Journal of Theoretical Probability (2024) 37:307–351
https://doi.org/10.1007/s10959-023-01251-y

Volterra Equations Driven by Rough Signals 3: Probabilistic Construction of the Volterra Rough Path for Fractional Brownian Motions

Fabian Harang[2] · Samy Tindel[1] · Xiaohua Wang[1]

Received: 13 June 2022 / Revised: 27 June 2022 / Accepted: 21 February 2023 /
Published online: 6 May 2023

Abstract
Based on the recent development of the framework of Volterra rough paths (Harang and Tindel in Stoch Process Appl 142:34–78, 2021), we consider here the probabilistic construction of the Volterra rough path associated to the fractional Brownian motion with $H > \frac{1}{2}$ and for the standard Brownian motion. The Volterra kernel $k(t, s)$ is allowed to be singular, and behaving similar to $|t - s|^{-\gamma}$ for some $\gamma \geq 0$. The construction is done in both the Stratonovich and Itô senses. It is based on a modified Garsia–Rodemich–Romsey lemma which is of interest in its own right, as well as tools from Malliavin calculus. A discussion of challenges and potential extensions is provided.

Keywords Volterra equation · Signature · Rough path · Malliavin calculus · Fractional Brownian motion

Mathematics Subject Classification (2020) Primary: 60L20 · 60L10; Secondary: 60H07 · 60H05 · 60G22

✉ Fabian Harang
fabian.a.harang@bi.no

Samy Tindel
stindel@purdue.edu

Xiaohua Wang
wang3296@purdue.edu

1 Department of Mathematics, Purdue University, 150 N. University Street, West Lafayette, IN 47907, USA

2 Department of Economics, BI Norwegian Business School, Handelshøyskolen BI, 0442 Oslo, Norway

1 Introduction

Volterra equations are used in a great variety of sciences to model evolution phenomena where memory effects are present in the dynamics. Volterra equations typically take the form

$$y_t = y_0 + \int_0^t k(t,s) f(y_s) \mathrm{d}x_s, \quad t \in [0,T], \tag{1.1}$$

where k is called Volterra kernel defined on $[0,T]^2$, and is possibly singular on the diagonal. The process x represents a potentially highly irregular control, such as a stochastic processes or otherwise nowhere differentiable path. In applications, the process x is typically stochastic, or a realized path associated to a stochastic process. Thus making sense of the integral appearing in (1.1) can be challenging, and the construction will often highly depend on the assumptions of both the kernel k and the driving process x. Applying Itô's theory for stochastic differential equations, construction of the integral as well as stochastic well-posedness of the equation has been well established in the case when the process x is a Brownian motion and the kernel k satisfies the integrability condition $k(t,\cdot) \in L^2([0,t])$ for all $t \in [0,T]$, see e.g. [20, 21].

Volterra equations have recently received much attention towards modeling of rough volatility. In this case it is desirable to allow x to be a more generic stochastic processes, such as a fractional Brownian motion, while at the same time allowing the Volterra kernel k to be singular, see e.g. [1, 5, 11] and the references therein. With this application in mind, the authors of [1] observed that for numerical computations related to rough volatility modeling, a pathwise approach to Volterra equations of the form (1.1) is highly useful. The singular Volterra kernel in combination with the driving Brownian motion created divergence in the covariation $\langle \int_0^\cdot k(\cdot,s)\mathrm{d}W_s, W\rangle$ appearing as an Itô-Stratonovich correction, requiring an infinite-type renormalization procedure. Based on the modern theory of Regularity Structures, the authors proved well-posedness of Eq. (1.1) under such renormalization.

An alternative approach to deal with Volterra equations in a pathwise manner was proposed in [15]. There, a new generic methodology based on the theory of rough paths was proposed to treat Banach-valued Volterra equations like (1.1) in the case when the Volterra kernel $k(t,s)$ is behaving similarly to $|t-s|^{-\gamma}$ for some $\gamma \geq 0$, and the driving signal x is only assumed to be Hölder continuous (with Hölder regularity possibly lower than $1/2$). The Volterra rough path framework is developed around a splitting of the arguments in a Volterra process, in the sense that one lifts the classical form of Volterra process $z_t := \int_0^t k(t,s)\mathrm{d}x_s$ defined on $[0,T]$, to a two parameter object defined on the simplex $\Delta_2[0,T] := \{(s,t) \in [0,T]^2;\ s \leq t\}$ given formally by

$$z_t^\tau := \int_0^t k(\tau,s)\mathrm{d}x_s, \quad t \leq \tau. \tag{1.2}$$

Clearly, when the two parameter object is restricted to the diagonal in $[0, T]^2$, we have $z_t^t = z_t$, obtaining the classical type of Volterra process. The advantage of viewing the Volterra process as this two parameter object is that one can easily distinguish between the regularity contributed by the driving signal versus the possible singularity obtained from the kernel k, thus making pathwise regularity analysis easier, and sewing based arguments more straightforward.

In a similar spirit as for classical rough paths, the idea is to lift the Volterra signal $(t, \tau) \mapsto z_t^\tau$ as defined in (1.2), to a signature type object (see e.g. [8, 19]) resembling a collection of iterated integrals, satisfying certain algebraic relations, which is called the Volterra signature. In this article we will focus on the second-order lift; $z \mapsto (\mathbf{z}^1, \mathbf{z}^2)$. In the case of a smooth signal x, the two components take the form

$$\mathbf{z}_{ts}^{1,\tau} = z_{ts}^\tau, \qquad \mathbf{z}_{ts}^{2,\tau} = \int_s^t k(\tau, r) \int_s^r k(r, u)\, \mathrm{d}x_u\, \mathrm{d}x_r. \tag{1.3}$$

In contrast to classical rough path theory, the Volterra signature does not satisfy Chen's relation with the tensor product, but a convolution type product is required in order to obtain an equivalent algebraic relation. Indeed, by definition of $\mathbf{z}^2$ above, one can readily check that for any $s \le u \le t \le \tau$

$$\mathbf{z}_{ts}^{2,\tau} - \mathbf{z}_{tu}^{2,\tau} - \mathbf{z}_{us}^{2,\tau} \neq \mathbf{z}_{tu}^{1,\tau} \otimes \mathbf{z}_{us}^{1,\tau}.$$

However, as observed in [15] the following generalized Chen's relation holds:

$$\mathbf{z}_{ts}^{2,\tau} - \mathbf{z}_{tu}^{2,\tau} - \mathbf{z}_{us}^{2,\tau} = \mathbf{z}_{tu}^{1,\tau} * \mathbf{z}_{us}^{1,\cdot}$$

where the convolution product $*$ used on the right hand side is defined by

$$\mathbf{z}_{tu}^{1,\tau} * \mathbf{z}_{us}^{1,\cdot} := \lim_{|\mathcal{P}| \to 0} \sum_{[u',v'] \in \mathcal{P}} \mathbf{z}_{v'u'}^{1,\tau} \otimes \mathbf{z}_{us}^{1,u'}. \tag{1.4}$$

Notice that in the classical rough paths setting where $k(t, s) \equiv 1$, then $z_t^\tau = z_t$, and $\mathbf{z}^1 * \mathbf{z}^1 = \mathbf{z}^1 \otimes \mathbf{z}^1$. It is the introduction of a Volterra kernel which requires an extension of the classical tensor product in order to obtain a suitable Chen's relation for the Volterra rough path. So far, the assumption has been that x is a smooth path, and in this case both the iterated integral in (1.3) and the convolution product constructed as an integral in (1.4) exist by standard integration arguments. However, in the theory of rough paths we are interested in irregular, nowhere differentiable signals x, requiring a careful analysis of the construction of these objects. Once this is in place, the Volterra signature in combination with certain controlled Volterra paths is used in [15] to prove existence and uniqueness of solutions to (1.1) in a purely pathwise manner.

Although [15] provides the basic framework for Volterra rough paths, two important problems relating to this theory was left open:

Analytic extension On the analytic side, [15] only deals with the case when $\alpha - \gamma \ge 1/3$ (where we recall that α is the regularity of the signal, while γ is the possible order of singularity from the kernel k). To get a complete analytic picture of the framework of Volterra rough paths, this regime must be extended to $\alpha - \gamma > 0$.

Probabilistic construction For completeness of the framework it is crucial to provide a complete probabilistic construction of the lift of a stochastic Volterra process into a Volterra rough path, analogues to the rough path lift for stochastic processes.

Regarding the analytic problem, extending this regime was dealt with in the article [16], where the algebraic framework was described for $\alpha - \gamma \geq 1/4$. In a very recent article [4], Bruned and Kastetsiadis extends this even further to all $\alpha - \gamma > 0$ by invoking algebraic theories similar to that used for non-geometric rough paths [12, 14] and regularity structures [13].

The problem of a probabilistic construction of the Volterra rough path is the main goal of the current article. More specifically, our main contribution is twofold:

(i) As the framework for Volterra rough paths relies on spaces for Volterra–Hölder paths with two parameters (one corresponding to regularity and one to singularity), a direct application of the classical Kolmogorov continuity theorem will not provide sufficient answers. Hence new arguments need to be developed, specifically suited for the type of Hölder spaces necessary to properly define rough Volterra equations. Extending the Garsia–Rodemich–Rumsey (GRR) inequality to suit Volterra paths is therefore the first aim of this article. This extension is not only highly useful for the probabilistic treatment in the context of Volterra rough paths, but could also prove valuable towards applications for other types of singular Hölder norms, such as those considered in [2].

(ii) With the singular GRR inequality in hand, we provide a construction of the Volterra rough path in the regime $\alpha - \gamma \geq \frac{1}{3}$ (requiring one iterated integral) by using tools from the theory of Malliavin calculus. This construction is both done in the case when the driving stochastic process is a fractional Brownian motion with $H > \frac{1}{2}$ together with a singular kernel. We also handle the case of classical Brownian motion with a singular kernel. Note that in both cases the construction of a Volterra rough path like (1.3) is required. Indeed, a singular kernel behaving similarly to $|t - s|^{-\gamma}$ pushes down the regularity of the Volterra process constructed from an fBm to be $H - \gamma$. This exponent can be smaller than $\frac{1}{2}$ even though $H > \frac{1}{2}$. This is in contrast to the classical rough path regime, where the rough path associated to a fractional Brownian motion with $H > \frac{1}{2}$ can simply be constructed by classical Young theory.

As the reader can see from the description above, our analysis will be a delicate combination of analytic and probabilistic techniques.

The article is organized as follows: Sect. 2 provides an overview of Volterra paths, Volterra-Hölder spaces, as well as a summary of central concepts from [15] regarding the convolution product and Volterra sewing. In Sect. 3 the extension of the GRR inequality is provided. Section 4 deals with the construction of the Volterra rough path for fractional Brownian motion with $H > \frac{1}{2}$. In Sect. 5 this construction is extended to the case of a regular Brownian motion. A discussion on further extensions to rough fractional Brownian motion is discussed in the end of Sect. 5.

2 Preliminary Results

In [15] and [16], the Volterra rough formalism was based on certain spaces of functions having specific regularity/singularity features. Before defining the proper spaces quantifying this type of regularity, let us introduce some notation:

Notation 2.1 Let $T > 0$ be a finite time horizon, and $n \geq 2$. Then the simplex Δ_n^T is defined by

$$\Delta_n^T = \left\{(s_1, \ldots, s_n) \in [0, T]^n; 0 \leq s_1 < \cdots < s_n \leq T\right\}.$$

When this causes no ambiguity, we will abbreviate Δ_n^T as Δ_n. For $(s, t) \in \Delta_2$, we designate $\mathcal{P}$ to be a generic partition of $[s, t]$. Two successive points forming an interval contained in this partition are written as $[u, v] \in \mathcal{P}$.

The functions quantifying our regularities are also labeled in the following notation.

Notation 2.2 Consider four parameters $\alpha, \gamma \in (0, 1)$ and $\zeta, \eta \in [0, 1]$ satisfying

$$\rho \equiv \alpha - \gamma > 0, \qquad 0 \leq \zeta \leq \inf(\rho, \eta). \tag{2.1}$$

For $(s, t, \tau', \tau) \in \Delta_4$, we set

$$\psi^1_{(\alpha,\gamma)}(\tau, t, s) = \left[|\tau - t|^{-\gamma}|t - s|^{\alpha}\right] \wedge |t - s|^{\rho}, \tag{2.2}$$

and

$$\psi^{1,2}_{(\alpha,\gamma,\eta,\zeta)}(\tau, \tau', t, s) = |\tau - \tau'|^{\eta}|\tau' - t|^{-(\eta-\zeta)} \left(\left[|\tau' - t|^{-\gamma-\zeta}|t - s|^{\alpha}\right] \wedge |t - s|^{\rho-\zeta}\right). \tag{2.3}$$

In the next definition the functional spaces called $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ are introduced, which are equivalent to those used in [15, 16]. As is evident from the analysis in [15, 16], these spaces are natural function sets when dealing with Volterra type regularities.

Definition 2.3 Consider four parameters $\alpha, \gamma \in (0, 1)$ and $\zeta, \eta \in [0, 1]$ satisfying relation (2.1), and fix $m \geq 1$. Throughout the article we consider functions $z : \Delta_2 \to \mathbb{R}^m$ of the form $(t, \tau) \mapsto z_t^\tau$, such that $z_0^\tau = z_0$ for all $\tau \in (0, T]$. We define the space of Volterra paths of index $(\alpha, \gamma, \eta, \zeta)$, denoted by $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_2; \mathbb{R}^m)$, as the set of such functions satisfying

$$\|z\|_{(\alpha,\gamma,\eta,\zeta)} = \|z\|_{(\alpha,\gamma),1} + \|z\|_{(\alpha,\gamma,\eta,\zeta),1,2} < \infty. \tag{2.4}$$

Recalling Notation 2.1 and 2.2, the 1-norms and 1,2-norms in (2.4) are respectively defined as follows:

$$\|z\|_{(\alpha,\gamma),1} = \sup_{(s,t,\tau)\in\Delta_3} \frac{|z_{ts}^{\tau}|}{\psi^1_{(\alpha,\gamma)}(\tau,t,s)}, \tag{2.5}$$

$$\|z\|_{(\alpha,\gamma,\eta,\zeta),1,2} = \sup_{(s,t,\tau',\tau)\in\Delta_4} \frac{|z_{ts}^{\tau\tau'}|}{\psi^{1,2}_{(\alpha,\gamma,\eta,\zeta)}(\tau,\tau',t,s)}, \tag{2.6}$$

with the convention $z_{ts}^{\tau} = z_t^{\tau} - z_s^{\tau}$ and $z_s^{\tau\tau'} = z_s^{\tau} - z_s^{\tau'}$. Notice that under the mapping

$$z \mapsto |z_0| + \|z\|_{(\alpha,\gamma,\eta,\zeta)},$$

the space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ is a Banach space.

Remark 2.4 As mentioned in [16, Remark 2.6], the spaces $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ enjoy embedding properties of the form $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)} \subset \mathcal{V}^{(\beta,\gamma,\eta,\zeta)}$ for $0 < \alpha < \beta < 1$. In addition, the norms defined by (2.4)-(2.6) verify the following relation on $[0, T]$:

$$\|y\|_{(\beta,\gamma),1} \le T^{\alpha-\beta}\|y\|_{(\alpha,\gamma),1}, \quad \|y\|_{(\beta,\gamma,\eta,\zeta),1,2} \le T^{\alpha-\beta}\|y\|_{(\alpha,\gamma,\eta,\zeta),1,2},$$
$$\|y\|_{(\beta,\gamma,\eta,\zeta)} \le T^{\alpha-\beta}\|y\|_{(\alpha,\gamma,\eta,\zeta)}.$$

Remark 2.5 The spaces given in Definition 2.3 are slightly different than those introduced in [15], as there is no supremum over the parameters η and ζ appearing here. It was observed in [16] that the supremum was unnecessary for the Volterra rough path methodology to work, but one must instead introduce an assumption that the Volterra paths of interest is contained in a suitable family Volterra spaces as those in Definition 2.3. Avoiding the original supremum also makes probabilistic analysis, as we will consider in the subsequent sections, more tractable. We therefore use the same types of norms and spaces as the ones introduced in [16]

Remark 2.6 Comparing (2.2) and (2.3), one can relate the functions ψ^1 and $\psi^{1,2}$ in the following way:

$$\psi^{1,2}_{(\alpha,\gamma,\eta,\zeta)}(\tau,\tau',t,s) = \left|\tau-\tau'\right|^{\eta}\left|\tau'-t\right|^{-(\eta-\zeta)}\psi^1_{(\alpha,\gamma+\zeta)}(\tau',t,s) \tag{2.7}$$

As explained in [16, Proposition 2.10], the parameter η above accounts for the regularity of a Volterra path in the upper variables τ, τ'. Then one plays with extra parameters ζ in order to get regularities for paths of the form $r \mapsto z_r^r$.

As illustrated in the introduction, convolution products plays a central role for the subsequent considerations of the Volterra rough path. Let us recall a proposition from [16] giving explicit meaning to this concept, and establishing the existence in a general setting.

Proposition 2.7 *We consider two Volterra paths $z \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\mathbb{R}^m)$ and $y \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\mathcal{L}(\mathbb{R}^m))$ as given in Definition 2.3. On top of condition (2.1), we assume* *that the exponents α, η are such that $\eta > 1-\alpha$. Otherwise stated, our parameters $\alpha, \gamma, \zeta, \eta$ satisfy*

$$\rho \equiv \alpha - \gamma > 0, \qquad 0 \le \zeta \le \inf(\rho, \eta), \quad \textit{and} \quad \eta > 1-\alpha. \tag{2.8}$$

Then the convolution product of the two Volterra paths y and z is a bilinear operation on $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\mathbb{R}^m)$ given by

$$z_{tu}^{\tau} * y_{us}^{\cdot} = \int_{t>r>u} dz_r^{\tau} y_{us}^{r} := \lim_{|\mathcal{P}|\to 0} \sum_{[u',v']\in\mathcal{P}} z_{v'u'}^{\tau} y_{us}^{u'}, \tag{2.9}$$

where $\mathcal{P}$ is a generic partition of $[u,t]$ for which we recall Notation 2.1. The integral in (2.9) is understood as a Volterra-Young integral for all $(s,u,t,\tau) \in \Delta_4$. Moreover, the following two inequalities hold for any tuple (s,u,t,τ,τ') lying in Δ_5:

$$\left|z_{tu}^{\tau} * y_{us}^{\cdot}\right| \lesssim \|z\|_{(\alpha,\gamma),1} \|y\|_{(\alpha,\gamma,\eta,\zeta),1,2}\, \psi^{1}_{(2\rho+\gamma,\gamma)}(\tau,t,s), \tag{2.10}$$

$$\left|z_{tu}^{\tau'\tau} * y_{us}^{\cdot}\right| \lesssim \|z\|_{(\alpha,\gamma,\eta,\zeta),1,2} \|y\|_{(\alpha,\gamma,\eta,\zeta),1,2}\, \psi^{1,2}_{(2\rho+\gamma,\gamma,\eta,\zeta)}(\tau,\tau',t,s). \tag{2.11}$$

Commonly for rough path based theories, we will also here work with the δ–operator. The next notation recalls this operator which will be significant for subsequent proofs.

