^{-\frac{1}{2}} C_0},$$

then

$$\left\| (u\zeta_z^{\varepsilon})_{xx} \right\|_p \leq 2C_0 \left\| g_z^{\varepsilon} \right\|_p$$
$$\leq 2C_0 \left(\left\| f\zeta_z^{\varepsilon} \right\|_p + \tilde{\delta}^{-1} \left\| |u_x| \left| (\zeta_z^{\varepsilon})_x \right| \right\|_p + \tilde{\delta}^{-1} \left\| |u| \left| (\zeta_z^{\varepsilon})_{xx} \right| \right\|_p \right). \quad (7.3)$$

According to Fubini's theorem, (7.2) and (7.3), it is easy to check

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left| (u\zeta_z^{\varepsilon})_{xx} \right|^p \, dx \, dz \leq C(p, \varepsilon, \tilde{\delta}^{-1}, C_0) \left(\|u_x\|_p^p + \|u\|_p^p + \|f\|_p^p \right).$$

Moreover, we have

$$\begin{aligned} \|u_{xx}\|_p^p &\lesssim \int_{\mathbb{R}^d} \left\| (u)_{xx} \cdot \zeta_z^{\varepsilon} \right\|_p^p \, dz \\ &\lesssim \int_{\mathbb{R}^d} \left\| (u\zeta_z^{\varepsilon})_{xx} - (u)_x (\zeta_z^{\varepsilon})_x - u(\xi_z^{\varepsilon})_{xx} \right\|_p^p \, dz \\ &\leq C(p, \varepsilon, \tilde{\delta}^{-1}, C_0) \left(\|u_x\|_p^p + \|u\|_p^p + \|f\|_p^p \right) \\ &\leq \frac{1}{2} \|u_{xx}\|_p^p + C(p, \varepsilon, \tilde{\delta}^{-1}, C_0)(\|u\|_p^p + \|f\|_p^p), \end{aligned}$$

where the third inequality is due to (7.2) and (7.3) and the last inequality is due to

$$\|u_x\|_p \leq C(\|u_{xx}\|_p + \|u\|_p), \quad (7.4)$$

and Young's inequality. Therefore, we proved

$$\|u_{xx}\|_p \leq C(p, \varepsilon, \tilde{\delta}^{-1}, C_0)(\|u\|_p + \|f\|_p).$$

Since $\lambda u = L^{\sigma^R(x)} u - f$, we have

$$\begin{aligned} \lambda \|u\|_p &\leq \left(\left\| L^{\sigma^R(x)} u \right\|_p + \|f\|_p \right) \\ &\leq C(d, \varpi, \tilde{\delta}, p) \left(\|u\|_p + \|f\|_p \right). \end{aligned}$$

Hence, we obtain

$$\|u_{xx}\|_p + \lambda \|u\|_p \leq C(d, \varpi, \tilde{\delta}, p) \left(\|u\|_p + \|f\|_p \right).$$

Notice that $\lambda > (C(d, \varpi, \tilde{\delta}, p) + 1)$, we obtain

$$\|u_{xx}\|_p + \|u\|_p \leq C(d, \varpi, \tilde{\delta}, p) \|f\|_p. \quad (7.5)$$

Combine (7.5) with (7.4), we obtain

$$\|u\|_{2,p} \leq C_1(d, \varpi, \tilde{\delta}, p) \|f\|_p.$$

Step (iv) Set

$$\mathcal{T}_t f(x) := \int_{\mathbb{R}^d} f(y)\rho(t,x,y)\,dy,$$

where $\rho(t,x,y)$ is the fundamental solution of the operator $\partial_t - L^{\sigma^R(x)}$. It is well known that

$$\left|\nabla_x^j \rho(t,x,y)\right| \le C_j(\varpi,\tilde{\delta},d)t^{-j/2}(2t)^{-d/2}e^{-k_j(\varpi,\tilde{\delta},d)|x-y|^2/(2t)}. \tag{7.6}$$

By [25, Lemma 3.4], for any $p, p' \in (1,\infty)$ and $\alpha \in [0,2)$, there exists a constant $C = C(d,\varpi,\tilde{\delta},p,\alpha,p')$ such that for any $f \in L^p(\mathbb{R}^d)$,

$$\|\mathcal{T}_t f\|_{\alpha,p'} \le Ct^{\left(-\frac{\alpha}{2}-\frac{d}{2p}+\frac{d}{2p'}\right)}\|f\|_p\,. \tag{7.7}$$

