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Limit Theorem for Self-intersection Local Time Derivative of Multidimensional Fractional Brownian Motion

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Abstract
The existence condition $H < 1/d$ for first-order derivative of self-intersection local time for $d \geq 3$ dimensional fractional Brownian motion was obtained in Yu (J Theoret Probab 34(4):1749–1774, 2021). In this paper, we establish a limit theorem under the nonexistence critical condition $H = 1/d$.

Keywords Self-intersection local time · Fractional Brownian motion · Limit theorem

Mathematics Subject Classification (2020) Primary 60G22; Secondary 60J55

1 Introduction

Consider a d-dimensional fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1)$, which is a d-dimensional centered Gaussian process $B^H = \{B_t^H,\ t \geq 0\}$ with component processes being independent copies of a 1-dimensional centered Gaussian process $B^{H,i}$, $i = 1, 2, \ldots, d$ and the covariance function given by

$$\mathbb{E}[B_t^{H,i} B_s^{H,i}] = \frac{1}{2}\left[t^{2H} + s^{2H} - |t-s|^{2H}\right].$$

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Note that $B_t^{\frac{1}{2}}$ is a classical standard Brownian motion. Let $D = \{(r, s) : 0 < r < s < t\}$. The self-intersection local time (SLT) of fBm was first investigated in Rosen [13] and formally defined as

$$\alpha_t(y) = \int_D \delta(B_s^H - B_r^H - y)\mathrm{d}r\mathrm{d}s,$$

where B^H is a 2-dimensional fBm and δ is the Dirac delta function. It was further investigated in Hu [4], Hu and Nualart [6]. In particular, Hu and Nualart [6] showed its existence whenever $Hd < 1$. Moreover, $\alpha_t(y)$ is Hölder continuous in time of any order strictly less than $1 - H$ which can be derived from Xiao [15].

The derivative of self-intersection local time (DSLT) for fBm was first considered in the works by Yan et al. [16, 17], where the ideas were borrowed form Rosen [14]. The DSLT for fBm has two versions. One is extended by the Tanaka formula (see in Jung and Markowsky [9]):

$$\widetilde{\alpha}_t'(y) = -H \int_D \delta'(B_s^H - B_r^H - y)(s-r)^{2H-1}\mathrm{d}r\mathrm{d}s.$$

The other is from the occupation-time formula (see Jung and Markowsky [10]):

$$\widehat{\alpha}_t'(y) = -\int_D \delta'(B_s^H - B_r^H - y)\mathrm{d}r\mathrm{d}s.$$

Motivated by the first-order DSLT for fBm in Jung and Markowsky [10] and the k-th-order derivative of intersection local time (ILT) for fBm in Guo et al. [3], we will consider the following k-th-order DSLT for fBm in this paper,

$$\begin{aligned}\widehat{\alpha}_t^{(k)}(y) &= \frac{\partial^k}{\partial_{y_1}^{k_1}\ldots\partial_{y_d}^{k_d}} \int_D \delta(B_s^H - B_r^H - y)\mathrm{d}r\mathrm{d}s \\ &= (-1)^{|k|}\int_D \delta^{(k)}(B_s^H - B_r^H - y)\mathrm{d}r\mathrm{d}s,\end{aligned}$$

where $k = (k_1, \ldots, k_d)$ is a multi-index with all k_i being nonnegative integers and $|k| = k_1 + k_2 + \cdots + k_d$, δ is the Dirac delta function of d variables and $\delta^{(k)}(y) = \frac{\partial^k}{\partial_{y_1}^{k_1}\ldots\partial_{y_d}^{k_d}}\delta(y)$ is the k-th-order partial derivative of δ.

Set

$$f_\varepsilon(x) = \frac{1}{(2\pi\varepsilon)^{\frac{d}{2}}}e^{-\frac{|x|^2}{2\varepsilon}} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\langle p,x\rangle}e^{-\varepsilon\frac{|p|^2}{2}}\mathrm{d}p,$$

where $\langle p, x\rangle = \sum_{j=1}^d p_j x_j$ and $|p|^2 = \sum_{j=1}^d p_j^2$.

Since the Dirac delta function δ can be approximated by $f_\varepsilon(x)$, we approximate $\delta^{(k)}$ and $\widehat{\alpha}_t^{(k)}(y)$ by

$$f_\varepsilon^{(k)}(x) = \frac{i^{|k|}}{(2\pi)^d}\int_{\mathbb{R}^d} p_1^{k_1}\dots p_d^{k_d} e^{i\langle p,x\rangle} e^{-\varepsilon\frac{|p|^2}{2}}\mathrm{d}p$$

and

$$\widehat{\alpha}_{t,\varepsilon}^{(k)}(y) = (-1)^{|k|}\int_D f_\varepsilon^{(k)}(B_s^H - B_r^H - y)\mathrm{d}r\mathrm{d}s, \tag{1.1}$$

respectively.

If $\widehat{\alpha}_{t,\varepsilon}^{(k)}(y)$ converges to a random variable in L^p as $\varepsilon \to 0$, we denote the limit by $\widehat{\alpha}_t^{(k)}(y)$ and call it the k-th DSLT of B^H.

Recently, Yu [18] studied the existence and Hölder continuity conditions of $\widehat{\alpha}_t^{(k)}(y)$ and related limit theorem in critical case. We recall the existence condition for $\widehat{\alpha}_t^{(k)}(y)$ in L^2 as follows.

Theorem 1.1 [18] *For $0 < H < 1$ and $\widehat{\alpha}_{t,\varepsilon}^{(k)}(y)$ defined in (1.1), let $\# := \#\{k_i \text{ is odd}, i = 1,2,\dots d\}$ denotes the odd number of k_i, for $i = 1,2,\dots,d$. If $H < \min\{\frac{2}{2|k|+d}, \frac{1}{|k|+d-\#}, \frac{1}{d}\}$ for $|k| = \sum_{j=1}^d k_j$, then $\widehat{\alpha}_t^{(k)}(0)$ exists in L^2.*

Note that, if $|k| = 1$, the existence condition of $\widehat{\alpha}_t^{(k)}(0)$ is $H < 1/d$, and $Hd = 1$ is the critical condition of $\widehat{\alpha}_t^{(k)}(y)$ for any $d \geq 2$. When $Hd = 1$ for $d = 2$, Markowsky [11] proved the limit theorem for $|k| = 1$.

Theorem 1.2 [11] *$\widehat{\alpha}_{t,\varepsilon}^{(k)}(y)$ is defined in (1.1) with $y = 0$. Suppose that $H = \frac{1}{2}$, $d = 2$ and $|k| = 1$, then as $\varepsilon \to 0$,*

$$\Big(\log 1/\varepsilon\Big)^{-1}\widehat{\alpha}_{t,\varepsilon}^{(k)}(0) \overset{law}{\to} N\left(0,\sigma_0^2\right).$$

In this paper, we will consider the case of $Hd = 1$ for any $d \geq 3$ and $|k| = 1$, and prove a limit theorem for $\widehat{\alpha}_{t,\varepsilon}^{(k)}(0)$. Without loss of generality, we assume that $k_1 = 1, k_2 = 0, \dots, k_d = 0$ for the multi-index $k = (k_1,\dots,k_d)$, and for the convenience of writing, we will abbreviate $\widehat{\alpha}_{t,\varepsilon}^{(1,0,\dots,0)}(0)$ as $\widehat{\alpha}_{t,\varepsilon}^{(1)}(0)$ in the subsequent content of this paper without causing confusion.

Theorem 1.3 *$\widehat{\alpha}_{t,\varepsilon}^{(k)}(y)$ is defined in (1.1) with $y = 0$. Suppose that $Hd = 1$ for any $d \geq 3$ and $|k| = 1$. Then, as $\varepsilon \to 0$, we have*

$$\left(\varepsilon^{-\frac{1}{H}}\log 1/\varepsilon\right)^{H-\frac{1}{2}}\widehat{\alpha}_{t,\varepsilon}^{(1)}(0) \overset{law}{\to} N(0,\sigma^2),$$

where $\sigma^2 = \frac{2Ht^{3-4H}}{(2\pi)^d(1-2H)^2}$.

When $|k| = 1$, under the condition $H > 1/d$, the behavior of $\widehat{\alpha}^{(1)}_{t,\varepsilon}(0)$ as $\varepsilon \to 0$ is also of interest. One would expect a central limit theorem to exist, but this remains unproved. Nevertheless, we venture the following conjecture

(1) If $H = \frac{2}{d+2} > \frac{1}{d}$ and $d \geq 3$, $(\log \frac{1}{\varepsilon})^{\gamma_1(H)}\widehat{\alpha}^{(1)}_{t,\varepsilon}(0)$ converges in distribution to a normal law for some $\gamma_1(H) < 0$;

(2) If $H > \frac{1}{2} \geq \frac{2}{d+2}$ and $d \geq 2$, $\varepsilon^{\gamma_2(H)}\widehat{\alpha}^{(1)}_{t,\varepsilon}(0)$ converges in distribution to a normal law for some $\gamma_2(H) > 0$;

(3) If $\frac{2}{d+2} < H < \frac{1}{2}$ and $d \geq 3$, $\varepsilon^{\gamma_3(H)}(\log \frac{1}{\varepsilon})^{\gamma_4(H)}\widehat{\alpha}^{(1)}_{t,\varepsilon}(0)$ converges in distribution to a normal law for some $\gamma_3(H) > 0$ and $\gamma_4(H) < 0$.