Notation 2.8 Let g be a path from Δ_2 to $\mathbb{R}^m$, and consider $(s,u,t) \in \Delta_3$. Then the quantity $\delta_u g_{ts}$ is defined by

$$\delta_u g_{ts} = g_{ts} - g_{tu} - g_{us}. \tag{2.12}$$

With Definition 2.3 and Proposition 2.7 in hand we are now ready to state the main assumption used in [16]. Namely the Volterra rough paths analysis relies on the ability to construct a family $\{z^{j,\tau};\, j \le n\}$ of Volterra iterated integrals according to the following definition:

Definition 2.9 Consider $\alpha, \gamma \in (0,1)$, and for an arbitrary finite integer $N \ge 1$, let $\{\zeta_k, \eta_k;\, 1 \le k \le N\}$ be a family of exponents satisfying the relation (2.8). Then for $n = \lfloor \rho^{-1} \rfloor$, $\{\mathbf{z}^{j,\tau};\, j \le n\}$ is assumed to enjoy the following properties:

(i) $\mathbf{z}^1 = z$ and $\mathbf{z}_{ts}^{j,\tau} \in (\mathbb{R}^m)^{\otimes j}$.
(ii) For all $j \le n$ and $(s,t,\tau) \in \Delta_3$ we have

$$\delta_u \mathbf{z}_{ts}^{j,\tau} = \sum_{i=1}^{j-1} \mathbf{z}_{tu}^{j-i,\tau} * \mathbf{z}_{us}^{i,\cdot} = \sum_{i=1}^{j-1} \int_s^t d\mathbf{z}_{tr}^{j-i,\tau}\, \mathbf{z}_{us}^{i,r}, \tag{2.13}$$

where the right hand side of (2.13) is given by Proposition 2.7.
(iii) For all $j = 1, \dots, n$, we have $\mathbf{z}^j \in \bigcap_{k=1}^{N} \mathcal{V}^{(j\rho+\gamma,\gamma,\eta_k,\zeta_k)}$.

As the reader might have observed, Definition 2.9 is a natural extension of the more classical definition of rough path, see e.g. [6], where convolution products naturally extends the tensor product for Volterra paths. In the decomposition (2.13), it is desirable to quantify the regularity of the objects depending on the variables $(s, u, t, \tau) \in \Delta_4$. Thus a small variation of Definition 2.3 is suitable for this quantification, illustrated in the next definition (see also in [16, Definition 2.9]).

Definition 2.10 As in Definition 2.3, consider $m \geq 1$, as well as four parameters $\alpha, \gamma \in (0, 1)$, $\eta, \zeta \in [0, 1]$ satisfying the relation (2.8). Let $\mathbf{z} : \Delta_4 \to \mathbb{R}^m$ be of the form $(s, u, t, \tau) \mapsto \mathbf{z}^\tau_{tus}$. The definition of $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_3; \mathbb{R}^m)$ can be extended in order to define a space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_4; \mathbb{R}^m)$, by using the same definition as (2.4). That is we have $z \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_4; \mathbb{R}^m)$ if

$$\|z\|_{(\alpha,\gamma,\eta,\zeta)} = \|z\|_{(\alpha,\gamma),1} + \|z\|_{(\alpha,\gamma,\eta,\zeta),1,2} < \infty. \tag{2.14}$$

The quantities $\|z\|_{(\alpha,\gamma),1}$ and $\|z\|_{(\alpha,\gamma,\eta,\zeta),1,2}$ in (2.14) are slight modifications of (2.5) and (2.6), respectively defined by

$$\|\mathbf{z}\|_{(\alpha,\gamma),1} = \sup_{(s,u,t,\tau)\in\Delta_4} \frac{|\mathbf{z}^\tau_{tus}|}{\psi^1_{(\alpha,\gamma)}(\tau, t, s)}, \tag{2.15}$$

and

$$\|\mathbf{z}\|_{(\alpha,\gamma,\eta,\zeta),1,2} = \sup_{(s,u,t,\tau',\tau)\in\Delta_5} \frac{|\mathbf{z}^{\tau\tau'}_{tus}|}{\psi^{1,2}_{(\alpha,\gamma,\eta,\zeta)}(\tau, \tau', t, s)}. \tag{2.16}$$

3 An Extension of Garsia–Rodemich–Rumsey's Inequality

This section is devoted to extend Garsia–Rodemich–Rumsey's celebrated result [10] to the Volterra space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ introduced in Definition 2.3. To this aim, we introduce two integral functionals resembling the role of a Sobolev norm, tailored for the regularity functions introduced in (2.2)–(2.3). These will be used to extend the Garsia–Rodemich–Rumsey inequality to Volterra paths.

Definition 3.1 Let $z : \Delta_3 \to \mathbb{R}^d$ be a continuous Volterra increment. Then for some parameters $p \geq 1$ and $\alpha, \gamma \in (0, 1)$, $\eta, \zeta \in [0, 1]$ satisfy the relation (2.1) we define

$$U^\tau_{(\alpha,\gamma),p,1}(z; \eta, \zeta) := \left(\int_{(v,w)\in\Delta_2^\tau} \frac{|z^\tau_{wv}|^{2p}}{|\tau - w|^{-2p(\eta-\zeta)} |\psi^1_{(\alpha,\gamma+\zeta)}(\tau, w, v)|^{2p} |w - v|^2} \mathrm{d}v\mathrm{d}w \right)^{\frac{1}{2p}} \tag{3.1}$$

$$U^\tau_{(\alpha,\gamma,\eta,\zeta),p,1,2}(z) := \left(\int_{(v,w,r',r)\in\Delta_4^\tau} \frac{|z^{rr'}_{wv}|^{2p}}{|\psi^{1,2}_{(\alpha,\gamma,\eta,\zeta)}(r, r', w, v)|^{2p} |w - v|^2 |r - r'|^2} \mathrm{d}v\mathrm{d}w dr' dr \right)^{\frac{1}{2p}}, \tag{3.2}$$

where recall that the functions ψ^1, $\psi^{1,2}$ are respectively defined in (2.2) and (2.3).

Remark 3.2 Notice that if we set

$$D^\tau(w,v) = \frac{|z^\tau_{wv}|^{2p}}{|\tau - w|^{-2p(\eta-\zeta)}|\psi^1_{(\alpha,\gamma)}(\tau,w,v)|^{2p}|w-v|^2},$$

then we trivially have $D^\tau(w,v) \geq 0$. Plugging this information in relation (3.1), we get that $\tau \mapsto U^\tau_{(\alpha,\gamma),p,1}(z;\eta,\zeta)$ is a non-decreasing function. Thus for $\tau \leq T$ we have $U^\tau_{(\alpha,\gamma),p,1}(z;\eta,\zeta) \leq U^T_{(\alpha,\gamma),p,1}(z;\eta,\zeta)$.

Remark 3.3 The quantity $U^\tau_{(\alpha,\gamma),p,1}(z;\eta,\zeta)$ evaluated at $\eta = \zeta = 0$, will be denoted by $U^\tau_{(\alpha,\gamma),p,1}(z)$ for notational sake.

Using the above integral functionals, we now state and prove the extension of Garsia–Rodemich–Rumsey's inequality for general Volterra increments on Δ_3. This will in turn be applied to provide an upper bound for the Volterra norms introduced in Definition 2.3 in terms of the integral functionals in Definition 3.1.

Lemma 3.4 *Let $\mathbf{z} : \Delta_3 \to \mathbb{R}^d$ be a continuous increment. Consider 4 parameters $\kappa, \gamma, \eta, \zeta$ such that*

$$0 \leq \gamma < \kappa < 1, \quad 0 \leq \zeta < \kappa - \gamma, \quad \textit{and} \quad \zeta \leq \eta \leq 1. \tag{3.3}$$

Then there exists a universal constant $C > 0$ such that for all $(s,t,\tau) \in \Delta_3$ we have

$$|\mathbf{z}^\tau_{ts}| \leq C|\tau - t|^{-(\eta-\zeta)}\psi^1_{\kappa,\gamma+\zeta}(\tau,t,s)\left(U^\tau_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta) + \|\delta\mathbf{z}\|^{[s,t]}_{(\kappa,\gamma,\eta,\zeta),1}\right), \tag{3.4}$$

where the quantity $\|\delta\mathbf{z}\|^{[s,t]}_{(\kappa,\gamma,\eta,\zeta),1}$ is defined as

$$\|\delta\mathbf{z}\|^{[s,t]}_{(\kappa,\gamma,\eta,\zeta),1} = \sup_{s\leq u<v\leq t} \frac{|\delta_u \mathbf{z}^\tau_{vs}|}{|\tau - v|^{-(\eta-\zeta)}\psi^1_{(\kappa,\gamma+\zeta)}(\tau,v,s)}. \tag{3.5}$$

In particular, for $\eta = \zeta = 0$, we have

$$|\mathbf{z}^\tau_{ts}| \lesssim \psi^1_{(\kappa,\gamma)}(\tau,t,s)\left(U^\tau_{(\kappa,\gamma),p,1}(\mathbf{z}) + \|\delta\mathbf{z}\|_{(\alpha,\gamma),1}\right), \tag{3.6}$$

where $\|\delta\mathbf{z}\|_{(\alpha,\gamma),1}$ is given by (2.15).

Proof Consider a tuple $(s,t,\tau) \in \Delta_3$, with $t - s < \frac{T}{2}$. First construct a sequence of points $(s_k)_{k\geq 0}$, such that $s_k \in [0,T]$ and s_k converges to s by induction. Namely, set $s_0 = t$, and suppose that $s_0, s_1, \ldots, s_k$ have been constructed, and let $D_k = (s, \frac{s_k+s}{2})$. Define the function I as follows:

$$I(w) := \int_s^w \frac{|\mathbf{z}^\tau_{wv}|^{2p}}{|\tau - w|^{-2p(\eta-\zeta)}|\psi^1_{(\kappa,\gamma+\zeta)}(\tau,w,v)|^{2p}|w-v|^2}\mathrm{d}v. \tag{3.7}$$

According to the value of I, define two subsets of the interval D_k:

$$A_k := \left\{ w \in D_k \middle| \quad I(w) > \frac{4(U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta))^{2p}}{|s_k - s|} \right\}, \tag{3.8}$$

$$B_k := \left\{ w \in D_k \middle| \quad \frac{|\mathbf{z}^{\tau}_{s_k w}|^{2p}}{|\tau - s_k|^{-2p(\eta-\zeta)} |\psi^{1}_{(\kappa,\gamma+\zeta)}(\tau, s_k, w)|^{2p} |s_k - w|^2} > \frac{4I(s_k)}{|s_k - s|} \right\}, \tag{3.9}$$

where we recall again that $\psi^1_{(\kappa,\gamma+\zeta)}(\tau, w, v)$ is given by (2.2). We claim that $A_k \cup B_k \subset D_k$, where the inclusion is strict. Toward proving this claim, observe that the set of (v, w) such that

$$s < v < w < \frac{s_k + s}{2}$$

is included in $[0, T)^2$. Hence due to the definition (3.1) of $U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)$ we get

$$\left(U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\right)^{2p} \geq \int_{A_k} \mathrm{d}v\, I(v). \tag{3.10}$$

Therefore thanks to relation (3.8) defining A_k, we get

$$\left(U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\right)^{2p} > \frac{4\left(U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\right)^{2p}}{|s_k - s|} \mu(A_k), \tag{3.11}$$

where $\mu(A_k)$ denotes the Lebesgue measure of set A_k. It is thus readily checked from (3.11) that

$$\mu(A_k) < \frac{|s_k - s|}{4} = \frac{\mu(D_k)}{2}. \tag{3.12}$$

Arguing similarly for the set B_k, we note that since the set B_k defined by (3.9) is a subset of (s, s_k), we have

$$I(s_k) \geq \int_{B_k} \frac{|\mathbf{z}^{\tau}_{s_k v}|^{2p}}{|\tau - s_k|^{-2p(\eta-\zeta)} |\psi^{1}_{(\kappa,\gamma+\zeta)}(\tau, s_k, v)|^{2p} |s_k - v|^2} \mathrm{d}v. \tag{3.13}$$

Thus plugging the definition (3.9) of B_k in the right hand side of (3.13), we get

$$I(s_k) > \frac{4\mu(B_k)}{|s_k - s|} I(s_k),$$

from which we obtain again that

$$\mu(B_k) < \frac{|s_k - s|}{4} = \frac{\mu(D_k)}{2} \tag{3.14}$$

Combining (3.12) and (3.14), we have thus obtained

$$\mu(A_k) < \frac{\mu(D_k)}{2}, \quad \text{and} \quad \mu(B_k) < \frac{\mu(D_k)}{2},$$

and it follows that

$$\mu(A_k) + \mu(B_k) < \mu(D_k), \tag{3.15}$$

from which we easily deduce that $A_k \cup B_k$ is a strict subset of D_k. Now we can choose s_{k+1} arbitrarily in $D_k \backslash (A_k \cup B_k)$. Summarizing our considerations so far; for all n we have constructed a family $\{s_0, \ldots, s_n\}$ such that for all $0 \le k \le n$, we have $0 \le s_k - s \le \frac{t-s}{2^k}$ and the following two conditions are met:

$$\frac{|\mathbf{z}^\tau_{s_k s_{k+1}}|^{2p}}{|\tau - s_k|^{-2p(\eta-\zeta)} |\psi^1_{(\kappa,\gamma+\zeta)}(\tau, s_k, s_{k+1})|^{2p} |s_k - s_{k+1}|^2} \le \frac{4I(s_k)}{|s_k - s|},$$
$$I(s_{k+1}) \le \frac{4\left(U^\tau_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\right)^{2p}}{|s_k - s|}. \tag{3.16}$$

With (3.16) in hand, decompose $\mathbf{z}^\tau_{ts}$ into

$$\mathbf{z}^\tau_{ts} = \mathbf{z}^\tau_{s_{n+1}s} + \sum_{k=0}^{n} \left(\mathbf{z}^\tau_{s_k s_{k+1}} + \delta_{s_{k+1}} \mathbf{z}^\tau_{s_k s} \right). \tag{3.17}$$

The aim is now to bound the term $\mathbf{z}^\tau_{s_k s_{k+1}}$ in (3.17). To this aim, notice that since $s_{k+1} \notin B_k$, we have

$$\frac{|\mathbf{z}^\tau_{s_k s_{k+1}}|^{2p}}{|\tau - s_k|^{-2p(\eta-\zeta)} |\psi^1_{(\kappa,\gamma)}(\tau, s_k, s_{k+1})|^{2p} |s_k - s_{k+1}|^2} \le 4\frac{I(s_k)}{|s_k - s|}. \tag{3.18}$$

Moreover, we also have $s_k \notin A_{k-1}$. Hence we obtain

$$I(s_k) < \frac{4\left(U^\tau_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\right)^{2p}}{|s_{k-1} - s|}. \tag{3.19}$$

Gathering (3.18) and (3.19) yields

$$\frac{|\mathbf{z}^\tau_{s_k s_{k+1}}|^{2p}}{|\tau - s_k|^{-2p(\eta-\zeta)} |\psi^1_{(\kappa,\gamma+\zeta)}(\tau, s_k, s_{k+1})|^{2p} |s_k - s_{k+1}|^2}$$

$$< \frac{16\left(U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\right)^{2p}}{|s_k-s||s_{k-1}-s|} \lesssim \frac{\left(U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\right)^{2p}}{|s_k-s|^2}, \tag{3.20}$$

where we have used the fact that $|s_k - s| \leq |s_{k-1} - s|$ for the second inequality. In addition, thanks to $|s_k - s_{k+1}| \leq |s_k - s|$, it is easily seen that we can recast (3.20) as

$$|\mathbf{z}^{\tau}_{s_k s_{k+1}}| \lesssim U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\,\psi^1_{(\kappa,\gamma+\zeta)}(\tau,s_k,s)|\tau-s_k|^{-(\eta-\zeta)}. \tag{3.21}$$

Next recall that η is assumed to be larger than ζ, and in addition we assume that $0 \leq \zeta < \kappa - \gamma$. Thus owing to the fact that $|\tau - s_k| \geq |\tau - t|$, $|s_k - s| \lesssim 2^{-k}(t - s)$, and recalling the expression (2.2) for ψ^1, we end up with

$$\left|\mathbf{z}^{\tau}_{s_k s_{k+1}}\right| \lesssim \frac{U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)}{2^{k(\kappa-\gamma-\zeta)}}\psi^1_{(\kappa,\gamma+\zeta)}(\tau,t,s)|\tau-t|^{-(\eta-\zeta)}$$

Summing this inequality over k (and using that $\kappa - \gamma - \zeta > 0$), we get the following bound for the right hand side of (3.17):

$$\left|\sum_{k=0}^{n}\mathbf{z}^{\tau}_{s_k s_{k+1}}\right| \lesssim U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z};\eta,\zeta)\,\psi^1_{(\kappa,\gamma+\zeta)}(\tau,t,s)|\tau-t|^{-(\eta-\zeta)}. \tag{3.22}$$

Now we turn to bound the second term $\delta_{s_{k+1}}\mathbf{z}^{\tau}_{s_k s}$ in the right hand side of (3.17). It is clear that

$$|\delta_{s_{k+1}}\mathbf{z}^{\tau}_{s_k s}| \lesssim \|\delta\mathbf{z}\|^{[s,t]}_{(\kappa,\gamma,\eta,\zeta),1}\,|\tau-t|^{-(\eta-\zeta)}\psi^1_{(\kappa,\gamma+\zeta)}(\tau,s_k,s),$$

recalling that $\|\delta\mathbf{z}\|^{[s,t]}_{(\kappa,\gamma,\eta,\zeta),1}$ as given in (3.5). Hence similarly to (3.22), we obtain

$$|\sum_{k=0}^{n}\delta_{s_{k+1}}\mathbf{z}^{\tau}_{s_k s}| \lesssim \|\delta\mathbf{z}\|^{[s,t]}_{(\kappa,\gamma,\eta,\zeta),1}\,|\tau-t|^{-(\eta-\zeta)}\psi^1_{(\kappa,\gamma+\zeta)}(\tau,t,s). \tag{3.23}$$

Plugging (3.22) and (3.23) into (3.17), and letting $n \to \infty$, we get relation (3.4) thanks to the continuity of $\mathbf{z}$. This completes the proof. □

In preparation for the next proposition, we recall here a classical Sobolev embedding inequality. The particular form of the inequality stated here is as a consequence of the classical Garsia–Rodemich–Rumsey inequality [9], and can be found stated in the form below in [17, pp. 2].

