Journal of Theoretical Probability (2024) 37:1–42
https://doi.org/10.1007/s10959-023-01242-z

Generalized Unimodality and Subordinators, With Applications to Stable Laws and to the Mittag-Leffler Function

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Received: 25 February 2022 / Revised: 11 January 2023 / Accepted: 20 January 2023 /
Published online: 4 May 2023

Abstract

Using the differential operator $\Omega_s := s\ I - x\frac{\mathrm{d}}{\mathrm{d}x},\ \ s > 0$, we build a new class of infinitely divisible distributions on the half-line. For this class, we give a stochastic interpretation and we provide several monotonicity properties for the associated subordinators. As an application, we solve a problem raised separately by Sendov and Shan in (J Theor Probab 28:1689–1725, 2015) and by Simon in (Math Nachr 285(4): 497–506, 2012) on the distribution of the stable subordinators. Finally, we provide a new complete monotonicity property for the Mittag-Leffler function.

Keywords Bernstein function · Complete monotonicity · Generalized unimodality · Mittag-Leffler function · Positive stable distribution · Subordinators

Mathematics Subject Classification (2020) Primary 26A48 · 26D07 · Secondary 30E20

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1 Introduction

Let $\mathcal{CM}$ denote the class of completely monotone functions, i.e., of those infinitely differentiable functions $f : (0,\infty) \to (0,\infty)$ satisfying $(-1)^n f^{(n)} \geq 0$ for all integer $n = 0, 1, 2, \dots$ Bernstein showed that the class $\mathcal{CM}$ corresponds to the Laplace transforms of measures on $[0,\infty)$. That is $f \in \mathcal{CM}$ if, and only if,

$$f(\lambda) = \mathcal{L}_\nu(\lambda) := \int_{[0,\infty)} e^{-\lambda x}\nu(dx), \quad \lambda > 0, \text{ for some Radon measure} \nu \text{ on}[0,\infty).$$

Let $\mathcal{CF}$ denote the class of cumulant functions,

$$\mathcal{CF} := \{\varphi_Z(\lambda) = -\log \mathbb{E}[e^{-\lambda Z}], \quad s.t.\ \lambda \geq 0, \text{ where} Z \geq 0 \text{ is a random variable}\} \tag{1}$$

Let φ_Z be the cumulant function of a non-negative random variable Z. Adapting [12, Theorem 2.1] (by looking at the Laplace transform of Z as the Mellin transform of e^{-Z}), we see that

$$\frac{\varphi_Z(\lambda)}{\lambda} \text{ decreases to } \inf\{x \geq 0, \text{ s.t. } \mathbb{P}(Z \leq x) > 0\}, \text{ as } \lambda \text{ increases to } \infty. \tag{2}$$

As a consequence of (1) and (2), one observes that any $\varphi \in \mathcal{CF}$ is infinitely differentiable on $(0,\infty)$, satisfies $\varphi(0) = 0$,

$$\mathrm{d} := \lim_{x\to\infty} \frac{\varphi(x)}{x} \in [0,\infty) \text{ and } \varphi(\lambda) - \mathrm{d}\lambda \geq 0, \quad \text{for all } \lambda \geq 0. \tag{3}$$

An application of Hölder's inequality shows that cumulant functions are concave. More generally, elementary considerations show that every concave function $\varphi : [0,\infty) \to [0,\infty)$, differentiable on $(0,\infty)$, satisfies

$$\mathrm{d} := \lim_{x\to\infty} \frac{\varphi(x)}{x} = \lim_{x\to\infty} \varphi'(x) \in [0,\infty). \tag{4}$$

Indeed, being concave and differentiable φ has non-increasing derivative and satisfies $\varphi(x) \leq \varphi(y) + \varphi'(y)(x-y)$ for all $x, y > 0$. Divide both sides by y and take limit infimum as y approach infinity. Then, divide both sides by x and take limit superior as x approaches infinity. Combine the two resulting inequalities to conclude. Observe that (2) provides the following property

$$\varphi \in \mathcal{CF} \Longrightarrow 0 \leq x\varphi'(x) \leq \varphi(x), \quad \forall x > 0 \Longrightarrow \lim_{x\to 0^+} x\varphi'(x) = 0. \tag{5}$$

The class of Bernstein functions, usually denoted $\mathcal{BF}$, consists of those functions

$$\phi : [0,\infty) \to [0,\infty), \text{ satisfying } \quad \phi' \in \mathcal{CM}.$$

Every Bernstein function can be represented, see the book by Schilling, Song and Vondraček [15, Eqs. (3.2), (3.3)], by

$$\phi(\lambda) = \mathrm{q} + \mathrm{d}\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\mu(dx) = \mathrm{q} + \lambda \left[\mathrm{d} + \int_0^\infty e^{-\lambda x}\mu(x,\infty)dx\right], \ \lambda \geq 0, \quad (6)$$

where $\mathrm{q},\ \mathrm{d} \geq 0$ are called the *killing rate* and the *drift term*, respectively. The so-called *Lévy measure* μ satisfies the integrability condition

$$\int_{(0,\infty)} (1 \wedge x)\mu(dx) < \infty. \quad (7)$$

The classes $\mathcal{CM}$ and $\mathcal{BF}$ are closed under pointwise limits and form closed convex cones. The former is also closed under multiplication, while the latter under composition. From (6), one also has the implication

$$\phi(\lambda) \in \mathcal{BF} \Longrightarrow \frac{\phi(\lambda)}{\lambda} \in \mathcal{CM}. \quad (8)$$

For justification of these facts, see [15, Corollarys 1.6 and 3.8]. Bernstein functions (with no killing rate) are in one-to-one correspondence with infinitely divisible non-negative random variables, see [15, Lemma 5.8]: If $X \geq 0$ has cumulant function φ_X, then the distribution of X is infinitely divisible if, and only if φ_X is a Bernstein function. In this case, X is embedded into a subordinator $(X_t)_{t\geq 0}$. That means that $(X_t)_{t\geq 0}$ is an increasing Lévy process starting from zero, and the celebrated Lévy–Khintchine formula holds:

$$X \stackrel{d}{=} X_1 \text{ and } \mathbb{E}[e^{-\lambda X_t}] = e^{-t\varphi_X(\lambda)}, \ \lambda \geq 0.$$

Let $\phi = \mathrm{q} + \varphi_X$ be the Bernstein function obtained by adding to φ_X a killing rate $\mathrm{q} > 0$. This Bernstein function is then associated by the Lévy–Khintchine formula to the killed process $(X_t^{(\mathrm{q})})_{t\geq 0}$ defined by

$$X_t^{(\mathrm{q})} := \begin{cases} X_t & \text{if } t < \mathbf{e}_{\mathrm{q}}, \\ \infty & \text{otherwise,} \end{cases}$$

where $\mathbf{e}_{\mathrm{q}}$ is an independent exponential random variable with parameter q. Bernstein and completely monotone functions are also connected by the relationship

$$\phi \in \mathcal{BF} \Longleftrightarrow e^{-t\phi} \in \mathcal{CM}, \quad \text{for all } \ t > 0,$$

as shown in [15, Theorem 3.7]. Other good references for these classes of functions and their properties are the books by Steutel and van Harn [17] and Bertoin [2]. Define

$$\varepsilon_t(\varphi)(\lambda) := 1 - e^{-t\varphi(\lambda)}, \ \varphi \in \mathcal{CF},$$

and observe that if $\varphi(0) = 0$, then

$$\varphi \in \mathcal{CF} \Longleftrightarrow e^{-\varphi} \in \mathcal{CM} \Longleftrightarrow 1 - e^{-\varphi} \in \mathcal{BF} \Longleftrightarrow \varepsilon_n(\varphi) \in \mathcal{BF}, \quad \text{for all} \quad n = 1, 2, \dots \tag{9}$$

(To see the necessity in the last implication, differentiate $\varepsilon_n(\varphi)$ and use the facts that $e^{-\varphi}$ and $e^{-\varphi}\varphi'$ are both completely monotone and that product of completely monotone functions is completely monotone.) Since

$$\{\phi \in \mathcal{BF}, \quad s.t. \quad \phi(0) = 0\} \subset \mathcal{CF}, \tag{10}$$

we also observe that

$$\begin{aligned} \phi \in \mathcal{BF} &\Longleftrightarrow t\ (\phi - \phi(0)) \in \mathcal{CF}, \quad \text{for all} \quad t > 0 \\ &\Longleftrightarrow \varepsilon_t(\phi - \phi(0)) \in \mathcal{BF}, \quad \text{for all} \quad t > 0. \end{aligned} \tag{11}$$

Finally, observe that any Bernstein function satisfies (3), (4), and (5) too.

In [7], Hansen introduced what he called the class of *reverse s-self-decomposable distributions*, as follows:

Definition 1.1 (Hansen, Definition 3.1 [7].) Let X be real-valued infinitely divisible random variable and $s > 0$. The distribution of X is called *reverse s-self-decomposable*, if its characteristic function Ψ_X is such that

$$\Psi_X'(u) \text{ exists for } u \neq 0, \ \lim_{u \to 0} u\ \Psi_X'(u) = 0 \tag{12}$$

and, for every $c \in (0, 1)$, there exists a characteristic function Ψ_c such that

$$\Psi_X(u) = \Psi_X^{c^s}(u/c)\ \Psi_c(u), \ \ u \in \mathbb{R}. \tag{13}$$

Taking into account properties (3) of cumulant functions and motivated by (13), we introduce for $c > 0$ and $s > 0$, the difference operator $\omega_{c,s}$ defined, for functions $\varphi : [0, \infty) \to [0, \infty)$, differentiable on $(0, \infty)$, such that the limit

$$\mathrm{d} := \lim_{x \to \infty} \frac{\varphi(x)}{x} \text{ exists and is in } [0, \infty), \quad \text{and} \quad \lim_{x \to 0^+} x\varphi'(x) = 0. \tag{14}$$

Let

$$\varphi_\diamond(\lambda) := \varphi(\lambda) - \varphi(0) - \mathrm{d}\lambda, \tag{15}$$

and note that $\varphi_\diamond(\lambda) = \varphi(\lambda) - \mathrm{d}\lambda \geq 0$, whenever $\varphi \in \mathcal{CF}$. Define

$$\omega_{c,s}(\varphi)(\lambda) := \varphi_\diamond(c\lambda) - c^s \varphi_\diamond(\lambda), \ \ \lambda \geq 0, \tag{16}$$

and note that

$$\omega_{c,s}(\varphi)(\lambda) = \omega_{c,s}(\varphi_\diamond)(\lambda) = \varphi(c\lambda) - c^s\varphi(\lambda) - (1-c^s)\,\varphi(0) - (c-c^s)\,\mathrm{d}\,\lambda.$$

Next, observe that

$$\varphi \in \mathcal{CF} \cup \mathcal{BF} \Longrightarrow \omega_{c,s}(\varphi)(0) = 0 \text{ and } \lim_{x\to\infty} \frac{\omega_{c,s}(\varphi)(x)}{x} = 0. \tag{17}$$

By analogy to Definition 1.1, we have the following one in terms of cumulant functions.

Definition 1.2 Let $s > 0$. We say that a non-negative random variable X has a distributions in the class $\mathbf{RSD_s}$, and we denote $X \sim \mathbf{RSD_s}$, if its cumulant function φ_X is such that for every $c \in (0,1)$, there exists a non-negative random variable Y_c such that

$$e^{-\omega_{c,s}(\varphi_X)(\lambda)} = \mathbb{E}[e^{-\lambda Y_c}], \quad \lambda \geq 0. \tag{18}$$

In other words, $X \sim \mathbf{RSD_s}$ if $\omega_{c,s}(\varphi_X) \in \mathcal{CF}$, for every $c \in (0,1)$.

Remark 1.3 (i) Note that condition (18) is analogous to (13) obtained by replacing the characteristic function of X with the normalized cumulant function $\varphi_X(\lambda) - \mathrm{d}\lambda$. Conditions similar to (12) are not needed in Definition 1.2, since, by (3), it is immediate that φ_X satisfy

$$\varphi_X'(\lambda) - \mathrm{d} \in (0,\infty), \quad \text{for all } \lambda > 0, \text{ and } \lim_{x\to 0+} x\,(\varphi_X'(x) - \mathrm{d}) = 0.$$

(ii) Theorem 2.6 below shows several important implications of (18). That is, if $X \sim \mathbf{RSD_s}$, then

- The cumulant function φ_X is a Bernstein function, that is X is infinitely divisible;
- The cumulant function $\omega_{c,s}(\varphi_X)$ is a Bernstein function, for every $c \in (0,1)$, that is, Y_c is infinitely divisible.

We will see that the difference operator $\omega_{c,s}$ is tightly linked to the differential operator $\Omega_s,\ s > 0$, defined for functions $\varphi : [0,\infty) \to [0,\infty)$, differentiable on $(0,\infty)$ and satisfying (14), by

$$\Omega_s(\varphi)(\lambda) := s\,\varphi_\diamond(\lambda) - \lambda\varphi_\diamond'(\lambda),\ \lambda > 0, \quad \Omega_s(\varphi)(0) = 0. \tag{19}$$

where $\varphi_\diamond$ is given by (15). Note that

$$\Omega_s(\varphi)(\lambda) = \Omega_s(\varphi_\diamond)(\lambda) = s\,\big(\varphi(\lambda) - \varphi(0)\big) - \lambda\varphi'(\lambda) - (s-1)\,\mathrm{d}\,\lambda,$$
$$\Omega_s(\varphi) \equiv 0 \Longleftrightarrow \varphi(\lambda) = \mathrm{q} + \mathrm{d}\,\lambda + c\lambda^s, \quad \text{for some q, d, } c \geq 0.$$

GENERALIZED UNIMODALITY AND SUBORDINATORS

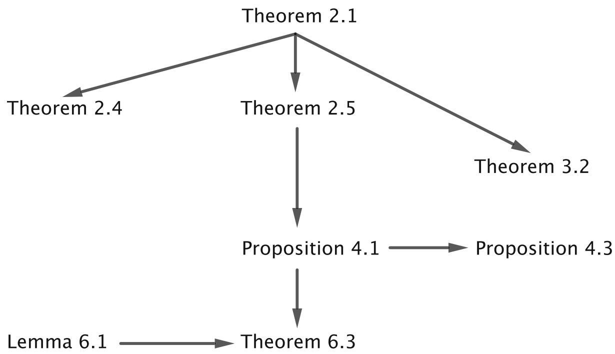

Fig. 1 Dependencies between the main results

Analogously to (17), we have:

$$\varphi \in \mathcal{CF} \cup \mathcal{BF} \Longrightarrow \lim_{x\to\infty} \frac{\Omega_s(\varphi)(x)}{x} = 0.$$

Adopting Hansen's terminology, keeping (11) and Definition 1.2 in mind, we introduce the following subclasses of $\mathcal{CF}$ and $\mathcal{BF}$. Recall the definition of $\phi_\diamond$, given in (15).

Definition 1.4 For any $s > 0$, define

(i) $\mathcal{CF}_s := \{\varphi \in \mathcal{CF}, \ \ s.t. \ \ \omega_{c,s}(\varphi) \in \mathcal{CF}, \ \ \forall\, c \in (0,1)\}$;
(ii) $\mathcal{BF}_s := \{\phi \in \mathcal{BF}, \ \ s.t. \ \ \Omega_s(\phi) \in \mathcal{BF}\}$;
(iii) $\mathcal{BF}_s^* := \{\phi \in \mathcal{BF}, \ \ s.t. \ \ 1 - e^{-t\phi_\diamond} \in \mathcal{BF}_s, \ \ \forall t > 0\}$.

Note that the class $\mathcal{BF}_1$ has already been investigated by the first two authors in [1, Sect. 5]. The goal of this work is to develop a comprehensive probabilistic theory of the classes $\mathcal{BF}_s$ and $\mathcal{BF}_s^*$, and to complete Hansen's results and those of [1]. This allows us, in Sect. 6, to answer a conjecture stated in [14, Open Problem 4.1] that is related to a problem previously raised by Simon [16]. To help the reader navigate through the results in this paper, we include Fig. 1, showing the dependencies between the main results.

Section 2 deals with several analytic properties of the class $\mathcal{BF}_s$. Theorem 2.6 below shows that

$$\mathcal{CF}_s = \mathcal{BF}_s \cap \{\phi \in \mathcal{BF}, \ \ s.t. \ \ \phi(0) = 0\}.$$

The former also shows that our extension of Hansen's classes to $\mathbf{RSD_s}$, $s > 0$, corresponds to the cumulant functions in the class $\mathcal{BF}_s$. A full characterization of this class is given in Theorem 2.1.