The paper has the following structure. We state some preliminary lemmas in Sect. 2. Section 3 is to prove the main result. Throughout this paper, if not mentioned otherwise, the letter C, with or without a subscript, denotes a generic positive finite constant and may change from line to line.

2 Preliminaries

In this section, we present two basic lemmas, which will be used in Sect. 3. The first lemma gives the bounds on the quantity of $\lambda\rho - \mu^2$, which could be obtained from the Appendix B in [9] or Lemma 3.1 in [4]. In fact, λ, ρ and μ represent the three quantities of the covariance matrix of the increment of fBm, and the bound estimation of $\lambda\rho - \mu^2$ is beneficial for the subsequent calculation of the convergence of multiple integrals, which will bring a lot of convenience to the proof in Sect. 3.

Lemma 2.1 *Let*

$$\lambda = |s - r|^{2H}, \quad \rho = |s' - r'|^{2H},$$

and

$$\mu = \frac{1}{2}\Big(|s' - r|^{2H} + |s - r'|^{2H} - |s' - s|^{2H} - |r - r'|^{2H}\Big).$$

Case (i) *Suppose that $D_1 = \{(r, r', s, s') \in [0, t]^4 \,|\, r < r' < s < s'\}$, let $r' - r = a$, $s - r' = b$, $s' - s = c$. Then, there exists a constant K_1 such that*

$$\lambda\rho - \mu^2 \geq K_1 \left((a+b)^{2H}c^{2H} + a^{2H}(b+c)^{2H}\right)$$

and

$$2\mu = (a+b+c)^{2H} + b^{2H} - a^{2H} - c^{2H}.$$

Case (ii) *Suppose that $D_2 = \{(r, r', s, s') \in [0, t]^4 \mid r < r' < s' < s\}$, let $r' - r = a$, $s' - r' = b$, $s - s' = c$. Then, there exists a constant K_2 such that*

$$\lambda\rho - \mu^2 \geq K_2\, b^{2H}\left(a^{2H} + c^{2H}\right)$$

and

$$2\mu = (a+b)^{2H} + (b+c)^{2H} - a^{2H} - c^{2H}.$$

Case (iii) *Suppose that* $D_3 = \{(r, r', s, s') \in [0,t]^4 \mid r < s < r' < s'\}$, *let* $s - r = a$, $r' - s = b$, $s' - r' = c$. *Then, there exists a constant* K_3 *such that*

$$\lambda\rho - \mu^2 \geq K_3(ac)^{2H}$$

and

$$2\mu = (a+b+c)^{2H} + b^{2H} - (a+b)^{2H} - (c+b)^{2H}.$$

The second lemma shows the Wiener chaos expansion of $\widehat{\alpha}^{(k)}_{t,\varepsilon}(0)$ with $|k| = 1$. Before that, we need to explain some notations. We will denote by $\mathcal{H}$ the Hilbert space obtained by taking the completion of the space of step functions endowed with the inner product

$$\langle \mathbb{1}_{[a,b]}, \mathbb{1}_{[c,d]} \rangle_{\mathcal{H}} = \mathbb{E}[(B^{H,1}_b - B^{H,1}_a)(B^{H,1}_d - B^{H,1}_c)], \tag{2.1}$$

where $B^{H,1}$ is a 1-dimensional fBm. The mapping $\mathbb{1}_{[0,t]} \to B^{H,1}_t$ can be extended to a linear isometry between $\mathcal{H}$ and a Gaussian subspace $L^2(\Omega, \mathcal{F}, \mathbb{P})$. For any integer $q \in \mathbb{N}$, we denote by $\mathcal{H}^{\otimes q}$ and $\mathcal{H}^{\odot q}$ the q-th tensor product of $\mathcal{H}$, and the q-th symmetric tensor product of $\mathcal{H}$, respectively.

Similarly, for d-dimensional fBm $B^H = (B^{H,1}, \ldots, B^{H,d})$, we can define corresponding Hilbert space $\mathcal{H}^d$ and tensor product spaces $(\mathcal{H}^d)^{\otimes q}$ and $(\mathcal{H}^d)^{\odot q}$. If $h = (h^1, \ldots, h^d) \in \mathcal{H}^d$, we set $B^H(h) = \sum_{j=1}^d B^{H,j}(h^j)$. Then $h \mapsto B^H(h)$ is a linear isometry between $\mathcal{H}^d$ and the Gaussian subspace of $L^2(\Omega_1 \times \cdots \times \Omega_d, \mathcal{F}_1 \times \cdots \times \mathcal{F}_d, \mathbb{P}_1 \times \cdots \times \mathbb{P}_d)$ generated by B^H. The q-th Wiener chaos of $L^2(\Omega_1 \times \cdots \times \Omega_d, \mathcal{F}_1 \times \cdots \times \mathcal{F}_d, \mathbb{P}_1 \times \cdots \times \mathbb{P}_d)$, denoted by $\mathfrak{H}_q$, is the closed subspace of $L^2(\Omega_1 \times \cdots \times \Omega_d, \mathcal{F}_1 \times \cdots \times \mathcal{F}_d, \mathbb{P}_1 \times \cdots \times \mathbb{P}_d)$ generated by the variables

$$\Big\{ \prod_{j=1}^d H_{q_j}(B^{H,j}(h^j)) | \sum_{j=1}^d q_j = q, h^j \in \mathcal{H}, \|h^j\|_{\mathcal{H}} = 1 \Big\},$$

where H_q is the q-th Hermite polynomial, defined by

$$H_q(x) = (-1)^q e^{x^2/2} \frac{d^q}{dx^q} e^{-x^2/2}.$$

For $q \in \mathbb{N}$, $q \geq 1$ and $h \in \mathcal{H}^d$ of the form $h = (h^1, \dots, h^d)$ with $\|h^j\|_{\mathcal{H}} = 1$, we can write

$$h^{\otimes q} = \sum_{i_1,\dots,i_q=1}^{d} h^{i_1} \otimes \cdots \otimes h^{i_q}.$$

For such h, we define the mapping

$$I_q(h^{\otimes q}) = \sum_{i_1,\dots,i_q=1}^{d} \prod_{j=1}^{d} H_{q_j(i_1,\dots,i_q)}(B^{H,j}(h^j)),\ j = 1, \dots, d$$

where $q_j(i_1, \dots, i_q)$ denotes the number of indices in $(i_1, \dots, i_q)$ equal to j. The range of I_q is contained in $\mathfrak{H}_q$. This mapping provides a linear isometry between $(\mathcal{H}^d)^{\odot q}$ (equipped with the norm $\sqrt{q!}\|\cdot\|_{(\mathcal{H}^d)^{\otimes q}}$) and $\mathfrak{H}_q$ (equipped with the L^2-norm). Here multiple stochastic integral I_n is the d-dimensional version see in Jaramillo and Nualart [8] (or in Flandoli and Tudor [2]).

It also holds that $I_n(f) = I_n(\widetilde{f})$, where $\widetilde{f}$ denotes the symmetrization of f. We recall that any square integrable random variable F which is measurable with respect to the σ-algebra generated by B^H can be expanded into an orthogonal sum of multiple stochastic integrals

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where $f_n \in (\mathcal{H}^d)^{\odot n}$ are (uniquely determined) symmetric functions and $I_0(f) = \mathbb{E}(F)$.

The proof process of Wiener chaos expansion also requires the knowledge of Malliavin derivative $\mathbb{D}$ with respect to fBm B^H. Denote by $C_b^{\infty}(\mathbb{R}^n)$ the space of bounded smooth functions on $\mathbb{R}^n$. Consider the space of random variables

$$\mathcal{S} := \{F = g(B^H(f_1), \dots, B^H(f_n)),\ g \in C_b^{\infty}(\mathbb{R}^n),\ f_j \in \mathcal{H}^d,\ j = 1, \dots, d\}.$$

The Malliavin derivative of $F \in \mathcal{S}$, denoted by $\mathbb{D}F$, is given by

$$\mathbb{D}F = \sum_{j=1}^{n} \partial_j g(B^H(f_1), \dots, B^H(f_n)) f_j.$$

By iteration, we can define the n-th derivatives D^n for every $n \geq 2$, which is an element of $L^2(\Omega, (\mathcal{H}^d)^{\otimes n})$. For example, we write for the smooth function f,

$$\begin{aligned}\mathbb{D}f(B_t^{H,1}, \dots, B_t^{H,d}) &= \mathbb{D}f(B^H(h_1), \dots, B^H(h_d)) \\ &= \sum_{j=1}^{d} \partial_j f(B^H(h_1), \dots, B^H(h_d))h_j,\quad h_j \in \mathcal{H}^d,\ j = 1, 2, \dots, d,\end{aligned}$$

where $h_j = \mathbb{1}_{[0,t]}e_j$, $j = 1, 2, \ldots, d$ and

$$e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1). \tag{2.2}$$

Similarly,

$$\begin{aligned} &\mathbb{D}^n f(B_t^{H,1}, \ldots, B_t^{H,d}) \\ &= \sum_{i_1,\ldots,i_n=1}^{d} \partial_{i_1} \cdots \partial_{i_n} f(B_t^{H,1}, \ldots, B_t^{H,d}) \bigotimes_{j=1}^{n} (\mathbb{1}_{[0,t]}e_{i_j}), \end{aligned} \tag{2.3}$$

where $\mathbb{1}_{[0,t]}e_{i_j} \in \mathcal{H}^d$, $\bigotimes_{j=1}^n (\mathbb{1}_{[0,t]}e_{i_j}) \in (\mathcal{H}^d)^{\otimes n}$, $i_j \in \{1, 2, \ldots, d\}$, $j = 1, 2, \ldots, n$.