Proposition 3.5 *Let $h : [a, b] \to \mathbb{R}^d$ be continuous. Then for any $p > \frac{1}{\alpha}$ the following inequality holds*

$$|h_{ts}| \lesssim_{\alpha,p} |t-s|^{\alpha}\left(\int_a^b\int_a^u \frac{|h_{uv}|^p}{|u-v|^{2+p\alpha}}\mathrm{d}v\mathrm{d}u\right)^{\frac{1}{p}}, \tag{3.24}$$

where we have set $h_{ts} = h_t - h_s$ *for* $(s, t) \in \Delta_2$.

We follow up with a technical lemma, combining Proposition 3.5 with Lemma 3.4.

Lemma 3.6 *Let* $\mathbf{z} : \Delta_3 \to \mathbb{R}^d$ *be continuous. Consider four parameters* $\alpha, \gamma \in (0, 1)$, $\eta, \zeta \in [0, 1]$ *that satisfy the relation* (2.1). *Recall that* $\psi^{1,2}$ *is defined by* (2.3) *and the quantities* U *are introduced in Definition 3.2. Then for any* $\alpha - \gamma > \frac{1}{p}$, *the following inequality holds for any* $(s, t, \tau', \tau) \in \Delta_4^T$,

$$\left(\frac{|\mathbf{z}_{ts}^{\tau\tau'}|}{\psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(\tau, \tau', t, s)} \right)^{2p} \lesssim U^T_{(\alpha,\gamma,\eta,\zeta),p,1,2}(\mathbf{z}) + \int_{\tau'}^{\tau} \int_{\tau'}^{r} \sup_{0 \le s < u < v \le t} \frac{|\delta_u \mathbf{z}_{vs}^{rr'}|^{2p}}{\psi^{1,2}_{(\alpha,\gamma,\eta,\zeta)}(r, r', v, s)^{2p} |r - r'|^2} \, dr' \, dr. \tag{3.25}$$

Proof First, since $\mathbf{z}$ is continuous, we apply Proposition 3.5 to the increment $\mathbf{z}_{ts}^{\tau} - \mathbf{z}_{ts}^{\tau'}$, and we get

$$\frac{\left|\mathbf{z}_{ts}^{\tau\tau'}\right|}{|\tau' - \tau|^{\eta}} \lesssim \left(\int_{\tau'}^{\tau} \int_{\tau'}^{r} \frac{\left|\mathbf{z}_{ts}^{rr'}\right|^{2p}}{|r - r'|^{2+2p\eta}} dr' dr \right)^{1/2p}. \tag{3.26}$$

Moreover, let us write again relation (2.7) for the reader's convenience:

$$\psi^{1,2}_{\alpha,\gamma,\eta,\zeta}(\tau, \tau', t, s) = \left|\tau - \tau'\right|^{\eta} \left|\tau' - t\right|^{-(\eta-\zeta)} \psi^{1}_{(\alpha,\gamma+\zeta)}(\tau', t, s). \tag{3.27}$$

Plugging (3.27) in (3.26), we end up with

$$\left(\frac{|\mathbf{z}_{ts}^{\tau\tau'}|}{\psi^{1,2}_{(\alpha,\gamma,\eta,\zeta)}(\tau, \tau', t, s)} \right)^{2p} \lesssim I(\tau, \tau', t, s), \tag{3.28}$$

where we have set

$$I(\tau, \tau', t, s) = \int_{\tau'}^{\tau} \int_{\tau'}^{r} \frac{\left|\mathbf{z}_{ts}^{rr'}\right|^{2p}}{|\tau' - t|^{-2p(\eta-\zeta)} \left|\psi^{1}_{(\alpha,\gamma+\zeta)}(\tau', t, s)\right|^{2p} |r - r'|^{2+2p\eta}} dr' \, dr.$$

Invoking the fact that $t \le \tau' \le r' \le \tau$ and $\eta - \zeta \ge 0$ we have $|\tau' - t|^{\eta-\zeta} \le |r' - t|^{\eta-\zeta}$. Hence it immediately follows that

$$I(\tau, \tau', t, s) \lesssim \int_{\tau'}^{\tau} \int_{\tau'}^{r} \frac{|\mathbf{z}_{ts}^{rr'}|^{2p}}{|r' - t|^{-2p(\eta-\zeta)} \left|\psi^{1}_{(\alpha,\gamma+\zeta)}(r', t, s)\right|^{2p} |r - r'|^{2+2p\eta}} \, dr' \, dr. \tag{3.29}$$

We thus fix r and apply inequality (3.4) to the Volterra path $(r', t, s) \mapsto \mathbf{z}_{ts}^{rr'}$. We get

$$\left|\mathbf{z}_{ts}^{rr'}\right| \lesssim \left|r' - t\right|^{-(\eta-\zeta)} \psi^1_{(\alpha,\gamma+\zeta)}\left(r', t, s\right)\left(U^{r'}_{(\alpha,\gamma),p,1}(\mathbf{z}^{r,\cdot}; \eta, \zeta) + \|\delta \mathbf{z}^{r,\cdot}\|^{[s,t]}_{(\alpha,\gamma,\eta,\zeta),1}\|\right). \tag{3.30}$$

We now plug (3.30) into (3.29), recall the definition (3.1) of $U^{r'}$, resort to (2.7) again and use the expression of (3.5) for $\|\delta \mathbf{z}^{r,\cdot}\|^{[s,t]}_{(\alpha,\gamma,\eta,\zeta),1}$. We end up with

$$I(\tau, \tau', t, s) \lesssim I_1(\tau, \tau') + I_2(\tau, \tau', t, s), \tag{3.31}$$

where I_1 and I_2 are respectively given by

$$I_1(\tau, \tau') = \int_{\tau'}^{\tau}\int_{\tau'}^{r}\int_0^{r'}\int_0^{v} \frac{|\mathbf{z}_{vu}^{rr'}|^{2p}}{\left|\psi^{1,2}_{(\alpha,\gamma,\eta,\zeta)}(r, r', v, u)\right|^{2p} |v-u|^2|r-r'|^2} \mathrm{d}u\, \mathrm{d}v\, \mathrm{d}r'\, \mathrm{d}r,$$

$$I_2(\tau, \tau', t, s) = \int_{\tau'}^{\tau}\int_{\tau'}^{r} \sup_{0\le s<u<v\le t} \frac{|\delta_u \mathbf{z}_{vs}^{rr'}|^{2p}}{\left|\psi^{1}_{(\alpha,\gamma+\zeta)}(r', v, s)\right|^{2p} |r'-v|^{-2p(\eta-\zeta)}|r-r'|^{2+2p\eta}} dr'\, dr.$$

Going back to (3.2), it is now readily checked that

$$I_1(\tau, \tau') \le \left(U^{\tau}_{(\alpha,\gamma,\eta,\zeta),p,1,2}(\mathbf{z})\right)^{2p} \le \left(U^{T}_{(\alpha,\gamma,\eta,\zeta),p,1,2}(\mathbf{z})\right)^{2p}. \tag{3.32}$$

Furthermore, another application of (2.7) reveals that

$$I_2(\tau, \tau', t, s) = \int_{\tau'}^{\tau}\int_{\tau'}^{r} \sup_{0\le s<u<v\le t} \frac{|\delta_u \mathbf{z}_{vs}^{rr'}|^{2p}}{\psi^{1,2}_{(\alpha,\gamma,\eta,\zeta)}(r, r', v, u)^{2p}|r-r'|^2} dr'\, dr. \tag{3.33}$$

Plugging (3.32)–(3.33) into (3.31) and then back to in (3.28), this achieves the proof of our claim (3.25). ◻

Now we will combine Lemma 3.4 and 3.6 to obtain a modified Garsia–Rodemich–Rumsey inequality tailored to Volterra rough paths.

Theorem 3.7 *Let $\mathbf{z} : \Delta_3 \to \mathbb{R}^d$. For $\alpha, \gamma \in (0, 1)$, $\eta, \zeta \in [0, 1]$ satisfy the relation (2.1), we assume that $\delta \mathbf{z} \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ where $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ is introduced in Definition 2.10. Suppose $\kappa \in (0, \alpha)$. Then for any $p > \frac{1}{\alpha-\kappa} \vee \frac{1}{\zeta}$, the following two bounds holds:*

$$\|\mathbf{z}\|_{(\kappa,\gamma),1} \lesssim U^{T}_{(\kappa,\gamma),1,p}(\mathbf{z}) + \|\delta \mathbf{z}\|_{(\kappa,\gamma),1}, \tag{3.34}$$

$$\|\mathbf{z}\|_{(\kappa,\gamma,\eta,\zeta),1,2} \lesssim U^{T}_{(\kappa,\gamma,\eta,\zeta),1,2,p}(\mathbf{z}) + \|\delta \mathbf{z}\|_{\left(\kappa,\gamma,\eta+\frac{1}{p},\zeta+\frac{1}{p}\right),1,2} T^{2+\alpha-\kappa-\frac{1}{p}}. \tag{3.35}$$

Proof We begin by proving (3.34). It follows directly from (3.6) that for any $0 < \kappa < \alpha$

$$|\mathbf{z}_{ts}^{\tau}| \lesssim \psi^1_{(\kappa,\gamma)}(\tau, t, s)\left(U^{\tau}_{(\kappa,\gamma),p,1}(\mathbf{z}) + \|\delta \mathbf{z}\|_{(\alpha,\gamma),1}\right).$$

Using that $\tau \mapsto U^\tau$ is increasing (see Remark 3.2) and taking supremum over τ on the right hand side above, it is easily seen that (3.34) holds. We now move on to prove (3.35). To this aim, we shall spell out the right hand side of (3.25) in a slightly different way. Namely note that for $\delta \mathbf{z} \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ and $\eta < \eta'$, we have

$$\int_{\tau'}^{\tau}\int_{\tau'}^{r} \sup_{0\le s<u<v\le t} \frac{|\delta_u \mathbf{z}_{vs}^{rr'}|^{2p}}{\psi^{1,2}_{(\kappa,\gamma,\eta,\zeta)}(r,r',v,s)^{2p}|r-r'|^2}\, dr'\, dr$$
$$\lesssim \|\delta \mathbf{z}\|^{2p}_{\left(\alpha,\gamma,\eta+\frac{1}{p},\zeta+\frac{1}{p}\right),1,2} \int_{\tau'}^{\tau}\int_{\tau'}^{r} \sup_{0\le s<u<v\le t} \frac{\psi^{1,2}_{(\alpha,\gamma,\eta+\frac{1}{p},\zeta+\frac{1}{p})}(r,r',v,s)^{2p}}{\psi^{1,2}_{(\kappa,\gamma,\eta,\zeta)}(r,r',v,s)^{2p}|r-r'|^2}\, dr'\, dr, \tag{3.36}$$

where we have used the Definition 2.6 of the $(1,2)$-norm. Furthermore, since we have assumed $p > \frac{1}{\alpha-\kappa}$ and $s, v \in [0, T]$ it is readily checked that

$$\frac{\psi^{1,2}_{(\alpha,\gamma,\eta+\frac{1}{p},\zeta+\frac{1}{p})}(r,r',v,s)^{2p}}{\psi^{1,2}_{(\kappa,\gamma,\eta,\zeta)}(r,r',v,s)^{2p}|r-r'|^2} \lesssim |v-s|^{2p(\alpha-\kappa)-2} \le T^{2p(\alpha-\kappa)-2}. \tag{3.37}$$

Hence the right hand side of (3.25) can be upper bounded by

$$C_{T,p,\alpha,\kappa}\|\delta \mathbf{z}\|^{2p}_{(\alpha,\gamma,\eta+\frac{1}{p},\zeta+\frac{1}{p}),1,2}.$$

Plugging this information into (3.25), the proof of (3.35) is now easily achieved. □

4 Volterra Rough Path Driven by Fractional Brownian Motion

In this section, we are going to construct the Volterra rough path driven by a fractional Brownian motion with Hurst parameter $H > 1/2$. As mentioned in the introduction, this regime leads to nontrivial rough paths development in the Volterra case, due to the singularity of the kernel k in (1.1). Indeed, this singularity pushes down the overall regularity of the Volterra path, so that a singularity of order γ yields a regularity $H-\gamma$ of the Volterra path constructed from the fBM (which is thus allowed to be smaller than $\frac{1}{2}$).

Let us first recall some basic facts about the stochastic calculus of variations with respect to fractional Brownian motion.

4.1 Malliavin Calculus Preliminaries

This section is devoted to review some elementary information on Malliavin calculus (mostly borrowed from [18]) that we will use in Sect. 4.2 and Sect. 4.3. We first introduce the notation for our main process of interest.

Notation 4.1 In the sequel we denote by $B = \left\{(B_t^1, \ldots, B_t^m),\ t \in [0,T]\right\}$ a standard m-dimensional fractional Brownian motion with Hurst parameter $H \in (1/2, 1)$. Recall that B is a centered Gaussian process with independent coordinates. For each component B^i, the covariance function R is defined by

$$R(s,t) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right). \tag{4.1}$$

We now say a few words about Cameron-Martin type spaces related to each component B^i in Notation 4.1. Namely let $\mathcal{H}$ be the Hilbert space defined as the closure of the set of step functions on the interval $[0,T]$ with respect to the scalar product

$$\langle \mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]}\rangle_{\mathcal{H}} = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

Under the assumption $H > 1/2$, it is easy to see that the covariance of the fBm (4.1) can be written as

$$R(s,t) = a_H \int_0^t \int_0^s |u-v|^{2H-2} \mathrm{d}u \mathrm{d}v,$$

where the constant a_H is defined by $a_H = H(2H-1)$. This implies that

$$\langle f, g\rangle_{\mathcal{H}} = a_H \int_0^T \int_0^T f_u g_v |u-v|^{2H-2} \mathrm{d}u \mathrm{d}v, \tag{4.2}$$

for any pair of step functions f and g on $[0,T]$. Therefore $\mathcal{H}$ can also be seen as the completion of step functions with respect to the inner product (4.2). We now introduce a family of additional spaces $|\mathcal{H}|^{\otimes l}$ which will be useful for our computations. Namely for $l \geq 1$ we define $|\mathcal{H}|^{\otimes l}$ as the linear space of measurable functions f on $[0,T]^l \subset \mathbb{R}^l$ such that

$$\|f\|^2_{|\mathcal{H}|^{\otimes l}} := a_H^l \int_{[0,T]^{2l}} |f_{\mathbf{u}}||f_{\mathbf{v}}||u_1 - v_1|^{2H-2} \cdots |u_l - v_l|^{2H-2} d\mathbf{u} d\mathbf{v} < \infty, \tag{4.3}$$

where we write $\mathbf{u} = (u_1, \cdots, u_l)$, $\mathbf{v} = (v_1, \ldots, v_l) \in [0,T]^l$. Notice that $|\mathcal{H}|^{\otimes l}$ is a subset of $\mathcal{H}^{\otimes l}$. The main interest of the spaces $|\mathcal{H}|$ is due to the fact that while $\mathcal{H}^{\otimes l}$ contains distributions, the space $|\mathcal{H}|^{\otimes l}$ is a space of functions.

For each component B^i, the mapping $\mathbb{1}_{[0,t]} \mapsto B_t^i$ can be extended to a linear isometry between $\mathcal{H}$ and the Gaussian space spanned by B^i. We denote this isometry by $h \mapsto B^i(h)$. In this way, $\{B^i(h), h \in \mathcal{H}\}$ is an isonormal Gaussian process indexed by the Hilbert space $\mathcal{H}$. Namely, we have

$$\mathbb{E}\left[B^i(f)\, B^i(g)\right] = \langle f, g\rangle_{\mathcal{H}}. \tag{4.4}$$

It is also worth mentioning that the Wiener integral can be approximated by Riemann type sums. Namely for $h \in \mathcal{H}$ the following limit holds true in $L^2(\Omega)$:

$$B^i(h) = \lim_{|\mathcal{P}|\to 0} \sum_{[r,v]\in\mathcal{P}} B^i_{vr}\, h(r), \tag{4.5}$$

where the Riemann sum is written similarly to (2.9) and we recall that $B^i_{vr} = B^i_v - B^i_r$.

Let $\mathcal{S}$ be the set of smooth and cylindrical random variables of the form

$$F = f(B_{s_1}, \dots, B_{s_N}),$$

where $N \geq 1$ and $f \in C_b^\infty(\mathbb{R}^{m\times N})$. For each $j = 1, \dots, m$ and $t \in [0, T]$, the partial Malliavin derivative of F with respect to the component B^j is defined for $F \in \mathcal{S}$ as the $\mathcal{H}$-valued random variable

$$D^j_t F = \sum_i^N \frac{\partial f}{\partial x^j_i}(B_{s_1}, \dots, B_{s_N})\mathbb{1}_{[0,s_i]}(t), \qquad t \in [0, T], \tag{4.6}$$

where x^j_i stands for the j-th component of x. We can iterate this procedure to define higher-order derivatives $D^{j_1,\dots,j_l} F$, which take values in $\mathcal{H}^{\otimes l}$. For any $p \geq 1$ and integer $k \geq 1$, we define the Sobolev space $\mathbb{D}^{k,p}$ as the closure of $\mathcal{S}$ with respect to the norm

$$\|F\|^p_{k,p} = \mathbb{E}[|F|^p] + \mathbb{E}\left[\sum_{i=1}^k \left(\sum_{j_1,\dots,j_l}^m \|D^{j_i,\dots,j_l} F\|^2_{\mathcal{H}^{\otimes l}}\right)^{p/2}\right]. \tag{4.7}$$

If V is Hilbert space, $\mathbb{D}^{k,p}(V)$ denotes the corresponding Sobolev space of V-valued random variables.