Sections 3 and 4 give various stochastic interpretations for the class $\mathbf{RSD_s}$. For instance, Corollary 3.1 provides the decomposability of the distributions of the associated subordinators: if X is a non-negative infinitely divisible random variable, with

Bernstein function φ_X, and is embedded into the subordinator $(X_t)_{t\geq 0}$, then

$$\varphi_X \in \mathcal{BF}_s \iff c\ X_t \stackrel{d}{=} X_{c^s t} + \mathrm{d}\ (c - c^s)\ t + c\ Z_t^{c,s},\ \text{ for all } t > 0,\ \ c \in (0,1), \tag{20}$$

where in the last identities, d is the drift term and $Z_t^{c,s}$ is a non-negative random variable whose distribution is necessarily infinitely divisible. This decomposition explains the name of reverse self-decomposability and produces a self-similar temporal property which mimics the one of the classical and well-known class of *self-decomposable distributions* and *self-similar processes*. To explain this resemblance, we remind that a random variable X is self-decomposable if it satisfies the identities in law

$$X \stackrel{d}{=} cX + Z_c, \quad \text{for all } c \in (0,1),$$

where Z_c is a random variable independent of X, necessarily infinitely divisible. By [13, Theorem 16.1], we know that for such X, and for any $\gamma > 0$, there exists a self-similar process with independent increments $(X_t)_{t\geq 0}$, i.e., a process satisfying

$$(a\ X_t)_{t\geq 0} \stackrel{d}{=} (X_{a^\gamma t})_{t\geq 0}, \quad \text{for any } a \geq 0, \tag{21}$$

and $X_{t_n} - X_{t_{n-1}}, \ldots, X_{t_2} - X_{t_1}$ are independent for any $t_n > \cdots > t_1 \geq 0$, such that X_1 and X have the same distribution.

In Sect. 4, we will see that the concept of reverse self-decomposability is intimately connected to the concept of generalized unimodality which is defined as follows:

Definition 1.5 [Olshen and Savage [11]] Let $s > 0$. A real-valued random variable Z is s-star unimodal, if it is of the form

$$Z \stackrel{d}{=} \mathbb{U}^{1/s} V,$$

where $\mathbb{U}$ is uniformly distributed on $(0,1)$ and independent of V.

Let $\mathcal{DF}$ be the collection of distribution functions of non-negative random variables:

$$\mathcal{DF} := \{F_Z(x) = \mathbb{P}(Z \leq x), \quad s.t.\ \ x \geq 0,\ \ Z \geq 0\}.$$

Proposition 4.1 below asserts, among other things, that if Z is a positive random variable with distribution function F_Z and cumulant function φ_Z, then

$$1 - e^{-\varphi_Z} \in \mathcal{BF}_s \iff \frac{1}{Z} \text{ is } s-\text{star unimodal} \tag{22}$$

$$\iff \omega_{c,s}(F_Z) \in \mathcal{DF}, \quad \forall c \in (0,1) \tag{23}$$

$$\iff Z \ \text{ has a probability density function } \frac{p_s(x)}{x^{s+1}}, \quad x > 0, \text{ where } p_s(x) \text{ is non-decreasing and right-continuous.} \tag{24}$$

In Sect. 5 we show that $\mathcal{BF}_s^*$ is a strict subclass of $\mathcal{BF}_s$. Corollary 5.2 shows that $\mathcal{BF}_s^*$ corresponds to the subset of stabilizers of $\mathcal{BF}_s$ with respect to composition, namely

$$\mathcal{BF}_s^* := \{\phi \in \mathcal{BF}_s, \quad s.t. \;\; \psi \circ \phi_\diamond \in \mathcal{BF}_s \text{ for all } \psi \in \mathcal{BF}\}. \tag{25}$$

There, we also show that if X is a non-negative infinitely divisible random variable, with Bernstein function φ_X, with drift term d, and X is embedded into the subordinator $(X_t)_{t\geq 0}$, then $\varphi_X \in \mathcal{BF}_s^*$ if, and only if, the r.v. $X_t^{(\diamond)} := X_t - \mathrm{d}\ t$ satisfies any of conditions (22), (23) or (24) for all $t > 0$. The interest of characterization (25) of $\mathcal{BF}_s^*$ is that $\phi \in \mathcal{BF}_s^*$, if, and only if, decomposition (20), applied to the subordinator $(X_t^{(\diamond)})_{t\geq 0}$, is preserved by subordination:

$$\phi \in \mathcal{BF}_s^* \iff \text{the subordinated process,} (X_{\eta_t}^{(\diamond)})_{t\geq 0}, \text{ also decomposes as in}(20),$$

for arbitrary subordinators $(\eta_t)_{t\geq 0}$, independent of $(X_t^{(\diamond)})_{t\geq 0}$. We stress that Sendov and Shan [14] were the first to introduce the class $\mathcal{BF}_1^*$ (and also $\mathcal{BF}_1$) without making use of the identities in law (43) and (44) and density shapes (31) and (49) below. The above discussion illustrates to what extent the classes $\mathcal{BF}_s$ and $\mathcal{BF}_s^*$ are rich from a stochastic point of view.

Finally, in Sects. 6 and 7, we draw attention to Simon's work [16], who focused on *positive stable distributions*, namely those associated with the Bernstein functions $\phi_\alpha(\lambda) := \lambda^\alpha, \ \alpha \in (0, 1)$. With a different approach and with techniques restricted to this special case, Simon studied the range of values $s > 0$, for which $\phi_\alpha \in \mathcal{BF}_s$. We emphasize that, for the function ϕ_α, we have, see (87):

$$\phi_\alpha \in \mathcal{BF}_s^* \iff \Omega_s(1 - e^{-\phi_\alpha}) \in \mathcal{BF}.$$

Then, the problem becomes to find the values of $s > 0$ for which $\Omega_s(1 - e^{-\phi_\alpha}) \in \mathcal{BF}$. This simple looking, but in our opinion non-trivial, question is completely answered in Theorem 6.3. As a consequence, Corollary 7.1 illustrates when the usual Mittag-Leffler function E_α is such that

$$\lambda \mapsto 1 - r\,\Gamma(1-\alpha)\,\lambda\,E_\alpha(-\lambda), \quad r > 0,$$
$$\text{is completely monotone, or at least is non-negative.}$$

Sections 1 and 1 consist of two appendices that clarify the structure of Lévy exponents and Bernstein functions.

2 The Classes $\mathcal{BF}_s$ for $s > 0$

We start this section with an additional account on several interesting subclasses of Bernstein and completely monotone functions that will be needed in the sequel. The subclass of complete Bernstein functions, $\mathcal{CBF}$, consists of those Bernstein functions that have associated Lévy measure of the form

$$\mu(dx) = m(x)\,dx, \quad x > 0, \text{ where } m(x) \in \mathcal{CM}. \tag{26}$$

Similarly, the class of Stieltjes functions, $\mathcal{S}$, consists of those completely monotone functions having a representation measure that satisfies (26), i.e., a double Laplace transform, also known as a Stieltjes transform. The class $\mathcal{CBF}$ is intimately related to the class $\mathcal{S}$ via the equivalences, see [15, Theorem 6.2, Proposition 7.1, Theorem 7.3 and (7.3)],

$$\phi(\lambda) \in \mathcal{CBF} \iff \frac{\phi(\lambda)}{\lambda} \in \mathcal{S} \iff \phi\left(\frac{1}{\lambda}\right) \in \mathcal{S} \iff \frac{1}{\phi(\lambda)} \in \mathcal{S}.$$

If in (26) we have $x\,m(x) \in \mathcal{CM}$, then we have a Thorin Bernstein function, and we denote $\phi \in \mathcal{TBF}$. The following equivalence is [15, Theorem 8.2]

$$\phi \in \mathcal{TBF} \iff \phi \geq 0 \text{ and } \phi' \in \mathcal{S}. \tag{27}$$

See Sect. 1 for the integral representations of the functions in these subclasses. The corresponding classes of infinitely divisible distributions are, respectively, the famous Bondesson class and the class of generalized Gamma convolutions popularized by Bondesson [3], see also [8, 15, 17] for more information.

Sendov and Shan [14] focused on another proper subclass of $\mathcal{BF}$, namely those ϕ such that in (6) the Lévy measure μ has an harmonically concave tail, that is

$$x \mapsto x\,\mu(x,\infty), \quad x > 0, \text{ is concave}.$$

The latter is equivalent to $\phi \in \mathcal{BF}_1$. After some analytical effort, the equivalence

$$\phi \in \mathcal{BF}_1 \iff \mu(dx) = \frac{p(x)}{x^2}dx,$$
$$\text{where } p(x) \text{ is a measurable, non-decreasing function,}$$

was shown in [1]. The class $\mathcal{BF}_s$, introduced in Definition 1.4, is not void and extends $\mathcal{BF}_1$. Indeed, consider the Bernstein function $\varphi_\alpha(\lambda) = \lambda^\alpha,\ 0 \leq \alpha \leq 1$, associated with the positive stable distribution, cf. (75) below. We have

$$\varphi_\alpha \in \mathcal{BF}_s \iff \Omega_s(\varphi_\alpha)(\lambda) = (s-\alpha)\lambda^\alpha \in \mathcal{BF} \iff s \geq \alpha. \tag{28}$$

The characterization of $\mathcal{BF}_s$, for all $s > 0$, is obtained by the following theorem. Recall notation (14), (15), (19), and define

$$\mathrm{a}_s := \lim_{\lambda\to\infty} \frac{\phi_\diamond(\lambda)}{\lambda^s}. \tag{29}$$

(The next theorem gives conditions under which a_s exists.)

Theorem 2.1 *Let $s > 0$ and let $\phi : [0, \infty) \to [0, \infty)$ be a differentiable function on $(0, \infty)$. Then, the following assertions hold.*

(1) *If $\Omega_1(\phi) \in \mathcal{BF}$, then d exists and is finite, $\phi_\diamond \geq 0$ and*

$$\lim_{\lambda\to\infty} \frac{\Omega_1(\phi)(\lambda)}{\lambda} = 0;$$

(2) *Assume that d exists and is finite. Assume $\phi_\diamond \geq 0$ and $\Omega_s(\phi) \in \mathcal{BF}$. Then,*

  (i) *$\phi_\diamond(\lambda)/\lambda^s \in \mathcal{CM}$;*
  (ii) *$\phi \in \mathcal{BF}_s$ (in particular $\phi \in \mathcal{BF}$), limits (14), and (29) exist, and ϕ has the representation*

$$\phi(\lambda) = \phi(0) + \mathrm{d}\,\lambda + \mathrm{a}_s\,\lambda^s + \int_1^\infty \frac{\Omega_s(\phi)(\lambda x)}{x^{s+1}} dx, \tag{30}$$

where $\mathrm{d} \geq 0$, $\mathrm{a}_s \in [0, \infty)$, if $s < 1$, and $\mathrm{a}_s = 0$, if $s \geq 1$.

(3) *The following conditions are equivalent:*

  (i) *$\phi \in \mathcal{BF}_s$;*
  (ii) *$\phi \in \mathcal{BF}$ and the Lévy measure μ of ϕ is of the form*

$$\mu(dx) = \frac{p_s(x)}{x^{s+1}} dx, \quad x > 0, \tag{31}$$

for some non-decreasing, right-continuous, function $p_s : (0, \infty) \to [0, \infty)$, such that

$$\int_0^1 \frac{p_s(x)}{x^s} dx + \int_1^\infty \frac{p_s(x)}{x^{s+1}} dx < \infty, \tag{32}$$

and $p_s(0+) = 0$, if $s \geq 1$;

  (iv) *ϕ has the representation*

$$\phi(\lambda) = \phi(0) + \mathrm{d}\,\lambda + \mathrm{a}_s\,\lambda^s + \int_1^\infty \frac{\varphi_s(\lambda x)}{x^{s+1}} dx, \quad \lambda \geq 0, \tag{33}$$

where $\mathrm{d} \geq 0$, $\mathrm{a}_s \in [0, \infty)$, if $s < 1$, and $\mathrm{a}_s = 0$, if $s \geq 1$, and $\varphi_s \in \mathcal{BF}$ has no killing term nor drift term.

(4) *The coefficient a_s can be expressed as $\mathrm{a}_s = p_s(0+)\Gamma(1-s)/s$, whenever $s < 1$.*
(5) *$\mathcal{BF}_s \subseteq \mathcal{BF}_r$, for all $r > s > 0$.*

Proof (1) By the definition of Ω_1 and the assumption $\Omega_1(\phi) \in \mathcal{BF}$, we see that ϕ is twice differentiable and $-\phi''(\lambda) = \Omega_1(\phi)'(\lambda)/\lambda \geq 0$. Thus, ϕ is concave and by (4), we conclude that, as λ approaches infinity, $\phi'(\lambda)$ decreases to $\mathrm{d} \in [0, \infty)$ and

$$\lim_{\lambda\to\infty} \frac{\Omega_1(\phi)(\lambda)}{\lambda} = \lim_{\lambda\to\infty} \Big(\frac{\phi(\lambda) - \phi(0)}{\lambda} - \phi'(\lambda) \Big) = 0.$$

The inequality $\phi_\diamond(\lambda) = \Omega_1(\phi)(\lambda) + \lambda \left(\phi'(\lambda) - \mathrm{d}\right) \geq 0$ follows.
(2) (i): Since $\Omega_s(\phi_\diamond) = \Omega_s(\phi)$ is infinitely often differentiable, then so is $\phi_\diamond$. Observe that

$$- \frac{d}{d\lambda} \left(\frac{\phi_\diamond(\lambda)}{\lambda^s} \right) = \frac{1}{\lambda^s} \, \frac{\Omega_s(\phi_\diamond)(\lambda)}{\lambda}, \;\; \lambda > 0 \tag{34}$$

is a product of two completely monotone functions, hence is completely monotone, see (8). Then, since $\phi_\diamond$ is non-negative, conclude that $\phi_\diamond(\lambda)/\lambda^s \in \mathcal{CM}$.

(2) (ii): We first discuss the conditions on a_s. Since $\phi_\diamond(\lambda)/\lambda^s \in \mathcal{CM}$, then necessarily $\mathrm{a}_s \in [0, \infty)$. On the one hand, since the limit d is assumed to exist, by (29) we see that $\mathrm{a}_1 = 0$. The assumption $\Omega_s(\phi_\diamond) \in \mathcal{BF}$ implies that

$$\lim_{\lambda \to \infty} \frac{\Omega_s(\phi_\diamond)(\lambda)}{\lambda} = s \; \mathrm{a}_1 - \lim_{\lambda \to \infty} \phi'_\diamond(\lambda) = - \lim_{\lambda \to \infty} \phi'_\diamond(\lambda) \in [0, \infty).$$

On the other hand, if $s > 1$, then

$$0 = \lim_{\lambda \to \infty} \frac{\Omega_s(\phi_\diamond)(\lambda)}{\lambda^s} = s \; \mathrm{a}_s - \lim_{\lambda \to \infty} \lambda^{1-s} \phi'_\diamond(\lambda) = s \; \mathrm{a}_s,$$

hence $\mathrm{a}_s = 0$. Next, integrating (34), we obtain

$$\phi(\lambda) = \phi(0) + \mathrm{d}\, \lambda + \phi_\diamond(\lambda) = \phi(0) + \mathrm{d}\, \lambda + \lambda^s \left(\mathrm{a}_s + \int_\lambda^\infty \frac{\Omega_s(\phi_\diamond)(y)}{y^{s+1}} dy \right), \;\; \lambda \geq 0.$$

Then, making the change of variable $y = \lambda x$, we arrive at (30). Observe that $\lambda^s \in \mathcal{BF}$ if $s < 1$ and if $s \geq 1$, then $\mathrm{a}_s = 0$. Since $\mathcal{BF}$ is a closed convex cone, see the comments above (8), formula (30) shows that $\phi \in \mathcal{BF}$. Finally, the assumption $\Omega_s(\phi) \in \mathcal{BF}$ implies that $\phi \in \mathcal{BF}_s$.
(3) $(i) \Longrightarrow (ii)$: By the definition of the class $\mathcal{BF}_s$, we have that ϕ and $\Omega_s(\phi)(\lambda)$ are both in $\mathcal{BF}$. By representation (6) of ϕ, it follows that

$$\Omega_s(\phi)(\lambda) = \lambda \int_{(0,\infty)} e^{-\lambda x} \big[s \; \mu(x, \infty) dx - x \; \mu(dx) \big] \in \mathcal{BF}.$$

Thus, necessarily

$$s \, \mu(x, \infty) \, dx - x \; \mu(dx) = \nu_s(x, \infty) \, dx, \text{ for some Lévy measure} \nu_s.$$

