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A Large Deviation Principle for the Stochastic Heat Equation with General Rough Noise

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Abstract
We study the Freidlin–Wentzell large deviation principle for the nonlinear one-dimensional stochastic heat equation driven by a Gaussian noise:

$$\frac{\partial u^\varepsilon(t,x)}{\partial t} = \frac{\partial^2 u^\varepsilon(t,x)}{\partial x^2} + \sqrt{\varepsilon}\sigma(t,x,u^\varepsilon(t,x))\dot{W}(t,x), \quad t>0,\ x\in\mathbb{R},$$

where $\dot{W}$ is white in time and fractional in space with Hurst parameter $H \in \left(\frac{1}{4}, \frac{1}{2}\right)$. Recently, Hu and Wang (Ann Inst Henri Poincaré Probab Stat 58(1):379–423, 2022) have studied the well-posedness of this equation without the technical condition of $\sigma(0)=0$ which was previously assumed in Hu et al. (Ann Probab 45(6):4561–4616, 2017). We adopt a new sufficient condition proposed by Matoussi et al. (Appl Math Optim 83(2):849–879, 2021) for the weak convergence criterion of the large deviation principle.

Keywords Stochastic heat equation · Fractional Brownian motion · Large deviation principle · Weak convergence approach

Mathematics Subject Classification (2020) Primary 60F10; Secondary 60H15

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Contents

1 Introduction

In this paper, we consider the following nonlinear stochastic heat equation (SHE, for short) driven by a Gaussian noise which is white in time and fractional in space:

$$\frac{\partial u^{\varepsilon}(t,x)}{\partial t}=\frac{\partial^2 u^{\varepsilon}(t,x)}{\partial x^2}+\sqrt{\varepsilon}\sigma(t,x,u^{\varepsilon}(t,x))\dot{W}(t,x),\quad t>0,\ x\in\mathbb{R}. \tag{1.1}$$

Here $\varepsilon>0$, $u^{\varepsilon}(0,\cdot)\equiv 1$, $W(t,x)$ is a centered Gaussian field with the covariance given by

$$\mathbb{E}[W(t,x)W(s,y)]=\frac{1}{2}(s\wedge t)\left(|x|^{2H}+|y|^{2H}-|x-y|^{2H}\right), \tag{1.2}$$

for some $\frac{1}{4}<H<\frac{1}{2}$, and $\dot{W}(t,x)=\frac{\partial^2 W}{\partial t\partial x}(t,x)$. The covariance of the noise $\dot{W}$ is given by

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)]=\delta_0(t-s)\Lambda(x-y),$$

where Λ is a distribution, whose Fourier transform is the measure $\mu(d\xi)=c_{1,H}|\xi|^{1-2H}d\xi$ with

$$c_{1,H}=\frac{1}{2\pi}\Gamma(2H+1)\sin(\pi H). \tag{1.3}$$

Λ can be formally written as $\Lambda(x-y)=H(2H-1)\times|x-y|^{2H-2}$. In addition, the measure μ satisfies the integrability condition $\int_{\mathbb{R}}\frac{\mu(d\xi)}{1+|\xi|^2}<\infty$. However, the corresponding covariance Λ is not locally integrable and fails to be nonnegative when $H\in(\frac{1}{4},\frac{1}{2})$. It does not satisfy the classical Dalang's condition in [11] (there Λ is given by a nonnegative locally integrable function). See [1, 19] for more details. For this reason, the approaches used in references [9, 11, 12, 30, 31] cannot handle such rough covariance.

Recently, many authors have studied the existence and uniqueness for the solutions of stochastic partial differential equations (SPDEs, for short) driven by Gaussian noises which are rough in space, see, e.g., [19, 20, 22, 24, 35]. We refer to [18, 34] for surveys. When the diffusion coefficient is an affine function $\sigma(x) = ax + b$, Balan et al. [1] proved the existence and uniqueness of the mild solution to SHE (1.1) by using the technique of Fourier analysis, and they established the Hölder continuity of the solution in [2].

For the general nonlinear coefficient σ, which has a Lipschitz derivative and satisfies $\sigma(0) = 0$, the problem of the existence and uniqueness of the solution was proved by Hu et al. [19]. Under this condition, the large deviations, the moderate deviations and the transportation inequalities were studied in [10, 21, 23], respectively.

In [22], Hu and Wang removed the technical and unusual condition of $\sigma(0) = 0$ and they proved the well-posedness of the solution to Eq. (1.1) under Condition (**H**) in Sect. 2.3. Without the condition of $\sigma(0) = 0$, the solution is no longer in the space $\mathcal{Z}_T^p$ (see [19] or (2.14) in Sect. 2.3 of this paper with $\lambda(x) \equiv 1$). To relax the restriction, Hu and Wang [22] introduced a decay weight (as the spatial variable x goes to infinity) to enlarge the solution space $\mathcal{Z}_T^p$ to a weighted space $\mathcal{Z}_{\lambda,T}^p$ for some suitable power decay function $\lambda(x)$, see Sect. 2.3 for details.

The aim of this paper is to establish a large deviation principle (LDP, for short) for the solution u^ε of (1.1) as $\varepsilon \to 0$. A useful approach of investigating the LDP is the well-known weak convergence method (see, e.g., [3–7, 14]), which is mainly based on the variational representation formula for measurable functionals of Brownian motion. For some relevant LDP results by using the weak convergence method, we refer to [25, 33, 39] and the references therein. In particular, Liu et al. [27] and Xiong and Zhai [38] proved the LDP for a large class of SPDEs with locally monotone coefficients driven by Brownian motions or Lévy noises, respectively. However, the frameworks of [27] and [38] cannot be applied to SHE (1.1) because of the spatial rough noise with $H \in (\frac{1}{4}, \frac{1}{2})$.

Comparing with the LDP result for SHE (1.1) under the assumption of $\sigma(0) = 0$ in Hu et al. [21], our result and its proof are in the weighted space $\mathcal{Z}_{\lambda,T}^p$. Notice that the introduction of the weight brings many difficulties. See Hu and Wang [22] for the excellent analysis. In this paper, we adopt a new sufficient condition for the LDP (see Condition 2.2) which was proposed by Matoussi et al. [29]. This method was successfully applied to the study of LDPs for SPDEs, see, e.g., [13, 16, 17, 26, 36, 37].

The rest of this paper is organized as follows. In Sect. 2, we first present the framework introduced in [22] and recall the weak convergence method given by [6, 29]; then, we formulate the main result of the present paper. In Sect. 3, the associated skeleton equation is studied. Sect. 4 and 5 are devoted to verifying the two conditions for the weak convergence criterion. Finally, we give some useful lemmas in Appendix.

We adopt the following notations throughout this paper. We always use C_α to denote a constant dependent on the parameter α, which may change from line to line. $A \lesssim B$ ($A \gtrsim B$, resp.) means that $A \leq CB$ ($A \geq CB$, resp.) for some positive generic constant C, the value of which might change from line to line, and $A \simeq B$ means that $A \lesssim B$ and $A \gtrsim B$.

2 Preliminaries and Main Result

In this section, we first give some preliminaries of SHE (1.1); then, we state the weak convergence criterion for the LDP in [6, 29] and give the main result of this paper.

2.1 Covariance Structure and Stochastic Integration

Recall some notations from [19] and [22]. Denote by $\mathcal{D} = \mathcal{D}(\mathbb{R})$ the space of real-valued infinitely differentiable functions with compact support on $\mathbb{R}$, and by $\mathcal{D}'$ the dual of $\mathcal{D}$ with respect to the $L^2(\mathbb{R}, \mathrm{d}x)$, where $L^2(\mathbb{R}, \mathrm{d}x)$ is the Hilbert space of square integrable functions with respect to Lebesgue measure on $\mathbb{R}$. The Fourier transform of a function $f \in \mathcal{D}$ is defined as

$$\mathcal{F} f(\xi) := \int_{\mathbb{R}} e^{-i\xi x} f(x)\mathrm{d}x.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$ be the space of real-valued infinitely differentiable functions with compact support on $\mathbb{R}_+ \times \mathbb{R}$. The noise $\dot{W}$ is a zero-mean Gaussian family $\{W(\phi), \phi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R})\}$ with the covariance structure given by

$$\mathbb{E}[W(\phi)W(\psi)] = c_{1,H} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathcal{F}\phi(s,\xi)\overline{\mathcal{F}\psi(s,\xi)} \cdot |\xi|^{1-2H} \mathrm{d}\xi \mathrm{d}s, \tag{2.1}$$

where $H \in (\frac{1}{4}, \frac{1}{2})$, $c_{1,H}$ is given in (1.3), and $\mathcal{F}\phi(s,\xi)$ is the Fourier transform with respect to the spatial variable x of the function $\phi(s,x)$. Then (2.1) defines a Hilbert scalar product on $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$. Denote $\mathfrak{H}$ the Hilbert space obtained by completing $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$ with respect to this scalar product.

Proposition 2.1 ([22, Proposition 2.1], [32, Theorem 3.1]) *The space $\mathfrak{H}$ is a Hilbert space equipped with the scalar product*

$$\begin{aligned}\langle \phi, \psi \rangle_{\mathfrak{H}} :=& c_{1,H} \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}} \mathcal{F}\phi(t,\xi)\overline{\mathcal{F}\psi(t,\xi)} \cdot |\xi|^{1-2H} \mathrm{d}\xi \right) \mathrm{d}t \\ =& c_{2,H} \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}^2} [\phi(t,x+y) - \phi(x)] \cdot [\psi(t,x+y) - \psi(x)] \cdot \right. \\ & \left. |y|^{2H-2} \mathrm{d}x\mathrm{d}y \right) \mathrm{d}t,\end{aligned} \tag{2.2}$$

where

$$\begin{aligned}c_{2,H} :=& \left(\frac{1}{2} - H\right)^{\frac{1}{2}} H^{\frac{1}{2}} \left[\Gamma\left(H + \frac{1}{2}\right)\right]^{-1} \\ & \left(\int_0^\infty \left[(1+t)^{H-\frac{1}{2}} - t^{H-\frac{1}{2}}\right]^2 \mathrm{d}t + \frac{1}{2H} \right)^{\frac{1}{2}}.\end{aligned} \tag{2.3}$$

The space $\mathcal{D}(\mathbb{R}_+ \times \mathbb{R})$ is dense in $\mathfrak{H}$.

For any $t \geq 0$, let $\mathcal{F}_t$ be the filtration generated by W, that is,

$$\mathcal{F}_t := \sigma\left\{W(\phi) : \phi \in \mathcal{D}([0,t] \times \mathbb{R})\right\},$$

where $\mathcal{D}([0,t] \times \mathbb{R})$ is the space of real-valued infinitely differentiable functions on $[0,t] \times \mathbb{R}$.

Definition 2.2 An elementary process g is a process given by

$$g(t,x) = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \mathbf{1}_{(a_i,b_i]}(t) \mathbf{1}_{(h_j,l_j]}(x),$$

where n and m are finite positive integers, $0 \leq a_1 < b_1 < \cdots < a_n < b_n < \infty$, and $h_j < l_j$ and $X_{i,j}$ are $\mathcal{F}_{a_i}$-measurable random variables for $i = 1, \ldots, n$, $j = 1, \ldots, m$. The stochastic integral of an elementary process with respect to W is defined as

$$\begin{aligned}\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(t,x) W(\mathrm{d}t, \mathrm{d}x) &= \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} W(\mathbf{1}_{(a_i,b_i]} \otimes \mathbf{1}_{(h_j,l_j]}) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \Big[W(b_i, l_j) - W(a_i, l_j) \\ &\qquad - W(b_i, h_j) + W(a_i, h_j) \Big].\end{aligned} \tag{2.4}$$

Hu et al. [19, Proposition 2.3] extended the notion of integral with respect to W to a broad class of adapted processes in the following way.