For any $j = 1, \dots, m$, we denote by $\delta^{\diamond,j}$ the adjoint of the derivative operator D^j. For a process $\{u_t;\ t \in [0, T]\}$, we say $u \in \text{Dom}\, \delta^{\diamond,j}$ if there is a $\delta^{\diamond,j}(u) \in L^2(\mathbb{R}^m)$ such that for any $F \in \mathbb{D}^{k,p}$ the following duality relation holds

$$\mathbb{E}\left[\langle u, D^j F\rangle_{\mathcal{H}}\right] = \mathbb{E}\left[\delta^{\diamond,j}(u) F\right]. \tag{4.8}$$

The random variable $\delta^{\diamond,j}(u)$ is also called the Skorohod integral of u with respect to the fBm B^j, and we use the notation $\delta^{\diamond,j}(u) = \int_0^T u_t \delta^\diamond B^j_t$. It is well known that $\mathbb{D}^{1,2}(\mathcal{H}) \subset \text{Dom}\,(\delta^{\diamond,j})$ for all $j = 1, \dots, m$.

We now introduce a pathwise type integral defined on the Wiener space, called Stratonovich integral. Namely let $u = \{u_t, t \in [0, T]\}$ be a continuous stochastic process, and let $\mathcal{P}$ be a generic partition of $[s, t]$. Following [18, Section3.1], we define

$$B_t^{i,\mathcal{P}} = \sum_{[r,v]\in\mathcal{P}} \frac{B_{vr}^i}{v-r} \mathbb{1}_{[r,v]}(t), \qquad \text{and} \qquad S_{ts}^{i,\mathcal{P}} = \int_s^t u_r\, B_r^{i,\mathcal{P}} dr. \tag{4.9}$$

Then the Stratonovich integral of u with respect to B^i is defined as

$$\int_s^t u_r\, dB_r^i = \lim_{|\mathcal{P}|\to 0} S_{ts}^{i,\mathcal{P}}, \tag{4.10}$$

where the limit is understood in probability. On the other hand, assume that u is C^κ-Hölder with $\kappa + H > 1$. Moreover we suppose that $u \in \mathbb{D}^{1,2}(\mathcal{H})$ and the derivative $D_s^j u_t$ exists and satisfies

$$\int_0^T \int_0^T |D_s^j u_t||t-s|^{2H-2} ds\, dt < \infty \quad \text{a.s,} \qquad \text{and} \qquad \mathbb{E}\left[\|D^j u\|^2_{|\mathcal{H}|^{\otimes 2}}\right] < \infty.$$

Then the Stratonovich integral $\int_0^T u_t dB_t^j$ exists, and we have the following relation between Skorohod and Stratonovich stochastic integrals:

$$\int_0^T u_t dB_t^j = \int_0^T u_t \delta^\diamond B_t^j + a_H \int_0^T \int_0^T D_s^j\, u_t |t-s|^{2H-2} ds\, dt. \tag{4.11}$$

We close this section by spelling out Meyer's inequality (see [18, Proposition 1.5.4]) for the Skorohod integral: given $p > 1$ and an integer $k \geq 1$, there is a constant $c_{k,p}$ such that the k-th iterated Skorohod integral satisfies

$$\|(\delta^\diamond)^k(u)\|_p \leq c_{k,p} \|u\|_{\mathbb{D}^{k,p}(\mathcal{H}^{\otimes k})}, \qquad \text{for all} \quad u \in \mathbb{D}^{k,p}(\mathcal{H}^{\otimes k}). \tag{4.12}$$

4.2 First Level of the Volterra Rough Path

In this section, we will construct the first level of the Volterra rough path driven by a fBm as introduced in Notation 4.1. We start by defining our main object of study.

Definition 4.2 Consider a fractional Brownian motion $B : [0, T] \to \mathbb{R}^m$ with Hurst parameter H as given in Notation 4.1, and a function h of the form $h_{ts}^\tau(r) = (\tau - r)^{-\gamma} \mathbb{1}_{[s,t]}(r)$. We assume that H, γ satisfy $H \in (1/2, 1)$, $\gamma \in (0, 2H-1)$. Then for $(s, t, \tau) \in \Delta_3$ we define the increment $\mathbf{z}_{ts}^{1,\tau,i} = \int_s^t (\tau - r)^{-\gamma}\, dB_r^i$ as a Wiener integral of the form

$$\mathbf{z}_{ts}^{1,\tau,i} := B^i(h_{ts}^\tau). \tag{4.13}$$

Remark 4.3 Note that for the particular type of integrand h considered in Definition 4.2, the process $B^i(h_{ts}^\tau)$ is additive in its lower variables, in the sense that

$$B^i(h_{ts}^\tau) = B^i(h_{t0}^\tau) - B^i(h_{s0}^\tau). \tag{4.14}$$

Thus defining $\mathbf{z}_t^{1,\tau} := \mathbf{z}_{t0}^{1,\tau}$ we have that $\mathbf{z}^1$ is defined on the simplex Δ_2.

With Definition 4.2 in hand, we now estimate the second moment of $\mathbf{z}_{ts}^{1,\tau,i}$ and $\mathbf{z}_{ts}^{1,\tau\tau',i}$.

Lemma 4.4 *Consider the Volterra rough path $\mathbf{z}^1$ as given in (4.13), and four parameters $H \in (1/2, 1)$, $\gamma \in (0, 1)$, η, $\zeta \in [0, 1]$ satisfying*

$$\gamma < 2H - 1, \qquad and \qquad 0 \le \zeta \le \inf(H - \gamma, \eta). \tag{4.15}$$

Then for $(s, t, \tau) \in \Delta_3$, we have

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,i})^2] \lesssim \left|\psi^1_{(H,\gamma)}(\tau, t, s)\right|^2. \tag{4.16}$$

In addition for $(s, t, \tau', \tau) \in \Delta_4$, we get

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau\tau',i})^2] \lesssim \left|\psi^{1,2}_{(H,\gamma,\eta,\zeta)}\left(\tau, , \tau', t, s\right)\right|^2, \tag{4.17}$$

where ψ^1 and $\psi^{1,2}$ are given in Notation 2.2.

Proof We first prove relation (4.16). According to (4.13) and (4.4), we can compute $\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,i})^2]$ as

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,i})^2] = \mathbb{E}\left[B^i(h_{ts}^\tau)\, B^i(h_{ts}^\tau)\right] = \langle h_{ts}^\tau, h_{ts}^\tau\rangle_{\mathcal{H}}. \tag{4.18}$$

Owing to relation (4.2) for the inner product in $\mathcal{H}$, we thus obtain

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,i})^2] = H(2H-1)\iint_{[s,t]\times[s,t]} (\tau - r)^{-\gamma}(\tau - l)^{-\gamma}\,|r - l|^{2H-2}\,drdl. \tag{4.19}$$

Notice that the function $(\tau - r)^{-\gamma}(\tau - l)^{-\gamma}\,|r - l|^{2H-2}$ is symmetric. Hence we can recast (4.19) as

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,i})^2] = 2H(2H-1)\int_s^t (\tau - r)^{-\gamma}dr \int_r^t (\tau - l)^{-\gamma}(l - r)^{2H-2}dl. \tag{4.20}$$

In the right hand side of (4.20), we first estimate the integral

$$\int_r^t (\tau - l)^{-\gamma}(l - r)^{2H-2}dl := J. \tag{4.21}$$

Since $l \in (r, t)$ in (4.21), we proceed to a change of variable $l = r + \theta(t - r)$. We obtain

$$J = (t - r)^{2H-1}\int_0^1 (\tau - r - \theta(t - r))^{-\gamma}\,\theta^{2H-2}d\theta$$

$$\leq (t-r)^{2H-1}(\tau-r)^{-\gamma}\int_0^1 (1-\theta)^{-\gamma}\theta^{2H-2}d\theta. \tag{4.22}$$

Recall that we have assumed that $\gamma < 2H-1 < 1$. Moreover $H > 1/2$ and thus $2H-2 > -1$. Hence the right hand side of (4.22) can be expressed in terms of Beta functions in the following way:

$$\int_0^1 (1-\theta)^{-\gamma}\theta^{2H-2}d\theta = \mathrm{Beta}(1-\gamma, 2H-1) < \infty.$$

Reporting this identity in the right hand side of (4.22), we end up with

$$J \lesssim (t-r)^{2H-1}(\tau-r)^{-\gamma}. \tag{4.23}$$

Plugging (4.23) into (4.19), we thus get

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,i})^2] \lesssim \int_s^t (\tau-r)^{-2\gamma}(t-r)^{2H-1}dr. \tag{4.24}$$

We now bound the right hand side of (4.24) in two different ways. First since $(\tau-r) > (t-r)$, we have

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,i})^2] \lesssim \int_s^t (t-r)^{2H-2\gamma-1}dr \lesssim (t-s)^{2H-2\gamma}, \tag{4.25}$$

where we have resorted to the fact that $\gamma < 2H-1 < H$ for the second inequality. Next we also use the fact that $(\tau-r) > (\tau-t)$ in the right hand side of (4.24), which allows to write

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,i})^2] \lesssim (\tau-t)^{-2\gamma}\int_s^t (t-r)^{2H-1}dr \lesssim (\tau-t)^{-2\gamma}(t-s)^{2H}. \tag{4.26}$$

Combining (4.25) and (4.26), we end up with the following estimate for the second moment of $\mathbf{z}_{ts}^{1,\tau,i}$:

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,i})^2] \lesssim \left[(\tau-t)^{-2\gamma}(t-s)^{2H}\right] \wedge (t-s)^{2H-2\gamma} = \left(\psi^1_{(H,\gamma)}(\tau,t,s)\right)^2, \tag{4.27}$$

where we have appealed to the definition (2.2) of ψ^1 for the second identity. Relation (4.27) is the desired result (4.16).

Next, we will prove inequality (4.17). To this aim, we first note that owing to (4.13), we have the following expression for $\mathbf{z}_{ts}^{1,\tau\tau',i}$,

$$\mathbf{z}_{ts}^{1,\tau\tau',i} = \mathbf{z}_{ts}^{1,\tau,i} - \mathbf{z}_{ts}^{1,\tau',i} = B^i(h_{ts}^{\tau} - h_{ts}^{\tau'}). \tag{4.28}$$

Similarly to (4.20), we can thus rewrite $\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau\tau',i})^2]$ as

$$\begin{aligned}\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau\tau',i})^2] &= \langle h_{ts}^{\tau} - h_{ts}^{\tau'}, h_{ts}^{\tau} - h_{ts}^{\tau'}\rangle_{\mathcal{H}} \\ &= 2H(2H-1)\int_s^t \left[(\tau'-r)^{-\gamma} - (\tau-r)^{-\gamma}\right] dr \int_r^t \left[(\tau'-l)^{-\gamma}\right. \\ &\quad \left. -(\tau-l)^{-\gamma}\right](l-r)^{2H-2} dl. \end{aligned} \tag{4.29}$$

We now recall an elementary inequality on increments of negative power functions. Namely for $\tau > \tau' > r$ and $\eta \in [0,1]$ we have

$$(\tau'-r)^{-\gamma} - (\tau-r)^{-\gamma} \lesssim (\tau-\tau')^{\eta}(\tau'-r)^{-\eta-\gamma}.$$

Plugging this upper bound into the right hand side of (4.29), we obtain

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau\tau',i})^2] \lesssim |\tau-\tau'|^{2\eta} \int_s^t (\tau'-r)^{-\eta-\gamma} \int_r^t (\tau'-l)^{-\eta-\gamma}(l-r)^{2H-2} dl. \tag{4.30}$$

The expression (4.30) is now very similar to (4.20). Therefore with the same steps as for (4.21)-(4.26), for some $\zeta \in [0, H-\gamma)$ and $\eta \in [\zeta, 1]$ we get

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau\tau',i})^2] \lesssim |\tau-\tau'|^{2\eta}|\tau'-t|^{-2(\eta-\zeta)} \left(\left[(\tau-t)^{-2\gamma-2\zeta}(t-s)^{2H}\right] \wedge (t-s)^{2H-2\gamma-2\zeta}\right). \tag{4.31}$$

According to the definition (2.3) of $\psi^{1,2}$, (4.31) is equivalent to

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau\tau',i})^2] \lesssim \left|\psi_{(H,\gamma,\eta,\zeta)}^{1,2}(\tau,\tau',t,s)\right|^2.$$

This finishes the proof of (4.17). □

Remark 4.5 One can easily extend the computation of Lemma 4.16 in order to get more general bounds for covariance functions. Namely for any $(s,u,v,\tau) \in \Delta_4$, and recalling the expression (2.2) for ψ^1 we have

$$\left|\mathbb{E}[\mathbf{z}_{us}^{1,\tau,i}\mathbf{z}_{vs}^{1,\tau,i}]\right| \lesssim \left|\psi_{(H,\gamma)}^{1}(\tau,v,s)\right|^2. \tag{4.32}$$

Similarly for any $(s,u,v,\tau',\tau) \in \Delta_5$ and recalling our definition (2.3) for $\psi^{1,2}$, we obtain

$$\left|\mathbb{E}[\mathbf{z}_{us}^{1,\tau\tau',i}\mathbf{z}_{vs}^{1,\tau\tau',i}]\right| \lesssim \left|\psi_{(H,\gamma,\eta,\zeta)}^{1,2}(\tau,\tau',v,s)\right|^2, \tag{4.33}$$

where $H \in (\frac{1}{2}, 1)$, $\gamma \in (0,1)$, and $\eta, \zeta \in [0,1]$ satisfy relation (4.15).

4.3 Second Level of the Volterra Rough Path

In this section we turn our attention to the construction of a nontrivial Volterra rough path above a fBm. More specifically our aim is to construct a family $\{\mathbf{z}^{1,\tau}, \mathbf{z}^{2,\tau}\}$ verifying Definition 2.9. Let us start with the definition of $\mathbf{z}^{2,\tau}$.

Definition 4.6 We consider a fractional Brownian motion $B : [0, T] \to \mathbb{R}^m$ as given in Notation 4.1, as well as the first level of the Volterra rough path $\mathbf{z}^{1,\tau}$ defined by (4.13). As in Definition 4.2, we assume that H, γ satisfy $H \in (\frac{1}{2}, 1)$ and $\gamma \in (0, 2H - 1)$. Then for $(s, r, t, \tau) \in \Delta_4$, we set

$$u_{ts}^{\tau,i}(r) = (\tau - r)^{-\gamma} \mathbf{z}_{rs}^{1,r,i} \mathbb{1}_{[s,t]}(r). \tag{4.34}$$

The increment $\mathbf{z}_{ts}^{2,\tau}$ is given as follows: if $i \neq j$ we define $\mathbf{z}_{ts}^{2,\tau,i,j}$ as

$$\mathbf{z}_{ts}^{2,\tau,i,j} = B^j(u_{ts}^{\tau,i}), \tag{4.35}$$

where (conditionally on B^i) the random variable $B^j(u_{ts}^{\tau,i})$ has to be interpreted as a Wiener integral. In the case $i = j$, we set

$$\mathbf{z}_{ts}^{2,\tau,i,i} = \int_s^t u_{ts}^{\tau,i}(r) dB_r^i, \tag{4.36}$$

where the right hand side of (4.36) is defined as a Stratonovich integral like (4.11).

Remark 4.7 With Definition 4.2 of $\mathbf{z}^{1,\tau}$ in mind when considering the process $u^{\tau,i}$ in (4.34), we get that $\mathbf{z}^{2,\tau}$ in (4.35)-(4.36) is formally interpreted as

$$\mathbf{z}_{ts}^{2,\tau,i,j} = \int_s^t (\tau - r)^{-\gamma} \int_s^r (r - l)^{-\gamma} \, dB_l^i dB_r^j. \tag{4.37}$$

Below we will show that $\mathbf{z}^{2,\tau}$ can indeed be considered as the double iterated integral in (4.37).

Similarly to what we did for $\mathbf{z}^1$, we will now estimate the second moment of $\mathbf{z}^{2,\tau}$.

Proposition 4.8 *Consider the second level $\mathbf{z}^{2,\tau}$ of the Volterra rough path, as defined in* (4.35)-(4.36)*. Recall that $H \in (\frac{1}{2}, 1)$, $\gamma \in (0, 1)$, and $\eta, \zeta \in [0, 1]$ satisfy the relation* (4.15)*. Then for $(s, t, \tau) \in \Delta_3$ and any $i, j = 1, \ldots, d$, we have*

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau,i,j}\right)^2\right] \lesssim \left|\psi_{(2H-\gamma,\gamma)}^1(\tau, t, s)\right|^2. \tag{4.38}$$

As far as the $(1, 2)$*-type increments are considered, we get*

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau\tau',i,j}\right)^2\right] \lesssim \left|\psi_{(2H-\gamma,\gamma,\eta,\zeta)}^{1,2}(\tau, \tau', t, s)\right|^2, \tag{4.39}$$

where ψ^1 and $\psi^{1,2}$ are given in Notation 2.2.