We deduce that μ is absolutely continuous and could be written in form (31) with some non-negative function p_s. Further,

$$\nu_s(x, \infty) = s \int_x^\infty \frac{p_s(u)}{u^{s+1}} du - \frac{p_s(x)}{x^s} \tag{35}$$

is non-increasing, i.e., $\nu_s(dx) = -d(\nu_s(x, \infty)) = dp_s(x)/x^s$ is a positive measure, or in other words, $p_s(x)$ is a non-decreasing function. Since the integral is continuous in x and $\nu_s(x,\infty)$ is right-continuous, then so is $p_s(x)$. For later use, we record that in this case, $\Omega_s(\phi) = \Omega_s(\phi_\diamond)$ is represented by

$$\Omega_s(\phi)(\lambda) = \int_{(0,\infty)} (1-e^{-\lambda x}) \frac{dp_s(x)}{x^s}, \ \ \lambda \geq 0. \tag{36}$$

Integrability condition (32) on $p_s(x)$ is a reformulation of (7). It is clear that since p_s is non-decreasing and right continuous, then $p_s(0+) \in [0,\infty)$. For the last assertion, note that the integrability of $p_s(x)/x^s$ on $(0, 1)$ implies that

$$\lim_{x\to 0+} xp_s(x)/x^s = \lim_{x\to 0+} p_s(x)/x^{s-1} = 0,$$

hence $p(0+) = 0$, if $s \geq 1$.
(3) $(ii) \Longrightarrow (i)$: is obtained by reading the arguments in $(i) \Longrightarrow (ii)$ in reverse.
(3) $(i) \Longrightarrow (iii)$: is true with $\varphi_s = \Omega_s(\phi)$ in (30).
(3) $(iii) \Longrightarrow (i)$: Change the variable under the integral $y = \lambda x$ in (33) and then differentiate with respect to λ to get $\Omega_s(\phi) = \varphi_s$. This implies that $\Omega_s(\phi) \in \mathcal{BF}$ and by part 2) (ii), we conclude that $\phi \in \mathcal{BF}_s$.
(4) For $s < 1$, we have

$$\begin{aligned} \mathrm{a}_s = \lim_{\lambda\to\infty} \frac{\phi_\diamond(\lambda)}{\lambda^s} = p_s(0+)\frac{\Gamma(1-s)}{s} + \lim_{\lambda\to\infty} \\ \frac{1}{\lambda^s}\int_0^\infty (1-e^{-\lambda x}) \frac{p_s(x)-p_s(0+)}{x^{s+1}}\, dx \end{aligned}$$

and

$$\begin{aligned} &\lim_{\lambda\to\infty} \frac{1}{\lambda^s}\int_0^\infty (1-e^{-\lambda x}) \frac{p_s(x)-p_s(0+)}{x^{s+1}}\, dx \\ &= \lim_{\lambda\to\infty} \int_0^\infty (1-e^{-y}) \frac{p_s(y/\lambda)-p_s(0+)}{y^{s+1}}\, dy = 0, \end{aligned}$$

follows from (32), from the monotonicity of p_s, and from the dominated convergence theorem.
(5) Use the fact that

$$\begin{aligned} \phi \in \mathcal{BF}_s &\Longleftrightarrow \phi \in \mathcal{BF} \ \text{ and } \ \Omega_s(\phi) \in \mathcal{BF} \\ &\Longrightarrow \phi \in \mathcal{BF} \ \text{ and } \ \Omega_r(\phi) = (r-s)\phi_\diamond + \Omega_s(\phi) \in \mathcal{BF} \\ &\Longleftrightarrow \phi \in \mathcal{BF}_r. \end{aligned}$$

This concludes the proof of the theorem. □

Remark 2.2 Note that the set $\mathcal{BF}_s$ is closed under pointwise limits and forms a closed convex cone. This is immediate from its definition with the help of [15, Corollary 3.9].

Theorem 6.2 in [15] states that the operator Δ defined by

$$\Delta\phi(\lambda) := \lambda^2 \int_{(0,\infty)} e^{-\lambda x}\phi(x)\,dx, \quad \phi \in \mathcal{BF},$$

is a bijection from $\mathcal{BF}$ onto $\mathcal{CBF}$. Let $\mathcal{BFT}_s,\ s > 0$, be the class of Bernstein functions φ represented by

$$\varphi(\lambda) = \mathrm{q} + \mathrm{d}\lambda + \int_{(0,\infty)} \frac{\lambda}{\lambda+u}\frac{p_s(u)}{u^s}du,\ \lambda \geq 0, \tag{37}$$

where $\mathrm{q},\ \mathrm{d} \geq 0$ and p_s satisfies conditions 3(ii) of Theorem 2.1. By (104), one sees that $\mathcal{BFT}_1 = \mathcal{TBF}$. The following extension of this bijection gives an additional interest to the class $\mathcal{BF}_s$.

Proposition 2.3 *The operator Δ is a bijection from $\mathcal{BF}_s$ onto $\mathcal{BFT}_s$. Moreover, $\mathcal{BFT}_s \subseteq \mathcal{BFT}_r$ if $s < r$.*

Proof If $\phi \in \mathcal{BF}_s$, then, by Theorem 2.1, there exist $\mathrm{q},\ \mathrm{d} \geq 0$ and a non-decreasing function p_s such that

$$\phi(\lambda) = \mathrm{q} + \mathrm{d}\lambda + \int_{(0,\infty)} (1-e^{-\lambda u})\frac{p_s(u)}{u^{s+1}}\,du,\ \lambda \geq 0,$$

and elementary computation shows that $\Delta\phi(\lambda)$ is given by the right-hand side of (37). The surjectivity is obtained by reversing the calculus. The inclusion $\mathcal{BFT}_s \subseteq \mathcal{BFT}_r$ is obvious since

$$\frac{p_s(u)}{u^{s+1}} = \frac{u^{r-s}p_s(u)}{u^{r+1}},$$

and $u \mapsto u^{r-s}p_s(u)$ remains non-decreasing. □

Remark 2.4 In [3, (9.5.1), pp. 150] and [15, Definition 8.9], a particular class of Bernstein functions was introduced. In [15], it is denoted by $\mathcal{TBF}_s,\ s > 0$, and it corresponds to those Bernstein functions such that the corresponding Lévy measure μ in (6) has a density of the form

$$\mu(dx) = m(x)\,dx \quad \text{and} \quad x \mapsto x^{2-s}\,m(x) \in \mathcal{CM}.$$

It is not difficult to show that

$$s \geq 1 \Longrightarrow \mathcal{BFT}_s \subset \mathcal{TBF}_s.$$

To check the latter, use point 3(ii) in Theorem 2.1, and the fact that $p_s(0+) = 0$ if $s \geq 1$, to obtain that any function $\varphi \in \mathcal{BFT}_s$, with $s \geq 1$, is represented by (37), hence has the form

$$\varphi(\lambda) = \mathrm{q} + \mathrm{d}\lambda + \int_0^\infty (1 - e^{-\lambda x})\, m_s(x)\, dx,$$

and the function

$$\begin{aligned} x \mapsto x^{2-s}\, m_s(x) = x^{2-s} \int_0^\infty u^{1-s}\, p_s(u)\, e^{-ux}\, du \\ = \int_{(0,\infty)} \left(\int_{vx}^\infty u^{1-s}\, e^{-u}\, du \right) dp_s(v) \end{aligned}$$

is completely monotone.

Point 3 in next proposition shows a nice sufficient condition for a function to be in $\mathcal{BF}_s$.

Proposition 2.5 *Let* $s > 0$ *and let* $\phi \in \mathcal{BF}$ *have drift term* d. *Then, the following hold.*

(1) *If* $\phi \in \mathcal{BF}_s$, *then* $\phi_\alpha(\lambda) := \phi(\lambda^\alpha) \in \mathcal{BF}_s$, *for every* $\alpha \in (0, \min(1, s)]$. *In the case when* $\mathrm{d} = 0$, *the implication holds for every* $\alpha \in (0, 1]$.

(2) *If* $\phi \in \mathcal{BF}_s,\ s \geq 1$, *then*

$$\psi_{a,s}(\lambda) := a\, s\, \big(\phi(\lambda) - \phi(0) - \mathrm{d}\,\lambda\big) - \lambda\big(\phi(\lambda + a) - \phi(\lambda) - a\,\mathrm{d}\big) \in \mathcal{BF},\ \textit{for every} a > 0. \tag{38}$$

(3) *If* ϕ *satisfies* (38) *for some* $s > 0$, *then* $\phi \in \mathcal{BF}_s$.

Proof (1) It suffices to use that $\mathrm{d} := \lim_{x\to\infty} \phi(x)/x \in [0, \infty)$ implies

$$\lim_{x\to\infty} \phi_\alpha(x)/x = 0$$

and to write

$$\begin{aligned} \Omega_s(\phi_\alpha)(\lambda) &= s\, \big(\phi(\lambda^\alpha) - \phi(0)\big) - \alpha\, \lambda^\alpha\, \phi'(\lambda^\alpha) \\ &= \alpha \left(s\big(\phi(\lambda^\alpha) - \phi(0)\big) - \lambda^\alpha\, \phi'(\lambda^\alpha) - (s-1)\,\mathrm{d}\,\lambda^\alpha \right) \\ &\quad + s\,(1-\alpha)\, \big(\phi(\lambda^\alpha) - \phi(0) - \mathrm{d}\,\lambda^\alpha\big) + (s-\alpha)\,\mathrm{d}\,\lambda^\alpha \\ &= \alpha\, \Omega_s(\phi)(\lambda^\alpha) + s\,(1-\alpha)\, \phi_\diamond(\lambda^\alpha) + (s-\alpha)\,\mathrm{d}\,\lambda^\alpha, \end{aligned}$$

then, use that non-negative linear combinations and compositions of Bernstein functions are Bernstein. In the case when $\mathrm{d} = 0$, the last term above disappears, leaving non-negative linear combination of Bernstein functions for every $\alpha \in (0, 1]$.

(2) There is no loss of generality to assume that ϕ has zero killing rate and drift term, and by Theorem 2.1 the Lévy measure of ϕ is given by (31). For $x > 0$, define

$$\epsilon(x) := \frac{1 - e^{-x}}{x} \text{ and } l_{a,s}(x) := s \int_x^\infty \frac{p_s(u)}{u^{s+1}}\, du - \epsilon(ax)\, \frac{p_s(x)}{x^s}. \tag{39}$$

Observe that $0 < \epsilon(x) < 1$ and that

$$\kappa_s(x) := s\big(1 - \epsilon(x)\big) + x\,\epsilon'(x) \geq \kappa_1(x) \geq 0.$$

Then, using (35), it is clear that

$$l_{a,s}(x) \geq s \int_x^\infty \frac{p_s(u)}{u^{s+1}} - \frac{p_s(x)}{x^s} \geq 0.$$

Integrating by parts, using that $\epsilon(ax)$ is a continuous function and [4, Theorem 6.2.2], gives

$$\begin{aligned}
\int_x^\infty \epsilon(au)\frac{dp_s(u)}{u^s} &= -\epsilon(ax)\frac{p_s(x)}{x^s} - \int_x^\infty \frac{p_s(u)}{u^{s+1}}(au\epsilon'(au) - s\epsilon(au))\,du \\
&= -\epsilon(ax)\frac{p_s(x)}{x^s} - \int_x^\infty \frac{p_s(u)}{u^{s+1}}(\kappa_s(au) - s)\,du \\
&= l_{a,s}(x) - \int_x^\infty \frac{p_s(u)}{u^{s+1}}\kappa_s(au)\,du.
\end{aligned}$$

(In the first equality, use (32) to conclude that $p_s(x)/x^{s+1}$ approaches zero at infinity.) This expresses $l_{a,s}$ as the right tail of a positive measure:

$$l_{a,s}(x) = \int_{[x,\infty)} \left(\kappa_s(au)\frac{p_s(u)}{u^{s+1}}\,du + \epsilon(au)\frac{dp_s(u)}{u^s}\right),$$

implying that $l_{a,s}$ is non-increasing. It is also right-continuous, because $p_s(x)$ is. The fact that

$$\nu_{a,s}(du) := \kappa_s(au)\frac{p_s(u)}{u^{s+1}}\,du + \epsilon(au)\frac{dp_s(u)}{u^s}$$

is a Lévy measure, follows from (39). Using both representations of ϕ in (6), one can see that

$$\lambda \mapsto \psi_{a,s}(\lambda) = a\int_0^\infty (1 - e^{-\lambda x})\,\nu_{a,s}(dx) \in \mathcal{BF}.$$

(3) Use the fact that $\mathcal{BF}$ is closed under taking point wise limits and notice that $\Omega_s(\phi) = \lim_{a\to 0+}\psi_{a,s}/a$. □

Recall the difference operators $\omega_{c,s}$ introduced in (16). It is easy to verify that we have

$$\omega_{c^{n+1},s}(\phi)(\lambda) = \omega_{c,s}(\phi)(c^n\lambda) + c^s\omega_{c^n,s}(\phi)(\lambda),\ \ n = 1, 2, \ldots$$

and by an induction, that

$$\omega_{c,s}(\phi) \in \mathcal{BF} \Longrightarrow \omega_{c^n,s}(\phi) \in \mathcal{BF}, \text{ for all } n = 1, 2, \ldots \tag{40}$$

Further $\omega_{c,s}$ is tightly linked to the differential operator Ω_s given in (19). Indeed, if $\phi : [0, \infty) \to [0, \infty)$ is differentiable on $(0, \infty)$, then

$$\omega_{c,s}(\phi)(\lambda) = c^s \int_c^1 \frac{\Omega_s(\phi)(\lambda x)}{x^{s+1}} dx, \quad c \in (0, 1), \tag{41}$$

$$\Omega_s(\phi) = \lim_{c \to 1^-} \frac{\omega_{c,s}(\phi)}{1-c} = \lim_{c \to 1^-} \frac{1 - e^{-\omega_{c,s}(\phi)}}{1-c}. \tag{42}$$

Indeed, to see (41), one needs to integrate by parts the second term in the middle of (19), while formula (42) follows from L'Hôpital's rule. These observations lead to the following result.

Theorem 2.6 *Let $s > 0$ and ϕ be a cumulant function.*

(1) *If $\omega_{c,s}(\phi) \in \mathcal{BF}$ for some $c \in (0, 1)$ and $s \geq 1$, then $\phi \in \mathcal{BF}$.*
(2) *The following conditions are equivalent.*

  (i) $\phi \in \mathcal{BF}_s$;
  (ii) $\omega_{c_n,s}(\phi) \in \mathcal{CF}$ *for some sequence* $c_n \in (0, 1)$, *such that* $\lim_{n\to\infty} c_n = 1$;
  (iii) $\omega_{c,s}(\phi) \in \mathcal{BF}$ *for all* $c \in (0, 1)$.

(3) *If* $r > s > 0$ *and* $X \sim \mathbf{RSD_s}$, *then* $X \sim \mathbf{RSD_r}$.

Proof (1) Recall that the class $\mathcal{BF}$ is a convex cone, closed under taking point-wise limits. If $\omega_{c,s}(\phi) \in \mathcal{BF}$, then (40) implies $\omega_{c^n,s}(\phi) \in \mathcal{BF}$ for all $n = 1, 2, \ldots$. Thus, $\phi(0) = 0$, $s \geq 1$ and (16) yield

$$\lim_{n\to\infty} \omega_{c^n,s}(\phi)\left(\frac{\lambda}{c^n}\right) = \lim_{n\to\infty}\left(\phi(\lambda) - c^{ns}\phi\left(\frac{\lambda}{c^n}\right) - (c^n - c^{ns})d\frac{\lambda}{c^n}\right)$$
$$= \phi(\lambda) - \mathbb{1}_{s=1}\, \mathrm{d}\lambda \in \mathcal{BF}.$$

Therefore, we conclude that $\phi \in \mathcal{BF}$.

(2) We use again, the fact that $\mathcal{BF}$ is closed under taking point-wise limits.
$(i) \Longrightarrow (ii)$: If $\phi \in \mathcal{BF}_s$, then by definition $\Omega_s(\phi) \in \mathcal{BF}$. By (41), we obtain that $\omega_{c,s}(\phi) \in \mathcal{BF}$ for every $c \in (0, 1)$. Then, (11) and $\omega_{c,s}(\phi)(0) = 0$ imply that $\omega_{c,s}(\phi) \in \mathcal{CF}$ for every $c \in (0, 1)$.
$(ii) \Longrightarrow (i)$: The assumption and (9) imply that $1 - e^{-\omega_{c_n,s}(\phi)} \in \mathcal{BF}$ for all n. Now the second equality in (42) shows that $\Omega_s(\phi) \in \mathcal{BF}$ and together with Theorem 2.1, part 2), we conclude that $\phi \in \mathcal{BF}_s$ as well.
$(i) \iff (iii)$: The necessity is done in $(i) \Longrightarrow (ii)$. The sufficiency is similar to $(ii) \Longrightarrow (i)$, using the first equality in (42).