Proposition 2.3 ([19, Proposition 2.3]) *Let Λ_H be the space of predictable processes g defined on $\mathbb{R}_+ \times \mathbb{R}$ such that almost surely $g \in \mathfrak{H}$ and $\mathbb{E}[\|g\|_{\mathfrak{H}}^2] < \infty$. Then, we have that:*

(i) *The space of the elementary processes defined in Definition 2.2 is dense in Λ_H;*
(ii) *For any $g \in \Lambda_H$, the stochastic integral $\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(s,x) W(\mathrm{d}s, \mathrm{d}x)$ is defined as the $L^2(\Omega)$-limit of Riemann sums along elementary processes approximating g in Λ_H, and we have*

$$\mathbb{E}\left[\left(\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(s,x) W(\mathrm{d}s, \mathrm{d}x)\right)^2\right] = \mathbb{E}\left[\|g\|_{\mathfrak{H}}^2\right]. \tag{2.5}$$

Let $\mathcal{H}$ be the Hilbert space obtained by completing $\mathcal{D}(\mathbb{R})$ with respect to the following scalar product:

$$\begin{aligned}\langle \phi, \psi \rangle_{\mathcal{H}} &= c_{1,H} \int_{\mathbb{R}} \mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)} \cdot |\xi|^{1-2H} \mathrm{d}\xi \\ &= c_{2,H} \int_{\mathbb{R}^2} [\phi(x+y) - \phi(x)] \cdot [\psi(x+y) \\ &\quad - \psi(x)] \cdot |y|^{2H-2} \mathrm{d}x\mathrm{d}y, \ \ \forall \phi, \psi \in \mathcal{D}(\mathbb{R}).\end{aligned} \tag{2.6}$$

By Proposition 2.3, for any orthonormal basis $\{e_k\}_{k\geq 1}$ of the Hilbert space $\mathcal{H}$, the family of processes

$$\left\{ B_t^k := \int_0^t \int_{\mathbb{R}} e_k(y) W(\mathrm{d}s, \mathrm{d}y) \right\}_{k \geq 1} \tag{2.7}$$

is a sequence of independent standard Wiener processes and the process $B_t := \sum_{k\geq 1} B_t^k e_k$ is a cylindrical Brownian motion on $\mathcal{H}$. It is well known that (see [9] or [12]) for any $\mathcal{H}$-valued predictable process $g \in L^2(\Omega \times [0, T]; \mathcal{H})$, we can define the stochastic integral with respect to the cylindrical Wiener process B as follows:

$$\int_0^T g(s) \mathrm{d}B_s := \sum_{k \geq 1} \int_0^T \langle g(s), e_k \rangle_{\mathcal{H}} \mathrm{d}B_s^k. \tag{2.8}$$

Note that the above series converges in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and the sum does not depend on the selected orthonormal basis. Moreover, each summand, in the above series, is a classical Itô integral with respect to a standard Brownian motion.

Let $(B, \|\cdot\|_B)$ be a Banach space with the norm $\|\cdot\|_B$. Let $H \in (\frac{1}{4}, \frac{1}{2})$ be a fixed number. For any function $f : \mathbb{R} \to B$, denote

$$\mathcal{N}^B_{\frac{1}{2}-H} f(x) := \left(\int_{\mathbb{R}} \|f(x+h) - f(x)\|_B^2 \cdot |h|^{2H-2} \mathrm{d}h \right)^{\frac{1}{2}}, \tag{2.9}$$

if the above quantity is finite. When $B = \mathbb{R}$, we abbreviate the notation $\mathcal{N}^{\mathbb{R}}_{\frac{1}{2}-H} f$ as $\mathcal{N}_{\frac{1}{2}-H} f$. As in [19], when $B = L^p(\Omega)$, we denote $\mathcal{N}^B_{\frac{1}{2}-H}$ by $\mathcal{N}_{\frac{1}{2}-H,\, p}$, that is,

$$\mathcal{N}_{\frac{1}{2}-H,\, p} f(x) := \left(\int_{\mathbb{R}} \|f(x+h) - f(x)\|_{L^p(\Omega)}^2 \cdot |h|^{2H-2} \mathrm{d}h \right)^{\frac{1}{2}}. \tag{2.10}$$

The following Burkholder–Davis–Gundy's inequality is well known (see, e.g., [19]).

Proposition 2.4 ([19, Proposition 3.2]) *Let W be the Gaussian noise with the covariance (2.1), and let $f \in \Lambda_H$ be a predictable random field. Then, we have that for any $p \geq 2$,*

$$\left\| \int_0^t \int_{\mathbb{R}} f(s, y) W(\mathrm{d}s, \mathrm{d}y) \right\|_{L^p(\Omega)} \leq \sqrt{4p} c_H \left(\int_0^t \int_{\mathbb{R}} \left[\mathcal{N}_{\frac{1}{2}-H,\, p} f(s, y) \right]^2 \mathrm{d}y \mathrm{d}s \right)^{\frac{1}{2}}, \tag{2.11}$$

where c_H is a constant depending only on H and $\mathcal{N}_{\frac{1}{2}-H,\,p} f(s,y)$ denotes the application of $\mathcal{N}_{\frac{1}{2}-H,\,p}$ to the spatial variable y.

2.2 Stochastic Heat Equation

Let $\mathcal{C}([0,T]\times\mathbb{R})$ be the space of all continuous real-valued functions on $[0,T]\times\mathbb{R}$, equipped with the following metric:

$$d_{\mathcal{C}}(u,v) := \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0\le t\le T, |x|\le n} (|u(t,x)-v(t,x)| \wedge 1). \tag{2.12}$$

Define the weight function

$$\lambda(x) := c_H (1+|x|^2)^{H-1}, \tag{2.13}$$

where c_H is a constant such that $\int_{\mathbb{R}} \lambda(x)\mathrm{d}x = 1$.

For any $p \ge 2$ and $\frac{1}{4} < H < \frac{1}{2}$, we introduce a norm $\|\cdot\|_{\mathcal{Z}^p_{\lambda,T}}$ for a random field $v = \{v(t,x)\}_{(t,x)\in[0,T]\times\mathbb{R}}$ as follows:

$$\|v\|_{\mathcal{Z}^p_{\lambda,T}} := \sup_{t\in[0,T]} \|v(t,\cdot)\|_{L^p_\lambda(\Omega\times\mathbb{R})} + \sup_{t\in[0,T]} \mathcal{N}^*_{\frac{1}{2}-H,\,p} v(t), \tag{2.14}$$

where

$$\|v(t,\cdot)\|_{L^p_\lambda(\Omega\times\mathbb{R})} := \left(\int_{\mathbb{R}} \mathbb{E}\left[|v(t,x)|^p\right] \lambda(x)\mathrm{d}x \right)^{\frac{1}{p}}, \tag{2.15}$$

and

$$\mathcal{N}^*_{\frac{1}{2}-H,\,p} v(t) := \left(\int_{\mathbb{R}} \|v(t,\cdot)-v(t,\cdot+h)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})} \cdot |h|^{2H-2}\mathrm{d}h \right)^{\frac{1}{2}}. \tag{2.16}$$

Denote $\mathcal{Z}^p_{\lambda,T}$ the space of all random fields $v = \{v(t,x)\}_{(t,x)\in[0,T]\times\mathbb{R}}$ such that $\|v\|_{\mathcal{Z}^p_{\lambda,T}}$ is finite.

For the well-posedness of the solution and the large deviation principle, we assume the following conditions.

(H) Assume that $\sigma(t,x,u) \in \mathcal{C}^{0,1,1}([0,T]\times\mathbb{R}^2)$ (the space of all continuous functions σ, with continuous partial derivatives σ'_x, σ'_u and σ''_{xu}), and there exists a constant $C > 0$ such that

$$\sup_{t\in[0,T],\, x\in\mathbb{R}} |\sigma(t,x,u)| \le C(1+|u|), \quad \forall u \in \mathbb{R}; \tag{2.17}$$

$$\sup_{t\in[0,T],\, x\in\mathbb{R}} |\sigma(t,x,u)-\sigma(t,x,v)| \le C|u-v|, \quad \forall u,v \in \mathbb{R}; \tag{2.18}$$

$$\sup_{t\in[0,T],\, x\in\mathbb{R},\, u\in\mathbb{R}} |\sigma'_u(t,x,u)| \le C\,; \tag{2.19}$$

$$\sup_{t\in[0,T],\, x\in\mathbb{R}} |\sigma'_x(t,x,0)| \le C\,; \tag{2.20}$$

$$\sup_{t\in[0,T],\, x\in\mathbb{R},\, u\in\mathbb{R}} |\sigma''_{xu}(t,x,u)| \le C\,. \tag{2.21}$$

Moreover, there exist some constants $p_0 > \frac{6}{4H-1}$ and $C > 0$ such that

$$\sup_{t\in[0,T],\, x\in\mathbb{R}} \lambda^{-\frac{1}{p_0}}(x)\left|\sigma'_u(t,x,u_1)-\sigma'_u(t,x,u_2)\right| \le C|u_1-u_2|, \quad \forall u_1,u_2\in\mathbb{R}, \tag{2.22}$$

where $\lambda(x)$ is the weight function defined by (2.13).

Remark 1 By the monotonicity of $\lambda(x)$, we know that if (2.22) holds for some $p_0 > \frac{6}{4H-1}$, then it also holds for all $p \ge p_0$.

Let $p_t(x) := \frac{1}{\sqrt{4\pi t}}\exp\left(-\frac{x^2}{4t}\right)$ be the heat kernel associated with the Laplacian operator Δ. Recall the following definition of the solution to SHE (1.1) from [22].

Definition 2.5 ([22, Definition 1.4]) Given the initial value $u_0(\cdot)\equiv 1$, a real-valued adapted stochastic process u^ε is called a mild solution to (1.1), if for all $t\ge 0$ and $x\in\mathbb{R}$,

$$u^\varepsilon(t,x) = 1+\sqrt{\varepsilon}\int_0^t\int_{\mathbb{R}} p_{t-s}(x-y)\sigma(s,y,u^\varepsilon(s,y))W(\mathrm{d}s,\mathrm{d}y), \quad \text{a.s.}. \tag{2.23}$$

The following theorem follows from [22].

Theorem 2.6 ([22, Theorem 1.6]) *Assume that σ satisfies the hypothesis* (**H**). *Then* (1.1) *admits a unique mild solution in $\mathcal{C}([0,T]\times\mathbb{R})$ almost surely.*

2.3 A General Criterion for the Large Deviation Principle

Let $\{u^\varepsilon\}_{\varepsilon>0}$ be a family of random variables defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ and taking values in a Polish space E.

Definition 2.7 A function $I: E\to[0,\infty]$ is called a rate function on E, if for each $M<\infty$ the level set $\{y\in E: I(y)\le M\}$ is a compact subset of E.

Definition 2.8 Let I be a rate function on E. The sequence $\{u^\varepsilon\}_{\varepsilon>0}$ is said to satisfy a large deviation principle on E with the rate function I, if the following two conditions hold:

(a) For each closed subset F of E,

$$\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}(u^\varepsilon\in F) \le -\inf_{y\in F} I(y);$$

(b) For each open subset G of E,

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(u^\varepsilon \in G) \geq -\inf_{y\in G} I(y).$$

Set $\mathbb{V} = C([0,T];\mathbb{R}^\infty)$ and let $\mathbb{U}$ be a Polish space. Let $\{\Gamma^\varepsilon\}_{\varepsilon>0}$ be a family of measurable maps from $\mathbb{V}$ to $\mathbb{U}$. We recall a criterion for the LDP of the family $Z^\varepsilon = \Gamma^\varepsilon(W)$ as $\varepsilon \to 0$, where and throughout this section W is the Gaussian process identified a sequence of independent, standard, real-valued Brownian motions, by using the representation formulae of (2.7).

Define the following space of stochastic processes:

$$\mathcal{L}_2 := \left\{\phi : \Omega \times [0,T] \to \mathcal{H} \text{ is predictable and } \int_0^T \|\phi(s)\|_{\mathcal{H}}^2 \mathrm{d}s < \infty,\ \ \mathbb{P}\text{-a.s.}\right\}, \tag{2.24}$$

and denote $L^2([0,T];\mathcal{H})$ the space of square integrable $\mathcal{H}$-valued functions on $[0,T]$. For each $N \geq 1$, let

$$S^N = \left\{g \in L^2([0,T];\mathcal{H}) : L_T(g) \leq N\right\}, \tag{2.25}$$

where

$$L_T(g) := \frac{1}{2}\int_0^T \|g(s)\|_{\mathcal{H}}^2 \mathrm{d}s, \tag{2.26}$$

and S^N is equipped with the topology of the weak convergence in $L^2([0,T];\mathcal{H})$. Set $\mathbb{S} = \bigcup_{N\geq 1} S^N$, and

$$\mathcal{U}^N = \left\{g \in \mathcal{L}_2 : g(\omega) \in S^N,\ \ \mathbb{P}\text{-a.s.}\right\}.$$

Condition 2.1 *There exists a measurable mapping $\Gamma^0 : \mathbb{V} \to \mathbb{U}$ such that the following two items hold:*

(a) *For every $N < +\infty$, the set $K_N = \left\{\Gamma^0\left(\int_0^\cdot g(s)\mathrm{d}s\right) : g \in S^N\right\}$ is a compact subset of $\mathbb{U}$;*
(b) *For every $N < +\infty$ and $\{g^\varepsilon\}_{\varepsilon>0} \subset \mathcal{U}^N$, if g^ε converges to g as S^N-valued random elements in distribution, then $\Gamma^\varepsilon\left(W + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot g^\varepsilon(s)\mathrm{d}s\right)$ converges in distribution to $\Gamma^0\left(\int_0^\cdot g(s)\mathrm{d}s\right)$.*

Let $I : \mathbb{U} \to [0,\infty]$ be defined by

$$I(\phi) := \inf_{\{g\in\mathbb{S};\ \phi=\Gamma^0\left(\int_0^\cdot g(s)\mathrm{d}s\right)\}} \{L_T(g)\},\ \ \phi \in \mathbb{U}, \tag{2.27}$$

with the convention $\inf \emptyset = \infty$.