Proof We will prove relation (4.38) in the following, (4.39) can be treated in a similar way and is left to the reader for sake of conciseness. According to Remark 4.7, we consider $\mathbf{z}_{ts}^{2,\tau,i,j}$ and $\mathbf{z}_{ts}^{2,\tau,i,i}$ as different integrals. Therefore we will split the proof of (4.38) into two parts: $i \neq j$ and $i = j$.
Step 1: Relation (4.38) *for* $i \neq j$. In this step, we will show that (4.38) holds for $\mathbf{z}_{ts}^{2,\tau,i,j}$ as given in (4.35). According to Definition 4.6, we consider the integral (4.35) as a conditional Wiener integral. Namely due to the independence of B^i and B^j we can write

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau,i,j}\right)^2\right] = \mathbb{E}\left\{\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau,i,j}\right)^2 \,\Big|\, B^i\right]\right\} = \mathbb{E}\left\{\mathbb{E}\left[\left(B^j(u_{ts}^{\tau,i})\right)^2 \,\Big|\, B^i\right]\right\}, \quad (4.40)$$

where we recall that $u_{ts}^{\tau,i}$ is defined by (4.34). Furthermore, relation (4.40) for Wiener integral reads

$$\mathbb{E}\left[\left(B^j(u_{ts}^{\tau,i})\right)^2 \,\Big|\, B^i\right] = \|u_{ts}^{\tau,i}\|_{\mathcal{H}}^2, \quad (4.41)$$

and thus

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau,i,j}\right)^2\right] = \mathbb{E}\left[\|u_{ts}^{\tau,i}\|_{\mathcal{H}}^2\right]. \quad (4.42)$$

In order to bound the right hand side of (4.42), we resort to the expression (4.2) for the inner product in $\mathcal{H}$. This yields

$$\mathbb{E}\left[\|u_{ts}^{\tau,i}\|_{\mathcal{H}}^2\right] = \mathbb{E}[\langle u_{ts}^{\tau,i}, u_{ts}^{\tau,i}\rangle_{\mathcal{H}}] = H(2H-1)\mathbb{E}\Big[\int_s^t \int_s^t \left((\tau - r_1)^{-\gamma} \int_s^{r_1} (r_1 - l_1)^{-\gamma} dB_{l_1}^i\right)$$
$$\times \left((\tau - r_2)^{-\gamma} \int_s^{r_2} (r_2 - l_2)^{-\gamma} dB_{l_2}^i\right) |r_1 - r_2|^{2H-2} dr_1 dr_2\Big].$$

Thanks to an easy application of Fubini's theorem, and invoking the symmetry of the integrand like in (4.20) we get

$$\mathbb{E}\left[\|u_{ts}^{\tau,i}\|_{\mathcal{H}}^2\right] = 2H(2H-1) \int_s^t \int_s^{r_1} (\tau - r_1)^{-\gamma} (\tau - r_2)^{-\gamma} |r_1 - r_2|^{2H-2}$$
$$\times \mathbb{E}\left[\int_s^{r_1} (r_1 - l_1)^{-\gamma} dB_{l_1}^i \int_s^{r_2} (r_2 - l_2)^{-\gamma} dB_{l_2}^i\right] dr_2 dr_1. \quad (4.43)$$

Moreover, owing to (4.32), and recalling the definition (2.2) of ψ^1, for $r_2 < r_1$ we have

$$\mathbb{E}\left[\int_s^{r_1} (r_1 - l_1)^{-\gamma} dB_{l_1}^i \int_s^{r_2} (r_2 - l_2)^{-\gamma} dB_{l_2}^i\right] = \mathbb{E}\left[\mathbf{z}_{r_1 s}^{1,r_1,i} \mathbf{z}_{r_2 s}^{1,r_2,i}\right] \lesssim |r_1 - s|^{2H-2\gamma}. \quad (4.44)$$

Plugging (4.44) into (4.43), we thus get

$$\mathbb{E}\left[\|u_{ts}^{\tau,i}\|_{\mathcal{H}}^2\right] \lesssim \int_s^t \int_s^{r_1} (\tau-r_1)^{-\gamma}(\tau-r_2)^{-\gamma}|r_1-r_2|^{2H-2}|r_1-s|^{2H-2\gamma}\,dr_1dr_2. \tag{4.45}$$

Similarly to what we did for (4.19)-(4.26) in the proof of Lemma 4.4, we evaluate the right hand side of (4.45) thanks to elementary integral bounds and the use of Beta functions. We let the patient reader check that we get

$$\mathbb{E}\left[\|u_{ts}^{\tau,i}\|_{\mathcal{H}}^2\right] \lesssim \left[|\tau-t|^{-2\gamma}|t-s|^{4H-2\gamma}\right] \wedge |t-s|^{4H-4\gamma}. \tag{4.46}$$

Plugging (4.46) into (4.41), we thus obtain

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau,i,j}\right)^2\right] \lesssim \left[|\tau-t|^{-2\gamma}|t-s|^{4H-2\gamma}\right] \wedge |t-s|^{4H-4\gamma} = \left|\psi^1_{(2H-\gamma,\gamma)}(\tau,t,s)\right|^2, \tag{4.47}$$

where we have invoked the definition (2.2) of ψ^1. This is the desired result (4.38).

Step 2: Relation (4.38) *for* $i=j$. In this step, we will show that relation (4.38) holds for $\mathbf{z}_{ts}^{2,\tau,i,i}$ defined by (4.36). According to Definition 4.6, we consider (4.36) as a Stratonovich integral like (4.11). We thus recast (4.36) as

$$\mathbf{z}_{ts}^{2,\tau,i,i} = \int_s^t u_{ts}^{\tau,i}(r)dB_r^i = \int_s^t u_{ts}^{\tau,i}(r)\,\delta^\diamond B_r^i + H(2H-1)\int_s^t\int_s^t D_l^i\left(u_{ts}^{\tau,i}(r)\right) |r-l|^{2H-2}dr\,dl, \tag{4.48}$$

where $u_{ts}^{\tau,i}(r)$ as given in (4.34). Taking square and expectation on both sides of (4.48), we obtain

$$\mathbb{E}\left[\left(\int_s^t u_{ts}^{\tau,i}(r)dB_r^i\right)^2\right] \lesssim J_1 + J_2, \tag{4.49}$$

where the terms J_1 and J_2 are respectively defined by

$$J_1 = \mathbb{E}\left[\left(\int_s^t u_{ts}^{\tau,i}(r)\,\delta^\diamond B_r^i\right)^2\right], \tag{4.50}$$

$$J_2 = (H(2H-1))^2\,\mathbb{E}\left[\left(\int_s^t\int_s^t D_l^i\left(u_{ts}^{\tau,i}(r)\right)|r-l|^{2H-2}dr\,dl\right)^2\right]. \tag{4.51}$$

In the following, we will estimate J_1 and J_2 separately.

In order to upper bound J_1, we recall that the integral $\int_s^t u_{ts}^{\tau,i}(r)\delta^\diamond B_r^i$ in the right hand side of (4.50) is interpreted as a Skorohod integral of the form $\delta^\diamond(u_{ts}^{\tau,i})$. Resorting to (4.12), we thus have

$$J_1 = \|\delta^\diamond\left(u_{ts}^{\tau,i}\right)\|_2^2 \lesssim \|u_{ts}^{\tau,i}\|_{\mathbb{D}^{1,2}(\mathcal{H}^{\otimes 2})}^2. \tag{4.52}$$

Let us now handle the right hand side of (4.52). Owing to (4.7), we get

$$J_1 \lesssim \mathbb{E}\left[\|u_{ts}^{\tau,i}\|_{\mathcal{H}}^2\right] + \mathbb{E}\left[\|D^i\left(u_{ts}^{\tau,i}\right)\|_{\mathcal{H}^{\otimes 2}}^2\right]. \tag{4.53}$$

Notice that the first term of the right hand side of (4.53) is what we upper bounded in Step 1. Thanks to (4.46), we obtain

$$\mathbb{E}\left[\|u_{ts}^{\tau,i}\|_{\mathcal{H}}^2\right] \lesssim \left[|\tau-t|^{-2\gamma}|t-s|^{4H-2\gamma}\right] \wedge |t-s|^{4H-4\gamma}. \tag{4.54}$$

In order to estimate the second term in the right hand side of (4.53), let us first compute the partial Malliavin derivative $D_l^i(u_{ts}^{\tau,i}(r))$ of $u_{ts}^{\tau,i}(r)$ with respect to B^i. Specifically, we gather (4.13) and (4.34) in order to get

$$u_{ts}^{\tau,i}(r) = (\tau-r)^{-\gamma} B^i(h_{rs}^r)\mathbb{1}_{[s,t]}(r), \qquad \text{with} \qquad h_{rs}^r(l) = (r-l)^{-\gamma}\mathbb{1}_{[s,r]}(l).$$

Thanks to (4.6), we thus get

$$D_l^i\left(u_{ts}^{\tau,i}(r)\right) = (\tau-r)^{-\gamma} h_{rs}^r(l)\mathbb{1}_{[s,t]}(r) = (\tau-r)^{-\gamma}(r-l)^{-\gamma}\mathbb{1}_{[s,r]}(l)\mathbb{1}_{[s,t]}(r). \tag{4.55}$$

Plugging (4.55) into the second term of the right hand side of (4.53), and having the definition (4.3) of $\mathcal{H}^{\otimes 2}$-norms in mind, we obtain

$$\begin{aligned}\|D^i u_{ts}^{\tau,i}\|_{\mathcal{H}^{\otimes 2}}^2 = (H(2H-1))^2 \int_s^t\int_s^t\int_s^{r_1}\int_s^{r_2} (\tau-r_1)^{-\gamma}(\tau-r_2)^{-\gamma} \\ \times(r_1-l_1)^{-\gamma}(r_2-l_2)^{-\gamma}|l_1-l_2|^{2H-2}|r_1-r_2|^{2H-2}dl_1dl_2dr_1dr_2.\end{aligned} \tag{4.56}$$

The right hand side of (4.56) can be estimated by elementary calculus similarly to (4.19)-(4.27). We let the patient reader check that whenever $\gamma < 2H-1$ we have

$$\mathbb{E}\left[\|D^i u_{ts}^{\tau,i}\|_{\mathcal{H}^{\otimes 2}}^2\right] \lesssim \left[|\tau-t|^{-2\gamma}|t-s|^{4H-2\gamma}\right] \wedge |t-s|^{4H-4\gamma}. \tag{4.57}$$

Eventually plugging (4.57) and (4.54) into (4.53), we end up with

$$J_1 \lesssim \left[|\tau-t|^{-2\gamma}|t-s|^{4H-2\gamma}\right] \wedge |t-s|^{4H-4\gamma}. \tag{4.58}$$

Next we upper bound J_2 as given in (4.51). Recalling that we have computed $D_l^i(u_{ts}^{\tau,i}(r))$ in (4.55) and plugging this identity into (4.51), we obtain

$$J_2 = (H(2H-1))^2 \left(\int_s^t \int_s^r (\tau - r)^{-\gamma} (r-l)^{-\gamma} |r-l|^{2H-2} dr\, dl \right)^2 . \tag{4.59}$$

Along the same lines as for the computations from (4.20) to (4.26), and recalling the fact that $\gamma < 2H - 1 < 1$, we get the following upper bound for J_2,

$$J_2 \lesssim \left[|\tau - t|^{-2\gamma} |t-s|^{4H-2\gamma} \right] \wedge |t-s|^{4H-4\gamma}. \tag{4.60}$$

Eventually plugging (4.60) and (4.58) into (4.49) and recalling again the definition (2.2) of ψ^1, we get

$$\mathbb{E}\left[\left(\int_s^t u_r dB_r^i \right)^2 \right] \lesssim \left[|\tau - t|^{-2\gamma} |t-s|^{4H-2\gamma} \right] \wedge |t-s|^{4H-4\gamma} = \left| \psi^1_{(2H-\gamma,\gamma)}(\tau, t, s) \right|^2 .$$

This completes the proof of Step 2.

Eventually, combining Step 1 and Step 2, relation (4.38) holds for the increment $\mathbf{z}_{ts}^{2,\tau}$ as given in (4.35)-(4.36). This concludes the proof of (4.38). ◻

Remark 4.9 The condition $\gamma < 2H - 1$, as stated in (4.15), is only invoked in order to properly bound the right hand side of (4.59).

4.4 Properties of the Volterra Rough Path Family $\{\mathbf{z}^{1,\tau}, \mathbf{z}^{2,\tau}\}$

We have constructed a Volterra rough path family $\{\mathbf{z}^{1,\tau}, \mathbf{z}^{2,\tau}\}$ and we have also upper bounded their moment in Sect. 4.2 and Sect. 4.3. In this section, we will verify that the family $\{\mathbf{z}^{1,\tau}, \mathbf{z}^{2,\tau}\}$ satisfies Definition 2.9. To this aim, we start by introducing the following notation:

Notation 4.10 Let $H \in (1/2, 1)$, and consider two parameters $\alpha \in (0, H)$, $\gamma \in (0, 2H-1)$, and a family (η_k, ζ_k) for $1 \le k \le N$ such that (2.8) is satisfied. The analytic property 2.9(iii) of Volterra rough path is also labeled in the following way:

$$\mathbf{z}^j \in \bigcap_{(\eta,\zeta)\in\mathcal{A}_N} \mathcal{V}^{(j\rho+\gamma,\gamma,\eta,\zeta)}, \tag{4.61}$$

where $\mathcal{A}_N$ is given by

$$\mathcal{A}_N = \{(\eta_k, \zeta_k);\ 1 \le k \le N\}. \tag{4.62}$$

Let us first check the analytic part of Definition 2.9 for $\mathbf{z}^{1,\tau}$.

Proposition 4.11 *Let $H \in (1/2, 1)$, and consider two parameters $\alpha \in (0, H)$, $\gamma \in (0, 2H-1)$ and a family (η_k, ζ_k) as in Notation 4.10. Then the increment $\mathbf{z}^{1,\tau}$ introduced in Definition 4.2 is almost surely in the Volterra space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_3; \mathbb{R}^m)$ for any $(\eta, \zeta) \in \mathcal{A}_N$, where $\mathcal{A}_N$ is given in (4.62) and $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_3; \mathbb{R}^m)$ is introduced in Definition 2.3. In addition, for any $p \geq 1$ and $(\eta, \zeta) \in \mathcal{A}_N$, we have that*

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta)}\right] < \infty. \tag{4.63}$$

Proof In this proof, we will turn to Theorem 3.7 in order to prove (4.63). According to the definition (2.4) of Volterra norms, it suffices to show that $\mathbb{E}[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}]$ and $\mathbb{E}[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta),1,2}]$ are finite. We will separate the study of those two moments.
Step 1: Estimate for the $2p$ moment of 1-norm. Let us first upper bound $\mathbb{E}[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}]$. Towards this aim, consider a fixed Volterra exponent $\gamma < \alpha < H$ and a parameter $p > 1$ to be determined later on. Relation (3.34) is then equivalent to

$$\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1} \lesssim \left(U^T_{(\alpha,\gamma),1,p}(\mathbf{z}^1)\right)^{2p} + \|\delta\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}. \tag{4.64}$$

Let us handle the term $\|\delta\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}$ in the right hand side of (4.64). Gathering the definitions in (4.13) and (2.12), for $(s, m, t) \in \Delta_3$ we have

$$\delta_m \mathbf{z}^{1,\tau,i}_{ts} = B^i(h^\tau_{ts}) - B^i(h^\tau_{tm}) - B^i(h^\tau_{ms}). \tag{4.65}$$

Moreover recalling that $h^\tau_{ts}(r) = (\tau - r)^{-\gamma}\mathbb{1}_{[s,t]}(r)$, it is readily checked that $h^\tau_{ts} - h^\tau_{tm} - h^\tau_{ms} = 0$. We thus get $\delta\mathbf{z}^{1,\tau,i} = 0$ and (4.64) is reduced to

$$\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1} \lesssim \left(U^T_{(\alpha,\gamma),1,p}(\mathbf{z}^1)\right)^{2p}. \tag{4.66}$$

Taking expectations on both sides of (4.66) and recalling the definition (3.1) of $U^T_{(\alpha,\gamma),1,p}$, we obtain

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}\right] \lesssim \mathbb{E}\left[\int_{(v,w)\in\Delta_2^\tau} \frac{|z^{1,\tau}_{wv}|^{2p}}{|\psi^1_{(\alpha,\gamma)}(\tau, w, v)|^{2p}|w-v|^2} \mathrm{d}v\, \mathrm{d}w\right]. \tag{4.67}$$

Invoking Fubini's theorem and the fact that $\mathbf{z}^{1,\tau}_{wv}$ is a Gaussian random variable, we thus get

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}\right] \lesssim \int_{(v,w)\in\Delta_2^\tau} \frac{\mathbb{E}\left[|z^{1,\tau}_{wv}|^2\right]^p}{|\psi^1_{(\alpha,\gamma)}(\tau, w, v)|^{2p}|w-v|^2} \mathrm{d}v\mathrm{d}w. \tag{4.68}$$

We can now apply (4.16), and hence relation (4.68) reads

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}\right] \lesssim \int_{(v,w)\in\Delta_2^\tau} \frac{|\psi^1_{(H,\gamma)}(\tau,w,v)|^{2p}}{|\psi^1_{(\alpha,\gamma)}(\tau,w,v)|^{2p}|w-v|^2} \mathrm{d}v\mathrm{d}w. \tag{4.69}$$

Recalling the definition (2.2) of ψ^1, we obtain

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}\right] \lesssim \int_{(v,w)\in\Delta_2^\tau} \frac{\left[|\tau-w|^{-2p\gamma}|w-v|^{2pH}\right]\wedge|w-v|^{2p(H-\gamma)}}{\left(\left[|\tau-w|^{-2p\gamma}|w-v|^{2p\alpha}\right]\wedge|w-v|^{2p(\alpha-\gamma)}\right)|w-v|^2} \mathrm{d}v\mathrm{d}w. \tag{4.70}$$

In order to upper bound the right hand side of (4.70), we split set Δ_2^τ into two subsets

$$E_1 = \left\{(v,w)\in\Delta_2^\tau \,\middle|\, |\tau-w|\le|w-v|\right\}, \quad \text{and} \quad E_2 = \left\{(v,w)\in\Delta_2^\tau \,\middle|\, |\tau-w|>|w-v|\right\}.$$

Then relation (4.70) is equivalent to

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}\right] \lesssim I_1 + I_2, \tag{4.71}$$

where I_1 and I_2 are respectively given by

$$I_1 = \int_{E_1} \frac{\left[|\tau-w|^{-2p\gamma}|w-v|^{2pH}\right]\wedge|w-v|^{2p(H-\gamma)}}{\left[|\tau-w|^{-2p\gamma}|w-v|^{2p\alpha+2}\right]\wedge|w-v|^{2p(\alpha-\gamma)+2}} \mathrm{d}v\mathrm{d}w, \tag{4.72}$$

$$I_2 = \int_{E_2} \frac{\left[|\tau-w|^{-2p\gamma}|w-v|^{2pH}\right]\wedge|w-v|^{2p(H-\gamma)}}{\left[|\tau-w|^{-2p\gamma}|w-v|^{2p\alpha+2}\right]\wedge|w-v|^{2p(\alpha-\gamma)+2}} \mathrm{d}v\mathrm{d}w. \tag{4.73}$$

In the following, we will estimate I_1 and I_2 separately.