More detailed introductions to Malliavin derivative and multiple stochastic integral can be found in Nualart [12], Hu [5] and the references therein.

Lemma 2.2 *Let $\widehat{\alpha}_{t,\varepsilon}^{(k)}(y)$ be defined in (1.1), then we have the Wiener chaos expansion for $k = (1, 0 \ldots, 0)$,*

$$\widehat{\alpha}_{t,\varepsilon}^{(k)}(0) = \sum_{q=1}^{+\infty} I_{2q-1}(f_{2q-1,\varepsilon}).$$

(i) If $d = 2$, $f_{2q-1,\varepsilon}$ is the element of $(\mathcal{H}^2)^{\otimes(2q-1)}$ given by

$$f_{2q-1,\varepsilon}(x_1, \ldots, x_{2q-1}) = \beta_q \int_{0<r<s<t} (|s-r|^{2H} + \varepsilon)^{-q-1} \bigotimes_{j=1}^{2q-1} (\mathbb{1}_{[r,s]}e_{i_j})(x_j) dr ds,$$

where $\beta_q = \frac{(-1)^q}{2\pi(2q-1)!} \sum_{q_1+q_2=q, q_1 \geq 1} \frac{(2q-1)!(2q_1)!}{(2q_1-1)!(q_1)!(q_2)!2^q}$ and $\mathbb{1}_{[r,s]}e_{i_j} \in \mathcal{H}^2$, $i_j \in \{1, 2\}$, $j = 1, 2, \ldots, 2q-1$, (e_{i_j} defined in (2.2)).

(ii) If $d \geq 3$, $f_{2q-1,\varepsilon} \in (\mathcal{H}^d)^{\otimes(2q-1)}$

$$f_{2q-1,\varepsilon}(x_1, \ldots, x_{2q-1}) = \beta_{q,d} \int_{0<r<s<t} (|s-r|^{2H} + \varepsilon)^{-q-d/2} \bigotimes_{j=1}^{2q-1} (\mathbb{1}_{[r,s]}e_{i_j})(x_j) dr ds,$$

where $\beta_{q,d} = \frac{(-1)^q}{(2q-1)!(2\pi)^{d/2}} \sum_{q_1+\cdots+q_d=q, q_1 \geq 1} \frac{(2q-1)!(2q_1)!}{(2q_1-1)!(q_1)!\cdots(q_d)!2^q}$ and $\mathbb{1}_{[r,s]}e_{i_j} \in \mathcal{H}^d$, $i_j \in \{1, 2, \ldots, d\}$, $j = 1, 2, \ldots, 2q-1$.

Proof The proof adopts a method similar to Lemma 7 in Hu and Nualart [6] (or the Appendix A in Das and Markowsky [1]).

(i) For the case $d = 2$, by Stroock's formula,

$$\widehat{\alpha}_{t,\varepsilon}^{(k)}(0) = \frac{i}{(2\pi)^2} \int_0^t \int_0^s \int_{\mathbb{R}^2} e^{i\langle \xi, B_s^H - B_r^H \rangle} \xi_1 e^{-\varepsilon|\xi|^2/2} d\xi \mathrm{d}r \mathrm{d}s$$

$$= \sum_{q=1}^{+\infty} I_{2q-1}(f_{2q-1,\varepsilon}),$$

where $f_{2q-1,\varepsilon} \in (\mathcal{H}^2)^{\otimes(2q-1)}$ and

$$f_{n,\varepsilon} \equiv f_{n,\varepsilon}(x_1, \dots, x_n) = \frac{1}{n!} \int_{0<r<s<t} \mathbb{E}[\mathbb{D}^n_{x_1,\dots,x_n} \partial_1 f_\varepsilon(B^H_s - B^H_r)] \mathrm{d}r \mathrm{d}s$$

with $x_j \in [0, t]$ for all $j = 1, 2, \dots, n$.

Let $i_j \in \{1, 2\}$ for all $j = 1, 2, \dots, n$. Then by (2.3), we can compute the expectation

$$\mathbb{E}[\mathbb{D}^n_{x_1,\dots,x_n} \partial_1 f_\varepsilon(B^H_s - B^H_r)] = \sum_{i_1,\dots,i_n=1}^{2} \mathbb{E}[\partial_{i_1} \cdots \partial_{i_n} \partial_1 f_\varepsilon(B^H_s - B^H_r)] \bigotimes_{j=1}^{n} (\mathbb{1}_{[r,s]} e_{i_j})(x_j),$$

where $(\mathbb{1}_{[r,s]} e_{i_j})(x_j) \in \mathcal{H}^2$, $i_j \in \{1, 2\}$, $j = 1, 2, \dots, n$, (e_{i_j} defined in (2.2)) and

$$\begin{aligned}
&\mathbb{E}[\partial_{i_1} \cdots \partial_{i_n} \partial_1 f_\varepsilon(B^H_s - B^H_r)] \\
&= \frac{i^{n+1}}{(2\pi)^2} \int_{\mathbb{R}^2} \xi_1 (\xi_{i_1} \xi_{i_2} \cdots \xi_{i_n}) \mathbb{E}[e^{i\langle \xi, B^H_s - B^H_r \rangle}] e^{-\varepsilon|\xi|^2/2} \mathrm{d}\xi \\
&= \frac{i^{n+1}}{(2\pi)^2} \int_{\mathbb{R}^2} \xi_1 (\xi_{i_1} \xi_{i_2} \cdots \xi_{i_n}) e^{-\frac{1}{2}(|s-r|^{2H}+\varepsilon)|\xi|^2} \mathrm{d}\xi \\
&= (i)^{n+1} (2\pi)^{-1} (|s-r|^{2H} + \varepsilon)^{-1-\frac{n+1}{2}} \mathbb{E}[X_1 X_{i_1} X_{i_2} \cdots X_{i_n}],
\end{aligned}$$

with the independent identical distribution standard Gaussian random variables X_{i_n} and

$$\mathbb{E}[X_1 X_{i_1} X_{i_2} \cdots X_{i_n}] = \begin{cases} \frac{(2m_1)!(2m_2)!}{(m_1)!(m_2)!2^m}, & \text{if} n = 2(m_1 + m_2) - 1, \\ & \text{ the number of } i_k = 1 \text{ is } 2m_1 - 1 \\ & \text{ and the number of } i_k = 2 \text{is } 2m_2, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $n = 2q - 1 = 2(q_1 + q_2) - 1$ with the number of $i_k = 1$ is $2q_1 - 1$ and the number of $i_k = 2$ is $2q_2$, the summation

$$\sum_{i_1,\dots,i_n=1}^{2} \mathbb{1}_{\{n=2(q_1+q_2)-1\}} \mathbb{1}_{\{\#\{i_k=1\}=2q_1-1\}} \mathbb{1}_{\{\#\{i_k=2\}=2q_2\}} = \sum_{q_1+q_2=q, q_1 \geq 1} \frac{(2q-1)!}{(2q_1-1)!(2q_2)!},$$

where $\#\{i_k = x\}$ denotes the number of $i_k = x$. This gives

$$\sum_{i_1,\dots,i_n=1}^{2} \mathbb{E}[X_1 X_{i_1} X_{i_2} \cdots X_{i_n}]$$

$$= \sum_{i_1,\dots,i_n=1}^{2} \mathbb{1}_{\{n=2(q_1+q_2)-1\}} \mathbb{1}_{\{\#\{i_k=1\}=2q_1-1\}} \mathbb{1}_{\{\#\{i_k=2\}=2q_2\}} \frac{(2q_1)!(2q_2)!}{(q_1)!(q_2)!2^q}$$
$$= \sum_{q_1+q_2=q, q_1\geq 1} \frac{(2q-1)!}{(2q_1-1)!(2q_2)!} \frac{(2q_1)!(2q_2)!}{(q_1)!(q_2)!2^q}.$$

Thus, we have

$$f_{2q-1,\varepsilon}(x_1,\dots,x_{2q-1}) = \beta_q \int_{0<r<s<t} (|s-r|^{2H}+\varepsilon)^{-q-1} \bigotimes_{j=1}^{2q-1} (\mathbb{1}_{[r,s]} e_{i_j})(x_j) \mathrm{d}r\mathrm{d}s,$$

where $\beta_q = \frac{(-1)^q}{2\pi(2q-1)!} \sum_{q_1+q_2=q, q_1\geq 1} \frac{(2q-1)!(2q_1)!}{(2q_1-1)!(q_1)!(q_2)!2^q}$.
(ii) Similarly, we can prove the case of $d \geq 3$.