The following result is due to Budhiraja et al. [6].

Theorem 2.9 ([6, Theorem 6]) *For any $\varepsilon > 0$, let $X^\varepsilon = \Gamma^\varepsilon(W)$ and suppose that Condition 2.1 holds. Then, the family $\{X^\varepsilon\}_{\varepsilon>0}$ satisfies an LDP with the rate function I defined by (2.27).*

Recently, a new sufficient condition (Condition 2.2) for verifying the assumptions in Condition 2.1 for LDPs has been proposed by Matoussi et al. [29]. It turns out that this new sufficient condition seems to be more suitable for establishing the LDP for SPDEs.

Condition 2.2 *There exists a measurable mapping $\Gamma^0 : \mathbb{V} \to \mathbb{U}$ such that the following two items hold:*

(a) *For every $N < +\infty$ and any family $\{g^n\}_{n\geq 1} \subset S^N$ that converges to some element g in S^N as $n \to \infty$, $\Gamma^0\left(\int_0^\cdot g^n(s)\mathrm{d}s\right)$ converges to $\Gamma^0\left(\int_0^\cdot g(s)\mathrm{d}s\right)$ in the space $\mathbb{U}$;*

(b) *For every $N < +\infty$, $\{g^\varepsilon\}_{\varepsilon>0} \subset \mathcal{U}^N$ and $\delta > 0$,*

$$\lim_{\varepsilon\to 0} \mathbb{P}\big(\rho(Y^\varepsilon, Z^\varepsilon) > \delta\big) = 0,$$

where $Y^\varepsilon = \Gamma^\varepsilon\left(W + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot g^\varepsilon(s)\mathrm{d}s\right)$, $Z^\varepsilon = \Gamma^0\left(\int_0^\cdot g^\varepsilon(s)\mathrm{d}s\right)$ and $\rho(\cdot,\cdot)$ stands for the metric in the space $\mathbb{U}$.

2.4 Main Result

To state our main result and give its proof, we need to introduce a map Γ^0 appeared in Condition 2.1 (or Condition 2.2). Given $g \in \mathbb{S}$, consider the following deterministic integral equation (the skeleton equation):

$$u^g(t,x) = 1 + \int_0^t \langle p_{t-s}(x-\cdot)\sigma(s,\cdot,u^g(s,\cdot)), g(s,\cdot)\rangle_{\mathcal{H}}\mathrm{d}s, \quad t \geq 0,\ x \in \mathbb{R}. \tag{2.28}$$

By Proposition 3.1, Eq. (2.28) admits a unique solution $u^g \in \mathcal{C}([0,T]\times\mathbb{R})$. For any $g \in \mathbb{S}$, we define

$$\Gamma^0\left(\int_0^\cdot g(s)\mathrm{d}s\right) := u^g(\cdot). \tag{2.29}$$

The following is the main result of this paper.

Theorem 2.10 *Assume that the hypothesis* (**H**) *holds. Then, the family $\{u^\varepsilon\}_{\varepsilon>0}$ in Eq.* (1.1) *satisfies an LDP in the space $\mathcal{C}([0,T]\times\mathbb{R})$ with the rate function I given by*

$$I(\phi) := \inf_{\{g\in\mathbb{S};\, \phi=\Gamma^0(\int_0^\cdot g(s)\mathrm{d}s)\}} \left\{\frac{1}{2}\int_0^T \|g(s)\|_{\mathcal{H}}^2\mathrm{d}s\right\}. \tag{2.30}$$

Proof According to Theorem 2.9 and [29, Theorem 3.2], it suffices to show that the conditions (a) and (b) in Condition 2.2 are satisfied. Condition (a) will be proved in Proposition 4.1, and Condition (b) will be verified in Proposition 5.1. The proof is complete. □

3 Skeleton Equation

In this section, we study the well-posedness of the skeleton equation (2.28).

For $p \geq 2$, $H \in (\frac{1}{4}, \frac{1}{2})$, recall the space $\mathcal{Z}^p_{\lambda,T}$ with the norm (2.14). The space of all non-random functions in $\mathcal{Z}^p_{\lambda,T}$ is denoted by $Z^p_{\lambda,T}$, with the following norm:

$$\|v\|_{Z^p_{\lambda,T}} := \sup_{t\in[0,T]} \|v(t,\cdot)\|_{L^p_\lambda(\mathbb{R})} + \sup_{t\in[0,T]} \mathcal{N}^*_{\frac{1}{2}-H,\,p} v(t), \tag{3.1}$$

where

$$\|v(t,\cdot)\|_{L^p_\lambda(\mathbb{R})} := \left(\int_{\mathbb{R}} |v(t,x)|^p \lambda(x)\mathrm{d}x\right)^{\frac{1}{p}}, \tag{3.2}$$

and

$$\mathcal{N}^*_{\frac{1}{2}-H,\,p} v(t) := \left(\int_{\mathbb{R}} \|v(t,\cdot) - v(t,\cdot+h)\|^2_{L^p_\lambda(\mathbb{R})} \cdot |h|^{2H-2}\mathrm{d}h\right)^{\frac{1}{2}}. \tag{3.3}$$

Proposition 3.1 *Assume that σ satisfies the hypothesis* (**H**). *Then, Eq.* (2.28) *admits a unique solution u^g in $\mathcal{C}([0,T]\times\mathbb{R})$. In addition, $\sup_{g\in S^N} \|u^g\|_{Z^p_{\lambda,T}} < \infty$ for any $N \geq 1$ and any $p \geq 2$.*

Due to the complexity of the space $\mathcal{H}$, it is difficult to prove Proposition 3.1 by using Picard's iteration directly. We use the approximation method by introducing a new Hilbert space $\mathcal{H}_\varepsilon$ as follows.

For every fixed $\varepsilon > 0$, let

$$f_\varepsilon(x) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x} e^{-\varepsilon|\xi|^2} |\xi|^{1-2H}\mathrm{d}\xi. \tag{3.4}$$

For any $\phi, \psi \in \mathcal{D}(\mathbb{R})$, we define

$$\begin{aligned}\langle\phi,\psi\rangle_{\mathcal{H}_\varepsilon} &:= c_{1,H}\int_{\mathbb{R}} \mathcal{F}\phi(\xi)\overline{\mathcal{F}\psi(\xi)} e^{-\varepsilon|\xi|^2}|\xi|^{1-2H}\mathrm{d}\xi \\ &= c_{1,H}\int_{\mathbb{R}^2} \phi(x)\psi(y) f_\varepsilon(x-y)\mathrm{d}x\mathrm{d}y,\end{aligned} \tag{3.5}$$

where $c_{1,H}$ is given by (1.3). Let $\mathcal{H}_\varepsilon$ be the Hilbert space obtained by completing $\mathcal{D}(\mathbb{R})$ with respect to the scalar product given by (3.5). For any $0 \leq \varepsilon_1 < \varepsilon_2$, we have that for any $\phi \in \mathcal{H}_{\varepsilon_1}$,

$$\|\phi\|_{\mathcal{H}_{\varepsilon_1}} \geq \|\phi\|_{\mathcal{H}_{\varepsilon_2}}, \tag{3.6}$$

and we have by the dominated convergence theorem,

$$\lim_{\varepsilon\to 0} \langle \phi, \psi\rangle_{\mathcal{H}_\varepsilon} = \langle \phi, \psi\rangle_{\mathcal{H}} \quad \text{for any } \phi, \psi \in \mathcal{H}.$$

For any $g \in \mathbb{S}$, let

$$u_\varepsilon^g(t,x) = 1 + \int_0^t \langle p_{t-s}(x-\cdot)\sigma(s,\cdot,u_\varepsilon^g(s,\cdot)), g(s,\cdot)\rangle_{\mathcal{H}_\varepsilon} \mathrm{d}s. \tag{3.7}$$

Since $|\xi|^{1-2H}e^{-\varepsilon|\xi|^2}$ is in $L^1(\mathbb{R})$, $|f_\varepsilon|$ is bounded. Due to the regularity in space, the existence and uniqueness of the solution u_ε^g to Eq. (3.7) are well known, see, e.g., [28, Section 4].

For any $t \geq 0, x, y, h \in \mathbb{R}$, let

$$D_t(x,h) := p_t(x+h) - p_t(x), \tag{3.8}$$

and

$$\square_t(x,y,h) := p_t(x+y+h) - p_t(x+y) - p_t(x+h) + p_t(x). \tag{3.9}$$

The following lemma asserts that the approximate solution u_ε^g is bounded in the space $Z_{\lambda,T}^p$ uniformly over $\varepsilon > 0$.

Lemma 3.1 *Let $H \in (\frac{1}{4}, \frac{1}{2})$ and $N \in \mathbb{N}$. Assume that σ satisfies the hypothesis* (**H**). *Then, for any $p \geq 2$,*

$$\sup_{g\in S^N} \sup_{\varepsilon>0} \|u_\varepsilon^g\|_{Z_{\lambda,T}^p} < \infty. \tag{3.10}$$

Proof We use the similar argument as in [22] replacing the stochastic integral by the deterministic integral. We first define Picard's iteration sequence. For any $t \geq 0, x \in \mathbb{R}$, let

$$u_\varepsilon^{g,0}(t,x) = 1,$$

and recursively for $n = 0, 1, 2, \cdots$,

$$u_\varepsilon^{g,n+1}(t,x) = 1 + \int_0^t \langle p_{t-s}(x-\cdot)\sigma(s,\cdot,u_\varepsilon^{g,n}(s,\cdot)), g(s,\cdot)\rangle_{\mathcal{H}_\varepsilon} \mathrm{d}s. \tag{3.11}$$

Since $g \in S^N$, we know that by (3.6),

$$\int_0^T \|g(s,\cdot)\|_{\mathcal{H}_\varepsilon}^2 \mathrm{d}s \leq 2N \text{ for any } \varepsilon > 0. \tag{3.12}$$

In Step 1, we prove the convergence of $u_\varepsilon^{g,n}(t,\cdot)$ in $L_\lambda^p(\mathbb{R})$ for any $t\in[0,T]$. In Steps 2 and 3, we give some quantitative estimates for $\|u_\varepsilon^{g,n}(t)\|_{L_\lambda^p(\mathbb{R})}$ and $\mathcal{N}^*_{\frac{1}{2}-H,p}u_\varepsilon^{g,n}(t)$ for each fixed $\varepsilon>0$. Step 4 is devoted to proving that u_ε^g is bounded in $\left(Z_{\lambda,T}^p,\ \|\cdot\|_{Z_{\lambda,T}^p}\right)$ uniformly over $\varepsilon>0$.

Step 1 In this step, we will prove that for any fixed $\varepsilon>0$, as n goes to infinity, the sequence $u_\varepsilon^{g,n}(t,\cdot)$ converges to $u_\varepsilon^g(t,\cdot)$ in $L_\lambda^p(\mathbb{R})$.