Let $f \in W^{2,p}(\mathbb{R}^d)$ and

$$u(x) := \int_0^\infty e^{-\lambda t}\,\mathcal{T}_t f(x)\,dt.$$

By (7.6) and the definition of $\mathcal{T}_t$, it is easy to check $u \in W^{2,p}(\mathbb{R}^d)$ and u satisfies (4.1). Indeed,

$$\begin{aligned}
L^{\sigma^R(x)}u(x) &= \int_0^\infty e^{-\lambda t}\int_{\mathbb{R}^d} f(y)L^{\sigma^R(x)}\rho(t,x,y)\,dy\,dt\\
&= \int_0^\infty e^{-\lambda t}\int_{\mathbb{R}^d} f(y)\partial_t\rho(t,x,y)\,dy\,dt\\
&= \int_{\mathbb{R}^d} f(y)\left(e^{-\lambda t}\rho(t,x,y)\Big|_0^\infty + \lambda\int_0^\infty e^{-\lambda t}\rho(t,x,y)\,dt\right)dy\\
&= f(x) + \lambda u(x).
\end{aligned}$$

By Jensen's inequality, we obtain

$$\begin{aligned}
\left|\Delta^{\frac{\alpha}{2}}u\right|^{p'} &= \left|\int_0^\infty e^{-\lambda t}\Delta^{\frac{\alpha}{2}}\mathcal{T}_t f(x)\,dt\right|^{p'}\\
&\le \left(\frac{1}{\lambda}\right)^{p'}\left(\int_0^\infty \lambda e^{-\lambda t}\left|\Delta^{\frac{\alpha}{2}}\mathcal{T}_t f(x)\right|^{p'}dt\right)
\end{aligned}$$

and

$$|u|^{p'} \le \left(\frac{1}{\lambda}\right)^{p'}\left(\int_0^\infty \lambda e^{-\lambda t}\,|\mathcal{T}_t f(x)|^{p'}\,dt\right).$$

By Fubini's theorem, we have

$$\left\| \Delta^{\frac{\alpha}{2}} u \right\|_{p'}^{p'} \leq \left(\frac{1}{\lambda} \right)^{p'} \left(\int_0^\infty \lambda e^{-\lambda t} \left\| \Delta^{\frac{\alpha}{2}} \mathcal{T}_t f(x) \right\|_{p'}^{p'} dt \right) \tag{7.8}$$

and

$$\|u\|_{p'}^{p'} \leq \left(\frac{1}{\lambda} \right)^{p'} \left(\int_0^\infty \lambda e^{-\lambda t} \left\| \mathcal{T}_t f(x) \right\|_{p'}^{p'} dt \right). \tag{7.9}$$

Moreover, by (2.1), (7.7), (7.8) and (7.9), if $(\frac{d}{p} + \alpha - \frac{d}{p'})/2 < \frac{1}{p'} \leq 1$, then

$$\begin{aligned}
\|u\|_{\alpha,p'}^{p'} &\lesssim \|f\|_p^{p'} \left(\frac{1}{\lambda} \right)^{p'} \lambda \int_0^\infty e^{-\lambda t}\, t^{(-\frac{\alpha}{2} - \frac{d}{2p} + \frac{d}{2p'})p'} dt \\
&\leq \|f\|_p^{p'} \lambda^{-p'} \frac{1}{\lambda^{\left(-\frac{\alpha}{2} - \frac{d}{2p} + \frac{d}{2p'}\right)p'}} \\
&= \|f\|_p^{p'} \lambda^{p'\left(\alpha - 2 + \frac{d}{p} - \frac{d}{p'}\right)/2},
\end{aligned}$$

where the second inequality is due to Laplace transformation.

Step (v) In this step, we will use weak convergence argument to prove the existence of (4.1). Let φ be a nonnegative smooth function in $\mathbb{R}^d$ which satisfies $\int_{\mathbb{R}^d} \varphi(x)\, dx = 1$ and support in $\{x \in \mathbb{R}^d : |x| \leq 1\}$. Let

$$\varphi_n(x) := n^d \varphi(nx), \quad \sigma_n := \sigma * \varphi_n, \quad f_n := f * \varphi_n,$$

where $*$ denotes the convolution.