$$\widehat{\alpha}_{t,\varepsilon}^{(k)}(0) = \sum_{q=1}^{+\infty} I_{2q-1}(f_{2q-1,\varepsilon}),$$

where $f_{2q-1,\varepsilon} \in (\mathcal{H}^d)^{\otimes(2q-1)}$ and

$$f_{n,\varepsilon}(x_1,\dots,x_n) = \frac{(\iota)^{n+1}}{n!} \frac{1}{(2\pi)^{d/2}} \int_{0<r<s<t} (|s-r|^{2H}+\varepsilon)^{-(n+1/2-d/2)} \bigotimes_{j=1}^{n} (\mathbb{1}_{[r,s]} e_{i_j})(x_j) \mathrm{d}r\mathrm{d}s$$
$$\times \sum_{i_1,\dots,i_n=1}^{d} \mathbb{E}[X_1 X_{i_1} X_{i_2} \cdots X_{i_n}],$$

with $(\mathbb{1}_{[r,s]} e_{i_j})(x_j) \in \mathcal{H}^d$, $i_j \in \{1,2,\dots,d\}$, $j = 1,2,\dots,n$.
Note that

$$\mathbb{E}[X_1 X_{i_1} X_{i_2} \cdots X_{i_n}] = \begin{cases} \frac{(2m_1)!\cdots(2m_d)!}{(m_1)!\cdots(m_d)!2^m}, & \text{if } n = 2(m_1+\cdots+m_2)-1, \\ & \text{the number of } i_k = 1 \text{ is } 2m_1 - 1 \\ & \text{and the number of } i_k = \ell \text{ is } 2m_\ell \\ & \text{for } \ell = 2,\dots,d, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $n = 2q-1 = 2(q_1+\cdots+q_d)-1$ with the number of $i_k = 1$ is $2q_1 - 1$ and the number of $i_k = \ell$ is $2q_\ell$, the summation

$$\sum_{i_1,\dots,i_n=1}^{d} \mathbb{1}_{\{n=2(q_1+\cdots+q_d)-1\}} \mathbb{1}_{\{\#\{i_k=1\}=2q_1-1\}} \mathbb{1}_{\{\#\{i_k=2\}=2q_2\}} \times \cdots \times \mathbb{1}_{\{\#\{i_k=d\}=2q_d\}}$$
$$= \sum_{q_1+\cdots+q_d=q, q_1\geq 1} \frac{(2q-1)!}{(2q_1-1)!(2q_2)!\cdots(2q_d)!}.$$

This gives

$$\sum_{i_1,\ldots,i_n=1}^{d} \mathbb{E}[X_1 X_{i_1} X_{i_2} \cdots X_{i_n}] = \sum_{q_1+\cdots+q_d=q, q_1\geq 1} \frac{(2q-1)!}{(2q_1-1)!(2q_2)!\cdots(2q_d)!} \frac{(2q_1)!\cdots(2q_d)!}{(q_1)!\cdots(q_d)!2^q}$$
$$= \sum_{q_1+\cdots+q_d=q, q_1\geq 1} \frac{(2q-1)!(2q_1)!}{(2q_1-1)!(q_1)!\cdots(q_d)!2^q}.$$

Thus,

$$f_{2q-1,\varepsilon}(x_1,\ldots,x_{2q-1}) = \beta_{q,d} \int_{0<r<s<t} (|s-r|^{2H}+\varepsilon)^{-q-d/2} \bigotimes_{j=1}^{2q-1} (\mathbb{1}_{[r,s]} e_{i_j})(x_j) \mathrm{d}r \mathrm{d}s,$$

where $\beta_{q,d} = \frac{(-1)^q}{(2q-1)!(2\pi)^{d/2}} \sum_{q_1+\cdots+q_d=q, q_1\geq 1} \frac{(2q-1)!(2q_1)!}{(2q_1-1)!(q_1)!\cdots(q_d)!2^q}$. □

Lemma 2.3 *If $Hd = 1$, as $\varepsilon \to 0$, we have*
(i)

$$\int_0^{\varepsilon^{-\frac{1}{H}}} x^{H-\frac{1}{2}}(1+x^H)^{-\frac{d}{2}-1} dx = O\left(\log \frac{1}{\varepsilon}\right)$$

and
(ii)

$$\int_0^1 x^{2H}(\varepsilon + x^{2H})^{-\frac{d}{2}-1} dx = O\left(\log \frac{1}{\varepsilon}\right).$$

Proof For (i), by L'Hôspital's rule, we have

$$\lim_{\varepsilon\to 0} \frac{1}{\log \frac{1}{\varepsilon}} \int_0^{\varepsilon^{-\frac{1}{H}}} x^{H-\frac{1}{2}}(1+x^H)^{-\frac{d}{2}-1} \mathrm{d}x = \lim_{\varepsilon\to 0} \frac{1}{H} \varepsilon^{-1-\frac{1}{2H}}(1+\varepsilon^{-1})^{-\frac{d}{2}-1}$$
$$= \lim_{\varepsilon\to 0} \frac{1}{H}(\varepsilon+1)^{-\frac{d}{2}-1}$$
$$= \frac{1}{H},$$

where we use the condition $Hd = 1$ in the second equality.

For (ii), take the variable transformation $x = y\varepsilon^{\frac{1}{2H}}$,

$$\lim_{\varepsilon\to 0} \frac{1}{\log \frac{1}{\varepsilon}} \int_0^1 x^{2H}(\varepsilon + x^{2H})^{-\frac{d}{2}-1} \mathrm{d}x = \lim_{\varepsilon\to 0} \frac{1}{\log \frac{1}{\varepsilon}} \int_0^{\varepsilon^{-\frac{1}{2H}}} y^{2H}(1+y^{2H})^{-\frac{d}{2}-1} \mathrm{d}y$$
$$= \lim_{\varepsilon\to 0} \frac{1}{2H}(\varepsilon+1)^{-\frac{d}{2}-1}$$

$$= \frac{1}{2H},$$

where we use L'Hôspital's rule and the condition $Hd = 1$ in the second equality.

□

3 Proof of Theorem 1.3

In this section, the proof of Theorem 1.3 is taken into account, we will consider the case of $Hd = 1$ for any $d \geq 3$ and $|k| = 1$. By Lemma 2.2, $\widehat{\alpha}^{(1)}_{t,\varepsilon}(0)$ has the following chaos decomposition

$$\widehat{\alpha}^{(1)}_{t,\varepsilon}(0) = \sum_{q=1}^{\infty} I_{2q-1}(f_{2q-1,\varepsilon}),$$

where

$$f_{2q-1,\varepsilon}(x_1, \ldots, x_{2q-1}) = \int_D f_{2q-1,\varepsilon,s,r}(x_1, \ldots, x_{2q-1})\mathrm{d}r\mathrm{d}s$$

with $D = \{(r, s) : 0 < r < s < t\}$, where

$$f_{2q-1,\varepsilon,s,r}(x_1, \ldots, x_{2q-1}) := \beta_{q,d}(|s-r|^{2H} + \varepsilon)^{-q-d/2} \bigotimes_{j=1}^{2q-1} (\mathbb{1}_{[r,s]} e_{i_j})(x_j).$$

For $q = 1$,

$$\mathbb{E}\Big[\Big|I_1(f_{1,\varepsilon})\Big|^2\Big] = \int_{D^2} \langle f_{1,\varepsilon,s,r}, f_{1,\varepsilon,s',r'}\rangle_{\mathcal{H}^d} \mathrm{d}r\mathrm{d}s\mathrm{d}r'\mathrm{d}s', \tag{3.1}$$

where $\mathcal{H}^d$ is the Hilbert space obtained by taking the completion of the step functions (see in Sect. 2).

For $q > 1$, we have to describe the terms $\langle f_{2q-1,\varepsilon,s_1,r_1}, f_{2q-1,\varepsilon,s_2,r_2}\rangle_{(\mathcal{H}^d)^{\otimes(2q-1)}}$, where $(\mathcal{H}^d)^{\otimes(2q-1)}$ is the $(2q-1)$-th tensor product of $\mathcal{H}^d$. For every $x,\ u_1,\ u_2 > 0$, we define

$$\mu(x, u_1, u_2) = |\mathbb{E}[B^{H,1}_{u_1}(B^{H,1}_{x+u_2} - B^{H,1}_x)]|. \tag{3.2}$$

For $j = 1, 2, \ldots, 2q-1$, $i_j \in \{1, 2, \ldots, d\}$,

$$\langle \mathbb{1}_{[r,s]} e_{i_j}, \mathbb{1}_{[r,s]} e_{i_j}\rangle_{\mathcal{H}^d} = \langle \mathbb{1}_{[r,s]}, \mathbb{1}_{[r,s]}\rangle_{\mathcal{H}}. \tag{3.3}$$

Then, we have

$$\langle f_{2q-1,\varepsilon,s_1,r_1}, f_{2q-1,\varepsilon,s_2,r_2}\rangle_{(\mathcal{H}^d)^{\otimes(2q-1)}}$$

$$
\begin{aligned}
&= \beta_{q,d}^2 (|s_1 - r_1|^{2H} + \varepsilon)^{-q-d/2} (|s_2 - r_2|^{2H} + \varepsilon)^{-q-d/2} \\
&\quad \times \langle \mathbb{1}_{[r_1,s_1]}^{2q-1}, \mathbb{1}_{[r_2,s_2]}^{2q-1} \rangle_{\mathcal{H}^{\otimes(2q-1)}} \\
&=: \beta_{q,d}^2 G_{\varepsilon, r_2 - r_1}^{(q,d)} (s_1 - r_1, s_2 - r_2),
\end{aligned} \tag{3.4}
$$

where

$$
G_{\varepsilon,x}^{(q,d)}(u_1, u_2) = \left(\varepsilon + u_1^{2H}\right)^{-\frac{d}{2}-q} \left(\varepsilon + u_2^{2H}\right)^{-\frac{d}{2}-q} \mu(x, u_1, u_2)^{2q-1}.
$$

Note that equations (3.2)–(3.4) here can degenerate into the case $d = 1$ of the equations (2.18)–(2.19) in Jaramillo and Nualart [7].