By the Cauchy–Schwarz inequality, (2.18), the boundedness of f_ε and Jensen's inequality with respect to $p_{t-s}(x-y)\mathrm{d}y$, we have that, for any $t\in[0,T]$ and $x\in\mathbb{R}$,

$$
\begin{aligned}
&|u_\varepsilon^{g,n+1}(t,x)-u_\varepsilon^{g,n}(t,x)|^2\\
=&\left|\int_0^t\left\langle p_{t-s}(x-\cdot)\left[\sigma(s,\cdot,u_\varepsilon^{g,n}(s,\cdot))-\sigma(s,\cdot,u_\varepsilon^{g,n-1}(s,\cdot))\right],g(s,\cdot)\right\rangle_{\mathcal{H}_\varepsilon}\mathrm{d}s\right|^2\\
\le&\int_0^t\|g(s,\cdot)\|^2_{\mathcal{H}_\varepsilon}\mathrm{d}s\cdot\int_0^t\left\|p_{t-s}(x-\cdot)\left[\sigma(s,\cdot,u_\varepsilon^{g,n}(s,\cdot))-\sigma(s,\cdot,u_\varepsilon^{g,n-1}(s,\cdot))\right]\right\|^2_{\mathcal{H}_\varepsilon}\mathrm{d}s\\
\le&2c_HN\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}}p_{t-s}(x-y)\left[\sigma(s,y,u_\varepsilon^{g,n}(s,y)-\sigma(s,y,u_\varepsilon^{g,n-1}(s,y))\right]\\
&\quad\cdot p_{t-s}(x-z)\left[\sigma(s,z,u_\varepsilon^{g,n}(s,z))-\sigma(s,z,u_\varepsilon^{g,n-1}(s,z))\right]f_\varepsilon(y-z)\mathrm{d}y\mathrm{d}z\mathrm{d}s\\
\le&2c_HN\|f_\varepsilon\|_\infty\cdot\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)\left[\sigma(s,y,u_\varepsilon^{g,n}(s,y)-\sigma(s,y,u_\varepsilon^{g,n-1}(s,y))\right]^2\mathrm{d}y\mathrm{d}s.
\end{aligned}
$$

Integrating with respect to the spatial variable with the weight $\lambda(x)$ and invoking (2.18), Jensen's inequality with respect to $p_{t-s}(x-y)\mathrm{d}y\mathrm{d}s$ and an application of Lemma 6.1 yield that for any $p\ge2$, $t\in[0,T]$,

$$
\begin{aligned}
&\|u_\varepsilon^{g,n+1}(t,\cdot)-u_\varepsilon^{g,n}(t,\cdot)\|^p_{L_\lambda^p(\mathbb{R})}\\
=&\int_{\mathbb{R}}\left|u_\varepsilon^{g,n+1}(t,x)-u_\varepsilon^{g,n}(t,x)\right|^p\lambda(x)\mathrm{d}x\\
\le&C_{\varepsilon,N}\int_{\mathbb{R}}\left|\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)\left|u_\varepsilon^{g,n}(s,y)-u_\varepsilon^{g,n-1}(s,y)\right|^2\mathrm{d}y\mathrm{d}s\right|^{\frac{p}{2}}\lambda(x)\mathrm{d}x\\
\le&C_{\varepsilon,N,p,T}\int_{\mathbb{R}}\left[\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)\left|u_\varepsilon^{g,n}(s,y)-u_\varepsilon^{g,n-1}(s,y)\right|^p\mathrm{d}y\mathrm{d}s\right]\lambda(x)\mathrm{d}x\\
=&C_{\varepsilon,N,p,T}\int_0^t\int_{\mathbb{R}}\frac{1}{\lambda(y)}\int_{\mathbb{R}}p_{t-s}(x-y)\lambda(x)\mathrm{d}x\left|u_\varepsilon^{g,n}(s,y)-u_\varepsilon^{g,n-1}(s,y)\right|^p\lambda(y)\mathrm{d}y\mathrm{d}s\\
\le&C_{\varepsilon,N,p,T}\int_0^t\|u_\varepsilon^{g,n}(s,\cdot)-u_\varepsilon^{g,n-1}(s,\cdot)\|^p_{L_\lambda^p(\mathbb{R})}\mathrm{d}s\\
\le&C^n_{\varepsilon,N,p,T}\frac{T^n}{n!}\sup_{s\in[0,T]}\|u_\varepsilon^{g,1}(s,\cdot)-u_\varepsilon^{g,0}(s,\cdot)\|^p_{L_\lambda^p(\mathbb{R})}.
\end{aligned}
\tag{3.13}
$$

Then, (3.13) implies that

$$\sup_{n\geq 1}\sup_{t\in[0,T]}\|u_\varepsilon^{g,n}(t,\cdot)\|_{L^p_\lambda(\mathbb{R})}<\infty,\quad \text{for each } \varepsilon>0,$$

and that $\{u_\varepsilon^{g,n}(t,\cdot)\}_{n\geq 1}$ is a Cauchy sequence in $L^p_\lambda(\mathbb{R})$ for any $t\in[0,T]$. Hence, for any fixed $t\in[0,T]$, there exists $u_\varepsilon^g(t,\cdot)\in L^p_\lambda(\mathbb{R})$ such that

$$u_\varepsilon^{g,n}(t,\cdot)\to u_\varepsilon^g(t,\cdot) \text{ in } L^p_\lambda(\mathbb{R}),\ \text{ as } n\to\infty. \tag{3.14}$$

Step 2 In this step, we will give a quantitative estimate for $\|u_\varepsilon^{g,n}(t,\cdot)\|_{L^p_\lambda(\mathbb{R})}$ for any fixed $\varepsilon>0$. By the Cauchy–Schwarz inequality, (2.6) and (3.5), we have

$$\begin{aligned}
|u_\varepsilon^{g,n+1}(t,x)|^p &\lesssim 1+\left(\int_0^t \left\|p_{t-s}(x-\cdot)\sigma(s,\cdot,u_\varepsilon^{g,n}(s,\cdot))\right\|^2_{\mathcal{H}_\varepsilon}\mathrm{d}s\right)^{\frac{p}{2}}\\
&\lesssim 1+\left(\int_0^t \left\|p_{t-s}(x-\cdot)\sigma(s,\cdot,u_\varepsilon^{g,n}(s,\cdot))\right\|^2_{\mathcal{H}}\mathrm{d}s\right)^{\frac{p}{2}}\\
&\simeq 1+\left(\int_0^t\int_{\mathbb{R}^2}\Big|p_{t-s}(x-y-h)\sigma(s,y+h,u_\varepsilon^{g,n}(s,y+h))\right.\\
&\qquad \left. -\,p_{t-s}(x-y)\sigma(s,y,u_\varepsilon^{g,n}(s,y))\Big|^2\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s\right)^{\frac{p}{2}}\\
&\lesssim 1+\mathcal{A}_1(t,x)+\mathcal{A}_2(t,x)+\mathcal{A}_3(t,x),
\end{aligned}\tag{3.15}$$

where

$$\begin{aligned}
\mathcal{A}_1(t,x)&:=\left(\int_0^t\int_{\mathbb{R}^2}p^2_{t-s}(x-y-h)\right.\\
&\qquad\cdot\left|\sigma(s,y+h,u_\varepsilon^{g,n}(s,y+h))-\sigma(s,y,u_\varepsilon^{g,n}(s,y+h))\right|^2\\
&\qquad\left.\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s\right)^{\frac{p}{2}};\\
\mathcal{A}_2(t,x)&:=\left(\int_0^t\int_{\mathbb{R}^2}p^2_{t-s}(x-y-h)\cdot\right|\\
&\qquad\sigma(s,y,u_\varepsilon^{g,n}(s,y+h))-\sigma(s,y,u_\varepsilon^{g,n}(s,y))\Big|^2\\
&\qquad\left.\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s\right)^{\frac{p}{2}};\\
\mathcal{A}_3(t,x)&:=\left(\int_0^t\int_{\mathbb{R}^2}|D_{t-s}(x-y,h)|^2\cdot\left|\sigma(s,y,u_\varepsilon^{g,n}(s,y))\right|^2\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s\right)^{\frac{p}{2}}.
\end{aligned}$$

If $|h| > 1$, then we have by (2.17)

$$\begin{aligned} &\left|\sigma(s, y+h, u_\varepsilon^{g,n}(s,y)) - \sigma(s, y, u_\varepsilon^{g,n}(s,y))\right|^2 \\ &\lesssim \left|\sigma(s, y+h, u_\varepsilon^{g,n}(s,y))\right|^2 + \left|\sigma(s, y, u_\varepsilon^{g,n}(s,y))\right|^2 \\ &\lesssim 1 + \left|u_\varepsilon^{g,n}(s,y)\right|^2 . \end{aligned} \tag{3.16}$$

If $|h| \leq 1$, then by (2.20) and (2.21) there exists some $\zeta \in (0,1)$ such that

$$\begin{aligned} &\left|\sigma(s, y+h, u_\varepsilon^{g,n}(s,y)) - \sigma(s, y, u_\varepsilon^{g,n}(s,y))\right|^2 \\ &\lesssim |\sigma(s, y+h, 0) - \sigma(s, y, 0)|^2 + \left|\int_0^{u_\varepsilon^{g,n}(s,y)} [\sigma'_\xi(s, y+h, \xi) - \sigma'_\xi(s, y, \xi) \mathrm{d}\xi\right|^2 \\ &\lesssim \left|\sigma'_x(s, y+\zeta h, 0)\right|^2 \cdot |h|^2 + \left|u_\varepsilon^{g,n}(s,y)\right|^2 \cdot |h|^2 \\ &\lesssim \left(1 + \left|u_\varepsilon^{g,n}(s,y)\right|^2\right) \cdot |h|^2. \end{aligned} \tag{3.17}$$

By a change of variable, (3.16) and (3.17), we have

$$\mathcal{A}_1(t,x) \lesssim \left(\int_0^t \int_{\mathbb{R}} p_{t-s}^2(x-y) \cdot \left(1 + \left|u_\varepsilon^{g,n}(s,y)\right|^2\right) \mathrm{d}y \mathrm{d}s\right)^{\frac{p}{2}}.$$

By a change of variable, Minkowski's inequality, Lemma 6.1 and Jensen's inequality with respect to

$$p_{t-s}^2(y)\mathrm{d}y \simeq (t-s)^{-\frac{1}{2}} p_{\frac{t-s}{2}}(y)\mathrm{d}y, \tag{3.18}$$

we have

$$\begin{aligned} &\left(\int_{\mathbb{R}} \mathcal{A}_1(t,x)\lambda(x)\mathrm{d}x\right)^{\frac{2}{p}} \\ &\lesssim \int_0^t \int_{\mathbb{R}} p_{t-s}^2(y) \cdot \left(\int_{\mathbb{R}} \left(1 + \left|u_\varepsilon^{g,n}(s,x)\right|^p\right) \lambda(x-y)\mathrm{d}x\right)^{\frac{2}{p}} \mathrm{d}y\mathrm{d}s \\ &\lesssim \int_0^t (t-s)^{-\frac{1}{2}} \cdot \left(\int_{\mathbb{R}^2} p_{\frac{t-s}{2}}(y) \left(1 + \left|u_\varepsilon^{g,n}(s,x)\right|^p\right) \lambda(x-y)\mathrm{d}x\mathrm{d}y\right)^{\frac{2}{p}} \mathrm{d}s \\ &\lesssim \int_0^t (t-s)^{-\frac{1}{2}} \cdot \left(1 + \|u_\varepsilon^{g,n}(s,\cdot)\|_{L_\lambda^p(\mathbb{R})}^2\right) \mathrm{d}s. \end{aligned} \tag{3.19}$$

By (2.18), a change of variable, Minkowski's inequality, Jensen's inequality and Lemma 6.1, we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}} \mathcal{A}_2(t,x)\lambda(x)\mathrm{d}x\right)^{\frac{2}{p}} \\
&\lesssim \left(\int_{\mathbb{R}} \left(\int_0^t \int_{\mathbb{R}^2} p_{t-s}^2(x-y)\cdot\right.\right. \\
&\quad \left.\left. \left|u_\varepsilon^{g,n}(s,y+h)-u_\varepsilon^{g,n}(s,y)\right|^2 \cdot |h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s\right)^{\frac{p}{2}} \lambda(x)\mathrm{d}x\right)^{\frac{2}{p}} \\
&\lesssim \int_0^t \int_{\mathbb{R}^2} p_{t-s}^2(y)\cdot\left(\int_{\mathbb{R}}\right. \\
&\quad \left.\left|u_\varepsilon^{g,n}(s,x+h)-u_\varepsilon^{g,n}(s,x)\right|^p \lambda(x-y)\mathrm{d}x\right)^{\frac{2}{p}} \cdot |h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s \\
&\lesssim \int_0^t \int_{\mathbb{R}} (t-s)^{-\frac{1}{2}}\cdot\left(\int_{\mathbb{R}^2} p_{\frac{t-s}{2}}(y)\right. \\
&\quad \left.\left|u_\varepsilon^{g,n}(s,x+h)-u_\varepsilon^{g,n}(s,x)\right|^p \lambda(x-y)\mathrm{d}y\mathrm{d}x\right)^{\frac{2}{p}} \cdot |h|^{2H-2}\mathrm{d}h\mathrm{d}s \\
&\lesssim \int_0^t (t-s)^{-\frac{1}{2}}\cdot\left[\mathcal{N}^*_{\frac{1}{2}-H,\,p} u_\varepsilon^{g,n}(s)\right]^2 \mathrm{d}s. \qquad (3.20)
\end{aligned}$$

By (2.17), a change of variable, Minkowski's inequality, Jensen's inequality with respect to $(t-s)^{1-H}\,|D_{t-s}(y,h)|^2\,|h|^{2H-2}\mathrm{d}y\mathrm{d}h$ and Lemma 6.5, we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}} \mathcal{A}_3(t,x)\lambda(x)\mathrm{d}x\right)^{\frac{2}{p}} \\
&\lesssim \int_0^t \int_{\mathbb{R}^2} |D_{t-s}(y,h)|^2\cdot\left(1+\int_{\mathbb{R}}\left|u_\varepsilon^{g,n}(s,x)\right|^p\lambda(x-y)\mathrm{d}x\right)^{\frac{2}{p}}\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s \\
&\lesssim \int_0^t (t-s)^{H-1}\cdot\left(\int_{\mathbb{R}^3}(t-s)^{1-H}\cdot\left|D_{t-s}(y,h)\right|^2\right. \\
&\quad \left.\cdot\left(1+\left|u_\varepsilon^{g,n}(s,x)\right|^p\right)\lambda(x-y)\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}x\right)^{\frac{2}{p}}\mathrm{d}s \\
&\lesssim \int_0^t (t-s)^{H-1}\cdot\left(1+\|u_\varepsilon^{g,n}(s,\cdot)\|^2_{L^p_\lambda(\mathbb{R})}\right)\mathrm{d}s. \qquad (3.21)
\end{aligned}$$