In order to upper bound I_1, we first note that for any $(v,w)\in E_1$, we have $|\tau-w|\le|w-v|$. Thus

$$|w-v|^{2p(H-\gamma)} = |w-v|^{2pH}|w-v|^{-2p\gamma} \le |\tau-w|^{-2p\gamma}|w-v|^{2pH}, \tag{4.74}$$

and we trivially get

$$\left[|\tau-w|^{-2p\gamma}|w-v|^{2pH}\right]\wedge|w-v|^{2p(H-\gamma)} = |w-v|^{2p(H-\gamma)}. \tag{4.75}$$

In the same way, on E_1 we can write

$$\left[|\tau-w|^{-2p\gamma}|w-v|^{2p\alpha+2}\right]\wedge|w-v|^{2p(\alpha-\gamma)+2} = |w-v|^{2p(\alpha-\gamma)+2}. \tag{4.76}$$

Plugging (4.75) and (4.76) into (4.72), we get

$$I_1 = \int_{E_1} \frac{|w-v|^{2p(H-\gamma)}}{|w-v|^{2p(\alpha-\gamma)+2}} = \int_{E_1} |w-v|^{2p(H-\alpha)-2} \mathrm{d}v \mathrm{d}w. \tag{4.77}$$

Similarly, reverting the inequality in (4.74) we get that

$$I_2 = \int_{E_2} \frac{|\tau-w|^{-2p\gamma}|w-v|^{2pH}}{|\tau-w|^{-2p\gamma}|w-v|^{2p\alpha+2}} \mathrm{d}v \mathrm{d}w = \int_{E_2} |w-v|^{2p(H-\alpha)-2} \mathrm{d}v \mathrm{d}w. \tag{4.78}$$

Now gathering (4.77) and (4.78) into (4.71), we end up with

$$\mathbb{E}\left[\|\mathbf{z}^1\|_{(\alpha,\gamma),1}^{2p}\right] \lesssim \int_{(v,w)\in\Delta_2^\tau} |w-v|^{2p(H-\alpha)-2} \mathrm{d}v \mathrm{d}w. \tag{4.79}$$

The right hand side above is easily checked to be finite as long as $\alpha < H - \frac{1}{2p}$.
Step 2: Estimate for the $2p$ *moment of* $(1,2)$*-norm.* Next, we will show that $\mathbb{E}[\|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2}]$ is finite. Similarly to the proof for the 1-norm in Step 1, considering again $p \geq 1$. Then resorting to (3.35), for $(\eta,\zeta) \in \mathcal{A}_N$ we get

$$\mathbb{E}\left[\|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2}^{2p}\right] \lesssim \mathbb{E}\left[\left(U^T_{(\alpha,\gamma,\eta,\zeta),1,2,p}(\mathbf{z}^1)\right)^{2p}\right].$$

As in Step 1, recalling the definition (3.2) of $U^T_{(\alpha,\gamma,\eta,\zeta),1,2,p}(\mathbf{z})$, invoking Fubini's theorem and thanks to the fact that $\mathbf{z}^{1,\tau\tau'}$ is a Gaussian random variable, we obtain

$$\mathbb{E}\left[\|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2}^{2p}\right] \lesssim \int_{(v,w,r',r)\in\Delta_4^\tau} \frac{\mathbb{E}^p\left[|z_{wv}^{1,rr'}|^2\right]}{|\psi^{1,2}_{(\alpha,\gamma,\eta,\zeta)}(r,r',w,v)|^{2p}|w-v|^2|r-r'|^2} \mathrm{d}v \mathrm{d}w \mathrm{d}r' \mathrm{d}r. \tag{4.80}$$

In addition, owing to (4.17), relation (4.80) yields

$$\mathbb{E}\left[\|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2}^{2p}\right] \lesssim \int_{(v,w,r',r)\in\Delta_4^\tau} \frac{|\psi^{1,2}_{(H,\gamma,\eta+\frac{1}{p},\zeta+\frac{1}{p})}(r,r',w,v)|^{2p}}{|\psi^{1,2}_{(\alpha,\gamma,\eta,\zeta)}(r,r',w,v)|^{2p}|w-v|^2|r-r'|^2} \mathrm{d}v \mathrm{d}w \mathrm{d}r' \mathrm{d}r. \tag{4.81}$$

We now recall the definition (2.3) of $\psi^{1,2}$ and plug this identity into (4.81). We get

$$\mathbb{E}\left[\|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2}^{2p}\right] \lesssim \int_{(v,w,r',r)\in\Delta_4^\tau} g_{(H,\alpha,\gamma,\eta,\zeta,p)}(r,r',w,v) \, \mathrm{d}v \mathrm{d}w \mathrm{d}r' \mathrm{d}r, \tag{4.82}$$

where $g_{(H,\alpha,\gamma,\eta,\zeta,p)}(r,r',w,v)$ is given by

$$g_{(H,\alpha,\gamma,\eta,\zeta,p)}(r,r',w,v)$$
$$= \frac{|r-r'|^{2p(\eta+\frac{1}{p})}|r'-w|^{-2p(\eta-\zeta)}\left(\left[|r'-w|^{-2p(\gamma+\zeta+\frac{1}{p})}|w-v|^{2pH}\right]\wedge|w-v|^{2p(H-\gamma-\zeta-\frac{1}{p})}\right)}{|r-r'|^{2p\eta}|r'-w|^{-2p(\eta-\zeta)}\left(\left[|r'-w|^{-2p(\gamma+\zeta)}|w-v|^{2p\alpha}\right]\wedge|w-v|^{2p(\alpha-\gamma-\zeta)}\right)|w-v|^2|r-r'|^2}. \tag{4.83}$$

Thanks to cancellations, we can simplify the right hand side of (4.83) as

$$g_{(H,\alpha,\gamma,\eta,\zeta,p)}(r,r',w,v) = \frac{\left[|r'-w|^{-2p(\gamma+\zeta)-2}|w-v|^{2pH}\right]\wedge|w-v|^{2p(H-\gamma-\zeta)-2}}{\left(\left[|r'-w|^{-2p(\gamma+\zeta)}|w-v|^{2p\alpha}\right]\wedge|w-v|^{2p(\alpha-\gamma-\zeta)}\right)|w-v|^2}. \tag{4.84}$$

Plugging (4.84) into (4.82), we thus get

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta),1,2}\right]$$
$$\lesssim \int_{(v,w,r',r)\in\Delta_4^\tau} \frac{\left[|r'-w|^{-2p(\gamma+\zeta)-2}|w-v|^{2pH}\right]\wedge|w-v|^{2p(H-\gamma-\zeta)-2}}{\left(\left[|r'-w|^{-2p(\gamma+\zeta)}|w-v|^{2p\alpha}\right]\wedge|w-v|^{2p(\alpha-\gamma-\zeta)}\right)|w-v|^2}\,\mathrm{d}v\mathrm{d}w\mathrm{d}r'\mathrm{d}r. \tag{4.85}$$

Notice that the right hand side of (4.85) is now very similar to the right hand side of (4.70). Therefore with the same steps as for (4.70)-(4.79), we obtain that

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta),1,2}\right] \lesssim \int_{(v,w,r',r)\in\Delta_4^\tau} |w-v|^{2p(H-\alpha)-4}\mathrm{d}v\mathrm{d}w\mathrm{d}r'\mathrm{d}r < \infty. \tag{4.86}$$

The right hand side of (4.86) is finite as long as $p > \frac{3}{2}(H-\alpha)^{-1}$, or equivalently $\alpha < H - \frac{3}{2p}$.

Step 3: (4.63) *holds for any* $p \geq 1$. Invoking (4.64) we immediately have

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta)}\right] \lesssim \mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma),1}\right] + \mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta),1,2}\right]. \tag{4.87}$$

Furthermore, combining (4.79) and (4.86) in the right hand side of (4.87), we end up with

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta)}\right] < \infty, \quad \text{for any} \quad p > \frac{3}{2(H-\alpha)}. \tag{4.88}$$

Next we observe that if (4.88) is satisfied under the constraint $p > \frac{3}{2(H-\alpha)}$, it is also verified for all $p \geq 1$. This yields the desired result (4.63). Moreover, it is easy to check that (4.88) implies

$$\|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta)} < \infty \quad \text{a.s.}$$

This means that $\mathbf{z}^1$ is almost surely in the Volterra space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_3;\mathbb{R}^m)$. $\square$

Remark 4.12 Note that the Volterra sewing lemma in [15] could indeed be used to construct $\mathbf{z}^1$ in a purely pathwise manner due to the Hölder regularity of the fBm combined with the assumption $H - \gamma > 0$. This would then be constructed as the pathwise integral given by

$$\mathbf{z}_{ts}^{1,\tau} = \lim_{|\mathcal{P}|\to 0} \sum_{[u,v]\in\mathcal{P}} k(\tau, u) B_{vu}(\omega),$$

where the sum converges by deterministic arguments and $\mathcal{P}$ is a partition of $[s, t]$. In fact, it follows then directly that $\mathbf{z}^1 \in \mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}$ for any $\alpha - \gamma > 0$ and $(\eta, \zeta) \in \mathcal{A}_N$. However we have chosen to construct $\mathbf{z}^1$ by probabilistic means, our motivation being twofold:

(i) As the reader will see, the construction of the second-order integral $\mathbf{z}^2$ is probabilistic by nature. Therefore it is natural and more consistent to construct $\mathbf{z}^1$ with the same kind of method.
(ii) The probabilistic construction of $\mathbf{z}^1$ is not only instructive, but also provides useful probabilistic bounds for the moments of $\mathbf{z}^1$. Those estimates are of interest on their own.

With Proposition 4.11 in hand, we finish the study of $\mathbf{z}^1$ by proving the algebraic relation (2.13) for $\mathbf{z}^{1,\tau}$ in more detail.

Proposition 4.13 *The increment $\mathbf{z}_{ts}^{1,\tau,i}$ as given in* (4.13) *satisfies relation* (2.13)*, that is almost surely we have*

$$\delta_m \mathbf{z}_{ts}^{1,\tau,i} = 0, \quad \textit{for all} \quad (s, m, t, \tau) \in \Delta_4. \tag{4.89}$$

Proof For fixed $(s, m, t, \tau) \in \Delta_4$, we have obtained in (4.65) that $\delta_m \mathbf{z}_{ts}^{1,\tau,i} = 0$ almost surely. We will now prove that

$$(t, \tau) \in \Delta_2 \mapsto \mathbf{z}_t^{1,\tau} \in \mathbb{R}^m \text{ is a continuous function.} \tag{4.90}$$

By a standard argument, which consists in taking limits on rational points, this will achieve our claim (4.89). The proof of (4.90) relies on Lemma 4.4. Indeed, according to (4.16) for (s, t, τ) in Δ_3, we have

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{1,\tau}\right)^2\right] \lesssim |t - s|^{2(H-\gamma)}. \tag{4.91}$$

In the same way thanks to (4.17) applied with $\zeta = \eta = H - \gamma - \epsilon$ with a small $\epsilon > 0$, we get

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{1,\tau\tau'}\right)^2\right] \lesssim \left|\tau - \tau'\right|^{H-\gamma-\epsilon} |t - s|^{\epsilon}. \tag{4.92}$$

Gathering (4.91) and (4.92), we end up with the following inequality, valid for $(s, t, \tau', \tau) \in \Delta_4$:

$$\|\mathbf{z}_{ts}^{1,\tau\tau'}\|_{L^2(\Omega)} \lesssim \left|\tau - \tau'\right|^{H-\gamma-\epsilon} + |t-s|^{H-\gamma}\,. \tag{4.93}$$

Moreover $\mathbf{z}_{ts}^{1,\tau\tau'}$ is a Gaussian random variable. Hence the upper bound (4.93) can be extended to arbitrary norms in $L^p(\Omega)$. Therefore a standard application of Kolmogorov's criterion yields the continuity property (4.91) for $\mathbf{z}^{1,\tau}$. This finishes our proof. □

We now turn to the analysis $\mathbf{z}^{2,\tau}$. We start this study by verifying the algebraic relation (2.13) for $\mathbf{z}^{2,\tau}$.

Proposition 4.14 *The increment $\mathbf{z}_{ts}^{2,\tau}$ as given in (4.35)–(4.36) satisfies relation (2.13), that is*

$$\delta_m \mathbf{z}_{ts}^{2,\tau,i,j} = \mathbf{z}_{tm}^{1,\tau,j} * \mathbf{z}_{ms}^{1,\cdot,i}, \quad \textit{for all } (s, m, t, \tau) \in \Delta_4 \quad \textit{a.s.} \tag{4.94}$$

Proof In order to show (4.94), we first prove that (4.94) holds for fixed $(s, m, t) \in \Delta_3^\tau$. According to Definition 4.6, we will separate the proof into two cases $i \neq j$ and $i = j$. *Step 1: (4.94) holds for fixed $(s, m, t) \in \Delta_3^\tau$ when $i \neq j$.* In this step, let us handle the case $i \neq j$. For any $(s, m, t) \in \Delta_3^\tau$, gathering (4.35) and (2.12), we have

$$\delta_m \mathbf{z}_{ts}^{2,\tau,i,j} = B^j\left(u_{ts}^{\tau,i}\right) - B^j\left(u_{tm}^{\tau,i}\right) - B^j\left(u_{ms}^{\tau,i}\right), \tag{4.95}$$

where we recall that the process u is defined by (4.34). In order to calculate the right hand side of (4.95), it is thus sufficient to compute $\delta_m u_{ts}^{\tau,i} = u_{ts}^{\tau,i} - u_{tm}^{\tau,i} - u_{ms}^{\tau,i}$. To this aim, according to the definition (4.34) of $u^{\tau,i}$, we obtain

$$\begin{aligned}\delta_m u_{ts}^{\tau,i}(r) &= u_{ts}^{\tau,i}(r) - u_{tm}^{\tau,i}(r) - u_{ms}^{\tau,i}(r)\\ &= (\tau-r)^{-\gamma}\,\mathbf{z}_{rs}^{1,r,i}\mathbb{1}_{[s,t]}(r) - (\tau-r)^{-\gamma}\,\mathbf{z}_{rm}^{1,r,i}\mathbb{1}_{[m,t]}(r) - (\tau-r)^{-\gamma}\,\mathbf{z}_{rs}^{1,r,i}\mathbb{1}_{[s,m]}(r).\end{aligned}$$

Resorting to the definition (4.13) of $\mathbf{z}_{rs}^{1,r,i}$, we thus get

$$\delta_m u_{ts}^{\tau,i}(r) = (\tau-r)^{-\gamma}\left(B^i(h_{rs}^r)\mathbb{1}_{[s,t]}(r) - B^i(h_{rm}^r)\mathbb{1}_{[m,t]}(r) - B^i(h_{rs}^r)\mathbb{1}_{[s,m]}(r)\right), \tag{4.96}$$

where the expression for h is given in Definition 4.2. The right hand side of (4.96) can be simplified by elementary calculus. We thus let the patient reader check that we have

$$\delta_m u_{ts}^{\tau,i}(r) = (\tau-r)^{-\gamma}\,B^i\left(h_{ms}^r\right)\mathbb{1}_{[m,t]}(r). \tag{4.97}$$

Furthermore, according to the definition of h in Definition 4.2, we have that $(\tau - r)^{-\gamma}\, \mathbb{1}_{[m,t]}(r) = h^{\tau}_{tm}(r)$. Hence (4.97) can be recast as

$$\delta_m u^{\tau,i}_{ts}(r) = h^{\tau}_{tm}(r) B^i\left(h^r_{ms}\right). \tag{4.98}$$

Plugging (4.98) into (4.95), we thus have

$$\delta_m \mathbf{z}^{2,\tau,i,j}_{ts} = B^j\left(h^{\tau}_{tm} B^i\left(h^r_{ms}\right)\right). \tag{4.99}$$

Resorting to the property (4.5) of $B^j(h)$, the right hand side of (4.99) can be written as

$$\delta_m \mathbf{z}^{2,\tau,i,j}_{ts} = \lim_{|\mathcal{P}|\to 0} \sum_{[r,v]} B^j_{vr}\, h^{\tau}_{tm}(r)\, B^i(h^r_{ms}), \tag{4.100}$$

where we recall that $\mathcal{P}$ is a generic partition of $[m,t]$ whose mesh $|\mathcal{P}|$ is converging to 0, and where the limit holds in $L^2(\Omega)$. We now consider a subsequence of partitions in order to get an almost sure convergence in (4.100). According to the definition (2.9) of convolution product we end up with

$$\delta_m \mathbf{z}^{2,\tau,i,j}_{ts} = \mathbf{z}^{1,\tau,j}_{tm} * \mathbf{z}^{1,\cdot,i}_{ms},$$

where we have used the definition (4.13) of $\mathbf{z}^{1,\tau}$.

Step 2: (4.94) holds for fixed $(s,m,t) \in \Delta^{\tau}_3$ when $i = j$. In this step, we will deal with the case $i = j$ for the second level of the Volterra rough path. For any $(s,m,t) \in \Delta^{\tau}_3$, according to the definition (4.36) of $\mathbf{z}^{2,\tau,i,i}_{ts}$, we obtain

$$\delta_m \mathbf{z}^{2,\tau,i,i}_{ts} = \delta_m\left(\int_s^t u^{\tau,i}_{ts}(r) dB^i_r\right), \tag{4.101}$$

where the integral above is understood in the Stratonovich sense. According to (4.10), we have

$$\int_s^t u^{\tau,i}_{ts}(r) dB^i_r = \lim_{|\mathcal{P}|\to 0} S^{i,\mathcal{P}}_{ts}, \quad \text{where} \quad S^{i,\mathcal{P}}_{ts} = \int_s^t u^{\tau,i}_{ts}(r) B^{i,\mathcal{P}}_r dr.$$

Now for a fixed $\mathcal{P}$, elementary algebraic manipulations show that

$$\delta_m S^{i,\mathcal{P}}_{ts} = \mathbf{z}^{1,\tau,i,\mathcal{P}}_{tm} * \mathbf{z}^{1,\cdot,i}_{ms}, \tag{4.102}$$

where $\mathbf{z}^{1,\tau,j,\mathcal{P}}_{tm}$ is defined by (4.10). Taking limits on both sides of (4.102) as $\mathcal{P} \to 0$, we get

$$\delta_m \mathbf{z}^{2,\tau,i,i}_{ts} = B^i\left(h^{\tau}_{tm} B^i(h^r_{ms})\right) = \mathbf{z}^{1,\tau,i}_{tm} * \mathbf{z}^{1,\cdot,i}_{ms},$$

which proves (4.94) for $i = j$.

Step 3: (4.94) *holds for all* $(s, m, t) \in \Delta_3^\tau$. The proof of this fact, based on Kolmogorov's criterion for continuity of stochastic processes, is very similar to the considerations in Proposition 4.13. For sake of conciseness, it is omitted here. The proof of (4.94) is now complete. □

Using the knowledge gained from Proposition 4.14, we are now ready to check the regularity of the object $\delta \mathbf{z}^{2,\tau}$.