(3) The assertion is an immediate consequence of point 5 in Theorem 2.1, the definition of the class $\mathbf{RSD_s}$ and the above equivalence $2)(i) \iff 2)(ii)$ of this theorem. □

3 Timescale Decomposability and Jurek–Vervaat-Type Stochastic Integral Representation for Subordinators Associated with the Class $\mathcal{BF}_s$

We will focus on the stochastic interpretation of the classes $\mathcal{BF}_s,\ s>0$ under several aspects. In Theorem 2.1, we have seen that for a cumulant function ϕ, $\Omega_s(\phi)\in\mathcal{BF}$ yields $\phi\in\mathcal{BF}$. Thus, using representations (6) and (31), we retrieve for any function $\phi\in\mathcal{BF}_s$ and $c\in(0,1)$, the following representation of $\omega_{c,s}(\phi)$:

$$\begin{aligned}\omega_{c,s}(\phi)(\lambda) &= c\lambda\int_0^\infty e^{-c\lambda x}\mu(x,\infty)\,dx - c^s\lambda\int_0^\infty e^{-\lambda x}\mu(x,\infty)\,dx\\ &= \lambda\int_0^\infty e^{-\lambda x}\left[\int_{x/c}^\infty \frac{p_s(u)}{u^{s+1}}du - c^s\int_x^\infty \frac{p_s(u)}{u^{s+1}}du\right]dx\\ &= c^s\lambda\int_0^\infty e^{-\lambda x}\int_x^\infty \frac{p_s(u/c)-p_s(u)}{u^{s+1}}du\,dx,\end{aligned}$$

where in the last equality we performed the changes of variables $u\mapsto u/c$. With Theorem 2.6, we deduce the following stochastic interpretation.

Corollary 3.1 *Let $s>0$. Let $\phi\in\mathcal{BF}$, $\phi(0)=0$, be associated to the subordinator $(X_t)_{t\geq 0}$. Under the notation of Theorem 2.1, the following two conditions are equivalent.*

(1) $\phi\in\mathcal{BF}_s$;

(2) *For all $c\in(0,1)$, there exists a subordinator $(Z_t^{(c,s)})_{t\geq 0}$, such that we have the identity in law*

$$c\,X_t \overset{d}{=} X_{c^s t} + \mathrm{d}\,(c-c^s)\,t + Z_t^{(c,s)}, \tag{43}$$

where $(X_{c^s t})_{t\geq 0}$ and $(Z_t^{(c,s)})_{t\geq 0}$ are assumed to be independent, and the Bernstein function of $Z^{(c,s)}$ equals $\omega_{c,s}(\phi)$.

Under the two conditions above, let $(Y_t^{(s)})_{t\geq 0}$ be the subordinator with Bernstein function $\phi_Y(\lambda):=\phi(\lambda)-\mathrm{d}\lambda-\mathrm{a}_s\,\lambda^s$, let $(S_t^{(s)})_{t\geq 0},\ s\in(0,1)$, be a $s-$standard stable subordinator (c.f. (75) below), independent of $(Y_t^{(s)})_{t\geq 0}$ and recall that $a_s=0$ for $s>1$. Then, we have the identities in law:

$$X_t \overset{d}{=} \mathrm{d}\,t + (\mathrm{a}_s)^{1/s}\,S_t^{(s)} + Y_t^{(s)} \text{ and } c\,Y_t^{(s)} \overset{d}{=} Y_{c^s t}^{(s)} + Z_t^{(c,s)}, \tag{44}$$

where the random variables $Y_{c^s t}^{(s)}$ and $Z_t^{(c,s)}$ are supposed to be independent.

Proof The assumptions $\phi\in\mathcal{BF}$ and $\phi(0)=0$ imply that ϕ is a cumulant function, cf. (10). By Theorem 2.6, we equivalently have $\phi\in\mathcal{BF}_s$ or $\omega_{c,s}(\phi)\in\mathcal{BF}$ for all $c\in(0,1)$. Identity (43) is justified by the decomposition

$$\phi(c\lambda) = c^s\,\phi(\lambda) + \mathrm{d}\,(c-c^s)\,\lambda + \omega_{c,s}(\phi).$$

The first identity in (44) is a consequence of representation (33), while the second one is obtained by the decomposition

$$\phi_Y(c\,\lambda) = c^s\ \phi_Y(\lambda) + \omega_{c,s}(\phi),$$

completing the proof. □

As already noticed before (21), identities (43) and (44) resemble the proper self-similarity property. The latter suggests conducting a deeper investigation into the stochastic interpretation of the class $\mathcal{BF}_s$; this is the main subject of the next section.

At this stage, we can propose the following Jurek-Vervaat-type stochastic integral representation associated with the class the class $\mathcal{BF}_s$. Recall the notations of Theorem 2.1 and of Corollary 3.1.

Theorem 3.2 *Let* $s > 0$ *and let* ϕ *be Bernstein function, with* $\phi(0) = 0$*, associated with a positive and infinitely divisible random variable* X*. Then,* $\phi \in \mathcal{BF}_s$*, if, and only if, the law of* X *is defined by the stochastic integral*

$$X \overset{d}{=} \mathrm{d} + \mathrm{a}_s^{1/s}\ S_1^{(s)} + \int_0^1 u^{-1/s} dZ_u^{(s)}, \tag{45}$$

where $S_1^{(s)}$ *has the standard positive stable distribution (c.f. (76) below) if* $0 < s < 1$*, independent of the subordinator* $Z^{(s)}$*. The term* a_s*, the Bernstein function* $\phi_{Z^{(s)}}$ *and the Lévy measure* ν_s *of* $Z^{(s)}$ *are given by*

$$\mathrm{a}_s = p_s(0+)\ \frac{\Gamma(1-s)}{s}\ 1\!\!1_{s<1},\quad \phi_{Z^{(s)}} = \frac{\Omega_s(\phi)}{s},\quad \textit{and}\quad \nu_s(dx) = \frac{dp_s(x)}{s\ x^s}, \tag{46}$$

where p_s *is a non-decreasing, right-continuous function, satisfying integrability conditions* (32)*.*

Proof By Theorem 2.1, $\phi \in \mathcal{BF}_s$ is equivalent to representation (33):

$$\phi(\lambda) = \mathrm{d}\lambda + \mathrm{a}_s\ \lambda^s + \frac{1}{s}\int_0^1 \varphi_s\left(\frac{\lambda}{u^{1/s}}\right) du,\ \ \lambda \geq 0. \tag{47}$$

for some Bernstein function φ_s which, by (30), necessarily equals to $\Omega_s(\phi)$. Observe that representation (47) of ϕ fits perfectly the injective one in [15, Lemma 10.1], when taking the θ-function there equal to $\theta(u) = u^{-1/s},\ u \in (0, 1)$, and then the Bernstein function of $Z^{(s)}$ is $\Omega_s(\phi)/s$. By (36), the Lévy measure of $Z^{(s)}$ is necessarily given by (46). This shows that (47) implies (45) and (46).

For the opposite direction, note that conditions (10.13) and (10.14) in [15, Proposition 10.4] are satisfied since the Bernstein function $\Omega_s(\phi)/s$ has no drift nor killing terms. We now check that ν_s satisfies integrability condition (10.15) in [15, Proposition 10.4]. After taking $\vartheta(y) := -(\theta^{-1})' = s/y^{s+1}$, when $y > 1$, and $\vartheta(y) := 0$

otherwise, the integrability condition is translated as follows:

$$
\begin{aligned}
I_s &:= \int_0^{\infty} \left(\int_0^{1/x} y\ \vartheta(y)\ dy \right) \nu_s(x,\infty)dx \\
&= \int_0^{1} \left(\int_1^{1/x} y\ \vartheta(y)\ dy \right) \nu_s(x,\infty)dx \\
&= \int_0^{1} \int_x^{\infty} \left(\int_1^{1/x} y\ \vartheta(y)\ dy \right)\ \nu_s(du)\ dx \\
&= \int_0^{\infty} \left(\int_0^{1\wedge u} \int_1^{1/x} y\ \vartheta(y)\ dy\ dx \right) \nu_s(du) \\
&= \int_0^{\infty} \eta_s(u)\ \frac{dp_s(u)}{u^s} < \infty,
\end{aligned} \tag{48}
$$

where

$$
\eta_s(u) := \int_0^{1\wedge u} \int_1^{1/x} \frac{1}{y^s}\ dy\ dx = \int_1^{\infty} \int_0^{(1/y)\wedge(1\wedge u)} \frac{1}{y^s}\ dx\ dy = \int_1^{\infty} \frac{(uy)\wedge 1}{y^{s+1}} dy.
$$

Now, observe that $J_s := \int_0^\infty (x\wedge 1)x^{-(s+1)}dx < \infty$ if $s < 1$, and recall that $p_s(0+) = 0$ if $s \geq 1$. Then, using Tonelli-Fubini's theorem and a change of variable, express the terms in (32) as

$$
\begin{aligned}
\int_0^1 \frac{p_s(x)}{x^s}dx + \int_1^{\infty} \frac{p_s(x)}{x^{s+1}}dx &= \int_0^{\infty} \frac{x\wedge 1}{x^{s+1}} \left(p_s(0+) + \int_{(0,x]} dp_s(u) \right) dx \\
&= p_s(0+)\ J_s\ \mathbb{1}_{s<1} + \int_{(0,\infty]} \left(\int_u^{\infty} \frac{x\wedge 1}{x^{s+1}} dx \right)\ dp_s(u) \\
&= p_s(0+)\ J_s\ \mathbb{1}_{s<1} + \int_{(0,\infty]} \eta_s(u)\ \frac{dp_s(u)}{u^s}.
\end{aligned}
$$

Thus, conditions (32) and (48) are equivalent and condition (10.15) in [15, Proposition 10.4] holds. Proposition and [15, Lemma 10.1] imply that $\lambda \mapsto s^{-1}\int_0^1 \Omega_s(\phi)(\lambda u^{-1/s})du$ is a Bernstein function that is the cumulant function of $\int_0^1 u^{-1/s} dZ_u^{(s)}$. Representation (47) follows. □

4 The Classes $\mathcal{BF}_s$ and Generalized Unimodality

Here, we provide another justification for the introduction of the class $\mathcal{BF}_s$ and also a stochastic interpretation for Hansen's result [7, Lemma 2.3, case $n = 1$].

Proposition 4.1 *Let $s > 0$. Let the random variable $\mathbb{U}$ be uniformly distributed on $(0, 1)$ and let $\mathbb{G}$ be exponentially distributed with parameter 1. Let Z be a positive random variable with cumulant function φ and distribution function F. Then, the following conditions are equivalent.*

(1) $1 - e^{-\varphi} \in \mathcal{BF}_s$;
(2) *Z has a probability density function of the form*

$$f(x) = \frac{p_s(x)}{x^{s+1}},\ x > 0,\ \text{where } p_s(x) \text{ is non-decreasing and right-continuous;} \tag{49}$$

(3) *For all* $c \in (0, 1)$,

$$F_c(x) := \frac{F(x/c) - c^s F(x)}{1 - c^s},\quad x \geq 0, \tag{50}$$

is the distribution function of some positive (continuous) random variable Z_c.
(4) $1/Z$ *is s-star unimodal in the sense of Definition 1.5, i.e., there exists a positive random variable* V_s *such that*

$$Z \stackrel{d}{=} \frac{V_s}{\mathbb{U}^{1/s}} \stackrel{d}{=} e^{\mathbb{G}/s}\ V_s,\ \text{where} \mathbb{U} \text{ and} \mathbb{G} \text{ are independent of} V_s; \tag{51}$$

(5) *For any positive random variable Y independent of Z, the quotient* Y/Z^s *has a non-increasing probability density function;*
(6) *For any positive random variable Y independent of Z,* $\lambda \mapsto \lambda\, \mathbb{E}[e^{-\lambda Y/Z^s}] \in \mathcal{BF}$;
(7) $\lambda \mapsto \lambda\, \mathbb{E}\left[e^{-\lambda/Z^s}\right] \in \mathcal{BF}$;
(8) $\lambda \mapsto \lambda\, \mathbb{E}\left[Z^s/(\lambda + Z^s)\right]$ *is a Thorin Bernstein function in the sense of* (27).

Proof We start with this observation: since

$$1 - e^{-\varphi(0)} = 0 \text{ and } \lim_{\lambda\to\infty} (1 - e^{-\varphi(\lambda)})/\lambda = 0,$$

then $(1 - e^{-\varphi(\lambda)})_\diamond = 1 - e^{-\varphi(\lambda)}$. Thus, by part 2)$(ii)$ of Theorem 2.1, we have

$$\Omega_s(1 - e^{-\varphi}) \in \mathcal{BF} \iff 1 - e^{-\varphi} \in \mathcal{BF}_s \implies 1 - e^{-\varphi} \in \mathcal{BF} \implies 1 - e^{-\varphi} \in \mathcal{CF}. \tag{52}$$

(1) $\iff$ (2): This is an application of the equivalence between (3i) and (3ii) in Theorem 2.1, applied to $1 - e^{-\varphi}$ with $\mu(dx) = dF(x)$, after noticing that

$$1 - e^{-\varphi(\lambda)} = \mathbb{E}[1 - e^{-\lambda Z}] = \int_0^\infty (1 - e^{-\lambda x})\, dF(x).$$

(1) $\iff$ (3): Recall the operator $\omega_{c,s}$ is defined in (16). The implication is obtained by writing

$$\frac{\omega_{c,s}(1 - e^{-\varphi})(\lambda)}{1 - c^s} = \frac{\mathbb{E}[1 - e^{-c\lambda Z}] - c^s \mathbb{E}[1 - e^{-\lambda Z}]}{1 - c^s}$$

$$= \frac{\lambda}{1-c^s} \int_0^\infty e^{-\lambda x}\big(\mathbb{P}(cZ > x) - c^s \mathbb{P}(Z > x)\big)\, dx$$

$$= \lambda \int_0^\infty e^{-\lambda x}\big(1 - F_c(x)\big) dx \tag{53}$$

$$= \int_0^\infty e^{-x}\big(1 - F_c(x/\lambda)\big) dx. \tag{54}$$

By definition, $F_c \geq 0$, since F is non-decreasing. Now, using (52), and Theorem 2.6, we get equivalences

$$1 - e^{-\varphi} \in \mathcal{BF}_s \iff \omega_{c,s}(1 - e^{-\varphi}) \in \mathcal{BF},\ \forall c \in (0,1)$$
$$\iff F_c \leq 1 \ \text{ and is non-decreasing, } \forall c \in (0,1),$$

while the second equivalence follows from representations (6) and (53). Since Z is a positive random variable, we have $F_c(0) = F(0) = 0$. Then, we use the dominated convergence theorem together with (54) and the fact that $\omega_{c,s}(1 - e^{-\varphi})(0) = 0$, to obtain that $\lim_{x\to\infty} F_c(x) = 1$. All this shows that F_c is the distribution function of a positive random variable for all $c \in (0,1)$.

(2) $\implies$ (4): Observe that if a positive random variable V is independent of $\mathbb{U}$, then the probability density function of the product $\mathbb{U}\ V$ is expressed by

$$f_{\mathbb{U}V}(x) = \mathbb{E}\left[\frac{f_{\mathbb{U}}(x/V)}{V}\right] = \int_{(0,\infty)} \frac{1}{v}\, \mathbb{1}_{\{\frac{x}{v}\in(0,1)\}} \frac{\mathbb{P}(V \in dv)}{v} = \int_{(x,\infty)} \frac{\mathbb{P}(V \in dv)}{v}. \tag{55}$$

By assumption, the probability density function of Z has form (49); thus, the one of Z^{-s} is expressed by

$$f_{Z^{-s}}(x) = \frac{f\left(x^{-\frac{1}{s}}\right)}{s\, x^{1+\frac{1}{s}}} = \frac{p_s\left(x^{-\frac{1}{s}}\right)}{s},\ \ x > 0. \tag{56}$$

Then, $f_{Z^{-s}}$ is a non-increasing function of form (55), i.e., $Z \overset{d}{=} \mathbb{U}^{-1/s}\ V_s$ for some positive random variable V_s independent of $\mathbb{U}$. The last identity in (51) is trivial.

(4) $\implies$ (5): This is simply seen by $Y/Z^s \overset{d}{=} \mathbb{U}\,(Y\,/V_s^s)$ and referring to formula (55).

(5) $\implies$ (6): If q_s is the non-increasing probability density function of Y/Z^s, then

$$\lambda\, \mathbb{E}[e^{-\lambda Y/Z^s}] = \lambda \int_0^\infty e^{-\lambda x} q_s(x) dx,$$

complies with the second representation of a Bernstein function in (6).

(6) $\implies$ (7): Take $Y = 1$.

(7) $\implies$ (2): Compare with the second representation in (6) to conclude that the Bernstein function $\lambda \mapsto \lambda\, \mathbb{E}[e^{-\lambda/Z^s}]$ needs to have killing rate and drift term zero.

Then, one sees that Z^{-s} has a non-increasing, right-continuous probability density function and we conclude by using the first equality in (56).