Denote u_n be the solution of

$$L^{\sigma_n^R(x)} u_n - \lambda u_n = f_n.$$

By the **Step (iii)** and **Step (iv)**, we have

$$\|u_n\|_{2,p} \leq C_1 \|f\|_p$$

and

$$\|u_n\|_{\alpha,p'} \leq C_2 \lambda^{\left(\alpha - 2 + \frac{d}{p} - \frac{d}{p'}\right)/2} \|f\|_p .$$

Since $W^{2,p}(\mathbb{R}^d)$ is weakly compact, we can find a subsequence still denoted by u_n and $u \in W^{2,p}(\mathbb{R}^d)$ such that $u_n \rightharpoonup u$ in $W^{2,p}(\mathbb{R}^d)$.

For any test function $\phi \in C_0^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left(L^{\sigma_m(x)} u_n - L^{\sigma(x)} u_n \right) \phi \, dx \\
&\quad \le C_\phi \left\| \sigma_m - \sigma \right\|_\infty \left\| (u_n)_{xx} \right\|_p \\
&\quad \le C_\phi \left\| \sigma_m - \sigma \right\|_\infty \left\| f \right\|_p \to 0 \quad (m \to 0) \quad \text{uniformly in } n,
\end{aligned}$$

and for fixed m

$$\int_{\mathbb{R}^d} \left(L^{\sigma_m(x)} u_n - L^{\sigma_m(x)} u \right) \phi \, dx \to 0, \quad \text{as} \quad n \to \infty.$$

Hence, we obtain

$$\int_{\mathbb{R}^d} \left(L^{\sigma_n(x)} u_n - L^{\sigma(x)} u \right) \phi \, dx \to 0, \quad \text{as} \quad n \to \infty.$$

Notice that

$$\langle L^{\sigma_n(x)} u_n, \phi \rangle - \langle \lambda u_n, \phi \rangle = \langle f_n, \phi \rangle.$$

Take $n \to \infty$, we obtain

$$\langle L^{\sigma(x)} u, \phi \rangle - \langle \lambda u, \phi \rangle = \langle f, \phi \rangle.$$

On the other hand, let $p_* := \frac{p'}{p'-1}$ and keep in mind $u_n \rightharpoonup u$ in $W^{2,p}(\mathbb{R}^d)$, we have

$$\begin{aligned}
\|u\|_{\alpha,p'} &= \left\| \left(I - \Delta^{\frac{\alpha}{2}} \right) u \right\|_{p'} = \sup_{\phi \in C_0^\infty(\mathbb{R}^d); \|\phi\|_{p_*} \le 1} \left| \int_{\mathbb{R}^d} \left\langle \left(I - \Delta^{\frac{\alpha}{2}} \right) u(x), \phi(x) \right\rangle dx \right| \\
&= \sup_{\phi \in C_0^\infty(\mathbb{R}^d); \|\phi\|_{p_*} \le 1} \lim_{n \to \infty} \left| \int_{\mathbb{R}^d} \left\langle u_n(x), \left(I - \Delta^{\frac{\alpha}{2}} \right) \phi(x) \right\rangle dx \right| \\
&= \sup_{\phi \in C_0^\infty(\mathbb{R}^d); \|\phi\|_{p_*} \le 1} \lim_{n \to \infty} \left| \int_{\mathbb{R}^d} \left\langle \left(I - \Delta^{\frac{\alpha}{2}} \right) u_n(x), \phi(x) \right\rangle dx \right| \\
&\le \sup_n \sup_{\phi \in C_0^\infty(\mathbb{R}^d); \|\phi\|_{p_*} \le 1} \left\| \left(I - \Delta^{\frac{\alpha}{2}} \right) u_n \right\|_{p'} \\
&= \sup_n \|u_n\|_{\alpha,p'} \le C_2 \lambda^{\left(\alpha - 2 + \frac{d}{p} - \frac{d}{p'} \right)/2} \|f\|_p .
\end{aligned}$$

We completed the proof. □

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