Thus, (3.15), (3.19), (3.20) and (3.21) yield that

$$\begin{aligned}
\|u_\varepsilon^{g,n+1}(t,\cdot)\|^2_{L^p_\lambda(\mathbb{R})} &= \left(\int_{\mathbb{R}} |u_\varepsilon^{g,n+1}(t,x)|^p\lambda(x)\mathrm{d}x\right)^{\frac{2}{p}} \\
&\lesssim 1+\int_0^t \left((t-s)^{H-1}+(t-s)^{-\frac{1}{2}}\right)\cdot
\end{aligned}$$

$$\|u_\varepsilon^{g,n}(s,\cdot)\|^2_{L^p_\lambda(\mathbb{R})}\mathrm{d}s$$
$$+\int_0^t (t-s)^{-1/2}\left[\mathcal{N}^*_{\frac{1}{2}-H,\,p}u_\varepsilon^{g,n}(s)\right]^2\mathrm{d}s. \tag{3.22}$$

Step 3 This step is devoted to estimating $\mathcal{N}^*_{\frac{1}{2}-H,\,p}u_\varepsilon^{g,n}(t)$. Using the Cauchy–Schwarz inequality and (3.6), we have

$$\begin{aligned}
&\left|u_\varepsilon^{g,n+1}(t,x)-u_\varepsilon^{g,n+1}(t,x+h)\right|^p\\
&\lesssim\left(\int_0^t\left\|D_{t-s}(x-\cdot,h)\sigma(s,\cdot,u_\varepsilon^{g,n}(s,\cdot))\right\|^2_{\mathcal{H}_\varepsilon}\mathrm{d}s\right)^{\frac{p}{2}}\\
&\lesssim\left(\int_0^t\left\|D_{t-s}(x-\cdot,h)\sigma(s,\cdot,u_\varepsilon^{g,n}(s,\cdot))\right\|^2_{\mathcal{H}}\mathrm{d}s\right)^{\frac{p}{2}}\\
&\simeq\Bigg(\int_0^t\int_{\mathbb{R}^2}\Big|D_{t-s}(x-y-z,h)\sigma(s,y+z,u_\varepsilon^{g,n}(s,y+z))\\
&\qquad-D_{t-s}(x-z,h)\sigma(s,z,u_\varepsilon^{g,n}(s,z))\Big|^2\cdot|y|^{2H-2}\mathrm{d}z\mathrm{d}y\mathrm{d}s\Bigg)^{\frac{p}{2}}\\
&\lesssim\mathcal{I}_1(t,x,h)+\mathcal{I}_2(t,x,h)+\mathcal{I}_3(t,x,h),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{I}_1(t,x,h):=&\Bigg(\int_0^t\int_{\mathbb{R}^2}|D_{t-s}(x-y-z,h)|^2\\
&\cdot\left|\sigma(s,y+z,u_\varepsilon^{g,n}(s,y+z))-\sigma(s,z,u_\varepsilon^{g,n}(s,y+z))\right|^2\cdot\\
&|y|^{2H-2}\mathrm{d}z\mathrm{d}y\mathrm{d}s\Bigg)^{\frac{p}{2}};\\
\mathcal{I}_2(t,x,h):=&\Bigg(\int_0^t\int_{\mathbb{R}^2}|D_{t-s}(x-y-z,h)|^2\\
&\cdot\left|\sigma(s,z,u_\varepsilon^{g,n}(s,y+z))-\sigma(s,z,u_\varepsilon^{g,n}(s,z))\right|^2\cdot\\
&|y|^{2H-2}\mathrm{d}z\mathrm{d}y\mathrm{d}s\Bigg)^{\frac{p}{2}};\\
\mathcal{I}_3(t,x,h):=&\Bigg(\int_0^t\int_{\mathbb{R}^2}|\Box_{t-s}(x-z,y,h)|^2\cdot\left|\sigma(s,z,u_\varepsilon^{g,n}(s,z))\right|^2\cdot\\
&|y|^{2H-2}\mathrm{d}z\mathrm{d}y\mathrm{d}s\Bigg)^{\frac{p}{2}}.
\end{aligned}$$

Therefore, by (3.3), we have

$$
\begin{aligned}
&\left[\mathcal{N}^*_{\frac{1}{2}-H,\,p} u_\varepsilon^{g,n+1}(t)\right]^2 \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| u_\varepsilon^{g,n+1}(t,x) - u_\varepsilon^{g,n+1}(t,x+h) \right|^p \lambda(x) \mathrm{d}x \right)^{\frac{2}{p}} \cdot |h|^{2H-2} \mathrm{d}h \\
&\lesssim \sum_{j=1}^{3} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathcal{I}_j(t,x,h) \lambda(x) \mathrm{d}x \right)^{\frac{2}{p}} \cdot |h|^{2H-2} \mathrm{d}h.
\end{aligned} \tag{3.23}
$$

For the first term of (3.23), by (3.16), (3.17) and a change of variable, we have

$$
\begin{aligned}
\mathcal{I}_1(t,x,h) \lesssim& \left(\int_0^t \int_{\mathbb{R}} \int_{|y|>1} |D_{t-s}(x-y-z,h)|^2 \cdot \left(1 + \left|u_\varepsilon^{g,n}(s,y+z)\right|^2\right) \cdot |y|^{2H-2} \mathrm{d}z\mathrm{d}y\mathrm{d}s \right)^{\frac{p}{2}} \\
&+ \left(\int_0^t \int_{\mathbb{R}} \int_{|y|\le 1} |D_{t-s}(x-y-z,h)|^2 \cdot \left(1 + \left|u_\varepsilon^{g,n}(s,y+z)\right|^2\right) \cdot |y|^{2H} \mathrm{d}z\mathrm{d}y\mathrm{d}s \right)^{\frac{p}{2}} \\
=& \left(\int_0^t \int_{\mathbb{R}} \int_{|y|>1} |D_{t-s}(x-z,h)|^2 \cdot \left(1 + \left|u_\varepsilon^{g,n}(s,z)\right|^2\right) \cdot |y|^{2H-2} \mathrm{d}z\mathrm{d}y\mathrm{d}s \right)^{\frac{p}{2}} \\
&+ \left(\int_0^t \int_{\mathbb{R}} \int_{|y|\le 1} |D_{t-s}(x-z,h)|^2 \cdot \left(1 + \left|u_\varepsilon^{g,n}(s,z)\right|^2\right) \cdot |y|^{2H} \mathrm{d}z\mathrm{d}y\mathrm{d}s \right)^{\frac{p}{2}} \\
\lesssim& \left(\int_0^t \int_{\mathbb{R}} |p_{t-s}(z-h) - p_{t-s}(z)|^2 \cdot \left(1 + \left|u_\varepsilon^{g,n}(s,x+z)\right|^2\right) \mathrm{d}z\mathrm{d}s \right)^{\frac{p}{2}}.
\end{aligned} \tag{3.24}
$$

By (3.24), a change of variable, Minkowski's inequality, Lemma 6.5 and Jensen's inequality with respect to $(t-s)^{1-H} |D_{t-s}(z,h)|^2 |h|^{2H-2} \mathrm{d}z\mathrm{d}h$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \mathcal{I}_1(t,x,h) \lambda(x) \mathrm{d}x \right|^{\frac{2}{p}} \cdot |h|^{2H-2} \mathrm{d}h \\
&\lesssim \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \left(\int_0^t \int_{\mathbb{R}} |D_{t-s}(z,h)|^2 \left(1 + \left|u_\varepsilon^{g,n}(s,x)\right|^2\right) \mathrm{d}z\mathrm{d}s \right)^{\frac{p}{2}} \lambda(x-z) \mathrm{d}x \right|^{\frac{2}{p}} \cdot \\
&\quad |h|^{2H-2} \mathrm{d}h \\
&\lesssim \int_0^t \int_{\mathbb{R}^2} |D_{t-s}(z,h)|^2 \cdot |h|^{2H-2} \left(\int_{\mathbb{R}} \left(1 + \left|u_\varepsilon^{g,n}(s,x)\right|^p\right) \lambda(x-z) \mathrm{d}x \right)^{\frac{2}{p}} \mathrm{d}z\mathrm{d}h\mathrm{d}s \\
&\lesssim \int_0^t (t-s)^{H-1} \cdot \left(\int_{\mathbb{R}^3} (t-s)^{1-H} \cdot |D_{t-s}(z,h)|^2 \cdot |h|^{2H-2} \cdot \left(1 + \left|u_\varepsilon^{g,n}(s,x)\right|^p\right) \right. \\
&\quad \left. \lambda(x-z) \mathrm{d}x\mathrm{d}z\mathrm{d}h \right)^{\frac{2}{p}} \mathrm{d}s \\
&\lesssim \int_0^t (t-s)^{H-1} \cdot \left(1 + \left\|u_\varepsilon^{g,n}(s,\cdot)\right\|^2_{L^p_\lambda(\mathbb{R})}\right) \mathrm{d}s.
\end{aligned} \tag{3.25}
$$

For the second term of (3.23), by (2.18) and a change of variable, we have

$$
\begin{aligned}
\mathcal{I}_2(t,x,h) &\lesssim \Bigg(\int_0^t\int_{\mathbb{R}^2}|D_{t-s}(x-y-z,h)|^2\cdot\left|u_\varepsilon^{g,n}(s,y+z)-u_\varepsilon^{g,n}(s,z)\right|^2\cdot\\
&\quad |y|^{2H-2}\mathrm{d}z\mathrm{d}y\mathrm{d}s\Bigg)^{\frac{p}{2}}\\
&= \Bigg(\int_0^t\int_{\mathbb{R}^2}|D_{t-s}(x-z,h)|^2\cdot\left|u_\varepsilon^{g,n}(s,z)-u_\varepsilon^{g,n}(s,z-y)\right|^2\cdot\\
&\quad |y|^{2H-2}\mathrm{d}z\mathrm{d}y\mathrm{d}s\Bigg)^{\frac{p}{2}}\\
&= \Bigg(\int_0^t\int_{\mathbb{R}^2}|D_{t-s}(x-z,h)|^2\cdot\left|u_\varepsilon^{g,n}(s,y+z)-u_\varepsilon^{g,n}(s,z)\right|^2\cdot\\
&\quad |y|^{2H-2}\mathrm{d}z\mathrm{d}y\mathrm{d}s\Bigg)^{\frac{p}{2}}\\
&= \Bigg(\int_0^t\int_{\mathbb{R}^2}|p_{t-s}(z-h)-p_{t-s}(z)|^2\\
&\quad \cdot\left|u_\varepsilon^{g,n}(s,x+y+z)-u_\varepsilon^{g,n}(s,x+z)\right|^2\cdot|y|^{2H-2}\mathrm{d}z\mathrm{d}y\mathrm{d}s\Bigg)^{\frac{p}{2}}.
\end{aligned}
\tag{3.26}
$$

By (3.26), a change of variable, Minkowski's inequality, Jensen's inequality and Lemma 6.5, we have

$$
\begin{aligned}
&\int_{\mathbb{R}}\left|\int_{\mathbb{R}}\mathcal{I}_2(t,x,h)\lambda(x)\mathrm{d}x\right|^{\frac{2}{p}}\cdot|h|^{2H-2}\mathrm{d}h\\
&\lesssim \int_{\mathbb{R}}\Bigg|\int_{\mathbb{R}}\Bigg(\int_0^t\int_{\mathbb{R}^2}|D_{t-s}(z,h)|^2\cdot\left|u_\varepsilon^{g,n}(s,x+y)-u_\varepsilon^{g,n}(s,x)\right|^2\cdot\\
&\quad |y|^{2H-2}\mathrm{d}z\mathrm{d}y\mathrm{d}s\Bigg)^{\frac{p}{2}}\cdot\lambda(x-z)\mathrm{d}x\Bigg|^{\frac{2}{p}}\cdot|h|^{2H-2}\mathrm{d}h\\
&\lesssim \int_0^t\int_{\mathbb{R}^3}|D_{t-s}(z,h)|^2\cdot|y|^{2H-2}\cdot|h|^{2H-2}\\
&\quad\cdot\left(\int_{\mathbb{R}}\left|u_\varepsilon^{g,n}(s,x+y)-u_\varepsilon^{g,n}(s,x)\right|^p\lambda(x-z)\mathrm{d}x\right)^{\frac{2}{p}}\mathrm{d}z\mathrm{d}y\mathrm{d}h\mathrm{d}s\\
&\lesssim \int_0^t\int_{\mathbb{R}}(t-s)^{H-1}\cdot\Bigg(\int_{\mathbb{R}^3}(t-s)^{1-H}\cdot|D_{t-s}(z,h)|^2\cdot|h|^{2H-2}\\
&\quad\cdot\left|u_\varepsilon^{g,n}(s,x+y)-u_\varepsilon^{g,n}(s,x)\right|^p\lambda(x-z)\mathrm{d}x\mathrm{d}z\mathrm{d}h\Bigg)^{\frac{2}{p}}\cdot|y|^{2H-2}\mathrm{d}y\mathrm{d}s
\end{aligned}
$$

$$\lesssim \int_0^t (t-s)^{H-1} \cdot \left[\mathcal{N}^*_{\frac{1}{2}-H,\,p} u_\varepsilon^{g,n}(s)\right]^2 \mathrm{d}s. \tag{3.27}$$