(2) $\Longrightarrow$ (8): Since Z^{-s} has a probability density function of form (49) we have

$$\mathbb{E}\left[\frac{\lambda Z^s}{\lambda + Z^s}\right] = \int_0^\infty \frac{\lambda x^s}{\lambda + x^s}\, \frac{p_s(x)}{x^{s+1}}\, dx,$$

then making the change of variable $x = y^{1/s}$, we obtain that

$$\mathbb{E}\left[\frac{\lambda Z^s}{\lambda + Z^s}\right] = \frac{1}{s}\int_0^\infty \frac{\lambda}{\lambda + y}\, \frac{p_s(y^{\frac{1}{s}})}{y}\, dy,$$

meets form (104) in Appendix, of a Thorin Bernstein function.

(8) $\Longrightarrow$ (2): By the uniqueness of representation (104), we see that the Thorin Bernstein function $\lambda \mapsto \mathbb{E}[\lambda Z^s/(\lambda + Z^s)]$ has no killing nor drift terms ($\mathrm{q} = \mathrm{d} = 0$) and that the probability density function of Z^s has the form $f_{Z^s}(z) = \sigma\big((0, z]\big)/z^2,\ z > 0$, for some positive measure σ. Since

$$f_{Z^s}(z) = \frac{1}{s}\, z^{\frac{1}{s}-1}\, f\left(z^{\frac{1}{s}}\right),$$

then, with the change of variable $x := z^{1/s}$, we obtain

$$f(x) = s\, x^{s-1}\, f_{Z^s}(x^s) = s\, \frac{\sigma\big((0, x^s]\big)}{x^{s+1}},\ \ x > 0.$$

This gives us (49) with $p_s(x) := s\,\sigma\big((0, x^s]\big)$. □

Remark 4.2 (i) Observe that the equivalent conditions in Proposition 4.1 do not hold if Z has degenerate distribution, that is if it has cumulant function of the form $\mathrm{d}\,\lambda$, for $\mathrm{d} > 0$. Indeed, for $s > 0$, we have

$$\Omega_s(1 - e^{-\mathrm{d}\lambda})(\lambda) = s - e^{-\mathrm{d}\lambda}(s + \mathrm{d}\,\lambda) \notin \mathcal{BF},$$

as can be verified by taking successive derivatives. Thus, $1 - e^{-\mathrm{d}\,\lambda} \notin \mathcal{BF}_s^*$.

(ii) The integrability of f in (49) on $(0, 1)$ implies that

$$\lim_{x \to 0^+} x\ f(x) = \lim_{x \to 0^+} p_s(x)/x^s = 0.$$

This is an improvement on the similar observation that one can make from the first integral in (32).

Example If $Z \overset{d}{=} e^{\mathbb{G}/s}$, then its probability density function is $f(x) = s/x^{s+1},\ x > 1$, and we have

$$1 - e^{-\varphi(\lambda)} = \mathbb{E}[1 - e^{-\lambda Z}] = 1 - s\,\lambda^s \int_\lambda^\infty \frac{e^{-u}}{u^{s+1}} du.$$

Thus, we obtain

$$\Omega_s\left(1-e^{-\varphi}\right)(\lambda)=s\ (1-e^{-\lambda}),\ \ \lambda\geq 0.$$

Let $\widetilde{\omega}_{c,s}$, $\widehat{\omega}_{c,s}$, and $\widetilde{\Omega}_s$ stand for the following modifications of the operators $\omega_{c,s}$ and Ω_s:

$$\widetilde{\omega}_{c,s}(h)(\lambda):=\frac{\omega_{c,s}(h)(\lambda)}{1-c^s},\ \text{for } c\in(0,1)$$
$$\widetilde{\Omega}_s(h)(x):=s\ h(x)+x\ h'(x).$$

The introduction of the operator $\widetilde{\omega}_{c,s}$ is justified by forms (53). The next result completes Proposition 4.1 by giving more information on the probability density function f_c (respectively, f) of the continuous random variable Z_c (respectively, Z) in Proposition 4.1.

Proposition 4.3 *Under the notation and the conditions of Proposition 4.1, we have the following assertions.*

(1) *The density of Z satisfies*

$$f(x)=\frac{s\ \mathbb{E}[V_s^s\ \mathbb{1}_{(V_s\leq x)}]}{x^{s+1}},\ \textit{for Lebesgue almost every} x>0; \tag{57}$$

(2) *The function f is almost everywhere differentiable, and we have the representation*

$$\Omega_s(1-e^{-\varphi})(\lambda)=s\int_0^\infty(1-e^{-\lambda x})\ \widetilde{f}_s(x)\,dx,\ \ \lambda\geq 0, \tag{58}$$

where

$$\widetilde{f}_s(x):=\frac{1}{s}\,\widetilde{\Omega}_{s+1}(f)(x)=\frac{p_s'(x)}{s\ x^s}\ \textit{is a probability density function};$$

(3) *The functions $\widetilde{\omega}_{c,s}(1-e^{-\varphi})$ and f_c, for $c\in(0,1)$, are represented by*

$$\widetilde{\omega}_{c,s}(1-e^{-\varphi})(\lambda)=\int_0^\infty(1-e^{-\lambda x})f_c(x)dx,\ \ \lambda\geq 0, \tag{59}$$
$$f_c(x)=\frac{c^s}{1-c^s}\int_c^1\frac{\widetilde{\Omega}_{s+1}(f)(x/u)}{u^{s+2}}\,du,\ \ x>0. \tag{60}$$

(4) *We have the identities in law*

$$c\ Z\overset{d}{=}\mathbb{B}_{c,s}Z+(1-\mathbb{B}_{c,s})Z_c,\ \textit{for all } c\in(0,1), \tag{61}$$

where $\mathbb{B}_{c,s}$ has the Bernoulli distribution with parameter c^s and the random variables $\mathbb{B}_{c,s}$, Z, Z_c are assumed to be independent. Furthermore, $Z_c\xrightarrow{d}Z$, as $c\to 1^-$.

(5) *We have the identities in law*

$$Z_c \stackrel{d}{=} \widetilde{Z}_s \; W_{c,s}, \; \textit{where } W_{c,s} \stackrel{d}{=} \mathcal{L}aw\big(c \, e^{\mathbb{G}/s} \mid c \, e^{\mathbb{G}/s} < 1\big), \; \textit{for all } c \in (0,1), \tag{62}$$

and $\widetilde{Z}_s$ *is a positive random variable (independent of* $W_{c,s}$*) with probability density function* $\widetilde{f}_s$.

Proof (1) The relationship is obtained by the change of variable $x := u^{-1/s} V_s$ and then exchanging the order of integration in:

$$\begin{aligned}\mathbb{E}[1 - e^{-\lambda Z}] &= \mathbb{E}[1 - e^{-\lambda \mathbb{U}^{-1/s} V_s}] = \mathbb{E}\left[\int_0^1 \left(1 - e^{-\lambda u^{-1/s} V_s}\right) du\right] \\ &= \int_0^\infty (1 - e^{-\lambda x}) \frac{s \, \mathbb{E}\left[V_s^s \, \mathbb{1}_{(V_s \le x)}\right]}{x^{s+1}} dx.\end{aligned}$$

(2) By (49), since p_s is non-decreasing, the function f is almost everywhere differentiable on $(0, \infty)$. The claim that $\widetilde{f}_s(x)$ is a probability density function on $[0, \infty)$ can be checked directly by integration by parts, utilizing (ii) in Remark 4.2. Using the fact that $\Omega_s(1 - e^{-\varphi})(0) = 0$ and an integration by parts, representation (58) follows from

$$\begin{aligned}\Omega_s(1 - e^{-\varphi})(\lambda) &= \Omega_s\big(u \mapsto \mathbb{E}[1 - e^{-uZ}]\big)(\lambda) \\ &= \Omega_s\left(u \mapsto \int_0^\infty (1 - e^{-ux}) \, f(x) \, dx\right)(\lambda) \\ &= \int_0^\infty \Omega_s(u \mapsto 1 - e^{-ux}) \, f(x) \, dx \\ &= \int_0^\infty \big(s\,(1 - e^{-\lambda x}) - \lambda x e^{-\lambda x}\big) \, f(x) \, dx \\ &= \int_0^\infty \big(s\,(1 - e^{-\lambda x}) \, f(x) - e^{-\lambda x} \, \big(x f'(x) + f(x)\big) \, dx \\ &= s - \int_0^\infty e^{-\lambda x} \, \big((s+1) \, f(x) + x \, f'(x)\big) \, dx \\ &= s - \int_0^\infty e^{-\lambda x} \, \widetilde{\Omega}_{s+1}(f)(x) \, dx = s \int_0^\infty (1 - e^{-\lambda x}) \, \widetilde{f}_s(x) \, dx.\end{aligned}$$

Finally, use (49) to observe that

$$\begin{aligned}\widetilde{\Omega}_{s+1}(f)(x) &= (s+1) f(x) + x f'(x) \\ &= (s+1) \frac{p_s(x)}{x^{s+1}} + x \left(\frac{p_s'(x)}{x^{s+1}} - (s+1) \frac{p_s(x)}{x^{s+2}}\right) = \frac{p_s'(x)}{x^s}.\end{aligned} \tag{63}$$

(3) Using (6) and (53), we obtain

$$\widetilde{\omega}_{c,s}(1-e^{-\varphi})(\lambda) = \lambda \int_0^\infty e^{-\lambda x}\big(1-F_c(x)\big)dx = \int_0^\infty (1-e^{-\lambda x}) f_c(x)dx.$$

In order to derive (60), we differentiate (50) and use (49), to find

$$\begin{aligned} f_c(x) &= \frac{c^s}{(1-c^s)x^{s+1}}\,[p_s(x/c)-p_s(x)] = \frac{c^s}{(1-c^s)x^{s+1}} \int_x^{x/c} p_s'(y)\,dy \\ &= \frac{c^s}{(1-c^s)x^s} \int_c^1 \frac{p_s'(x/u)}{u^2}\,du, \end{aligned}$$

where we made the change of variable $y = x/u$. Finally, expressing $p_s'(x)$ from (63) and substituting it in the latter integral, we conclude

$$\begin{aligned} f_c(x) &= \frac{c^s}{(1-c^s)x^s} \int_c^1 \frac{(x/u)^s\,\widetilde{\Omega}_{s+1}(f)(x/u)}{u^2}\,du \\ &= \frac{c^s}{(1-c^s)} \int_c^1 \frac{\widetilde{\Omega}_{s+1}(f)(x/u)}{u^{s+2}}\,du. \end{aligned}$$

(4) Noting that $F(x/c) = \mathbb{P}(cZ \le x)$ and differentiating (50), we obtain

$$f_{cZ}(x) = c^s\; f(x) + (1-c^s)\; f_c(x),\;\; x > 0,$$

which immediately proves identity (61).

(5) Expressing (60) in the form

$$f_c(x) = s\frac{c^s}{1-c^s}\int_c^1 \frac{\widetilde{f}_s(x/u)}{u}\,\frac{du}{u^{s+1}},$$

and using the formula for the density of the product of two independent random variables, gives that the probability density function of $W_{c,s}$ is $s\; c^s\; (1-c^s)^{-1}\; u^{-s-1},\;\; c < u < 1$. It is identified in (62) by the expression

$$\mathbb{P}(c\,e^{\mathbb{G}/s} \le u \mid c\,e^{\mathbb{G}/s} < 1) = \frac{\mathbb{P}(\mathbb{G} \le s\log(c/u))}{\mathbb{P}(\mathbb{G} \le s\log c)} = \frac{1-(c/u)^s}{1-c^s},\;\; c < u < 1,$$

after differentiating with respect to u. □

Remark 4.4 Note that (59) can be re-written as

$$\widetilde{\omega}_{c,s}(1-e^{-\varphi})(\lambda) = 1 - e^{-\varphi_c(\lambda)},$$

where φ_c is the cumulant function of the non-negative random variable Z_c with probability density function f_c.

5 The Class $\mathcal{BF}_s^*$ and Generalized Unimodality

Let $s > 0$. Let ϕ be a Bernstein function with killing term $\mathrm{q} = \phi(0)$ and drift term in d. The Bernstein function $\phi - \mathrm{q}$ corresponds to a non-killed subordinator $(X_t)_{t\geq 0}$, whereas $\phi_\diamond(\lambda) = \phi(\lambda) - \mathrm{d}\lambda - \mathrm{q}$ is the Bernstein function of subordinator $(X_t - \mathrm{d}\ t)_{t\geq 0}$. Observing that the function

$$\varepsilon_t(\phi)(\lambda) = 1 - e^{-t\phi(\lambda)} = 1 - e^{-t\mathrm{q}} + e^{-t\mathrm{q}} \int_{(0,\infty)} (1 - e^{-\lambda x})\mathbb{P}(X_t \in dx)$$

is also a Bernstein function for every $t > 0$, it is natural to introduce the class

$$\mathcal{BF}_s^* := \{\phi \in \mathcal{BF},\ s.t.\ \Omega_s\big(\varepsilon_t(\phi_\diamond)\big) \in \mathcal{BF},\quad \forall t > 0\}, \tag{64}$$

as was done in Definition 1.4. Since $\lim_{x\to\infty}(1 - e^{-t\phi_\diamond(x)})/x = 0$, we have $(\varepsilon_t(\phi_\diamond))_\diamond = \varepsilon_t(\phi_\diamond) \geq 0$, and Theorem 2.1 guarantees that Definition (64) is equivalent to

$$\mathcal{BF}_s^* = \{\phi \in \mathcal{BF},\ s.t.\ \varepsilon_t(\phi_\diamond) \in \mathcal{BF}_s,\quad \forall t > 0\}.$$

Using the fact that the class $\mathcal{BF}$ is closed under pointwise limits and the fact that

$$\lim_{t\to 0^+} \frac{\Omega_s(\varepsilon_t(\phi_\diamond))}{t} = \Omega_s(\phi_\diamond),$$

we see that $\mathcal{BF}_s^* \subset \mathcal{BF}_s$. The inclusion could be strict. Indeed, if $\phi(\lambda) = \lambda^{0.9}$, then

$$\Omega_1(\phi)(\lambda) = 0.1\ \lambda^{0.9} \in \mathcal{BF},\ \ \Omega_1(1 - e^{-\phi})(\lambda) = 1 - e^{-\lambda^{0.9}}(1 + 0.9\ \lambda^{0.9}) \notin \mathcal{BF}.$$

Therefore, we have

$$\phi \in \mathcal{BF}_1 \setminus \mathcal{BF}_1^*.$$

The fact that $1 - e^{-\lambda^{0.9}}(1 + 0.9\ \lambda^{0.9}) \notin \mathcal{BF}$ is checked by the sign change of its second derivative. By definition of the classes $\mathcal{BF}_s^*$, and by the relation

$$\Omega_r(\varepsilon_t(\phi_\diamond)) = \Omega_s(\varepsilon_t(\phi_\diamond)) + (r - s)\ \varepsilon_t(\phi_\diamond),$$

we have that

$$\mathcal{BF}_s^* \subset \mathcal{BF}_r^* \iff 0 < s < r.$$

We now present a simple characterization of the functions in $\mathcal{BF}_s^*$. Due to the representation

$$\Omega_s\big(\varepsilon_t(\phi_\diamond)\big)(\lambda) = s\ \varepsilon_t(\phi_\diamond)(\lambda) - \lambda\varepsilon_t(\phi_\diamond)'(\lambda) = s - e^{-t\phi_\diamond(\lambda)}\big(s + t\lambda\phi_\diamond'(\lambda)\big), \tag{65}$$

we see that if $\phi \in \mathcal{BF}_s^*$, then both functions $\Omega_s\big(\varepsilon_t(\phi_\diamond)\big)$ and $s - \Omega_s\big(\varepsilon_t(\phi_\diamond)\big)$ are non-negative, and by (9), we obtain that

$$\lambda \mapsto \varphi_{s,t}(\lambda) := s - \Omega_s\big(\varepsilon_t(\phi_\diamond)\big)(\lambda) = e^{-t\phi_\diamond(\lambda)}\big(s + t\lambda\phi'_\diamond(\lambda)\big) \in \mathcal{CM}, \ \forall t > 0. \tag{66}$$

Conversely, assume (66) holds and $\phi \in \mathcal{BF}$, the latter implies that $\phi_\diamond \in \mathcal{BF}$. Since $\lim_{\lambda\to 0+} \lambda\phi'_\diamond(\lambda) = 0$, see for example (2.11) in [14], the non-increasing function $\varphi_{s,t}$ is bounded by s, thus $\Omega_s\big(\varepsilon_t(\phi_\diamond)\big) = s - \varphi_{s,t} \in \mathcal{BF}$, we have the equivalences

$$\begin{aligned}\phi \in \mathcal{BF}_s^* \iff \phi \in \mathcal{BF} \text{ and } \lambda \mapsto s - \Omega_s\big(\varepsilon_t(\phi_\diamond(\lambda))\big)\\ = e^{-t\phi_\diamond}\big(s + t\lambda\phi'_\diamond(\lambda)\big) \in \mathcal{CM}, \quad \forall t > 0.\end{aligned} \tag{67}$$

Remark 5.1 We now show that $\mathcal{BF}_s^*$ is not void. Consider the Bernstein function $\varphi(\lambda) := \sqrt{\lambda} - \log(1 + \sqrt{\lambda}) \in \mathcal{BF}$ and let ϕ be such that

$$\phi(0) = \lim_{x\to\infty} \frac{\phi(x)}{x} = 0, \ \ \phi \in \mathcal{BF}_s, \ \text{ and } \lambda \mapsto \big(\lambda\phi'(\lambda)\big)^2 \in \mathcal{BF}.$$

For instance, one can take $\phi(\lambda) = \lambda^\alpha, \ \alpha \in (0, 1/2], \ s \geq \alpha$. We show that $\phi \in \mathcal{BF}_s^*$. Indeed, observing that

$$\lambda \mapsto \log s + t\phi(\lambda) - \log\big(s + t\lambda\phi'(\lambda)\big) = \frac{t}{s}\Omega_s(\phi) + \varphi\big((t\,\lambda\,\phi'(\lambda)/s)^2\big) \in \mathcal{BF}, \ \ \forall t > 0,$$

we conclude that

$$\lambda \mapsto e^{-\left(t\phi(\lambda) - \log(s + t\lambda\phi'(\lambda))\right)} = e^{-t\phi(\lambda)}\,\big(s + t\lambda\phi'(\lambda)\big) \in \mathcal{CM}, \ \ \forall t > 0.$$

As a consequence of Proposition 4.1, we connect the class $\mathcal{BF}_s^*$ to generalized unimodality, a relation that reads on the level of the subordinators.