Before completing the proof of the main result, we give some useful lemmas below.

Lemma 3.1

$$
\lim_{\varepsilon \to 0} \mathbb{E}\Big[\Big(\frac{1}{\log \frac{1}{\varepsilon}} \widehat{\alpha}_{t,\varepsilon}^{(1)}(0)\Big)^2\Big] = \sigma^2,
$$

where $\sigma^2 = \frac{2Ht^{3-4H}}{(2\pi)^d (1-2H)^2}$.

Proof Let $(X, Y) \in \mathbb{R} \times \mathbb{R}$ be a jointly Gaussian vector with mean zero and covariance $A = (A_{i,j})_{i,j=1,2}$, let f_A is the density of (X, Y) and $f_{1,\varepsilon}(x) = \frac{1}{\sqrt{2\pi\varepsilon}} e^{-\frac{x^2}{2\varepsilon}}$, $x \in \mathbb{R}$ be a 1-dimensional density function. Then,

$$
\begin{aligned}
\mathbb{E}[XY f_{1,\varepsilon}(X) f_{1,\varepsilon}(Y)] &= \int_{\mathbb{R}^2} xy f_{1,\varepsilon}(x) f_{1,\varepsilon}(y) f_A(x, y) \mathrm{d}x\mathrm{d}y \\
&= (2\pi)^{-2} \varepsilon^{-1} |A|^{-1/2} \int_{\mathbb{R}^2} xy e^{-\frac{1}{2}(x,y)(\varepsilon^{-1} I + A^{-1})(x,y)^T} \mathrm{d}x\mathrm{d}y \\
&= (2\pi)^{-1} \varepsilon^{-1} |A|^{-1/2} |\widetilde{A}|^{1/2} \int_{\mathbb{R}^2} xy f_{\widetilde{A}}(x, y) \mathrm{d}x\mathrm{d}y \\
&= (2\pi)^{-1} |\varepsilon I + A|^{-1/2} \widetilde{A}_{1,2} \\
&= (2\pi)^{-1} \varepsilon^2 |\varepsilon I + A|^{-\frac{3}{2}} A_{1,2},
\end{aligned}
$$

where $\widetilde{A} := (\varepsilon^{-1} I + A^{-1})^{-1}$, $f_{\widetilde{A}}$ denotes the density of a Gaussian vector with mean zero and covariance $\widetilde{A} = (\widetilde{A}_{i,j})_{i,j=1,2}$.

Similarly, let $\Sigma = (\Sigma_{i,j})_{i,j=1,2}$ be the covariance matrix of $(B_s^{H,1} - B_r^{H,1}, B_{s'}^{H,1} - B_{r'}^{H,1})$, and Σ^{d-1} be the covariance matrix of $(\widetilde{B}_s^H - \widetilde{B}_r^H, \widetilde{B}_{s'}^H - \widetilde{B}_{r'}^H)$ ($\widetilde{B}^H$ denotes the $(d-1)$-dimensional fBm). The notations f_Σ and $f_{\Sigma^{d-1}}$ represent their density functions, respectively. It is easy to find that Σ^{d-1} is a block diagonal matrix, and that the dimension of it is $2(d-1) \times 2(d-1)$. Then,

$$
(2\pi\varepsilon)^{-(d-1)} \int_{\mathbb{R}^{2(d-1)}} e^{-\frac{x_2^2 + y_2^2 + \cdots + x_d^2 + y_d^2}{2\varepsilon}} f_{\Sigma^{d-1}}(x_2, \ldots, x_d, y_2, \ldots, y_d) \mathrm{d}x_2 \cdots \mathrm{d}x_d \mathrm{d}y_2 \cdots \mathrm{d}y_d
$$

$$= \left((2\pi\varepsilon)^{-1} \int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2\varepsilon}} f_\Sigma(x,y) \mathrm{d}x \mathrm{d}y \right)^{d-1}$$
$$= \left((2\pi)^{-1} |\varepsilon I + \Sigma|^{-1/2}\right)^{d-1}.$$

Thus, for any Gaussian vector $(X, Y) \in \mathbb{R}^d \times \mathbb{R}^d$ and k-th ($k = (1, 0, \dots, 0)$) order derivative, we have

$$\begin{aligned}
\mathbb{E}[f_\varepsilon^{(1)}(X) f_\varepsilon^{(1)}(Y)] &= \frac{1}{\varepsilon^2} (2\pi\varepsilon)^{-d} \mathbb{E}\left[X_1 Y_1 e^{-\frac{X_1^2+\cdots+X_d^2+Y_1^2+\cdots+Y_d^2}{2\varepsilon}} \right] \\
&= \frac{1}{\varepsilon^2} (2\pi\varepsilon)^{-1} \int_{\mathbb{R}^2} x_1 y_1 e^{-\frac{x_1^2+y_1^2}{2\varepsilon}} f_\Sigma(x_1, y_1) \mathrm{d}x_1 \mathrm{d}y_1 \\
&\quad \times (2\pi\varepsilon)^{-(d-1)} \int_{\mathbb{R}^{2(d-1)}} e^{-\frac{x_2^2+y_2^2+\cdots+x_d^2+y_d^2}{2\varepsilon}} f_{\Sigma^{d-1}}(\widetilde{x}, \widetilde{y}) d\widetilde{x} d\widetilde{y} \\
&= \varepsilon^{-2} (2\pi)^{-1} \varepsilon^2 |\varepsilon I + \Sigma|^{-\frac{3}{2}} \Sigma_{1,2} \times (2\pi)^{-(d-1)} |\varepsilon I + \Sigma|^{-\frac{d-1}{2}} \\
&= (2\pi)^{-d} |\varepsilon I + \Sigma|^{-\frac{d}{2}-1} \Sigma_{1,2},
\end{aligned}$$

where $\widetilde{x} = (x_2, \dots, x_d)$, $\widetilde{y} = (y_2, \dots, y_d)$.

Thus,

$$\mathbb{E}\left[\left| \widehat{\alpha}_{t,\varepsilon}^{(1)}(0) \right|^2 \right] = V_1(\varepsilon) + V_2(\varepsilon) + V_3(\varepsilon)$$

with

$$V_i(\varepsilon) = \frac{2}{(2\pi)^d} \int_{D_i} |\varepsilon I + \Sigma|^{-\frac{d}{2}-1} |\mu| \mathrm{d}r \mathrm{d}s \mathrm{d}r' \mathrm{d}s', \tag{3.5}$$

where D_i (i=1, 2, 3) defined in Lemma 2.1 and Σ is a covariance matrix with $\Sigma_{1,1} = \lambda$, $\Sigma_{2,2} = \rho$, $\Sigma_{1,2} = \mu$ given in Lemma 2.1.

Next, we will split the proof into three parts to consider $V_1(\varepsilon)$, $V_2(\varepsilon)$ and $V_3(\varepsilon)$, respectively.

For the $V_1(\varepsilon)$ term, changing the coordinates (r, r', s, s') by $(r, a = r' - r, b = s - r', c = s' - s)$ and integrating the r variable, we get

$$\begin{aligned}
V_1(\varepsilon) &\le C \int_{[0,t]^4} |\varepsilon I + \Sigma|^{-\frac{d}{2}-1} |\mu| \mathrm{d}r \mathrm{d}a \mathrm{d}b \mathrm{d}c \\
&= C \int_{[0,t]^3} |\varepsilon I + \Sigma|^{-\frac{d}{2}-1} |\mu| \mathrm{d}a \mathrm{d}b \mathrm{d}c \\
&=: \widetilde{V}_1(\varepsilon).
\end{aligned}$$

Applying Lemma 2.1 Case (i), for some $C > 0$, we get

$$|\varepsilon I + \Sigma| = (\varepsilon + \Sigma_{1,1})(\varepsilon + \Sigma_{2,2}) - \Sigma_{1,2}^2 = \varepsilon^2 + \varepsilon(\Sigma_{1,1} + \Sigma_{2,2}) - |\Sigma|$$

$$\geq C\Big[\varepsilon^2 + \varepsilon((a+b)^{2H} + (b+c)^{2H}) + a^{2H}(c+b)^{2H} + c^{2H}(a+b)^{2H}\Big]$$
$$\geq C\Big[\varepsilon^2 + (a+b)^H(b+c)^H(\varepsilon + (ac)^H)\Big]$$
$$\geq C(a+b)^H(b+c)^H(\varepsilon + (ac)^H), \tag{3.6}$$

where we use the Young's inequality in the second to last inequality.