For the last term of (3.23), by (2.17) and a change of variable, we have

$$\begin{aligned}
\mathcal{I}_3(t,x,h) &\lesssim \left(\int_0^t \int_{\mathbb{R}^2} |\Box_{t-s}(x-z,y,h)|^2 \cdot \left(1+\left|u_\varepsilon^{g,n}(s,z)\right|^2\right) \cdot |y|^{2H-2} \mathrm{d}z\mathrm{d}y\mathrm{d}s\right)^{\frac{p}{2}} \\
&= \left(\int_0^t \int_{\mathbb{R}^2} |\Box_{t-s}(z,y,h)|^2 \cdot \left(1+\left|u_\varepsilon^{g,n}(s,x-z)\right|^2\right) \cdot |y|^{2H-2} \mathrm{d}z\mathrm{d}y\mathrm{d}s\right)^{\frac{p}{2}} \\
&\le \left(\int_0^t \int_{\mathbb{R}^2} |\Box_{t-s}(-z,y,h)|^2 \cdot |u_\varepsilon^{g,n}(s,x+z)-u_\varepsilon^{g,n}(s,x)|^2 \cdot |y|^{2H-2} \mathrm{d}z\mathrm{d}y\mathrm{d}s\right)^{\frac{p}{2}} \\
&+ \left(\int_0^t \int_{\mathbb{R}^2} |\Box_{t-s}(-z,y,h)|^2 \cdot \left(1+\left|u_\varepsilon^{g,n}(s,x)\right|^2\right) \cdot |y|^{2H-2} \mathrm{d}z\mathrm{d}y\mathrm{d}s\right)^{\frac{p}{2}} \\
&=: \mathcal{I}_{31}(t,x,h) + \mathcal{I}_{32}(t,x,h).
\end{aligned}$$

By a change of variable, Minkowski's inequality and Lemma 6.4, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \left|\int_{\mathbb{R}} \mathcal{I}_{31}(t,x,h)\lambda(x)\mathrm{d}x\right|^{\frac{2}{p}} \cdot |h|^{2H-2}\mathrm{d}h \\
&\lesssim \int_0^t \int_{\mathbb{R}^3} |\Box_{t-s}(z,y,h)|^2 \cdot \left(\int_{\mathbb{R}} |u_\varepsilon^{g,n}(s,x+z)-u_\varepsilon^{g,n}(s,x)|^p \lambda(x)\mathrm{d}x\right)^{\frac{2}{p}} \\
&\quad \cdot |h|^{2H-2} \cdot |y|^{2H-2} \mathrm{d}h\mathrm{d}z\mathrm{d}y\mathrm{d}s \\
&\lesssim \int_0^t (t-s)^{H-1} \cdot \left[\mathcal{N}^*_{\frac{1}{2}-H,\,p} u_\varepsilon^{g,n}(s)\right]^2 \mathrm{d}s.
\end{aligned} \tag{3.28}$$

By a change of variable, Minkowski's inequality and Lemma 6.2, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \left|\int_{\mathbb{R}} \mathcal{I}_{32}(t,x,h)\lambda(x)\mathrm{d}x\right|^{\frac{2}{p}} \cdot |h|^{2H-2}\mathrm{d}h \\
&\lesssim \int_0^t \int_{\mathbb{R}^3} |\Box_{t-s}(z,y,h)|^2 \cdot \left(1+\int_{\mathbb{R}} \left|u_\varepsilon^{g,n}(s,x)\right|^p \lambda(x)\mathrm{d}x\right)^{\frac{2}{p}} \cdot \\
&\quad |h|^{2H-2} \cdot |y|^{2H-2} \mathrm{d}h\mathrm{d}z\mathrm{d}y\mathrm{d}s \\
&\lesssim \int_0^t (t-s)^{2H-\frac{3}{2}} \cdot \left(1+\|u_\varepsilon^{g,n}(s,\cdot)\|^2_{L^p_\lambda(\mathbb{R})}\right) \mathrm{d}s.
\end{aligned} \tag{3.29}$$

Thus, by (3.23), (3.25), (3.27), (3.28) and (3.29), we have

$$\begin{aligned}&\left[\mathcal{N}^*_{\frac{1}{2}-H,\,p}u^{g,n+1}_\varepsilon(t)\right]^2\\&\quad\lesssim 1+\int_0^t\left((t-s)^{2H-\frac{3}{2}}+(t-s)^{H-1}\right)\cdot\left\|u^{g,n}_\varepsilon(s,\cdot)\right\|^2_{L^p_\lambda(\mathbb{R})}\mathrm{d}s\\&+\int_0^t(t-s)^{H-1}\cdot\left[\mathcal{N}^*_{\frac{1}{2}-H,\,p}u^{g,n}_\varepsilon(s)\right]^2\mathrm{d}s.\end{aligned}\tag{3.30}$$

Step 4 Let

$$\Psi^n_\varepsilon(t):=\|u^{g,n}_\varepsilon(t,\cdot)\|^2_{L^p_\lambda(\mathbb{R})}+\left[\mathcal{N}^*_{\frac{1}{2}-H,\,p}u^{g,n}_\varepsilon(t)\right]^2.$$

Putting (3.22) and (3.30) together, there exists a constant $C_{T,p,H,N}>0$ such that

$$\Psi^{n+1}_\varepsilon(t)\le C_{T,p,H,N}\left(1+\int_0^t(t-s)^{2H-\frac{3}{2}}\Psi^n_\varepsilon(s)\mathrm{d}s\right).$$

By the extension of Grönwall's lemma ([11, Lemma 15]), we have

$$\sup_{n\ge1}\sup_{t\in[0,T]}\Psi^n_\varepsilon(t)\le C,\tag{3.31}$$

where C is a constant independent of ε and $g\in S^N$.

For any fixed $\varepsilon>0$, by (3.14) and (3.31), we have that, for any $t\in[0,T]$,

$$\|u^g_\varepsilon(t,\cdot)\|_{L^p_\lambda(\mathbb{R})}=\lim_{n\to\infty}\left(\int_{\mathbb{R}}\left[|u^{g,n}_\varepsilon(t,x)|^p\right]\lambda(x)\mathrm{d}x\right)^{\frac{1}{p}}\le C.\tag{3.32}$$

For any fixed t and h, we have by (3.14),

$$\|u^{g,n}_\varepsilon(t,\cdot+h)-u^{g,n}_\varepsilon(t,\cdot)\|_{L^p_\lambda(\mathbb{R})}\to\|u^g_\varepsilon(t,\cdot+h)-u^g_\varepsilon(t,\cdot)\|_{L^p_\lambda(\mathbb{R})},\ \text{as } n\to\infty.$$

By Fatou's lemma and (3.31), we have

$$\begin{aligned}\mathcal{N}^*_{\frac{1}{2}-H,\,p}u^g_\varepsilon(t)&=\int_{\mathbb{R}}\|u^g_\varepsilon(t,\cdot+h)-u^g_\varepsilon(t,\cdot)\|^2_{L^p_\lambda(\mathbb{R})}|h|^{2H-2}\mathrm{d}h\\&\le\liminf_{n\to\infty}\int_{\mathbb{R}}\|u^{g,n}_\varepsilon(t,\cdot+h)-u^{g,n}_\varepsilon(t,\cdot)\|^2_{L^p_\lambda(\mathbb{R})}|h|^{2H-2}\mathrm{d}h\le C.\end{aligned}\tag{3.33}$$

By (3.32) and (3.33), we obtain that

$$\sup_{g\in S^N}\sup_{\varepsilon>0}\|u^g_\varepsilon\|_{Z^p_{\lambda,T}}\le\sup_{g\in S^N}\sup_{\varepsilon>0}\sup_{t\in[0,T]}\|u^g_\varepsilon(t,\cdot)\|_{L^p_\lambda(\mathbb{R})}+\sup_{g\in S^N}\sup_{\varepsilon>0}\sup_{t\in[0,T]}\mathcal{N}^*_{\frac{1}{2}-H,\,p}u^g_\varepsilon(t)<\infty.$$

The proof is complete. □

Lemma 3.2 *Let u_ε^g be the approximate process defined by (3.7).*

(i) *If $p > \frac{6}{4H-1}$, then there exists a constant $C_{T,p,H,N} > 0$ such that*

$$\sup_{t\in[0,T],\, x\in\mathbb{R}} \lambda^{\frac{1}{p}}(x)\cdot \mathcal{N}_{\frac{1}{2}-H} u_\varepsilon^g(t,x) \le C_{T,p,H,N}\left(1+\|u_\varepsilon^g\|_{Z^p_{\lambda,T}}\right). \tag{3.34}$$

(ii) *If $p > \frac{3}{H}$ and $0<\gamma<\frac{H}{2}-\frac{3}{2p}$, then there exists a constant $C_{T,p,H,N,\gamma} > 0$ such that*

$$\begin{aligned}\sup_{\substack{t,\,t+h\in[0,T],\\ x\in\mathbb{R}}} & \lambda^{\frac{1}{p}}(x)\cdot |u_\varepsilon^g(t+h,x)-u_\varepsilon^g(t,x)| \\ &\le C_{T,p,H,N,\gamma}|h|^\gamma\cdot\left(1+\|u_\varepsilon^g\|_{Z^p_{\lambda,T}}\right).\end{aligned} \tag{3.35}$$

(iii) *If $p > \frac{3}{H}$ and $0<\gamma<H-\frac{3}{p}$, then there exists a constant $C_{T,p,H,N,\gamma} > 0$ such that*

$$\sup_{\substack{t\in[0,T],\\ x,\,y\in\mathbb{R}}} \frac{|u_\varepsilon^g(t,x)-u_\varepsilon^g(t,y)|}{\lambda^{-\frac{1}{p}}(x)+\lambda^{-\frac{1}{p}}(y)} \le C_{T,p,H,N,\gamma}|x-y|^\gamma\cdot\left(1+\|u_\varepsilon^g\|_{Z^p_{\lambda,T}}\right). \tag{3.36}$$

Proof (i) By (3.7), we have

$$\begin{aligned}u_\varepsilon^g(t,x+h)-u_\varepsilon^g(t,x) &= \int_0^t \left\langle p_{t-s}(x+h-\cdot)\sigma(s,\cdot,u_\varepsilon^g(s,\cdot)), g(s,\cdot)\right\rangle_{\mathcal{H}_\varepsilon}\mathrm{d}s \\ &\quad - \int_0^t \left\langle p_{t-s}(x-\cdot)\sigma(s,\cdot,u_\varepsilon^g(s,\cdot)), g(s,\cdot)\right\rangle_{\mathcal{H}_\varepsilon}\mathrm{d}s \\ &=: \Phi(t,x+h)-\Phi(t,x).\end{aligned}$$