Corollary 5.2 *Let $s > 0$ and let ϕ be a Bernstein function, such that $\phi_\diamond$ is not identically equal to 0. Let $(X_t^{(\diamond)})_{t\geq 0}$ be the subordinator corresponding to $\phi_\diamond$. Then, the following conditions are equivalent.*

(1) $\phi \in \mathcal{BF}_s^*$;
(2) $\psi(\phi_\diamond) \in \mathcal{BF}_s$, *for all* $\psi \in \mathcal{BF}$;
(3) *for every $t > 0$, the random variable $Z := X_t^{(\diamond)}$ satisfies any of the equivalent conditions of Proposition* 4.1.

Proof Take $\varphi = t\phi_\diamond$ in Proposition 4.1, for $t > 0$, and deduce the equivalence between parts (1) and (3). To see the equivalence of parts (1) and (2), we use the representation

$$\psi(\phi_\diamond(\lambda)) = \mathrm{q}_\psi + \mathrm{d}_\psi\phi_\diamond(\lambda) + \int_{(0,\infty)} \varepsilon_t(\phi_\diamond)(\lambda)\ \mu_\psi(dt).$$

The facts that $\mathcal{BF}_s$ is a closed convex cone and that the integral is a limit of a sequence of Bernstein functions, ensure that we can use the linearity of the operator Ω_s to exchange it with the integral, see [15, Corollary 3.9]. □

Remark 5.3 Let ϕ be Bernstein function associated with the subordinator $(X_t)_{t\geq 0}$.

(i) With the observation

$$\lim_{t\to 0^+} \frac{\Omega_s(\varepsilon_t(\phi))}{t} = \Omega_s(\phi) + (s-1)\,\mathrm{d}\lambda,$$

we deduce that, if $s \leq 1$ and $\Omega_s\big(\varepsilon_t(\phi)\big) \in \mathcal{BF}$, for all $t > 0$, then $\phi \in \mathcal{BF}_s$.

(ii) One can notice that

$$\begin{aligned} \text{for } s > 0 :&\Omega_s\big(\psi(\phi)\big) \in \mathcal{BF}, \text{ for all } \psi \in \mathcal{BF} \Longrightarrow \Omega_s\big(\varepsilon_t(\phi)\big) \in \mathcal{BF}, \text{ for all } \ t > 0; \\ \text{for } s \in (0,1] :&\Omega_s\big(\psi(\phi)\big) \in \mathcal{BF}, \text{ for all } \psi \in \mathcal{BF} \Longleftarrow \Omega_s\big(\varepsilon_t(\phi)\big) \in \mathcal{BF}, \text{ for all } t > 0. \end{aligned}$$

Indeed, for the first implication take $\psi(\lambda) = 1 - e^{-t\lambda}$, while the second uses the same argument as in the proof of Corollary 5.2 and part (i) of this remark.

(iii) The *harmonic* and *potential harmonic* measures are given, in the vague sense, by

$$U(dx) = \int_0^\infty \mathbb{P}(X_t \in dx)\, dt \text{ and } H(dx) = \int_0^\infty \mathbb{P}(X_t \in dx)\, \frac{dt}{t},$$

respectively. They are linked to ϕ via

$$\frac{1}{\phi(\lambda)} = \int_{[0,\infty)} e^{-\lambda x}\, U(dx) \text{ and } \frac{\phi'(\lambda)}{\phi(\lambda)} = \int_{[0,\infty)} e^{-\lambda x}\, x\, H(dx),\ \ \lambda > 0. \tag{68}$$

Thus, due to the shape of the distribution of $\mathbb{P}(X_t^{(\diamond)} \in dx)$ provided by Corollary 5.2 in conjunction with (49), it is immediate that if $\phi \in \mathcal{BF}_s^*$, then the harmonic and potential harmonic measure, associated with $\phi_\diamond$, have form (31), i.e.,

$$U_\diamond(dx) = \frac{u(x)}{x^{s+1}}dx \text{ and } H_\diamond(dx) = \frac{h(x)}{x^{s+1}}dx,$$

where both u and h are non-decreasing functions. Writing,

$$\begin{aligned} \psi_\diamond(\lambda) &:= \frac{s}{\phi_\diamond(\lambda)} + \lambda\frac{\phi_\diamond'(\lambda)}{\phi_\diamond^2(\lambda)} = \frac{s}{\phi_\diamond(\lambda)} - \lambda\left(\frac{1}{\phi_\diamond(\lambda)}\right)' \\ &= s\int_0^\infty e^{-\lambda x}\frac{u(x)}{x^{s+1}}\, dx - \lambda\int_0^\infty e^{-\lambda x}\frac{u(x)}{x^s}\, dx \end{aligned}$$

$$= \lambda \int_0^\infty e^{-\lambda x} v(x)\,dx = \int_{(0,\infty)} e^{-\lambda x}\,dv(x),\ \ \lambda > 0,$$

where v is the non-decreasing function

$$v(x) := s\int_0^x \frac{u(y)}{y^{s+1}}\,dy - \frac{u(x)}{x^s},$$

we retrieve that $\psi_\diamond$ is a completely monotone function. Further, by an evident change of variable, we may use second representation (68) for $\phi_\diamond$ and affirm that

$$\lambda \mapsto \frac{\phi'_\diamond(\lambda)}{\lambda^{s-1}\,\phi_\diamond(\lambda)} = \int_{(0,\infty)} e^{-x}\,\frac{h(x/\lambda)}{x^s}dx \text{ is non-increasing.}$$

6 A Solution to a Conjecture by Sendov, Shan and its Relationship to a Result by Simon

In this section, we give an answer to a conjecture stated by Sendov and Shan [14, Open Problem 4.1], that has a strong connection with stable laws. The authors proved that if $\lambda^{\alpha_0} \in \mathcal{BF}_1^*$, then $\lambda^\alpha \in \mathcal{BF}_1^*$, for all $\alpha \le \alpha_0$. Thus, it is natural to ask

$$\text{what is the largest value of } \alpha_0 \in (0,1),\ \text{ such that } \lambda^\alpha \in \mathcal{BF}_1^*,\ \text{ for all} \alpha \le \alpha_0? \tag{69}$$

It is shown in [14] that (69) holds for $\alpha_0 \le 2/3$ and was conjectured that $\alpha_0 = 1/\sqrt{2} = 0.70710678118$. In this section we will find the exact optimal value, which turns out to be larger. We do this by resolving the more general problem of

$$\text{finding all pairs } (s,\alpha) \in (0,\infty) \times (0,1),\ \text{ such that } \lambda^\alpha \in \mathcal{BF}_s^*? \tag{70}$$

According to (65), this is equivalent to

$$\begin{aligned} &\text{finding all pairs } (s,\alpha) \in (0,\infty) \times (0,1),\\ &\text{such that } s - e^{-t\lambda^\alpha}(s + t\alpha\lambda^\alpha) \in \mathcal{BF},\ \text{ for all } t > 0. \end{aligned}$$

Since a function $\phi(\lambda)$ is a Bernstein function, if and only if $\phi(t\lambda) \in \mathcal{BF}$ for all $t > 0$, the problem simplifies to the equivalent one of

$$\text{finding all pairs } (s,\alpha) \in (0,\infty) \times (0,1),\ \text{ such that } s - e^{-\lambda^\alpha}(s + \alpha\lambda^\alpha) \in \mathcal{BF}. \tag{71}$$

By (67), the problem is also equivalent to

$$\text{finding all pairs } (s,\alpha) \in (0,\infty) \times (0,1),\ \text{ such that } e^{-\lambda^\alpha}(s + \alpha\lambda^\alpha) \in \mathcal{CM}. \tag{72}$$

This problem has been previously raised by Simon [16]. He showed that there exists an increasing function $R : [0, 1] \to [0, \infty]$, such that

$$e^{-\lambda^\alpha}(s + \alpha\lambda^\alpha) \in \mathcal{CM} \iff \alpha \le \frac{1}{2} \text{ or } s \ge R(\alpha), \tag{73}$$

where the function R satisfies

$$R(\alpha) = \alpha \text{ if } \alpha \in [0, 1/2] \text{ and } \frac{1}{4(1-\alpha)} \le R(\alpha) \le \frac{\alpha}{\sin^2(\pi\alpha)} \text{ if } \alpha \in [1/2, 1]. \tag{74}$$

In particular, taking $s = 1$, it is clear that (73), hence (69), holds true whenever $\alpha \in [0, 1/2]$ or whenever $\alpha \in [1/2, 1]$ and $1 \ge \alpha/\sin^2(\pi\alpha)$. Due to the inclusion $\mathcal{BF}_s^* \subset \mathcal{BF}_s$ and due to (28), we see that a pair (s, α) does not satisfy (70) if $s < \alpha$.

The importance of problems (69) and (70) is highlighted by the fact that $\phi(\lambda) = \lambda^\alpha$, for $0 < \alpha < 1$, is the Thorin Bernstein function associated with the stable subordinator $(S_t^{(\alpha)})_{t\ge0}$ through the following representations:

$$\lambda^\alpha = \int_0^\infty (1 - e^{-\lambda x}) \frac{c_\alpha}{x^{\alpha+1}}\, dx, \ \ c_\alpha := \frac{\alpha}{\Gamma(1-\alpha)} = \Gamma(\alpha+1)\frac{\sin(\pi\alpha)}{\pi}, \tag{75}$$

$$e^{-t\lambda^\alpha} = \mathbb{E}\big[e^{-\lambda S_t^{(\alpha)}}\big] = \int_0^\infty e^{-\lambda s} f_{t,\alpha}(s)\, ds, \ \ t > 0, \tag{76}$$

where $f_{t,\alpha}$ is the probability density function of the positive stable random variable $S_t^{(\alpha)}$. The so-called *scaling property* for stable processes could be noticed from (76) and gives

$$S_t^{(\alpha)} \stackrel{d}{=} t^{1/\alpha}\mathbb{S}_\alpha, \ \text{where we denote from now on,} \quad \mathbb{S}_\alpha := S_1^{(\alpha)}.$$

Notice that the probability density function $f_\alpha := f_{1,\alpha}$ of $\mathbb{S}_\alpha$ is explicit only for the value $\alpha = 1/2$ and

$$f_{1/2}(x) \ = \ \frac{e^{-1/4x}}{2\sqrt{\pi}x^{3/2}}, \ \ x > 0,$$

corresponds to the inverse-Gaussian distribution, whereas in general, it is only evaluated by the series expansion given by [18, formula (2.4.8), p. 90]:

$$f_\alpha(x) = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\frac{\Gamma(n\alpha+1)}{\Gamma(n+1)} \sin(\pi n\alpha)\, x^{-(n\alpha+1)}. \tag{77}$$

The Mellin transform of $\mathbb{S}_\alpha$ is given by

$$\mathbb{E}[\mathbb{S}_\alpha^{-\lambda}] = \frac{\Gamma(1+\frac{\lambda}{\alpha})}{\Gamma(1+\lambda)}$$

$$= \exp\left\{\frac{\beta}{\alpha}\psi(1)\lambda + \int_0^\infty (e^{-\lambda x} - 1 + \lambda x)\,\frac{e^{-\alpha x}(1-e^{-\beta x})}{x(1-e^{-x})(1-e^{-\alpha x})}dx\right\}, \tag{78}$$

where the Digamma function is defined by $\psi(u) = \Gamma'(u)/\Gamma(u),\ u > 0$. The stable random variable $\mathbb{S}_\alpha$ is also linked to the Mittag-Leffler function via

$$E_\alpha(x) := \sum_{k\ge 0} \frac{x^k}{\Gamma(k\alpha+1)} = \mathbb{E}\big[e^{x/(\mathbb{S}_\alpha)^\alpha}\big],\ x \in \mathbb{C}. \tag{79}$$

An important and known fact is that when we take two independent and identically distributed random variables $\mathbb{S}'_\alpha$, $\mathbb{S}_\alpha$, then $\mathbb{T}_\alpha := \mathbb{S}'_\alpha/\mathbb{S}_\alpha$ has the explicit probability density function, see [5, (4.23.3)]:

$$\begin{aligned} f_{\mathbb{T}_\alpha}(x) &= \frac{\sin(\pi\alpha)}{\pi}\frac{x^{\alpha-1}}{x^{2\alpha} + 2\cos(\pi\alpha)x^\alpha + 1} \qquad (80)\\ &= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\frac{x^{\alpha-1}}{\big(x^\alpha + \cos(\pi\alpha)\big)^2 + \sin^2(\pi\alpha)},\quad x > 0. \end{aligned}$$

Hence, using (76), one has the representation

$$\begin{aligned} E_\alpha(-\lambda^\alpha) &= \mathbb{E}\left[e^{-\lambda^\alpha/(\mathbb{S}_\alpha)^\alpha}\right] = \mathbb{E}\left[\mathbb{E}\big[e^{-\lambda\mathbb{S}'_\alpha/\mathbb{S}_\alpha}|\mathbb{S}_\alpha\big]\right] = \mathbb{E}\left[e^{-\lambda\mathbb{S}'_\alpha/\mathbb{S}_\alpha}\right] \qquad (81)\\ &= \int_0^\infty e^{-\lambda x}\, f_{\mathbb{T}_\alpha}(x)dx,\quad \lambda \ge 0. \end{aligned}$$

In Sect. 7 below we will show that there is much more consistent links between stable distributions and the Mittag-Leffler functions. If one takes $\mathbb{G}$ exponentially distributed, with scale parameter 1, and independent from $\mathbb{S}_\alpha$, then we have the well-known, see [5, (4.21.2)], identity

$$\mathbb{G} \overset{d}{=} \left(\frac{\mathbb{G}}{\mathbb{S}_\alpha}\right)^\alpha. \tag{82}$$

In order to state the main result of this section, we need the following technical lemma.

Lemma 6.1 *For $\alpha \in (0,1),\ \beta = 1-\alpha$, $s \ge 0$ and $x > 0$, let*

$$a_s(x) := \frac{e^x - 1}{1 + e^{-sx}(e^x - 1)} \ \textit{and}\ A(s) := \inf_{x>0}\frac{\log(1 + a_s(x))}{x}, \tag{83}$$

$$r_\alpha(x) := \log\frac{(1-e^{-\alpha x})(1-e^{-x})}{(1-e^{-\beta x})} \ \textit{and}\ R(\alpha) := \alpha + \max_{x>0}\frac{r_\alpha(x)}{x}, \tag{84}$$

$$e_{\alpha,s}(x) := \frac{e^{-\alpha x}(1-e^{-\beta x})}{(1-e^{-x})(1-e^{-\alpha x})} - e^{-sx}. \tag{85}$$

Then,

(1) *Function A is an increasing homeomorphism from* $[0, \infty]$ *onto* $[0, 1]$*, with inverse* R*. Together with* $e_{\alpha,s}$*, they satisfy*

$$\alpha \leq A(s) \iff s \geq R(\alpha) \iff e_{\alpha,s} \geq 0;$$

(2) *We have* $A(s) \leq \min(s, 1)$ *and* $A(s) = s$ *if* $s \leq 1/2$.