Proposition 4.15 *Let* $H \in (\frac{1}{2}, 1)$*, and consider the second level* $\mathbf{z}^{2,\tau}$ *of the Volterra rough path, as defined in* (4.35)- (4.36)*. Recall that* $\delta \mathbf{z}^{2,\tau}$ *is defined on* Δ_4*, and we refer to Definition 2.10 for the definition of* $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_4; \mathbb{R}^m)$*. Consider four parameters* $\alpha \in (0, H)$*,* $\gamma \in (0, 2H - 1)$ *and* $\eta, \zeta \in [0, 1]$*, satisfying relation* (2.8)*. Let also* $\mathcal{A}_N$ *be the set defined by* (4.62)*. Then almost surely, for all* $(\eta, \zeta) \in \mathcal{A}_N$ *we have*

$$\delta \mathbf{z}^{2,\tau} \in \mathcal{V}^{(2\rho+\gamma,\gamma,\eta,\zeta)}(\Delta_4; \mathbb{R}^m), \tag{4.103}$$

where we recall that $\rho = \alpha - \gamma$*. Moreover, for all* $p \geq 1$ *we have*

$$\mathbb{E}\left[\|\delta \mathbf{z}^{2,\tau}\|_{(2\rho+\gamma,\gamma,\eta,\zeta)}^{2p} \right] < \infty, \tag{4.104}$$

where the norm above is understood as in (2.14)*.*

Proof In this proof, we will show that (4.104) holds for any $p \geq 1$, and it is easy to check that (4.103) is a direct consequence of (4.104). According to the definition (2.14), it is necessary to prove that $\mathbb{E}[\|\delta \mathbf{z}^{2,\tau}\|_{(2\rho+\gamma,\gamma),1}^{2p}]$ and $\mathbb{E}[\|\delta \mathbf{z}^{2,\tau}\|_{(2\rho+\gamma,\gamma,\eta,\zeta),1,2}^{2p}]$ are finite. Thanks to (4.94), for any $(s, u, t, \tau) \in \Delta_4$ we have

$$\delta_u \mathbf{z}_{ts}^{2,\tau} = \mathbf{z}_{tu}^{1,\tau} * \mathbf{z}_{us}^{1,\cdot}. \tag{4.105}$$

Hence resorting to (2.10), we get

$$\left| \delta_u \mathbf{z}_{ts}^{2,\tau} \right| = \left| \mathbf{z}_{tu}^{1,\tau} * \mathbf{z}_{us}^{1,\cdot} \right| \lesssim \|\mathbf{z}^1\|_{(\alpha,\gamma),1} \|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2}\, \psi_{(2\rho+\gamma,\gamma)}^1(\tau, t, s). \tag{4.106}$$

Dividing by $\psi_{2\rho+\gamma,\gamma}^1(\tau, t, s)$ on both sides of (4.106), and then taking supremum over $(s, u, t, \tau) \in \Delta_4$, we obtain

$$\|\delta \mathbf{z}^2\|_{(2\rho+\gamma,\gamma),1} \leq \|\mathbf{z}^1\|_{(\alpha,\gamma),1} \|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2}, \tag{4.107}$$

where we have used the definition (2.15) of 1−norm for the Volterra space $\mathcal{V}^{(2\rho+\gamma,\gamma,\eta,\zeta)}(\Delta_4; \mathbb{R}^m)$. Similarly, resorting to (2.11) and (2.16), for any $(s, u, t, \tau', \tau) \in \Delta_5$ and $(\eta, \zeta) \in \mathcal{A}_N$. We let the patient reader check that we have

$$\|\delta \mathbf{z}^2\|_{(2\rho+\gamma,\gamma,\eta,\zeta),1,2} \leq \|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2} \|\mathbf{z}^1\|_{(\alpha,\gamma,\eta,\zeta),1,2}. \tag{4.108}$$

Combining (4.107) and (4.108), and recalling the definition (2.14) again, we thus obtain

$$\|\delta \mathbf{z}^2\|_{(2\rho+\gamma,\gamma,\eta,\zeta)} = \|\delta \mathbf{z}^2\|_{(2\rho+\gamma,\gamma),1} + \|\delta \mathbf{z}^2\|_{(2\rho+\gamma,\gamma,\eta,\zeta),1,2} \lesssim \|\mathbf{z}^1\|^2_{(\alpha,\gamma,\eta,\zeta)}. \tag{4.109}$$

Taking $2p$ moments on both sides of (4.109), we thus get

$$\mathbb{E}\left[\|\delta \mathbf{z}^2\|^{2p}_{(2\rho+\gamma,\gamma,\eta,\zeta)}\right] \lesssim \mathbb{E}\left[\|\mathbf{z}^1\|^{4p}_{(\alpha,\gamma,\eta,\zeta)}\right]. \tag{4.110}$$

According to (4.63), the right hand side of (4.110) is finite. This means that we have

$$\mathbb{E}\left[\|\delta \mathbf{z}^2\|^{2p}_{(2\rho+\gamma,\gamma,\eta,\zeta)}\right] < \infty, \qquad \text{for any} \qquad p \geq 1. \tag{4.111}$$

This is the desired result. □

Finally, let us close this section by giving the proof of the regularity result for $\mathbf{z}^{2,\tau}$.

Proposition 4.16 *Under the same assumption as for Proposition 4.15, the second level of the Volterra rough path $\mathbf{z}^{2,\tau}$ introduced in (4.35)-(4.36) is almost surely an element of the space $\mathcal{V}^{(2\rho+\gamma,\gamma,\eta,\zeta)}(\Delta_3;\mathbb{R}^m)$ for any $\alpha,\gamma \in (0,1)$ and $\eta,\zeta \in [0,1]$ satisfying relation (2.8). Furthermore, for $H-\alpha > \frac{1}{4p}$ and any $(\eta,\zeta) \in \mathcal{A}_N$ (where $\mathcal{A}_N$ is given in (4.62)), we have that*

$$\mathbb{E}\left[\|\mathbf{z}^2\|^{2p}_{(2\rho+\gamma,\gamma,\eta,\zeta)}\right] < \infty. \tag{4.112}$$

Proof Our strategy to prove this Proposition is the same as for the proof of Proposition 4.11, that is we will appeal to the Volterra GRR Lemma 3.7 to show that $\mathbb{E}[\|\mathbf{z}^2\|^{2p}_{(2\rho+\gamma,\gamma),1}]$ and $\mathbb{E}[\|\mathbf{z}^2\|^{2p}_{(2\rho+\gamma,\gamma,\eta,\zeta),1,2}]$ are both finite. Let us first show that $\mathbb{E}[\|\mathbf{z}^2\|^{2p}_{(2\rho+\gamma,\gamma),1}]$ is finite. To this aim, consider a fixed Volterra exponent $\alpha \in (\gamma, H)$ and a parameter $p \geq 1$ to be determined later. Then relation (3.34) reads

$$\|\mathbf{z}^2\|^{2p}_{(2\rho+\gamma,\gamma),1} \lesssim \left(U^T_{(2\rho+\gamma,\gamma),1,p}(\mathbf{z}^2)\right)^{2p} + \|\delta \mathbf{z}^2\|^{2p}_{(2\rho+\gamma,\gamma),1}. \tag{4.113}$$

Taking expectations on both sides of (4.113), we obtain

$$\mathbb{E}\left[\|\mathbf{z}^2\|^{2p}_{(2\rho+\gamma,\gamma),1}\right] \lesssim \mathbb{E}\left[\left(U^T_{(2\rho+\gamma,\gamma),1,p}(\mathbf{z}^2)\right)^{2p}\right] + \mathbb{E}\left[\|\delta \mathbf{z}^2\|^{2p}_{(2\rho+\gamma,\gamma),1}\right]. \tag{4.114}$$

Recalling (4.104), the second term of the right hand side of (4.114) is finite. In order to upper bound the left hand side of (4.114), it is thus sufficient to estimate the first term $\mathbb{E}[(U^T_{(2\rho+\gamma,\gamma),1,p}(\mathbf{z}^2))^{2p}]$. Toward this aim, we set

$$A_{\gamma,\rho,p} = \mathbb{E}\left[\left(U^T_{(2\rho+\gamma,\gamma),1,p}(\mathbf{z}^2)\right)^{2p}\right].$$

Recalling the definition (3.2) of $U^T_{(2\rho+\gamma,\gamma),1,p}$, we have

$$A_{\gamma,\rho,p} = \mathbb{E}\left[\int_{(v,w)\in\Delta_2^\tau} \frac{|z^{2,\tau}_{wv}|^{2p}}{|\psi^1_{(2\rho+\gamma,\gamma)}(\tau,w,v)|^{2p}|w-v|^2}\,\mathrm{d}v\,\mathrm{d}w\right]. \tag{4.115}$$

Observe that $\mathbf{z}^{2,\tau}_{wv}$ is an element of the second chaos of the fBm B, on which all L^p norms are equivalent. Hence invoking Fubini's theorem, we get

$$A_{\gamma,\rho,p} \lesssim \int_{(v,w)\in\Delta_2^\tau} \frac{\mathbb{E}^p\left[|z^{2,\tau}_{wv}|^2\right]}{|\psi^1_{(2\alpha-\gamma,\gamma)}(\tau,w,v)|^{2p}|w-v|^2}\,\mathrm{d}v\,\mathrm{d}w. \tag{4.116}$$

We now apply (4.38) to the right hand side of (4.116), we obtain

$$A_{\gamma,\rho,p} \lesssim \int_{(v,w)\in\Delta_2^\tau} \frac{|\psi^1_{(2H-\gamma,\gamma)}(\tau,w,v)|^{2p}}{|\psi^1_{(2\rho+\gamma,\gamma)}(\tau,w,v)|^{2p}|w-v|^2}\,\mathrm{d}v\mathrm{d}w. \tag{4.117}$$

Notice that relation (4.117) is very similar to (4.67). Hence we can carry out the same procedure going from (4.67) to (4.79) in the proof of Proposition 4.11. We end up with

$$A_{\gamma,\rho,p} \lesssim \int_{(v,w)\in\Delta_2^\tau} |w-v|^{4p(H-\alpha)-2}\,\mathrm{d}v\mathrm{d}w. \tag{4.118}$$

Eventually plugging (4.118) into (4.114), we get

$$\mathbb{E}\left[\|\mathbf{z}^2\|^{2p}_{(2\rho+\gamma,\gamma),1}\right] \lesssim \int_{(v,w)\in\Delta_2^\tau} |w-v|^{4p(H-\alpha)-2}\,\mathrm{d}v\mathrm{d}w + \mathbb{E}\left[\|\delta\mathbf{z}^2\|^{2p}_{(2\rho+\gamma,\gamma),1}\right]. \tag{4.119}$$

Recalling (4.104) again, the right hand side above is easily checked to be finite as long as $p > \frac{1}{4}(H-\alpha)^{-1}$. Considering such a p (which is allowed since $z^{2,\tau}_{wv}$ admits moments of all orders), we thus obtain

$$\mathbb{E}\left[\|\mathbf{z}^2\|^{2p}_{(2\rho+\gamma,\gamma),1}\right] < \infty. \tag{4.120}$$

Next we will show that $\mathbb{E}[\|\mathbf{z}^2\|^{2p}_{(2\rho+\gamma,\gamma,\eta,\zeta),1,2}]$ is finite for any $(\eta,\zeta)\in\mathcal{A}_N$. Similarly to the steps going from (4.113) to (4.119), we resort to (3.35) in order to get

$$\mathbb{E}\left[\|\mathbf{z}^2\|^{2p}_{(2\rho+\gamma,\gamma,\eta,\zeta),1,2}\right]\lesssim\int_{(v,w,r',r)\in\Delta_4^\tau}|w-v|^{4p(H-\alpha)-4}\,\mathrm{d}v\mathrm{d}w\mathrm{d}r'dr$$
$$+\mathbb{E}\left[\|\delta\mathbf{z}^{2,\tau}\|^{2p}_{(2\rho+\gamma,\gamma,\eta+\frac{1}{p},\zeta+\frac{1}{p}),1,2}\right]. \tag{4.121}$$

Owing to (4.104), the right hand side of (4.121) is finite as long as $p>\frac{3}{4(H-\alpha)}$. Eventually combining (4.120) and (4.121), and recalling our definition (2.4) of $(\alpha,\gamma,\eta,\zeta)$-norm, we trivially get that

$$\mathbb{E}\left[\|\mathbf{z}^2\|^{2p}_{(2\rho+\gamma,\gamma,\eta,\zeta)}\right]<\infty. \tag{4.122}$$

This completes the proof. □

5 Volterra Rough Path Driven by Brownian Motion

In this section we will construct Volterra type iterated integrals with respect to a Brownian motion. In this case our stochastic integrals will be interpreted in the Itô sense. The reason for this is that when the singularity exponent γ is strictly positive, the standard Itô-Stratonovich correction diverge (as will be illustrated in more detail later in this section). However, in order to take advantage of the computations performed in Sect. 4, we will stick to a Malliavin calculus setting. We start by highlighting in Sect. 5.1 the differences between basic stochastic analysis notions in the fBm context with $H>1/2$ and $H=1/2$ (representing the Brownian motion).

5.1 Analysis on the Wiener Space

The Malliavin calculus preliminaries for a Brownian motion are similar to what we wrote in Sect. 4.1 for a fBm. Keeping most of our previous notation, let us just highlight the main differences between the two situations.

(i) Our notation for the Brownian driving process is $W=(W^1,\ldots,W^m)$. The covariance function for each independent component is $R(s,t)=s\wedge t$.
(ii) The space $\mathcal{H}$ is $L^2([0,T])$, with inner product

$$\langle f,g\rangle_{\mathcal{H}}=\int_0^T f_u g_u\mathrm{d}u. \tag{5.1}$$

(iii) Let u be an adapted process in $L^2([0,T]\times\Omega)$. Then the Itô integral $\int_0^T u_t\,\delta^\diamond W_t^j$ is well defined for all $j=1,\dots,m$. It enjoys the Itô isometry property

$$\mathbb{E}\left[\left(\int_0^T u_t\,\delta^\diamond W_t^j\right)^2\right]=\int_0^T \mathbb{E}\left[u_t^2\right]dt. \tag{5.2}$$

Observe that for L^2-adapted processes, Itô and Skorokhod's integrals coincide. This explains why we still use the symbol $\delta^\diamond$ in the left hand side of (5.2).

5.2 Definition of the Volterra Rough Path

In this section we will construct and estimate iterated integrals in case of a driving noise given by a m-dimensional Brownian motion W. This case is rougher than in Sect. 4, although Volterra stochastics differential equations are arguably already addressed in the classical reference [18]. Nevertheless, it should be noticed that a rough path point of view on equation (1.1) driven by a Brownian motion is still useful, due to convenient continuity properties of the solution map with respect to the Volterra signature. We first introduce the definition of the first level Volterra rough path over Brownian motion W, which is a mere elaboration of Definition 4.2.

Definition 5.1 Consider a Brownian motion $W:[0,T]\to\mathbb{R}^m$ and a function h of the form $h_{ts}^\tau(r)=(\tau-r)^{-\gamma}\mathbb{1}_{[s,t]}(r)$ with $\gamma<1/2$. Then for $(s,t,\tau)\in\Delta_3$ we define the increment $\mathbf{z}_{ts}^{1,\tau,i}=\int_s^t(\tau-r)^{-\gamma}\,\mathrm{d}W_r^i$ as a Wiener integral of the form

$$\mathbf{z}_{ts}^{1,\tau,i}:=W^i(h_{ts}^\tau). \tag{5.3}$$

Similarly to what we did in Lemma 4.4, let us find a bound for second moment of $\mathbf{z}^1$.

Lemma 5.2 *Consider the Volterra rough path $\mathbf{z}^1$ as given in (5.3), and three parameters $\gamma\in(0,1)$, and $\eta,\ \zeta\in[0,1]$ satisfying*

$$0<\gamma<\frac{1}{2},\qquad \textit{and}\ \ 0\le\zeta\le\inf\left(\frac{1}{2}-\gamma,\eta\right). \tag{5.4}$$

Then for $(s,t,\tau)\in\Delta_3$, we have

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,i})^2]\lesssim\left|\psi^1_{(1/2,\gamma)}(\tau,t,s)\right|^2, \tag{5.5}$$

while for $(s,t,\tau',\tau)\in\Delta_4$, we get

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau\tau',i})^2]\lesssim\left|\psi^{1,2}_{(1/2,\gamma,\eta,\zeta)}\left(\tau,,\tau',t,s\right)\right|^2, \tag{5.6}$$

where ψ^1 and $\psi^{1,2}$ are given in Notation 2.2.

Proof In this proof, we will show that (5.5) holds for any $(s, t, \tau) \in \Delta_3$. Then relation (5.6) can be proved in a similar way. Toward to this aim, according to Definition 5.1, we have

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,i})^2] = \mathbb{E}\left[\left(\int_s^t (\tau - r)^{-\gamma} \, \mathrm{d}W_r^i\right)^2\right].$$

Furthermore, recalling that W is a Brownian motion and resorting to (5.1), we have

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{1,\tau,i}\right)^2\right] = \int_s^t (\tau - r)^{-2\gamma} \, dr.$$

Thanks to some elementary calculations similar to (4.25)-(4.26) in Sect. 4 and recalling definition (2.2) for the function $\psi^1_{1/2,\gamma}$, we now obtain

$$\mathbb{E}[(\mathbf{z}_{ts}^{1,\tau,i})^2] \lesssim \left[|\tau - t|^{-2\gamma}|t - s|\right] \wedge |t - s|^{1-2\gamma} = \left|\psi^1_{\frac{1}{2},\gamma}(\tau, t, s)\right|^2. \tag{5.7}$$

This is the desired result (5.5). □

Next we turn our attention to construct the second level Volterra rough path over a Brownian motion.

Definition 5.3 We consider a Brownian motion $W : [0, T] \to \mathbb{R}^m$, and the first level of the Volterra rough path $\mathbf{z}^{1,\tau}$ defined by (5.3). As in Definition 5.1, we assume that $\gamma < \frac{1}{2}$. Then for $(s, r, t, \tau) \in \Delta_4$, we set

$$u_{ts}^{\tau,i}(r) = (\tau - r)^{-\gamma} \mathbf{z}_{rs}^{1,r,i} \mathbb{1}_{[s,t]}(r). \tag{5.8}$$

With this notation in hand, we define the increment $\mathbf{z}_{ts}^{2,\tau}$ as an Itô integral of the form

$$\mathbf{z}_{ts}^{2,\tau,i,j} = \int_s^t u_{ts}^{\tau,i}(r) \, \delta^\diamond W_r^j, \quad \text{for} \quad i, j \in \{1, 2, \ldots, m\}. \tag{5.9}$$

Having the Definition 5.1 of $\mathbf{z}^{1,\tau}$ in mind when considering the process $u^{\tau,i}$ in (5.8), we get that $\mathbf{z}^{2,\tau}$ in (5.9) is rewritten as

$$\mathbf{z}_{ts}^{2,\tau,i,j} = \int_s^t \int_s^r (\tau - r)^{-\gamma} (r - l)^{-\gamma} \, \delta^\diamond W_l^i \, \delta^\diamond W_r^j, \quad \text{for} \quad i, j \in \{1, 2, \ldots, m\}.$$

Remark 5.4 Observe that in Definition 5.3 we have chosen to introduce $\mathbf{z}_{ts}^{2,\tau}$ as an Itô-type integral. This is in contrast with the fBm case with $H > \frac{1}{2}$, for which (4.36) had to be understood in the Stratonovich sense. As mentioned at the beginning of this section, this is due to the fact that the Stratonovich correction terms for $\mathbf{z}^2$ are diverging, which again is a consequence of the fact that the covariation between a singular fractional Brownian motion and a Brownian motion is diverging. This has also been noted in [1], where infinite renormalization procedures was proposed to deal with this problem in a regularity structures framework. We illustrate the issue in the following computations.