Substituting (3.6) and

$$|\mu| = \frac{1}{2}\left|(a+b+c)^{2H} + b^{2H} - a^{2H} - c^{2H}\right| \leq \sqrt{\lambda\rho} = (a+b)^H(b+c)^H$$

into the integrand of $\widetilde{V}_1(\varepsilon)$,

$$\widetilde{V}_1(\varepsilon) \leq C\int_{[0,t]^3}(a+b)^{-\frac{Hd}{2}}(b+c)^{-\frac{Hd}{2}}\Big(\varepsilon + (ac)^H\Big)^{-\frac{d}{2}-1}\mathrm{d}a\mathrm{d}b\mathrm{d}c$$
$$\leq C\int_{[0,t]^3}(a+b)^{H-\frac{Hd}{2}}(a+b)^{-H}(b+c)^{-\frac{Hd}{2}}\Big(\varepsilon + (ac)^H\Big)^{-\frac{d}{2}-1}\mathrm{d}a\mathrm{d}b\mathrm{d}c$$
$$\leq C\int_{[0,t]^3}b^{-H-\frac{Hd}{2}}a^{H-\frac{Hd}{2}}\Big(\varepsilon + (ac)^H\Big)^{-\frac{d}{2}-1}\mathrm{d}a\mathrm{d}b\mathrm{d}c$$
$$\leq C\varepsilon^{\frac{1}{H}-\frac{d}{2}-1}\int_0^{t\varepsilon^{-\frac{1}{H}}}\int_0^t a^{H-\frac{Hd}{2}}\Big(1 + (ac)^H\Big)^{-\frac{d}{2}-1}\mathrm{d}a\mathrm{d}c,$$

where we make the change of variable $c = c\,\varepsilon^{-\frac{1}{H}}$ in the last inequality.

By L'Hôspital's rule, we have

$$\lim_{\varepsilon\to 0}\widetilde{V}_1(\varepsilon) \leq \lim_{\varepsilon\to 0}\frac{-\frac{Ct}{H}\varepsilon^{-1-\frac{1}{H}}\int_0^t a^{H-\frac{1}{2}}(1+t^H a^H\varepsilon^{-1})^{-\frac{d}{2}-1}da}{(1-\frac{1}{2H})\varepsilon^{-\frac{1}{2H}}}$$
$$= \lim_{\varepsilon\to 0}\frac{\frac{Ct}{H}}{\frac{1}{2H}-1}\int_0^{t\varepsilon^{-\frac{1}{H}}}a^{H-\frac{1}{2}}(1+t^H a^H)^{-\frac{d}{2}-1}\mathrm{d}a$$
$$= O\left(\log\frac{1}{\varepsilon}\right),$$

where we have used Lemma 2.3 in the last equality.

So, we can obtain

$$\lim_{\varepsilon\to 0}\left(\varepsilon^{-\frac{1}{H}}\log 1/\varepsilon\right)^{2H-1}V_1(\varepsilon) = 0. \tag{3.7}$$

For the $V_2(\varepsilon)$ term, changing the coordinates (r, r', s, s') by $(r, a = r' - r, b = s' - r', c = s - s')$ and integrating the r variable, then by (3.5), we get

$$V_2(\varepsilon) \leq C \int_{[0,t]^3} |\varepsilon I + \Sigma|^{-\frac{d}{2}-1} |\mu| \mathrm{d}a\mathrm{d}b\mathrm{d}c =: \widetilde{V_2}(\varepsilon).$$

By Lemma 2.1 Case (ii),

$$\begin{aligned}|\varepsilon I + \Sigma| = (\varepsilon + \Sigma_{1,1})(\varepsilon + \Sigma_{2,2}) - \Sigma_{1,2}^2 \geq \varepsilon^2 + \varepsilon((a+b+c)^{2H} + b^{2H}) \\ + K_2 b^{2H}(a^{2H} + c^{2H}).\end{aligned}$$

Then, we have

$$\widetilde{V_2}(\varepsilon) \leq C \int_{[0,t]^3} |\mu| \Big(\varepsilon((a+b+c)^{2H} + b^{2H}) + b^{2H}(a^{2H} + c^{2H})\Big)^{-\frac{d}{2}-1} \mathrm{d}a\mathrm{d}b\mathrm{d}c.$$

Next, we need to estimate this integral over the regions $\{b \leq (a \vee c)\}$ and $\{b > (a \vee c)\}$ separately, and denote these two integrals by $\widetilde{V_{2,1}}(\varepsilon)$ and $\widetilde{V_{2,2}}(\varepsilon)$, respectively. Note that

$$\begin{aligned}|\mu| &= \frac{1}{2}\Big((a+b)^{2H} + (b+c)^{2H} - a^{2H} - c^{2H}\Big) \\ &= Hb \int_0^1 \Big((a+bv)^{2H-1} + (c+bv)^{2H-1}\Big) \mathrm{d}v \\ &\leq b^{2H} \wedge \Big(2Hb(a \wedge c)^{2H-1}\Big).\end{aligned}$$

If $b \leq (a \vee c)$, we choose $|\mu| \leq b^{2H}$. Thus,

$$\begin{aligned}\widetilde{V_{2,1}}(\varepsilon) &\leq C \int_{[0,t]^3} b^{2H} \Big(\varepsilon(a \vee c)^{2H} + b^{2H}(a \vee c)^{2H}\Big)^{-\frac{d}{2}-1} \mathrm{d}a\mathrm{d}b\mathrm{d}c \\ &\leq C \int_{[0,t]^3} (a \vee c)^{-1-2H} b^{2H} \Big(\varepsilon + b^{2H}\Big)^{-\frac{d}{2}-1} \mathrm{d}a\mathrm{d}b\mathrm{d}c \\ &\leq C \int_0^t b^{2H} \Big(\varepsilon + b^{2H}\Big)^{-\frac{d}{2}-1} \mathrm{d}b \\ &= O(\log \frac{1}{\varepsilon}), \ \ as\ \varepsilon \to 0,\end{aligned}$$

where we have used Lemma 2.3 in the last equality and the following fact

$$\varepsilon^2 + \varepsilon((a+b+c)^{2H} + b^{2H}) + b^{2H}(a^{2H} + c^{2H}) \geq \varepsilon(a \vee c)^{2H} + b^{2H}(a \vee c)^{2H}.$$

If $b > (a \vee c)$, we choose $|\mu| \le 2Hb(a \wedge c)^{2H-1}$. Similarly, we have

$$\begin{aligned}\limsup_{\varepsilon\to 0}\frac{\widetilde{V_{2,2}}(\varepsilon)}{\log\frac{1}{\varepsilon}} &\le \limsup_{\varepsilon\to 0}\frac{C}{\log\frac{1}{\varepsilon}}\int_{[0,t]^3} b(a\wedge c)^{2H-1}[b^{2H}(\varepsilon+(a\vee c)^{2H})]^{-\frac{d}{2}-1}\mathrm{d}a\mathrm{d}b\mathrm{d}c\\ &\le \limsup_{\varepsilon\to 0}\frac{C}{\log\frac{1}{\varepsilon}}\int_0^t b^{-2H}db\int_{[0,t]^2}(a\wedge c)^{2H-1}(\varepsilon+(a\vee c)^{2H})]^{-\frac{d}{2}-1}\mathrm{d}c\mathrm{d}a\\ &\le \limsup_{\varepsilon\to 0}\frac{C}{\log\frac{1}{\varepsilon}}\int_0^t\int_0^a c^{2H-1}[\varepsilon+a^{2H}]^{-\frac{d}{2}-1}\mathrm{d}c\mathrm{d}a\\ &= \limsup_{\varepsilon\to 0}\frac{C}{\log\frac{1}{\varepsilon}}\int_0^t a^{2H}(\varepsilon+a^{2H})^{-\frac{d}{2}-1}\mathrm{d}a<\infty,\end{aligned}$$

where we have used the following fact

$$\varepsilon^2+\varepsilon((a+b+c)^{2H}+b^{2H})+b^{2H}(a^{2H}+c^{2H})\ge \varepsilon b^{2H}+b^{2H}(a\vee c)^{2H}.$$

So, by the above result, we can obtain

$$\lim_{\varepsilon\to 0}\left(\varepsilon^{-\frac{1}{H}}\log 1/\varepsilon\right)^{2H-1}V_2(\varepsilon)=0. \tag{3.8}$$

For the $V_3(\varepsilon)$ term.

$$V_3(\varepsilon)=\frac{2}{(2\pi)^d}\int_{D_3}|\varepsilon I+\Sigma|^{-d/2-1}|\mu|\mathrm{d}s\mathrm{d}r\mathrm{d}s'\mathrm{d}r'.$$