Proof (1) The function A is non-decreasing since $s \mapsto a_s$ is such. Since $a_0(x) = 1 - e^{-x}$ is increasing and $a_0(x)/x$ is decreasing, then the product function

$$x \mapsto \frac{\log(1 + a_0(x))}{x} = \frac{\log(1 + a_0(x))}{a_0(x)} \frac{a_0(x)}{x}$$

decreases to its limit at infinity which is $0 = A(0)$. Clearly, $A(\infty) = 1$ because $a_\infty(x) = e^x - 1$ and then $\log(1 + a_\infty(x))/x = 1$. We deduce that $A(s) \in [0, 1]$, for all $s \geq 0$.

The continuity of A is justified as follows: suppose that the function A is not continuous at s, where s is a fixed finite non-negative number. We have that either $A(s-) < A(s)$ or $A(s+) > A(s)$.

(a) Assume $A(s-) < A(s)$ and choose any $\alpha \in \big(A(s-), A(s)\big)$. Since $\alpha < A(s)$, we have $e^{\alpha x} < 1 + a_s(x)$ for all $x > 0$. Substituting the expression for $a_s(x)$ and solving the inequality for e^{-sx} we arrive at the equivalent $e_{\alpha,s}(x) > 0$ for all $x > 0$. In fact, these arguments show that $\alpha \leq A(s)$ if, and only if $e_{\alpha,s}(x) \geq 0$ for all $x > 0$. Next, since $\alpha > A(s - 1/n)$, for every $n = 1, 2, \ldots$, one may build a sequence $x_n > 0$, such that $e_{\alpha,s-1/n}(x_n) < 0$. Putting the two together, we have

$$e_{\alpha,s}(x_n) \geq 0 \text{ and } e_{\alpha,s-1/n}(x_n) < 0.$$

By continuity of the function $s \mapsto e_{\alpha,s}(x)$ for all $x > 0$, we deduce that there exists a sequence $s_n \in (s - 1/n, s]$ such that $e_{\alpha,s_n}(x_n) = 0$. It is clear that $\lim_{n\to\infty} s_n = s$. We are going to show now that as $n \to \infty$, x_n converges to ∞. If that is not the case, then, taking a subsequence, if necessary, we may assume that x_n converges to a finite limit x_* which satisfies

$$e_{\alpha,s}(x_*) = \frac{1}{e^{\alpha x_*} - 1} - \frac{1}{e^{x_*} - 1} - \frac{1}{e^{s x_*}} = 0 \Longrightarrow e^{s x_*} = \frac{(e^{x_*} - 1)(e^{\alpha x_*} - 1)}{e^{x_*} - e^{\alpha x_*}}.$$

Substituting $e^{s x_*}$ into $a_s(x_*)$, we obtain

$$\log(1 + a_s(x_*)) = \log(e^{\alpha x_*}) = \alpha x_* \implies A(s) \leq \alpha.$$

This leads to the contradiction $\alpha < A(s) \leq \alpha$, confirming that $\lim_{n\to\infty} x_n = \infty$. Finally, let $y_n := e^{-x_n}$ and from $e_{\alpha,s_n}(x_n) = 0$, we obtain

$$\frac{1}{y_n^{-\alpha} - 1} - \frac{1}{y_n{}^{-1} - 1} - y_n^{s_n}$$

$$= 0 \implies s_n = \alpha + \frac{1}{\log(y_n)} \log\left(\frac{1 - y_n^{1-\alpha}}{(1 - y_n)(1 - y_n^{\alpha})}\right).$$

Using $\lim_{n\to\infty} y_n = 0$ and $\alpha \le 1$ (recall that $\alpha < A(s)$), one can see that the right-hand side approaches α, while the left-hand side approaches s, hence $s = \alpha$. But s is fixed, while we can choose α freely in the interval $(A(s-), A(s))$, a contradiction.

(b) Assume $A(s) < A(s+)$ and choose any $\alpha \in (A(s), A(s+))$. Then for any $\epsilon > 0$, $\alpha < A(s + \epsilon)$ implies that $e_{\alpha,s+\epsilon}(x) \ge 0$ for all $x > 0$. In other words, for any fixed $x > 0$, we have $e_{\alpha,s+\epsilon}(x) \ge 0$ for all $\epsilon > 0$. Taking the limit as ϵ goes to 0 gives $e_{\alpha,s}(x) \ge 0$ for all $x > 0$. In other words, $\alpha \le A(s)$, a contradiction.

By representation (85) of $e_{\alpha,s}$, it is also readily seen that

$$e_{\alpha,s}(x) \ge 0, \ \forall x > 0 \iff s \ge R(\alpha) \text{ and } e_{\alpha,s}(x) > 0, \ \forall x > 0 \iff s > R(\alpha).$$

Thus, we have

$$\alpha \le A(s) \iff s \ge R(\alpha) \text{ and } \alpha < A(s) \iff s > R(\alpha).$$

Hence, R is the inverse of A. The continuity of R and the strict monotonicity of A follow.

(2) If $\alpha \le 1/2$, then $r_\alpha \le 0$, implying that $R(\alpha) = \alpha$, which yields $A(s) = s$. If $\alpha > 1/2$, then r_α has sign changes, implying that $R(\alpha) \ge \alpha$, which yields $A(s) \le s$. □

We now straightforwardly retrieve the following preliminary observation.

Proposition 6.2 *The function λ^α belongs to $\mathcal{BF}_\alpha^*$, if and only if, $\alpha \le 1/2$. In this case, $\lambda^\alpha \in \mathcal{BF}_s^*$ for all $s \ge \alpha$.*

Proof By the equivalence between (70) and (71), we know that $\lambda^\alpha \in \mathcal{BF}_\alpha^*$, if and only if, $1 - e^{-\lambda^\alpha}(1 + \lambda^\alpha) \in \mathcal{BF}$. Since $e^{-\lambda^\alpha}(1 + \lambda^\alpha) \le 1$, the latter is equivalent to $e^{-\lambda^\alpha}(1 + \lambda^\alpha) \in \mathcal{CM}$. By (73), the latter holds, if and only if, $\alpha \le 1/2$ or $\alpha \ge R(\alpha)$. Using (74), we see that if $\alpha > 1/2$ then $\alpha < 1/(4(1 - \alpha)) \le R(\alpha)$. Thus, $e^{-\lambda^\alpha}(1 + \lambda^\alpha) \in \mathcal{CM}$, if and only if, $\alpha \le 1/2$. The last claim is obtained by part 5 of Theorem 2.1. □

We can now improve Proposition 6.2 by solving Sendov and Shans's problem [14] stated in (69) and expliciting the homeomorphism R in Simon's result [16] described in (73).

Theorem 6.3 *Let $\mathbb{U}$ denote a random variable with uniform distribution on $(0, 1)$ and let A, R be given by (83) and (84), respectively. Then, following statements are equivalent.*

(1) *There exists a real-valued random variable W_s, independent of $\mathbb{U}$, such that the following factorization in law holds:*

$$\frac{1}{\mathbb{S}_\alpha} \stackrel{d}{=} \mathbb{U}^{1/s} e^{W_s}; \tag{86}$$

(2) $\lambda^\alpha \in \mathcal{BF}_s^*$;
(3) $\alpha \le A(s)$ *or equivalently* $s \ge R(\alpha)$;
(4) *The probability density function of the random variable* $1/\mathbb{S}_\alpha^s$,

$$f_{1/\mathbb{S}_\alpha^s}(x) := \frac{1}{s\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n\alpha+1)}{\Gamma(n+1)} \sin(\pi n \alpha)\, x^{\frac{\alpha n}{s}-1},\ x > 0,$$

is non-increasing.

In particular, the random variable W_s *is infinitely divisible. Finally,* $\lambda^\alpha \in \mathcal{BF}_1^*$ *if, and only if,* $\alpha \le A(1) \approx 0.717461058844$.

Proof (1) $\iff$ (2): In (71), we have already noticed that

$$\lambda^\alpha \in \mathcal{BF}_s^* \iff \Omega_s(x \mapsto 1 - e^{-x^\alpha}) \in \mathcal{BF}, \tag{87}$$

and by (76), λ^α is the cumulant function of the positive random variable $\mathbb{S}_\alpha$. Thus, by the equivalence between (1) and (4) in Proposition 4.1, there is a positive random variable V_s, independent of $\mathbb{U}$, such that $\mathbb{S}_\alpha \stackrel{d}{=} V_s/\mathbb{U}^{1/s}$. Thus, $1/\mathbb{S}_\alpha \stackrel{d}{=} \mathbb{U}^{1/s}/V_s$ and the result follows by letting $W_s := -\log(V_s)$.

(1) $\iff$ (3): Using Mellin transform (78) and the Frullani representation of the logarithm,

$$\log\left(1+\frac{\lambda}{s}\right) = \int_0^\infty (1-e^{-\lambda x})\frac{e^{-sx}}{x}dx,\ \lambda \ge 0,\ s > 0,$$

we get $\mathbb{E}[\mathbb{U}^{\lambda/s}] = \left(1+\frac{\lambda}{s}\right)^{-1}$ and then,

$$\begin{aligned}\frac{\mathbb{E}[\mathbb{S}_\alpha^{-\lambda}]}{\mathbb{E}[\mathbb{U}^{\lambda/s}]} &= \exp\left\{\int_0^\infty (1-e^{-\lambda x})\frac{e^{-sx}}{x}dx + \frac{\beta}{\alpha}\psi(1)\lambda \right.\\ &\quad \left. + \int_0^\infty (e^{-\lambda x}-1+\lambda x)\,\frac{e^{-\alpha x}(1-e^{-\beta x})}{x(1-e^{-x})(1-e^{-\alpha x})}dx\right\}\\ &= \exp\left\{\gamma_{\alpha,s}\,\lambda + \int_0^\infty (e^{-\lambda x}-1+\lambda x)\,\frac{e_{\alpha,s}(x)}{x}dx\right\},\ \lambda \ge 0,\end{aligned} \tag{88}$$

where $\gamma_{\alpha,s} = (\frac{\beta}{\alpha}\psi(1) + \frac{1}{s})$ and $e_{\alpha,s}$ is given by (85). Observe that the expression in the last exponent is of form (99), given in Appendix 1. Suppose that $e_{\alpha,s}$ takes strictly negative values. Decompose $e_{\alpha,s}$, in a standard way, into the difference of two non-negative functions $e_{\alpha,s} = e_+ - e_-$. Then, the measures $\mu_+(dx) := e_+(x)\,dx/x$ and $\mu_-(dx) := e_-(x)\,dx/x$ are Lévy measures, that is, they satisfy integrability condition (7), because they are both dominated by the Lévy measure

$$\left(\frac{e^{-\alpha x}(1-e^{-\beta x})}{(1-e^{-x})(1-e^{-\alpha x})} + e^{-sx}\right)\frac{dx}{x}.$$

Moreover, for the same reason the measures $\mu_+(dx)$ and $\mu_-(dx)$ satisfy integrability conditions (100). Thus, by the discussion at the end of Appendix 1 and (102), there exist two independent random variables X^+ and X^-, with infinitely divisible distributions of the spectrally negative type, associated with the Lévy measures are μ_+ and μ_-, and such that we have

$$\frac{\mathbb{E}[\mathbb{S}_\alpha^{-\lambda}]}{\mathbb{E}[\mathbb{U}^{\lambda/s}]} = \mathbb{E}[e^{\lambda(X^+ - X^-)}], \ \lambda \geq 0.$$

Choosing X^+ and X^- to be independent of $\mathbb{S}_\alpha$ and $\mathbb{U}$, the last identity becomes

$$\mathbb{E}\left[\left(\frac{e^{X^-}}{\mathbb{S}_\alpha}\right)^\lambda\right] = \mathbb{E}[(\mathbb{U}^{1/s} e^{X^+})^\lambda], \ \lambda \geq 0,$$

giving an identity in law of the form

$$\frac{e^{X^-}}{\mathbb{S}_\alpha} \stackrel{d}{=} \mathbb{U}^{1/s} \, e^{X^+}.$$

Last identity can be simplified to form (86), if, and only if, $e_{\alpha,s}$ is non-negative. By Lemma 6.1, the non-negativity condition is equivalent to (3). Finally, thanks to representations (99) and (101), we see that the exponent in the right-hand side of (88) corresponds to the (bilateral) Laplace transform of a random variable W_s, whose distribution is necessarily infinitely divisible of a spectrally negative Lévy type. The latter is equivalent to the factorization in law (86).

(2) $\Longleftrightarrow$ (4): Since λ^α is the cumulant function of $\mathbb{S}_\alpha$, then (87) and the equivalence between parts 1 and 2 in Proposition 4.1, show that $\lambda^\alpha \in \mathcal{BF}_s^*$ if, and only if, the probability density function, $f_\alpha(x)$, of $\mathbb{S}_\alpha$ has the form $f_\alpha(x) = p_s(x)/x^{s+1}$ for some non-decreasing and right-continuous $p_s(x)$. That is, $\lambda^\alpha \in \mathcal{BF}_s^*$ is equivalent to

$$x^{s+1} f_\alpha(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(n\alpha+1)}{\Gamma(n+1)} \sin(\pi n \alpha) \, x^{s-n\alpha} \text{ is non-decreasing.}$$

It is simple to verify that $f_{1/\mathbb{S}_\alpha^s}(x) = s^{-1} x^{-1-1/s} f_\alpha(x^{-1/s}) = s^{-1} p_s(x^{-1/s})$, showing that $\lambda^\alpha \in \mathcal{BF}_s^*$ is equivalent to $f_{1/\mathbb{S}_\alpha^s}(x)$ being a non-increasing function.

Infinite divisibility of W_s is shown at the end of the proof of 1) $\Longleftrightarrow$ 3). The last assertion is evident and the evaluation of $A(1)$, using formula (83), was done by Maple. □

Remark 6.4 By (77), for every $\alpha < 1$, we have

$$f_\alpha(x) \sim \frac{c_\alpha}{x^{\alpha+1}}, \text{ as } x \to \infty,$$

where c_α is defined by (75). Since $\lim_{x\to 0+} x^{\alpha+1} f_\alpha(x) = 0$ (see [9], for instance), the function $x^{\alpha+1} f_\alpha(x)$ has a continuous extension to $[0, \infty)$. For $\alpha \leq 1/2$, we know that

$\lambda^\alpha \in \mathcal{BF}^*_\alpha$. The last inclusion is equivalent to $x^{\alpha+1} f_\alpha(x)/c_\alpha$ being non-decreasing. Since it is positive and converging to one as $x \to \infty$, we discover the remarkable fact that for $\alpha \le 1/2$, $x^{\alpha+1}\ f_\alpha(x)/c_\alpha$ is a cumulative distribution function. By (87), (57) and (76), we deduce that

$$\frac{x^{\alpha+1}}{c_\alpha} f_\alpha(x) = \Gamma(1-\alpha)\, \mathbb{E}\left[V_\alpha^\alpha\ \mathbb{1}_{(V_\alpha \le x)}\right],\ \ x \ge 0, \tag{89}$$

where comparing (51) and (86) we see that $V_\alpha = e^{-W_\alpha}$ for an infinitely divisible random variable W_s. In particular, letting x approach infinity in (89), we see that $\mathbb{E}\left[V_\alpha^\alpha\right] = 1/\Gamma(1-\alpha)$.

7 The Mittag-Leffler Function and the Class $\mathcal{BF}^*_s$

Proposition 6.2 and Theorem 6.3 illustrate the extent to which the case $\alpha > 1/2$ is more intricate than the case $\alpha \le 1/2$. The explicitness of the probability density function $f_{\mathbb{T}_\alpha}$, given in (80), will be helpful for problem (69). Recall (87), that $\lambda^\alpha \in \mathcal{BF}^*_s$ precisely when $1 - e^{-\lambda^\alpha} \in \mathcal{BF}_s$. Since, λ^α is the cumulant function of the positive random variable $\mathbb{S}_\alpha$, we can use the equivalence between parts (1) and (5) in Proposition 4.1. More precisely, applying part (5) in Proposition 4.1 with $Y = \mathbb{S}'_\alpha$, that is independent from, but identically distributed with $\mathbb{S}_\alpha$, we conclude that the probability density function $f_{\mathbb{T}^s_\alpha}$ is non-increasing. Thus, after an elementary calculation using (80), we see that

$$f_{\mathbb{T}^s_\alpha}(x^{s/\alpha}) = \frac{\sin(\pi\alpha)}{s\ \pi} \frac{x^{(\alpha-s)/\alpha}}{x^2 + 2\cos(\pi\alpha)x + 1} \text{ is non-increasing in } x \in (0,\infty).$$

The derivative of $-f_{\mathbb{T}^s_\alpha}(x^{s/\alpha})$ has the same sign as

$$(s+\alpha)x^2 + 2s\cos(\pi\alpha)x + s - \alpha.$$

We deduce that

$$\lambda^\alpha \in \mathcal{BF}^*_s \Longrightarrow (s+\alpha)x^2 + 2s\cos(\pi\alpha)x + s - \alpha \ge 0,\ \forall x > 0 \Longleftrightarrow \alpha \le s\ \sin(\pi\alpha),$$

and in particular

$$\lambda^\alpha \in \mathcal{BF}^*_1 \Longrightarrow \alpha < \ \sin(\pi\alpha) \Longleftrightarrow \alpha \le \alpha_1 := 0.736484448242.$$

The inequality $A(1) < \alpha_1$ comforts the last assertion of Theorem 6.3. As a consequence of Theorem 6.3, we can state the following result which improves Lemma 2.3 in [16].