For $i = 1, \ldots, m$ assume that $\mathbf{z}_{ts}^{2,\tau,i,i}$ is defined in the Stratonovich sense, written as

$$\mathbf{z}_{ts}^{2,\tau,i,\mathrm{S}} = \int_s^t u_{ts}^{\tau,i}(r)\,\mathrm{d}W_r^i, \tag{5.10}$$

with $u_{ts}^{\tau,i}$ given in (5.8) and $\mathrm{d}W^i$ denoting the Stratonovich differential. Then standard considerations about Itô-Stratonovich corrections reveal that

$$\int_s^t u_{ts}^{\tau,i}(r)\,\mathrm{d}W_r^i = \int_s^t u_{ts}^{\tau,i}(r)\,\delta^\diamond W_r^i + \frac{1}{2}\langle u_{ts}^{\tau,i}, W^i\rangle_{ts}, \tag{5.11}$$

where $\langle u_{ts}^{\tau,i}, W^i\rangle_{ts}$ denotes the quadratic variation of $u_{ts}^{\tau,i}$ and W^i over the interval $[s,t]$.

Let us now analyze the quadratic variation term in (5.11), which can be defined through a discretization procedure. Namely let $\mathcal{P}$ designate a generic partition of $[s,t]$, and $[\tilde{r}, r]$ a typical interval of the partition $\mathcal{P}$. Then a classical way to define the quadratic variation is through the following limit in $L^2(\Omega)$:

$$\langle u_{ts}^{\tau,i}, W^i\rangle_{ts} = \lim_{|\mathcal{P}|\to 0} \sum_{[\tilde{r},r]\in\mathcal{P}} \left(u_{ts}^{\tau,i}(r) - u_{ts}^{\tau,i}(\tilde{r})\right) W_{r\tilde{r}}^i. \tag{5.12}$$

Next we can decompose the right hand side of (5.12) in order to get

$$\langle u_{ts}^{\tau,i}, W^i\rangle_{ts} = \lim_{|\mathcal{P}|\to 0} \sum_{[\tilde{r},r]\in\mathcal{P}} (M_{r\tilde{r}} W_{r\tilde{r}}^i + V_{r\tilde{r}} W_{r\tilde{r}}^i), \tag{5.13}$$

where $M_{r\tilde{r}}$ and $V_{r\tilde{r}}$ are respectively defined by

$$M_{r\tilde{r}} = \int_{\tilde{r}}^r (\tau - r)^{-\gamma}(r-l)^{-\gamma}\mathrm{d}W_l^i, \quad V_{r\tilde{r}} =$$
$$\int_s^{\tilde{r}} \left[(\tau - r)^{-\gamma}(r-l)^{-\gamma} - (\tau-\tilde{r})^{-\gamma}(\tilde{r}-l)^{-\gamma}\right]\mathrm{d}W_l^i.$$

Starting from (5.13), we let the patient reader check that the relation below holds in $L^2(\Omega)$:

$$\lim_{|P|\to 0} \sum_{[\tilde{r},r]\in\mathcal{P}} V_{r\tilde{r}} W_{r\tilde{r}}^i = 0.$$

However the term $\sum_{[\tilde{r},r]\in\mathcal{P}} M_{r\tilde{r}} W_{r\tilde{r}}^i$ in (5.13) is more problematic. Specifically, it can be shown (tedious details are left again to the reader for sake of conciseness) that the following quantity converges in $L^2(\Omega)$ as $|\mathcal{P}| \to 0$:

$$\sum_{[\tilde{r},r]\in\mathcal{P}} \left(M_{r\tilde{r}} W^i_{r\tilde{r}} - c_\gamma (r-\tilde{r})^{1-\gamma}(\tau - r)^{-\gamma} \right),$$

where $c_\gamma = (1-\gamma)^{-1}$. Nevertheless, one can simply check that

$$\lim_{|\mathcal{P}|\to 0} \sum_{[\tilde{r},r]\in\mathcal{P}} (r-\tilde{r})^{1-\gamma}(\tau - r)^{-\gamma} = \infty.$$

Hence the quantity $\sum_{[\tilde{r},r]\in\mathcal{P}} M_{r\tilde{r}} W^i_{r\tilde{r}}$ is also divergent in $L^2(\Omega)$. This proves that the quadratic variation in (5.13) is divergent, and thus going back to (5.11) we get that $\mathbf{z}^{2,\tau,i,\mathrm{S}}_{ts}$ cannot be defined in the Stratonovich sense.

We now adapt the computations of Proposition 4.8 in order to estimate the second moment of the increment $\mathbf{z}^{2,\tau,i,j}$.

Proposition 5.5 *Consider the second level $\mathbf{z}^{2,\tau}_{ts}$ of the Volterra rough path, as defined in (5.9). Recall that the parameters $\gamma \in (0,1)$, and $\eta, \zeta \in [0,1]$ satisfy relation (5.4). Then for $(s,t,\tau) \in \Delta_3$, we have*

$$\mathbb{E}\left[\left(\mathbf{z}^{2,\tau}_{ts} \right)^2 \right] \lesssim \left| \psi^1_{(1-\gamma,\gamma)}(\tau,t,s) \right|^2. \tag{5.14}$$

For $(s,t,\tau',\tau) \in \Delta_4$, we get

$$\mathbb{E}\left[\left(\mathbf{z}^{2,\tau\tau'}_{ts} \right)^2 \right] \lesssim \left| \psi^{1,2}_{(1-\gamma,\gamma,\eta,\zeta)}(\tau,\tau',t,s) \right|^2, \tag{5.15}$$

For both (5.14) and (5.15), we recall that ψ^1 and $\psi^{1,2}$ are given in Notation 2.2.

Proof This proof is very similar to the proof of Lemma 5.2. We will prove (5.14), and let the patient reader show that (5.15) holds for $(s,t,\tau',\tau) \in \Delta_4$. For $(s,t,\tau) \in \Delta_3$ we have

$$\mathbb{E}\left[\left(\mathbf{z}^{2,\tau,i,j}_{ts} \right)^2 \right] = \mathbb{E}\left[\left(\int_s^t u^{\tau,i}_{ts}(r) \mathrm{d}W^j_r \right)^2 \right].$$

Hence according to Itô's isometry, we obtain

$$\mathbb{E}\left[\left(\mathbf{z}^{2,\tau,i,j}_{ts} \right)^2 \right] = \int_s^t \mathbb{E}\left[\left(u^{\tau,i}_{ts} \right)^2 \right] \mathrm{d}r. \tag{5.16}$$

Moreover recalling the definition (5.8) of u, we get

$$\mathbb{E}\left[\left(u^{\tau,i}_{ts} \right)^2 \right] = \mathbb{E}\left[(\tau - r)^{-2\gamma} \left(\mathbf{z}^{1,r,i}_{rs} \right)^2 \mathbb{1}_{[s,t]}(r) \right] = (\tau - r)^{-2\gamma} \mathbb{E}\left[\left(\mathbf{z}^{1,r,i}_{rs} \right)^2 \right] \mathbb{1}_{[s,t]}(r). \tag{5.17}$$

Thanks to (5.7), we have $\mathbb{E}\left[\left(\mathbf{z}_{rs}^{1,r,i}\right)^2\right] \lesssim (r-s)^{1--2\gamma}$. Then relation (5.17) reads

$$\mathbb{E}\left[\left(u_{ts}^{\tau,i}\right)^2\right] \lesssim (\tau-r)^{-2\gamma}(r-s)^{1-2\gamma}\mathbb{1}_{[s,t]}(r). \tag{5.18}$$

Eventually plugging (5.18) into (5.16), we get

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau}\right)^2\right] \lesssim \int_s^t (\tau-r)^{-2\gamma}(r-s)^{1-2\gamma}\mathrm{d}r. \tag{5.19}$$

In order to find a bound for the right hand side of (5.19), the procedure is very similar to (4.24)-(4.26) in Proposition 4.4. We finally obtain

$$\mathbb{E}\left[\left(\mathbf{z}_{ts}^{2,\tau}\right)^2\right] \lesssim \left[|\tau-t|^{-2\gamma}|t-s|^{2-2\gamma}\right] \wedge |t-s|^{2-4\gamma} = \left|\psi^1_{(1-\gamma,\gamma)}(\tau,t,s)\right|^2,$$

where we have appealed the definition (2.2) of ψ^1 for the second identity of the above equation. This is the desired result (5.14). □

With Definitions 5.1 and 5.3 in hand, we have constructed a Volterra rough path family $\{\mathbf{z}^{1,\tau}, \mathbf{z}^{2,\tau}\}$ over a Brownian motion, and we have also upper bounded their second moment in Lemma 5.2 and Proposition 5.5. In the following, we close this paper with verifying that $\{\mathbf{z}^{1,\tau}, \mathbf{z}^{2,\tau}\}$ satisfies Definition 2.9. Let us first state that $\mathbf{z}^{1,\tau}$ satisfies all properties that mentioned in Definition 2.9.

Proposition 5.6 *Consider the increment $\mathbf{z}^{1,\tau}$ introduced in Definition 5.1. Then for any $\alpha \in (0, \frac{1}{2})$, and $\zeta, \eta \in [0,1]$ satisfying the relation (2.8), we have*

(i) *$\mathbf{z}^{1,\tau}$ is almost surely in the Volterra space $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_3;\mathbb{R}^m)$, where $\mathcal{V}^{(\alpha,\gamma,\eta,\zeta)}(\Delta_3;\mathbb{R}^m)$ is introduced in Definition 2.3.*
(ii) *For all $p \geq 1$ we have that*

$$\mathbb{E}\left[\|\mathbf{z}^1\|^{2p}_{(\alpha,\gamma,\eta,\zeta)}\right] < \infty. \tag{5.20}$$

(iii) *Recalling the definition (2.12) of δ, then $\delta_m \mathbf{z}_{ts}^{1,\tau}$ satisfies relation (2.13). Namely almost surely we have*

$$\delta_m \mathbf{z}_{ts}^{1,\tau,i} = 0, \ \textit{for all } (s,m,t,\tau) \in \Delta_4. \tag{5.21}$$

The proof is very similar to the proof as for Proposition 4.11–4.13, we let the patient reader check the details. Similarly, we obtain the following Proposition for the second-order integral $\mathbf{z}^2$.

Proposition 5.7 *Consider the second level $\mathbf{z}^{2,\tau}$ of the Volterra rough path as defined in (5.9). Then the following properties hold for any $\alpha \in (0,\frac{1}{2})$, and $\zeta, \eta \in [0,1]$ satisfying the relation (2.8).*

(i) $\mathbf{z}^{2,\tau}$ *is almost surely an element of* $\mathcal{V}^{(2\rho+\gamma,\gamma,\eta,\zeta)}$ *for all* $(\eta,\zeta)\in\mathcal{A}_N$ *as defined in* (4.62).

(ii) *For all* $p\geq 1$ *and* $(\eta,\zeta)\in\mathcal{A}_N$ *we have that*

$$\mathbb{E}\left[\|\mathbf{z}^2\|^{2p}_{(2\rho+\gamma,\gamma,\eta,\zeta)}\right]<\infty. \tag{5.22}$$

(iii) *Recalling the definition* (2.12) *of* δ*, then* $\delta_m\mathbf{z}^{2,\tau}_{ts}$ *satisfies relation* (2.13)*, that is*

$$\delta_m\mathbf{z}^{2,\tau,i,j}_{ts}=\mathbf{z}^{1,\tau,j}_{tm}*\mathbf{z}^{1,\cdot,i}_{ms},\quad \textit{for all } (s,m,t,\tau)\in\Delta_4\quad \textit{a.s.}. \tag{5.23}$$

Proof As the proof is very similar to various proofs Sect. 4.4 we only give a sketch of the method here, and refer to equivalent proofs in this section for more details.

First it is readily seen that (i) follows from (ii). To prove (ii) we will resort to the Volterra GRR lemma 3.4, in combination with the moment estimates obtained in Proposition 5.5. Recall that since $\mathbf{z}^2$ is an element of the second chaos of the fBm, the L^p norms are equivalent. Therefore from the moment estimates obtained in Proposition 5.5, we have that

$$\begin{aligned}\mathbb{E}[(\mathbf{z}^{2,\tau}_{ts})^{2p}]&\lesssim\left|\psi^1_{(1-\gamma,\gamma)}(\tau,t,s)\right|^{2p}\\ \mathbb{E}[(\mathbf{z}^{2,\tau\tau'}_{ts})^{2p}]&\lesssim\left|\psi^{1,2}_{(1-\gamma,\gamma,\eta,\zeta)}(\tau,\tau',t,s)\right|^{2p}.\end{aligned} \tag{5.24}$$

Invoking relation (3.34), we can proceed directly in the same way as in the proof of Proposition 4.16 (note that in this case $\rho=\frac{1}{2}$). Combining with the bounds in (5.24), this proves the claim in (ii). Claim (iii) can be shown in the same spirit as Proposition 4.14, although the integral must now be interpreted in the Itô sense. Thus Step 1 of the proof of Proposition 4.14 is exactly the same, while in Step 2 one must consider the integration argument in the Itô sense. This follows by classical Itô integration considerations. Step 3 follows by exactly the same arguments. This concludes the proof. □

5.3 Further Extensions and Concluding Remarks

We have provided a construction of the Volterra rough path $(\mathbf{z}^1,\mathbf{z}^2)$ when the driving process is a fractional Brownian motion with $H>\frac{1}{2}$ or a Brownian motion, and the Volterra kernel is allowed to be singular. This corresponds to the Volterra rough path needed in order to deal with the regularity regime $\alpha-\gamma\geq\frac{1}{3}$, constructed in [15]. It is desirable to extend this construction further to also include higher-order components of the signature. As illustrated in the article [16] and [4], in such an extension one will need to deal with several different types of iterated integrals. This abundance of necessary iterated integrals stems from the non-geometric nature of the Volterra rough path. A more systematic analysis based on related algebraic structures, together with the tools based on Malliavin calculus invoked in the current article, is therefore needed to deal with this problem.

It is also natural to consider the case of rough fractional Brownian motions as the driving noise (i.e. $H < \frac{1}{2}$). However, the techniques used here, based on the integrability of the mixed partial derivative of the covariance function $R(s, t)$, will no longer work. One will therefore need to use new tools to handle this issue. We expect that techniques inspired by the results in [7], in combination with sewing techniques for Volterra covariance functions developed in [3], would prove useful to this aim. However, we leave this problem for future consideration.

Funding Open access funding was provided by Norwegian Business School. S. Tindel is supported by NSF Grant DMS-1952966

Data Availability This is a theoretical mathematical paper, not based on any data sets.

Declarations

Conflict of interest This is a theoretical mathematical paper with no potential conflict of interests or ethical challenges.

References

1. Bayer, C., Friz, P.K., Gassiat, P., Martin, J., Stemper, B.: A regularity structure for rough volatility. Math. Financ. **30**(3), 782–832 (2020)
2. Bellingeri, C., Friz, P.K., Gerencsér, M.: Singular paths spaces and applications. Stoch. Anal. Appl. (2021)
3. Benth, F.E., Harang, F.A.: Infinite dimensional pathwise Volterra processes driven by Gaussian noise—probabilistic properties and applications. Electron. J. Probab., 26:Paper No. 114, 42 (2021)
4. Bruned, Y., Katsetsiadis, F.: Ramification of volterra-type rough paths (2021)
5. El Euch, O., Rosenbaum, M.: The characteristic function of rough Heston models. Math. Finance **29**(1), 3–38 (2019)
6. Friz, P., Hairer, M.: A Course on Rough Paths with An Introduction to Regularity Structures, pp. 137–143. Springer, Berlin (2014)
7. Friz, P.K., Gess, B., Gulisashvili, A., Riedel, S.: The Jain–Monrad criterion for rough paths and applications to random Fourier series and non-Markovian Hörmander theory. Ann. Probab. **44**(1), 684–738 (2016)
8. Friz, P.K., Victoir, N.B.: Multidimensional Stochastic Processes as Rough Paths Theory and Applications, pp. 405–408. Cambridge University Press, Cambridge (2010)
9. Garsia, A.M., Rodemich, E., Rumsey, H. Jr.: A real variable lemma and the continuity of paths of some Gaussian processes. Indiana Univ. Math. J., 20:565–578 (1970/71)
10. Garsia, A.M.: Continuity properties of Gaussian processes with multidimensional time parameter. In: Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. II: Probability theory, pp. 369–374 (1972)
11. Gatheral, J., Jaisson, T., Rosenbaum, M.: Volatility is rough. Quant. Finance **18**(6), 933–949 (2018)
12. Gubinelli, M.: Ramification of rough paths. J. Differ. Equ. **248**(4), 693–721 (2010)

13. Hairer, M.: A theory of regularity structures. Invent. Math. **198**(2), 269–504 (2014)
14. Hairer, M., Kelly, D.: Geometric versus non-geometric rough paths. Ann. Inst. Henri Poincaré Probab. Stat. **51**(1), 207–251 (2015)
15. Harang, F.A., Tindel, S.: Volterra equations driven by rough signals. Stoch. Process. Appl. **142**, 34–78 (2021)
16. Harang, F.A., Tindel, S., Wang, X.: Volterra equations driven by rough signals 2: higher order expansions (2021)
17. Yaozhong, H., Le, K.: A multiparameter Garsia–Rodemich–Rumsey inequality and some applications. Stoch. Process. Appl. **123**(9), 3359–3377 (2013)
18. Nualart, D.: The Malliavin Calculus and Related Topics. Springer, Berlin (2010)
19. Nualart, D., Tindel, S.: A construction of the rough path above fractional Brownian motion using Volterra's representation. Ann. Probab. **39**(3), 1061–1096 (2011)
20. Oksendal, B., Zhang, T. S.: The stochastic Volterra equation. In: Barcelona Seminar on Stochastic Analysis (St. Feliu de Guíxols, 1991), volume 32 of Progr. Probab., pages 168–202. Birkhäuser, Basel, (1993)
21. Zhang, X.: Stochastic Volterra equations in Banach spaces and stochastic partial differential equation. J. Funct. Anal. **258**(4), 1361–1425 (2010)