By changing the coordinates (r, r', s, s') by $(r, a = s - r, b = r' - s, c = s' - r')$, then from (3.2) and Lemma 2.1 Case (iii), we can write

$$\begin{aligned}\mu(a+b,a,c)=|\mu| &= \frac{1}{2}\Big|(a+b+c)^{2H}+b^{2H}-(b+c)^{2H}-(a+b)^{2H}\Big|\\ &= H(1-2H)ac\int_0^1\int_0^1(b+ax+cy)^{2H-2}\mathrm{d}x\mathrm{d}y.\end{aligned}$$

and $|\varepsilon I + \Sigma| = \varepsilon^2 + \varepsilon(a^{2H} + c^{2H}) + (ac)^{2H} - \mu(a + b, a, c)^2$. It is not hard to see that

$$\begin{aligned}V_3(\varepsilon) &= \frac{2}{(2\pi)^d}\int_{[0,t]^3}\mathbb{1}_{(0,t)}(a+b+c)(t-a-b-c)|\varepsilon I+\Sigma|^{-d/2-1}|\mu|\mathrm{d}a\mathrm{d}b\mathrm{d}c\\ &= \frac{2}{(2\pi)^d}\int_{[0,t\varepsilon^{-\frac{1}{2H}}]^2\times[0,t]}\mathbb{1}_{(0,t)}(b+\varepsilon^{\frac{1}{2H}}(a+c))\\ &\quad\times\frac{(t-b-\varepsilon^{\frac{1}{2H}}(a+c))\mu(\varepsilon^{\frac{1}{2H}}a+b,\varepsilon^{\frac{1}{2H}}a,\varepsilon^{\frac{1}{2H}}c)}{\left[(1+a^{2H})(1+c^{2H})-\varepsilon^{-2}\mu(\varepsilon^{\frac{1}{2H}}a+b,\varepsilon^{\frac{1}{2H}}a,\varepsilon^{\frac{1}{2H}}c)^2\right]^{\frac{d}{2}+1}}\varepsilon^{\frac{1}{H}-2(d/2+1)}\mathrm{d}a\mathrm{d}c\mathrm{d}b,\end{aligned}$$

where we change the coordinates (a, b, c) by $(\varepsilon^{-\frac{1}{2H}} a, b, \varepsilon^{-\frac{1}{2H}} c)$ in the last equality. Denote

$$\widetilde{V}_3(\varepsilon) = \frac{2}{(2\pi)^d} \int_{O_{\varepsilon,3}} \mathbb{1}_{(0,t)}(b + \varepsilon^{\frac{1}{2H}}(a+c)) \times \frac{(t-b-\varepsilon^{\frac{1}{2H}}(a+c))\mu(\varepsilon^{\frac{1}{2H}} a + b, \varepsilon^{\frac{1}{2H}} a, \varepsilon^{\frac{1}{2H}} c)}{\Big[(1+a^{2H})(1+c^{2H}) - \varepsilon^{-2}\mu(\varepsilon^{\frac{1}{2H}} a + b, \varepsilon^{\frac{1}{2H}} a, \varepsilon^{\frac{1}{2H}} c)^2\Big]^{\frac{d}{2}+1}} \varepsilon^{\frac{1}{H}-2(d/2+1)} \mathrm{d}a\mathrm{d}c\mathrm{d}b,$$

where $O_{\varepsilon,3} = \{[0, t\varepsilon^{-\frac{1}{2H}}]^2 \times [(\log \frac{1}{\varepsilon})^{-1}, t]\}$.

We conclude that

$$\lim_{\varepsilon\to 0} \left(\varepsilon^{-\frac{1}{H}} \log 1/\varepsilon\right)^{2H-1} V_3(\varepsilon) = \lim_{\varepsilon\to 0} \left(\varepsilon^{-\frac{1}{H}} \log 1/\varepsilon\right)^{2H-1} \widetilde{V}_3(\varepsilon). \tag{3.9}$$

Indeed,

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \left(\varepsilon^{-\frac{1}{H}} \log 1/\varepsilon\right)^{2H-1} |V_3(\varepsilon) - \widetilde{V}_3(\varepsilon)| \\
&\le \limsup_{\varepsilon\to 0} C_{H,d,t} \left(\varepsilon^{-\frac{1}{H}} \log 1/\varepsilon\right)^{2H-1} \int_{[0,t\varepsilon^{-\frac{1}{2H}}]^2\times[0,(\log\frac{1}{\varepsilon})^{-1}]} \varepsilon\mu(a,a,c) \\
&\quad \times \Big[(1+a^{2H})(1+c^{2H}) - \mu^2(a,a,c)\Big]^{-\frac{d}{2}-1} \varepsilon^{\frac{1}{H}-2(d/2+1)} \mathrm{d}a\mathrm{d}c\mathrm{d}b \\
&\le \limsup_{\varepsilon\to 0} C_{H,d,t} \left(\varepsilon^{-\frac{1}{H}} \log 1/\varepsilon\right)^{2H-1} (\varepsilon \log \frac{1}{\varepsilon})^{-1} \\
&\quad \times \int_{[0,t\varepsilon^{-\frac{1}{2H}}]^2} (a\wedge c)^{2H} \Big[(1+a^{2H})(1+c^{2H})\Big]^{-\frac{d}{2}-1} \mathrm{d}a\mathrm{d}c \\
&\le \limsup_{\varepsilon\to 0} C_{H,d,t} \varepsilon^{\frac{1}{H}-3} \Big(\log 1/\varepsilon\Big)^{2H-2} \int_{[0,t\varepsilon^{-\frac{1}{2H}}]^2} (ac)^{H} \Big[(1+a^{2H})(1+c^{2H})\Big]^{-\frac{d}{2}-1} \mathrm{d}a\mathrm{d}c \\
&\le \limsup_{\varepsilon\to 0} C_{H,d,t} \varepsilon^{\frac{1}{H}-3} \Big(\log 1/\varepsilon\Big)^{2H-2} \\
&= 0,
\end{aligned}$$

where we use

$$\mu(\varepsilon^{\frac{1}{2H}} a + b, \varepsilon^{\frac{1}{2H}} a, \varepsilon^{\frac{1}{2H}} c) \le \mu(\varepsilon^{\frac{1}{2H}} a + 0, \varepsilon^{\frac{1}{2H}} a, \varepsilon^{\frac{1}{2H}} c) = \varepsilon\mu(a,a,c)$$

in the first inequality and use $\mu(a,a,c) \le (a\wedge c)^{2H}$,

$$(1+a^{2H})(1+c^{2H}) - \mu^2(a,a,c) \ge \frac{3}{4}(1+a^{2H})(1+c^{2H})$$

in the second inequality.

By the definition of $\mu(a+b,a,c)$, it is easy to find

$$\mu(\varepsilon^{\frac{1}{2H}}a+b,\varepsilon^{\frac{1}{2H}}a,\varepsilon^{\frac{1}{2H}}c)=H(1-2H)\varepsilon^{\frac{1}{H}}ac\int_{[0,1]^2}(b+\varepsilon^{\frac{1}{2H}}av_1+\varepsilon^{\frac{1}{2H}}cv_2)^{2H-2}\mathrm{d}v_1\mathrm{d}v_2$$

and use the Taylor's theorem for integrand,

$$\varepsilon^{-\frac{1}{H}}\mu(\varepsilon^{\frac{1}{2H}}a+b,\varepsilon^{\frac{1}{2H}}a,\varepsilon^{\frac{1}{2H}}c)=H(1-2H)acb^{2H-2}+O(\varepsilon^{\frac{1}{2H}}ac(a+c)).$$

Similarly, the denominator of the integrand in $V_3(\varepsilon)$ can be rewritten as

$$\begin{aligned}&\Big[(1+a^{2H})(1+c^{2H})-\varepsilon^{-2}\mu(\varepsilon^{\frac{1}{2H}}a+b,\varepsilon^{\frac{1}{2H}}a,\varepsilon^{\frac{1}{2H}}c)^2\Big]^{-\frac{d}{2}-1}\\&\quad=\Big[(1+a^{2H})(1+c^{2H})\Big]^{-\frac{d}{2}-1}+O\Big(\varepsilon^{\frac{2}{H}-2}a^2c^2[(1+a^{2H})(1+c^{2H})]^{-\frac{d}{2}-3}\Big).\end{aligned}$$

It is easy to see that

$$\lim_{\varepsilon\to 0}\Big(\log 1/\varepsilon\Big)^{2H-1}\int_{(\log\frac{1}{\varepsilon})^{-1}}^{t}b^{2H-2}\mathrm{d}b=\frac{1}{1-2H},$$

and

$$\begin{aligned}&\lim_{\varepsilon\to 0}\Big(\varepsilon^{-\frac{1}{H}}\Big)^{2H-1}\int_{[0,t\varepsilon^{-\frac{1}{2H}}]^2}\varepsilon^{\frac{1}{2H}+\frac{2}{H}-d-2}ac(a+c)\Big[(1+a^{2H})(1+c^{2H})\Big]^{-\frac{d}{2}-1}\mathrm{d}a\mathrm{d}c\\&\quad+\lim_{\varepsilon\to 0}\Big(\varepsilon^{-\frac{1}{H}}\Big)^{2H-1}\int_{[0,t\varepsilon^{-\frac{1}{2H}}]^2}\varepsilon^{\frac{2}{H}-2+\frac{2}{H}-d-2}a^3c^3\Big[(1+a^{2H})(1+c^{2H})\Big]^{-\frac{d}{2}-3}\mathrm{d}a\mathrm{d}c\\&=0.\end{aligned}\tag{3.10}$$

Then, by L'Hôspital's rule, we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}\Big(\varepsilon^{-\frac{1}{H}}\log 1/\varepsilon\Big)^{2H-1}\widetilde{V}_3(\varepsilon)\\&=H(1-2H)\frac{2}{(2\pi)^d}\lim_{\varepsilon\to 0}\Big(\log 1/\varepsilon\Big)^{2H-1}\int_{(\log\frac{1}{\varepsilon})^{-1}}^{t}(t-b)b^{2H-2}\mathrm{d}b\\&\quad\times\lim_{\varepsilon\to 0}\Big(\varepsilon^{-\frac{1}{H}}\Big)^{2H-1}\int_{[0,t\varepsilon^{-\frac{1}{2H}}]^2}ac\Big[(1+a^{2H})(1+c^{2H})\Big]^{-\frac{d}{2}-1}\mathrm{d}a\mathrm{d}c\\&=H(1-2H)\frac{2}{(2\pi)^d}\times\frac{t}{1-2H}\times\frac{t^{2-4H}}{(1-2H)^2}.\end{aligned}\tag{3.11}$$