Applying (3.5) and the Fubini theorem, we have that, for any $\alpha\in(0,1)$,

$$\begin{aligned}\Phi(t,x) &= \int_0^t \langle p_{t-s}(x-\cdot)\sigma(s,\cdot,u_\varepsilon^g(s,\cdot)), g(s,\cdot)\rangle_{\mathcal{H}_\varepsilon}\mathrm{d}s \\ &\simeq \int_0^t\int_{\mathbb{R}^2} p_{t-s}(x-y)\sigma(s,y,u_\varepsilon^g(s,y))g(s,\tilde{y})f_\varepsilon(y-\tilde{y})\mathrm{d}y d\tilde{y}\mathrm{d}s \\ &= \frac{\sin(\pi\alpha)}{\pi}\int_0^t\int_{\mathbb{R}^2} p_{t-s}(x-y)\left[\int_s^t (t-r)^{\alpha-1}(r-s)^{-\alpha}\mathrm{d}r\right] \\ &\quad \cdot\sigma(s,y,u_\varepsilon^g(s,y))g(s,\tilde{y})f_\varepsilon(y-\tilde{y})\mathrm{d}y d\tilde{y}\mathrm{d}s \\ &= \frac{\sin(\pi\alpha)}{\pi}\int_0^t\int_{\mathbb{R}^2}\left(\int_{\mathbb{R}} p_{t-r}(x-z)p_{r-s}(z-y)\mathrm{d}z\right)\end{aligned}$$

$$
\begin{aligned}
&\left(\int_s^t (t-r)^{\alpha-1}(r-s)^{-\alpha}\mathrm{d}r\right)\\
&\cdot\sigma(s,y,u_\varepsilon^g(s,y))g(s,\tilde{y})f_\varepsilon(y-\tilde{y})\mathrm{d}yd\tilde{y}\mathrm{d}s\\
\simeq&\int_0^t\int_{\mathbb{R}}(t-r)^{\alpha-1}\cdot p_{t-r}(x-z)\\
&\cdot\left(\int_0^r (r-s)^{-\alpha}\cdot\int_{\mathbb{R}^2} p_{r-s}(z-y)\sigma(s,y,u_\varepsilon^g(s,y))\right.\\
&g(s,\tilde{y})f_\varepsilon(y-\tilde{y})\mathrm{d}yd\tilde{y}\mathrm{d}s)\,\mathrm{d}z\mathrm{d}r\\
\simeq&\int_0^t\int_{\mathbb{R}}(t-r)^{\alpha-1}\cdot p_{t-r}(x-z)\\
&\cdot\left(\int_0^r (r-s)^{-\alpha}\langle p_{r-s}(z-\cdot)\sigma(s,\cdot,u_\varepsilon^g(s,\cdot)),g(s,\cdot)\rangle_{\mathcal{H}_\varepsilon}\mathrm{d}s\right)\mathrm{d}z\mathrm{d}r\\
=&\int_0^t\int_{\mathbb{R}}(t-r)^{\alpha-1}\cdot p_{t-r}(x-z)J_\alpha(r,z)\mathrm{d}z\mathrm{d}r, \qquad (3.37)
\end{aligned}
$$

where

$$
J_\alpha(r,z):=\int_0^r (r-s)^{-\alpha}\langle p_{r-s}(z-\cdot)\sigma(s,\cdot,u_\varepsilon^g(s,\cdot)),g(s,\cdot)\rangle_{\mathcal{H}_\varepsilon}\mathrm{d}s.
$$

Set

$$
\Delta_h J_\alpha(t,x):=J_\alpha(t,x+h)-J_\alpha(t,x).
$$

Applying a change of variable, we have

$$
\begin{aligned}
&\Phi(t,x+h)-\Phi(t,x)\\
\simeq&\int_0^t\int_{\mathbb{R}}(t-r)^{\alpha-1}\cdot p_{t-r}(x+h-z)\cdot J_\alpha(r,z)\mathrm{d}z\mathrm{d}r\\
&-\int_0^t\int_{\mathbb{R}}(t-r)^{\alpha-1}\cdot p_{t-r}(x-z)\cdot J_\alpha(r,z)\mathrm{d}z\mathrm{d}r\\
=&\int_0^t\int_{\mathbb{R}}(t-r)^{\alpha-1}\cdot p_{t-r}(x-z)\cdot J_\alpha(r,z+h)\mathrm{d}z\mathrm{d}r\\
&-\int_0^t\int_{\mathbb{R}}(t-r)^{\alpha-1}\cdot p_{t-r}(x-z)\cdot J_\alpha(r,z)\mathrm{d}z\mathrm{d}r\\
=&\int_0^t\int_{\mathbb{R}}(t-r)^{\alpha-1}\cdot p_{t-r}(x-z)\cdot\Delta_h J_\alpha(r,z)\mathrm{d}z\mathrm{d}r.
\end{aligned}
$$

Invoking Minkowski's inequality and Hölder's inequality with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$
\int_{\mathbb{R}}|\Phi(t,x+h)-\Phi(t,x)|^2\cdot|h|^{2H-2}\mathrm{d}h
$$

$$
\begin{aligned}
&\simeq \int_{\mathbb{R}} \left| \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} \cdot p_{t-r}(x-z) \Delta_h J_\alpha(r,z) \mathrm{d}z \mathrm{d}r \right|^2 \cdot |h|^{2H-2} \mathrm{d}h \\
&\lesssim \left(\int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} \cdot p_{t-r}(x-z) \left[\int_{\mathbb{R}} |\Delta_h J_\alpha(r,z)|^2 \cdot |h|^{2H-2} \mathrm{d}h \right]^{\frac{1}{2}} \mathrm{d}z \mathrm{d}r \right)^2 \\
&\lesssim \left(\int_0^t \int_{\mathbb{R}} (t-r)^{q(\alpha-1)} \cdot p_{t-r}^q(x-z) \lambda^{-\frac{q}{p}}(z) \mathrm{d}z \mathrm{d}r \right)^{\frac{2}{q}} \\
&\quad \cdot \left(\int_0^T \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |\Delta_h J_\alpha(r,z)|^2 \cdot |h|^{2H-2} \mathrm{d}h \right]^{\frac{p}{2}} \lambda(z) \mathrm{d}z \mathrm{d}r \right)^{\frac{2}{p}} \\
&\lesssim \lambda^{-\frac{2}{p}}(x) \left(\int_0^t (t-r)^{q(\alpha-\frac{3}{2}+\frac{1}{2q})} \mathrm{d}r \right)^{\frac{2}{q}} \\
&\quad \cdot \left(\int_0^T \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |\Delta_h J_\alpha(r,z)|^2 \cdot |h|^{2H-2} \mathrm{d}h \right]^{\frac{p}{2}} \lambda(z) \mathrm{d}z \mathrm{d}r \right)^{\frac{2}{p}}, \qquad (3.38)
\end{aligned}
$$

where in the last step we have used Lemma 6.1 and the following fact:

$$
p_{t-r}^q(x-z) \simeq (t-r)^{\frac{1-q}{2}} p_{\frac{t-r}{q}}(x-z). \qquad (3.39)
$$

If $q(\alpha - \frac{3}{2} + \frac{1}{2q}) > -1$, namely if $\alpha > \frac{3}{2p}$, then

$$
\begin{aligned}
&\sup_{t\in[0,T],x\in\mathbb{R}} \lambda^{\frac{1}{p}}(x) \left(\int_{\mathbb{R}} |\Phi(t,x+h) - \Phi(t,x)|^2 \cdot |h|^{2H-2} \mathrm{d}h \right)^{\frac{1}{2}} \\
&\quad \lesssim \left(\int_0^T \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |\Delta_h J_\alpha(r,z)|^2 \cdot |h|^{2H-2} \mathrm{d}h \right]^{\frac{p}{2}} \lambda(z) \mathrm{d}z \mathrm{d}r \right)^{\frac{1}{p}}.
\end{aligned} \qquad (3.40)
$$

Thus, to prove part (i) we only need to prove that there exists some positive constant C, independent of $r \in [0,T]$, such that for $\frac{3}{2p} < \alpha < 1$,

$$
\int_{\mathbb{R}} \left[\int_{\mathbb{R}} |\Delta_h J_\alpha(r,z)|^2 \cdot |h|^{2H-2} \mathrm{d}h \right]^{\frac{p}{2}} \lambda(z) \mathrm{d}z \le C \left(1 + \left\| u_\varepsilon^g \right\|_{Z_{\lambda,T}^p}^p \right). \qquad (3.41)
$$

Now, it remains to show (3.41). Recall $D_t(x,h)$ defined by (3.8). Applying the Cauchy–Schwarz inequality, (2.6), (3.5) and (3.12), we have

$$
\Delta_h J_\alpha(r,z) \le \int_0^r (r-s)^{-\alpha} \cdot \left\| D_{r-s}(z-\cdot,h)\sigma(s,\cdot,u_\varepsilon^g(s,\cdot)) \right\|_{\mathcal{H}_\varepsilon} \cdot \left\| g(s,\cdot) \right\|_{\mathcal{H}_\varepsilon} \mathrm{d}s
$$

$$\begin{aligned}
&\lesssim \left(\int_0^r (r-s)^{-2\alpha} \cdot \left\| D_{r-s}(z-\cdot,h)\sigma(s,\cdot,u_\varepsilon^g(s,\cdot)) \right\|_{\mathcal{H}_\varepsilon}^2 \mathrm{d}s\right)^{\frac{1}{2}} \cdot \\
&\quad \left(\int_0^r \left\| g(s,\cdot) \right\|_{\mathcal{H}_\varepsilon} \mathrm{d}s\right)^{\frac{1}{2}} \\
&\lesssim \left(\int_0^r (r-s)^{-2\alpha} \cdot \left\| D_{r-s}(z-\cdot,h)\sigma(s,\cdot,u_\varepsilon^g(s,\cdot)) \right\|_{\mathcal{H}_\varepsilon}^2 \mathrm{d}s\right)^{\frac{1}{2}} \\
&\lesssim \Bigg(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \cdot \Big| D_{r-s}(z-y-l,h)\sigma(s,y+l,u_\varepsilon^g(s,y+l)) \\
&\quad - D_{r-s}(z-y,h)\sigma(s,y,u_\varepsilon^g(s,y)) \Big|^2 \cdot |l|^{2H-2} \mathrm{d}l\mathrm{d}y\mathrm{d}s\Bigg)^{\frac{1}{2}}.
\end{aligned}$$

Set

$$\begin{aligned}
\mathcal{J}_1(r,z,h) :=& \Bigg(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \cdot \left| D_{r-s}(z-y-l,h) \right|^2 \\
& \cdot \left| \sigma(s,y+l,u_\varepsilon^g(s,y+l)) - \sigma(s,y,u_\varepsilon^g(s,y+l)) \right|^2 \\
& \cdot |l|^{2H-2} \mathrm{d}l\mathrm{d}y\mathrm{d}s\Bigg)^{\frac{p}{2}}; \\
\mathcal{J}_2(r,z,h) :=& \Bigg(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \cdot \left| D_{r-s}(z-y-l,h) \right|^2 \\
& \cdot \left| \sigma(s,y,u_\varepsilon^g(s,y+l)) - \sigma(s,y,u_\varepsilon^g(s,y)) \right|^2 \cdot |l|^{2H-2} \mathrm{d}l\mathrm{d}y\mathrm{d}s\Bigg)^{\frac{p}{2}}; \\
\mathcal{J}_3(r,z,h) :=& \Bigg(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \cdot \left| \Box_{r-s}(z-y,l,h) \right|^2 \cdot \\
& \left| \sigma(s,y,u_\varepsilon^g(s,y)) \right|^2 \cdot |l|^{2H-2} \mathrm{d}l\mathrm{d}y\mathrm{d}s\Bigg)^{\frac{p}{2}}.
\end{aligned}$$

By Minkowski's inequality, we have

$$\int_{\mathbb{R}} \left[\int_{\mathbb{R}} |\Delta_h J_\alpha(r,z)|^2 \cdot |h|^{2H-2} dh\right]^{\frac{p}{2}} \lambda(z)dz \lesssim \sum_{i=1}^3 \left[\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathcal{J}_i(r,z,h)\lambda(z)\mathrm{d}z\right)^{\frac{2}{p}} |h|^{2H-2}\mathrm{d}h\right]^{\frac{p}{2}}. \tag{3.42}$$

By the same technique as that in Step 2 of the proof for Lemma 3.1, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathcal{J}_1(r,z,h)\lambda(z)\mathrm{d}z\right)^{\frac{2}{p}} |h|^{2H-2}\mathrm{d}h \\
&\quad \lesssim \int_0^r (t-s)^{-2\alpha+H-1} \cdot \left(1 + \|u_\varepsilon^g(s,\cdot)\|_{L_\lambda^p(\mathbb{R})}^2\right) \mathrm{d}s;
\end{aligned} \tag{3.43}$$

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\mathcal{J}_2(r,z,h)\lambda(z)\mathrm{d}z\right)^{\frac{2}{p}}|h|^{2H-2}\mathrm{d}h \lesssim \int_0^r (t-s)^{-2\alpha+H-1}\cdot\left[\mathcal{N}^*_{\frac{1}{2}-H,\,p}u^g_\varepsilon(s)\right]^2\mathrm{d}s; \tag{3.44}$$

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\mathcal{J}_3(r,z,h)\lambda(z)\mathrm{d}z\right)^{\frac{2}{p}}|h|^{2H-2}\mathrm{d}h \lesssim \int_0^r (t-s)^{-2\alpha+2H-\frac{3}{2}}\cdot\left(1+\|u^g_\varepsilon(s,\cdot)\|^2_{L^p_\lambda(\mathbb{R})}\right)\mathrm{d}s + \int_0^r (t-s)^{-2\alpha+H-1}\cdot\left[\mathcal{N}^*_{\frac{1}{2}-H,\,p}u^g_\varepsilon(s)\right]^2\mathrm{d}s. \tag{3.45}$$

By (3.42), (3.43), (3.44), (3.45) and by using the same method as that in the proof of (4.27) in [22], we have

$$\int_{\mathbb{R}}\left[\int_{\mathbb{R}}|\Delta_h J_\alpha(r,z)|^2\cdot|h|^{2H-2}\mathrm{d}h\right]^{\frac{p}{2}}\lambda(z)\mathrm{d}z \le C\left(\int_0^r (r-s)^{-2\alpha+2H-\frac{3}{2}}+(r-s)^{-2\alpha+H-1}\mathrm{d}s\right)^{\frac{p}{2}}\left(1+\|u^g_\varepsilon\|^p_{Z^p_{\lambda,T}}\right). \tag{3.46}$$

If $-2\alpha+2H-\frac{3}{2}>-1$ and $-2\alpha+H-1>-1$, namely if $\alpha<H-\frac{1}{4}$, then we see that (3.41) follows from (3.46). This condition on α is combined with $\alpha>\frac{3}{2p}$ to become $\frac{3}{2p}<\alpha<H-\frac{1}{4}$. Therefore, we have proved that for any $p>\frac{6}{4H-1}$, (3.34) holds.