Corollary 7.1 *For $r > 0$ and $\alpha \in (0, 1)$, let*

$$\eta_{\alpha,r}(x) := 1 - r\,\Gamma(1-\alpha)\; x\; E_\alpha(-x) \text{ and } \phi_{\alpha,r}(\lambda) := \lambda^\alpha - \log(1 + r\lambda^\alpha),\;\; x,\; \lambda \geq 0. \quad (90)$$

Recall that E_α stands for the Mittag-Leffler function given in (79).

(1) *The following assertions are equivalent.*

- (i) $\alpha \leq 1/2$;
- (ii) $\lambda^\alpha \in \mathcal{BF}^*_\alpha$;
- (iii) $x\; E_\alpha(-x) \in \mathcal{BF}$;
- (iv) $\eta_{\alpha,1} \geq 0$;
- (v) $\eta_{\alpha,r} \in \mathcal{CM}$, *for all* $r \leq 1$;
- (vi) $\phi_{\alpha,r} \in \mathcal{TBF}$, *for all* $r \leq 1$.

(2) *More generally, below we have $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Rightarrow (iv)$.*

- (i) $r \leq \sin^2(\pi\alpha)$;
- (ii) $\eta_{\alpha,r} \geq 0$;
- (iii) $\phi_{\alpha,r} \in \mathcal{BF}$;
- (iv) $\lambda^\alpha \in \mathcal{BF}^*_{\alpha/r}$.

Remark 7.2 Let $\alpha_2 \approx 0.688483504697$ be the zero of the function $x \mapsto \sin^2(\pi x) - x$ in $(1/2, 1)$. For $\alpha \in (1/2, \alpha_2)$, we have $\sin^2(\pi\alpha) \geq \alpha$. So, taking $r = \sin^2(\pi\alpha)$ and applying of point (2) of last corollary, we obtain $\lambda^\alpha \in \mathcal{BF}^*_{\alpha/r} \subset \mathcal{BF}^*_1$. Of course, this is not optimal compared to what we obtained in Theorem 6.3.

Proof Recall $\beta = 1 - \alpha$. With the help of the asymptotic [6, (3.4.15)], observe first that

$$\eta_{\alpha,r} = 1 - r + r\;\eta_{\alpha,1} \text{ and } \lim_{x\to\infty} \eta_{\alpha,1}(x) = \lim_{x\to\infty} \big(1 - \Gamma(\beta)\, x\, E_\alpha(-x)\big) = 0. \quad (91)$$

Thus, the inequality $\eta_{\alpha,r} \geq 0$ fails for $r > 1$. Second, since

$$\phi'_{\alpha,r}(\lambda) = \frac{d}{d\lambda}\big(\lambda^\alpha - \log(1 + r\;\lambda^\alpha)\big) = \alpha\; c_{\alpha,r}(\lambda), \quad (92)$$

where

$$c_{\alpha,r}(\lambda) := \frac{1}{\lambda^\beta} - \frac{r\;\lambda^{\alpha-1}}{1 + r\;\lambda^\alpha},$$

and thanks to (75), we obtain the representation

$$c_{\alpha,r}(\lambda) = \frac{1}{\lambda^\beta} - \frac{1}{\lambda}\left(1 - \frac{1}{1 + r\;\lambda^\alpha}\right) = \frac{1}{\Gamma(\beta)} \int_0^\infty \frac{e^{-\lambda x}}{x^\alpha} dx - \frac{1}{\lambda}\Big(1 - \mathbb{E}\big[e^{-\lambda^\alpha r\,\mathbb{G}}\big]\Big).$$

Then, taking $\mathbb{S}'_\alpha$ to be an independent copy of $\mathbb{S}_\alpha$, both independent from the exponentially distributed r.v. $\mathbb{G}$, and using (76) followed by the identity in law for positive stable distributions (82), we obtain

$$1 - \mathbb{E}\big[e^{-\lambda^\alpha r\,\mathbb{G}}\big] = 1 - \mathbb{E}\big[e^{-\lambda \mathbb{S}_\alpha (r\,\mathbb{G})^{1/\alpha}}\big] = 1 - \mathbb{E}\big[e^{-\lambda \mathbb{G}\, r^{1/\alpha}\,(\mathbb{S}_\alpha/\mathbb{S}'_\alpha)}\big]$$
$$= \lambda \int_0^\infty e^{-\lambda x}\, \mathbb{P}\left(\mathbb{G} > \frac{x\,\mathbb{S}'_\alpha}{r^{1/\alpha}\,\mathbb{S}_\alpha}\right) dx.$$

Replacing in $c_{\alpha,r}$, we arrive at the simplified expression

$$c_{\alpha,r}(\lambda) = \int_0^\infty e^{-\lambda x} \rho_{\alpha,r}(x) dx, \tag{93}$$

where by Mittag-Leffler function representation (79),

$$\rho_{\alpha,r}(x) := \frac{1}{\Gamma(\beta)x^\alpha} - \mathbb{P}\left(\mathbb{G} > \frac{x\,\mathbb{S}'_\alpha}{r^{1/\alpha}\,\mathbb{S}_\alpha}\right) = \frac{1}{\Gamma(\beta)x^\alpha} - \mathbb{E}\big[e^{-x\,\mathbb{S}'_\alpha/(r^{1/\alpha}\,\mathbb{S}_\alpha)}\big]$$
$$= \frac{1}{\Gamma(\beta)x^\alpha} - \mathbb{E}\big[e^{-x^\alpha/(r\,\mathbb{S}^\alpha_\alpha)}\big]$$
$$= \frac{1 - \Gamma(\beta)\,x^\alpha\,E_\alpha(-x^\alpha/r)}{\Gamma(\beta)x^\alpha} = \frac{\eta_{\alpha,r}(x^\alpha/r)}{\Gamma(\beta)x^\alpha}, \quad x > 0. \tag{94}$$

Finally, integrating (92) between 0 and λ, and using (93) and (94), we obtain the following representation for $\phi_{\alpha,r}$, valid for all $r > 0,\ \lambda \geq 0$;

$$\phi_{\alpha,r}(\lambda) = \alpha \int_0^\lambda c_{\alpha,r}(u)du = \int_0^\infty (1 - e^{-\lambda x})\frac{\rho_{\alpha,r}(x)}{x} dx$$
$$= \frac{\alpha}{\Gamma(\beta)} \int_0^\infty (1 - e^{-\lambda x})\frac{\eta_{\alpha,r}(x^\alpha/r)}{x^{\alpha+1}} dx. \tag{95}$$

At this point, thanks to the equivalence between (70) and (72), we have found that for fixed $r \in (0, 1]$,

$$\eta_{\alpha,r} \geq 0 \Longleftrightarrow \phi_{\alpha,r} \in \mathcal{BF} \Longrightarrow \frac{\alpha}{r} e^{-\phi_{\alpha,r}(\lambda)} = e^{-\lambda^\alpha}\left(\frac{\alpha}{r} + \alpha\lambda^\alpha\right) \in \mathcal{CM} \Longleftrightarrow \lambda^\alpha \in \mathcal{BF}^*_{\alpha/r}, \tag{96}$$

and, by (95), that

$$x \mapsto \eta_{\alpha,r}(x^\alpha)/x^\alpha \in \mathcal{CM} \Longleftrightarrow \phi_{\alpha,r} \text{ has form(103) of a Thorin Bernstein function.} \tag{97}$$

Further, due to (91), we have

$$\lambda E_\alpha(-\lambda) \in \mathcal{BF} \Longleftrightarrow \eta_{\alpha,1}(\lambda) = 1 - \Gamma(\beta)\,\lambda\,E_\alpha(-\lambda) \in \mathcal{CM}$$
$$\Longleftrightarrow \eta_{\alpha,r}(\lambda) = 1 - r + r\,\eta_{\alpha,1}(\lambda) \in \mathcal{CM}, \quad \text{for all } r \leq 1. \tag{98}$$

(1) By Proposition 6.2, part (7) of Proposition 4.1, (91), (96), (97) and (98), we obtain the equivalences between $(i), \ldots, (vi)$, via the following scheme

$$\begin{array}{ccl}
\alpha \leq \frac{1}{2} \Longleftrightarrow \lambda^{\alpha} \in \mathcal{BF}^{*}_{\alpha} \Longleftrightarrow \lambda^{\alpha} \in \mathcal{BF}^{*}_{\alpha/r}, \quad \forall r \leq 1 \Longleftarrow \eta_{\alpha,1} \geq 0 & & \\
\Updownarrow \qquad\qquad\qquad\qquad \Uparrow & & \\
\lambda \mathbb{E}[e^{-\lambda/\mathbb{S}_{\alpha}^{\alpha}}] = \lambda E_{\alpha}(-\lambda) \in \mathcal{BF} & \Longleftrightarrow & \eta_{\alpha,r} \in \mathcal{CM}, \quad \forall r \leq 1 \\
 & \Longrightarrow & \eta_{\alpha,r}(x^{\alpha})/x^{\alpha} \in \mathcal{CM}, \quad \forall r \leq 1 \\
 & \Longleftrightarrow & \phi_{\alpha,r} \in \mathcal{TBF}, \quad \forall r \leq 1 \\
 & \Longrightarrow & \lambda^{\alpha} \in \mathcal{BF}^{*}_{\alpha/r}, \quad \forall r \leq 1.
\end{array}$$

(2) The implications $(ii) \Leftrightarrow (iii) \Rightarrow (iv)$ are given in (96). For the implication $(i) \Rightarrow (ii)$, if $\alpha \leq 1/2$, then by Proposition 6.2, we have $\lambda^{\alpha} \in \mathcal{BF}^{*}_{s}$ and we are done by part (1). Thus, it is enough to check the claim for $1/2 < \alpha < 1$, where it holds that $\alpha \leq \alpha_2 \Longleftrightarrow \alpha \leq \sin^2(\pi\alpha)$. Using representation (80) and (81), then performing the change of variable $v = xu/r^{1/\alpha}$, we arrive at

$$\begin{aligned}
\eta_{\alpha,r}(x^{\alpha}/r) &= 1 - \Gamma(\beta)\, x^{\alpha} E_{\alpha}(-x^{\alpha}/r) = 1 - \Gamma(\beta)\, x^{\alpha} \int_0^{\infty} e^{-xu/r^{1/\alpha}} f_{\mathbb{T}_{\alpha}}(u) du \\
&= 1 - r \int_0^{\infty} g_{\alpha}(r^{1/\alpha}\, v/x)\, e^{-v} \frac{v^{\alpha-1}}{\Gamma(\alpha)} dv \\
&= 1 - r\mathbb{E}[g_{\alpha}(r^{1/\alpha}\, \mathbb{G}_{\alpha}/x)],
\end{aligned}$$

where

$$g_{\alpha}(w) := \frac{1}{\left(w^{\alpha} + \cos(\pi\alpha)\right)^2 + \sin^2(\pi\alpha)},$$

and $\mathbb{G}_{\alpha}$ has the Gamma distribution with shape parameter α and rate parameter 1. Observing that

$$g_{\alpha} \leq \frac{1}{\sin^2(\pi\alpha)} \Longrightarrow \eta_{\alpha,r}(x^{\alpha}/r) \geq 1 - \frac{r}{\sin^2(\pi\alpha)}, \text{ for all } x > 0,$$

completes the proof. □

Acknowledgements We are grateful to anonymous referees for their valuable comments and suggestions that improved the presentation and the results of this paper.

Funding The work of the second author was supported by the "Researchers Supporting Project Number (RSP2023R162), King Saud University, Riyadh, Saudi Arabia." The work of the third author was supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada, Project Number RGPIN-2020-06425.

Data Availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Appendix 1: Spectrally Negative Lévy Processes and Lévy–Laplace Exponents

A *Lévy–Laplace exponent* Ψ is a function represented by

$$\Psi(\lambda) = \mathrm{a} + \mathrm{b}\lambda + \mathrm{c}\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda h(x)\right) \mu(dx), \ \ \lambda \geq 0, \tag{99}$$

where $\mathrm{a}, \ \mathrm{c} \geq 0$, $\mathrm{b} \in \mathbb{R}$, and h is a truncation function, i.e., any bounded function such $\lim_{x \to 0^+} (h(x) - x)/x$ exists, and the *Lévy measure* ν satisfies

$$\int_{(0,\infty)} \left(x^2 \wedge 1\right) \nu(dx) < \infty, \ \text{or} \int_{(0,\infty)} \left(x^2 \wedge x\right) \nu(dx) < \infty, \tag{100}$$

if the integral in (99) is finite when we take $h(x) = x$. The Lévy–Laplace exponents have the following stochastic interpretation: there is a bijection between the class of (non-killed) of the *spectrally negative Lévy processes*, i.e., processes $Z = (Z_t)_{t \geq 0}, \ \ Z_0 = 0$, with stationary and independent increments and non-positive jumps satisfying $\mathbb{E}[Z_1] \geq 0$, and the class of Lévy–Laplace exponents Ψ, with $\mathrm{a} = \Psi(0) = 0$, via the Lévy–Khintchine formula:

$$\mathbb{E}[e^{\lambda Z_t}] = e^{t\Psi(\lambda)}, \ \text{for } t \geq 0, \ \ \lambda \geq 0, \tag{101}$$

where Ψ is represented by (99). In fact, the distributions of Z_t, $t > 0$, are entirely determined by the *infinitely divisible* random variable Z_1. In (99), it is customary to label, as in (6), the quantity b the *drift term* and c the *Brownian coefficient* and killing the process Z, amounts to adding the *killing rate* a in Ψ, cf. the beginning of [10, Sect. 8.1]. Observe that splitting μ into the sum of two Lévy measures $\mu = \mu' + \mu''$ amounts to split Ψ into the sum of two Lévy–Laplace exponent $\Psi = \Psi' + \Psi''$. Equivalently, the random variable $X := Z_1$ splits into the sum of two independent random variables with infinitely divisible distributions:

$$X = X' + X'', \tag{102}$$

where the Lévy measures associated with X' and X'' are $\mu', \ \mu''$, respectively.

Appendix 2: Complete Bernstein Functions ($\mathcal{CBF}$) and Thorin Bernstein Functions ($\mathcal{TBF}$)

The set $\mathcal{CBF}$ of complete Bernstein functions [15, p. 69] corresponds to those functions $\phi \in \mathcal{BF}$ represented by

$$\phi(\lambda) = \mathrm{q} + \mathrm{d}\,\lambda + \int_{(0,\infty)} \frac{\lambda}{\lambda + u}\,\Delta(du) = \mathrm{q} + \mathrm{d}\,\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\,\mathcal{L}_{\Delta}(x)\,dx,\ \lambda \geq 0,$$

for some constants $\mathrm{q},\ \mathrm{d} \geq 0$, where $\Delta(du)/u,\ u > 0$, is necessarily a Lévy measure, and $\mathcal{L}_{\Delta}$ stands for the Laplace transform of Δ. The class of Stieltjes functions consists of those of the form

$$f(\lambda) = \mathrm{d} + \frac{\mathrm{q}}{\lambda} + \int_{(0,\infty)} \frac{1}{\lambda + u}\,\Delta(du),\ \lambda > 0,$$

where $\mathrm{d}, \mathrm{q} \geq 0$ are constants and $\int_{(0,\infty)} (1+u)^{-1}\,\Delta(du) < \infty$. The set $\mathcal{TBF}$ is the subclass of $\mathcal{CBF}$ formed by those functions ϕ of the form

$$\phi(\lambda) = \mathrm{q} + \mathrm{d}\,\lambda + \int_{(0,\infty)} \log\left(1 + \frac{\lambda}{u}\right)\sigma(du) = \mathrm{q} + \mathrm{d}\,\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\frac{\mathcal{L}_{\sigma}(x)}{x}dx \tag{103}$$

$$= \mathrm{q} + \mathrm{d}\,\lambda + \int_{(0,\infty)} \frac{\lambda}{\lambda + u}\,\frac{\sigma(0,u]}{u}du, \tag{104}$$

for some measure σ on $(0, \infty)$, such that

$$\int_{(0,\infty)} \frac{\sigma(0,u]}{u(1+u)}\,du < \infty.$$

See [15, Theorem 8.2].

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