Together (3.7), (3.8), (3.9) and (3.11), we can see

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\Big|\Big(\varepsilon^{-\frac{1}{H}}\log 1/\varepsilon\Big)^{H-1/2}\widehat{\alpha}^{(1)}_{t,\varepsilon}(0)\Big|^2\Big]=\frac{2Ht^{3-4H}}{(2\pi)^d(1-2H)^2}=:\sigma^2.$$

□

Lemma 3.2 *For $I_1(f_{1,\varepsilon})$ given in (3.1), then*

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\Big|\Big(\varepsilon^{-\frac{1}{H}}\log 1/\varepsilon\Big)^{H-1/2}I_1(f_{1,\varepsilon})\Big|^2\Big]=\sigma^2.$$

Proof From (3.1), we can find

$$\mathbb{E}\Big[\Big|I_1(f_{1,\varepsilon})\Big|^2\Big]=\Big(V_1^{(1)}(\varepsilon)+V_2^{(1)}(\varepsilon)+V_3^{(1)}(\varepsilon)\Big), \tag{3.12}$$

where $V_i^{(1)}(\varepsilon) = 2\int_{D_i}\langle f_{1,\varepsilon,s_1,r_1}, f_{1,\varepsilon,s_2,r_2}\rangle_{\mathcal{H}^d}dr_1dr_2ds_1ds_2$ for $i = 1,2,3$, and $\langle f_{1,\varepsilon,s_1,r_1}, f_{1,\varepsilon,s_2,r_2}\rangle_{\mathcal{H}^d}$ was defined in (3.4). Then we have

$$0\le V_i^{(1)}(\varepsilon)\le V_i(\varepsilon). \tag{3.13}$$

Combining (3.13) with (3.7) and (3.8), we can see

$$\lim_{\varepsilon\to 0}\Big(\varepsilon^{-\frac{1}{H}}\log 1/\varepsilon\Big)^{2H-1}\Big(V_1^{(1)}(\varepsilon)+V_2^{(1)}(\varepsilon)\Big)=0.$$

Thus, we only need to consider $\Big(\varepsilon^{-\frac{1}{H}}\log 1/\varepsilon\Big)^{2H-1}V_3^{(1)}(\varepsilon)$ as $\varepsilon\to 0$.

By (3.1), (3.4) and (3.12) we have

$$\begin{aligned}
V_3^{(1)}(\varepsilon) &= 2\beta_{1,d}^2\int_{D_3}G^{(1)}_{\varepsilon,r'-r}(s-r,s'-r')\mathrm{d}r\mathrm{d}s\mathrm{d}r'\mathrm{d}s'\\
&= 2\beta_{1,d}^2\int_{[0,t]^3}\int_0^{t-(a+b+c)}\mathbb{1}_{(0,t)}(a+b+c)(\varepsilon+a^{2H})^{-d/2-1}\\
&\quad\times(\varepsilon+c^{2H})^{-d/2-1}\mu(a+b,a,c)ds_1\mathrm{d}a\mathrm{d}b\mathrm{d}c\\
&= 2H(1-2H)\beta_{1,d}^2\int_0^t\int_{[0,t\varepsilon^{-\frac{1}{2H}}]^2}\int_{[0,1]^2}\mathbb{1}_{(0,t)}\Big((b+\varepsilon^{\frac{1}{2H}}(a+c)\Big)\\
&\quad\times\Big(t-b-\varepsilon^{\frac{1}{2H}}(a+c)\Big)\\
&\quad\times\Big[(1+a^{2H})(1+c^{2H})\Big]^{-d/2-1}ac\Big(b+\varepsilon^{\frac{1}{2H}}(av_1+cv_2)\Big)^{2H-2}dv_1dv_2\mathrm{d}a\mathrm{d}c\mathrm{d}b.
\end{aligned}$$

Note that

$$\int_{[0,1]^2}\Big(b+\varepsilon^{\frac{1}{2H}}(av_1+cv_2)\Big)^{2H-2}\mathrm{d}v_1\mathrm{d}v_2=b^{2H-2}+O(\varepsilon^{\frac{1}{2H}}(a+c))$$

and

$$\int_{[0,1]^2}\Big(t-b-\varepsilon^{\frac{1}{2H}}(a+c)\Big)\Big[(1+a^{2H})(1+c^{2H})\Big]^{-d/2-1}$$

$$
\begin{aligned}
&ac\Big(b+\varepsilon^{\frac{1}{2H}}(av_1+cv_2)\Big)^{2H-2}\mathrm{d}v_1\mathrm{d}v_2\\
&=(t-b)b^{2H-2}ac\Big[(1+a^{2H})(1+c^{2H})\Big]^{-d/2-1}\\
&+O\left(\varepsilon^{\frac{1}{2H}}(a+c)ac\Big[(1+a^{2H})(1+c^{2H})\Big]^{-d/2-1}\right).
\end{aligned}
$$

Similar to (3.9), we get

$$
\lim_{\varepsilon\to 0}\left(\varepsilon^{-\frac{1}{H}}\log 1/\varepsilon\right)^{2H-1}V_3^{(1)}(\varepsilon)=\lim_{\varepsilon\to 0}\left(\varepsilon^{-\frac{1}{H}}\log 1/\varepsilon\right)^{2H-1}\widetilde{V}_3^{(1)}(\varepsilon),
$$

where

$$
\begin{aligned}
\widetilde{V}_3^{(1)}(\varepsilon)&=2H(1-2H)\beta_{1,d}^2\int_{(\log\frac{1}{\varepsilon})^{-1}}^t\int_{[0,t\varepsilon^{-\frac{1}{2H}}]^2}\int_{[0,1]^2}\mathbb{1}_{(0,t)}\Big((b+\varepsilon^{\frac{1}{2H}}(a+c)\Big)\\
&\Big(t-b-\varepsilon^{\frac{1}{2H}}(a+c)\Big)\\
&\times\Big[(1+a^{2H})(1+c^{2H})\Big]^{-d/2-1}ac\Big(b+\varepsilon^{\frac{1}{2H}}(av_1+cv_2)\Big)^{2H-2}\mathrm{d}v_1\mathrm{d}v_2\mathrm{d}a\mathrm{d}c\mathrm{d}b.
\end{aligned}
$$

According to (3.10) and (3.11), we can find that

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\left(\varepsilon^{-\frac{1}{H}}\log 1/\varepsilon\right)^{2H-1}\widetilde{V}_3^{(1)}(\varepsilon)\\
&=2H(1-2H)\beta_{1,d}^2\lim_{\varepsilon\to 0}\Big(\log 1/\varepsilon\Big)^{2H-1}\int_{(\log\frac{1}{\varepsilon})^{-1}}^t(t-b)b^{2H-2}\mathrm{d}b\\
&\quad\times\lim_{\varepsilon\to 0}\left(\varepsilon^{-\frac{1}{H}}\right)^{2H-1}\int_{[0,t\varepsilon^{-\frac{1}{2H}}]^2}ac\Big[(1+a^{2H})(1+c^{2H})\Big]^{-\frac{d}{2}-1}\mathrm{d}a\mathrm{d}c\\
&=H(1-2H)\frac{2}{(2\pi)^d}\times\frac{t}{1-2H}\times\frac{t^{2-4H}}{(1-2H)^2}=\sigma^2,
\end{aligned}
$$

where we use $\beta_{1,d}^2=\frac{1}{(2\pi)^d}$ in the second equality.

Thus,

$$
\lim_{\varepsilon\to 0}\left(\varepsilon^{-\frac{1}{H}}\log 1/\varepsilon\right)^{2H-1}V_3^{(1)}(\varepsilon)=\sigma^2.
$$

□

Proof of Theorem 1.3 By Lemmas 3.1–3.2 and

$$
\widehat{\alpha}_{t,\varepsilon}^{(1)}(0)=I_1(f_{1,\varepsilon})+\sum_{q=2}^{\infty}I_{2q-1}(f_{2q-1,\varepsilon}),
$$

we can see

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\Big|\Big(\varepsilon^{-\frac{1}{H}}\log 1/\varepsilon\Big)^{H-1/2}\sum_{q=2}^{\infty} I_{2q-1}(f_{2q-1,\varepsilon})\Big|^2\Big]=0.$$

Since $I_1(f_{1,\varepsilon})$ is Gaussian, we have, as $\varepsilon\to 0$,

$$\Big(\varepsilon^{-\frac{1}{H}}\log 1/\varepsilon\Big)^{H-1/2} I_1(f_{1,\varepsilon}) \overset{law}{\to} N(0,\sigma^2).$$

Thus,

$$\Big(\varepsilon^{-\frac{1}{H}}\log 1/\varepsilon\Big)^{H-1/2} \widehat{\alpha}^{(1)}_{t,\varepsilon}(0) \overset{law}{\to} N(0,\sigma^2),$$

as $\varepsilon\to 0$. This completes the proof. □

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Data Availability The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declarations

Conflict of interest The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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