(ii) By (3.7) and (3.37), we have

$$\begin{aligned}
&\Phi(t+h,x)-\Phi(t,x)\\
&\simeq \int_0^t\int_{\mathbb{R}}(t+h-r)^{\alpha-1}p_{t+h-r}(x-z)J_\alpha(r,z)\mathrm{d}z\mathrm{d}r\\
&\quad-\int_0^t\int_{\mathbb{R}}(t-r)^{\alpha-1}p_{t-r}(x-z)J_\alpha(r,z)\mathrm{d}z\mathrm{d}r\\
&\quad+\int_t^{t+h}\int_{\mathbb{R}}(t+h-r)^{\alpha-1}p_{t+h-r}(x-z)J_\alpha(r,z)\mathrm{d}z\mathrm{d}r\\
&=\int_0^t\int_{\mathbb{R}}(t+h-r)^{\alpha-1}\left(p_{t+h-r}(x-z)-p_{t-r}(x-z)\right)J_\alpha(r,z)\mathrm{d}z\mathrm{d}r\\
&\quad+\int_0^t\int_{\mathbb{R}}\left((t+h-r)^{\alpha-1}-(t-r)^{\alpha-1}\right)p_{t-r}(x-z)J_\alpha(r,z)\mathrm{d}z\mathrm{d}r\\
&\quad+\int_t^{t+h}\int_{\mathbb{R}}(t+h-r)^{\alpha-1}p_{t+h-r}(x-z)J_\alpha(r,z)\mathrm{d}z\mathrm{d}r\\
&=:\mathcal{K}_1(t,h,x)+\mathcal{K}_2(t,h,x)+\mathcal{K}_3(t,h,x).
\end{aligned} \tag{3.47}$$

In the following, we give estimates for $\mathcal{K}_i(t,h,x)$, $i=1,2,3$, respectively. By Hölder's inequality, we have

$$\begin{aligned}\mathcal{K}_1(t,h,x) &= \int_0^t \int_{\mathbb{R}} (t+h-r)^{\alpha-1} \left(p_{t+h-r}(x-z) - p_{t-r}(x-z)\right) \\ &\qquad \lambda^{-\frac{1}{p}}(z) J_\alpha(r,z) \lambda^{\frac{1}{p}}(z) \mathrm{d}z\mathrm{d}r \\ &\leq \Big(\int_0^t \int_{\mathbb{R}} (t+h-r)^{q(\alpha-1)} \big| p_{t+h-r}(x-z) \\ &\qquad - p_{t-r}(x-z) \big|^q \lambda^{-\frac{q}{p}}(z) \mathrm{d}z\mathrm{d}r \Big)^{\frac{1}{q}} \\ &\qquad \cdot \Big(\int_0^t \int_{\mathbb{R}} \big| J_\alpha(r,z) \big|^p \lambda(z) \mathrm{d}z\mathrm{d}r \Big)^{\frac{1}{p}} \\ &=: |\mathcal{K}_{11}(t,h,x)|^{\frac{1}{q}} \cdot \Big(\int_0^t \big\| J_\alpha(r,\cdot) \big\|^p_{L^p_\lambda(\mathbb{R})} \mathrm{d}r \Big)^{\frac{1}{p}}. \end{aligned} \tag{3.48}$$

Applying the Cauchy–Schwarz inequality, (2.6), (3.5) and (3.12), we have

$$\begin{aligned} &\big\| J_\alpha(r,\cdot) \big\|^p_{L^p_\lambda(\mathbb{R})} \\ &= \int_{\mathbb{R}} \left| \int_0^r (r-s)^{-\alpha} \cdot \langle p_{r-s}(z-\cdot)\sigma(s,\cdot,u^g_\varepsilon(s,\cdot)), g(s,\cdot) \rangle_{\mathcal{H}_\varepsilon} \mathrm{d}s \right|^p \lambda(z)\mathrm{d}z \\ &\lesssim \int_{\mathbb{R}} \left| \int_0^r (r-s)^{-2\alpha} \cdot \big\| p_{r-s}(z-\cdot)\sigma(s,\cdot,u^g_\varepsilon(s,\cdot)) \big\|^2_{\mathcal{H}} \mathrm{d}s \right|^{\frac{p}{2}} \lambda(z)\mathrm{d}z \\ &\simeq \int_{\mathbb{R}} \bigg| \int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \cdot \Big(p_{r-s}(z-y-l)\sigma(s,y+l,u^g_\varepsilon(s,y+l)) \\ &\qquad - p_{r-s}(z-y)\sigma(s,y,u^g_\varepsilon(s,y)) \Big)^2 \cdot |l|^{2H-2} \mathrm{d}y\mathrm{d}l\mathrm{d}s \bigg|^{\frac{p}{2}} \lambda(z)\mathrm{d}z \\ &\lesssim \int_{\mathbb{R}} [\mathcal{B}_1(r,z) + \mathcal{B}_2(r,z) + \mathcal{B}_3(r,z)] \lambda(z)\mathrm{d}z, \end{aligned} \tag{3.49}$$

where

$$\begin{aligned} \mathcal{B}_1(r,z) &:= \Big(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \cdot p^2_{r-s}(z-y-l) \\ &\qquad \cdot \big| \sigma(s,y+l,u^g_\varepsilon(s,y+l)) - \sigma(s,y,u^g_\varepsilon(s,y+l)) \big|^2 \cdot \\ &\qquad |l|^{2H-2} \mathrm{d}y\mathrm{d}l\mathrm{d}s \Big)^{\frac{p}{2}}; \\ \mathcal{B}_2(r,z) &:= \Big(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \cdot p^2_{r-s}(z-y-l) \end{aligned}$$

$$\cdot \left|\sigma(s,y,u_\varepsilon^g(s,y+l)-\sigma(s,y,u_\varepsilon^g(s,y))\right|^2 \cdot |l|^{2H-2}\mathrm{d}y\mathrm{d}l\mathrm{d}s\Big)^{\frac{p}{2}};$$

$$\mathcal{B}_3(r,z) := \left(\int_0^r \int_{\mathbb{R}^2} (r-s)^{-2\alpha} \cdot \left|D_{r-s}(z-y,l)\right|^2 \cdot \left|\sigma(s,y,u_\varepsilon^g(s,y))\right|^2 \cdot |l|^{2H-2}\mathrm{d}y\mathrm{d}l\mathrm{d}s\right)^{\frac{p}{2}}.$$

By using the same technique as that in Step 3 of the proof for Lemma 3.1, we have

$$\int_{\mathbb{R}} \mathcal{B}_1(r,z)\lambda(z)\mathrm{d}z \lesssim \left(\int_0^r (r-s)^{-2\alpha-\frac{1}{2}} \cdot \left(1+\left\|u_\varepsilon^g(s,\cdot)\right\|^2_{L^p_\lambda(\mathbb{R})}\right)\mathrm{d}s\right)^{\frac{p}{2}}; \tag{3.50}$$

$$\int_{\mathbb{R}} \mathcal{B}_2(r,z)\lambda(z)\mathrm{d}z \lesssim \left(\int_0^r (r-s)^{-2\alpha-\frac{1}{2}} \cdot \left(N^*_{\frac{1}{2}-H,\,p} u_\varepsilon^g(s)\right)^2\mathrm{d}s\right)^{\frac{p}{2}}; \tag{3.51}$$

$$\int_{\mathbb{R}} \mathcal{B}_3(r,z)\lambda(z)\mathrm{d}z \lesssim \left(\int_0^r (r-s)^{-2\alpha+H-1} \cdot \left(1+\left\|u_\varepsilon^g(s,\cdot)\right\|^2_{L^p_\lambda(\mathbb{R})}\right)\mathrm{d}s\right)^{\frac{p}{2}}. \tag{3.52}$$

Putting (3.49), (3.50), (3.51) and (3.52) together, we have

$$\left\|J_\alpha(r,\cdot)\right\|^p_{L^p_\lambda(\mathbb{R})} \lesssim \left(\int_0^r (r-s)^{-2\alpha-\frac{1}{2}} + (r-s)^{-2\alpha+H-1}\mathrm{d}s\right)^{\frac{p}{2}}\left(1+\|u_\varepsilon^g\|^p_{\mathcal{Z}^p_{\lambda,T}}\right). \tag{3.53}$$

If $-2\alpha-\frac{1}{2} > -1$ and $-2\alpha+H-1 > -1$, namely if $\alpha < \frac{H}{2}$, then we have

$$\left\|J_\alpha(r,\cdot)\right\|^p_{L^p_\lambda(\mathbb{R})} \lesssim 1+\|u_\varepsilon^g\|^p_{Z^p_{\lambda,T}}. \tag{3.54}$$

By (3.39), (6.9) and Lemma 6.1, for any $\gamma \in (0,1)$, we have

$$\begin{aligned}\left|\mathcal{K}_{11}(t,h,x)\right| &\lesssim |h|^{q\gamma}\int_0^t\int_{\mathbb{R}}(t-r)^{q(\alpha-1-\gamma)+\frac{1-q}{2}}p_{\frac{2(t+h-r)}{q\gamma}}(x-z)\lambda^{-\frac{q}{p}}(z)\mathrm{d}z\mathrm{d}r\\ &\quad + |h|^{q\gamma}\int_0^t\int_{\mathbb{R}}(t-r)^{q(\alpha-1-\gamma)+\frac{1-q}{2}}p_{\frac{2(t-r)}{q\gamma}}(x-z)\lambda^{-\frac{q}{p}}(z)\mathrm{d}z\mathrm{d}r\\ &\lesssim |h|^{q\gamma}\lambda^{-\frac{q}{p}}(x)\int_0^t(t-r)^{q(\alpha-1-\gamma)+\frac{1-q}{2}}\mathrm{d}r.\end{aligned}$$

Hence, if $\alpha < \frac{H}{2}$ and $q(\alpha-1-\gamma)+\frac{1-q}{2} > -1$, namely if $0<\gamma<\frac{H}{2}-\frac{3}{2p}$, then we have

$$\left|\mathcal{K}_{11}(t,h,x)\right|^{\frac{1}{q}} \lesssim |h|^{\gamma}\lambda^{-\frac{1}{p}}(x). \tag{3.55}$$

If $0 < \gamma < \frac{H}{2} - \frac{3}{2p}$, then by putting (3.48), (3.54) and (3.55) together, we have

$$\sup_{\substack{t,\, t+h\in[0,T],\\ x\in\mathbb{R}}} \lambda^{\frac{1}{p}}(x)\cdot\left|\mathcal{K}_1(t,h,x)\right| \le C_{T,p,H,N,\gamma}\cdot|h|^{\gamma}\cdot\left(1+\|u_\varepsilon^g\|_{Z^p_{\lambda,T}}\right). \tag{3.56}$$

Applying Hölder's inequality, (3.39) and Lemma 6.1, we have

$$\begin{aligned}\mathcal{K}_2(t,h,x) &\le \left(\int_0^t\int_{\mathbb{R}}\left|(t+h-r)^{\alpha-1}-(t-r)^{\alpha-1}\right|^q\cdot p^q_{t-r}(x-z)\lambda^{-\frac{q}{p}}(z)\mathrm{d}z\mathrm{d}r\right)^{\frac{1}{q}}\\ &\quad\cdot\left(\int_0^t\int_{\mathbb{R}}\left|J_\alpha(r,z)\right|^p\lambda(z)\mathrm{d}z\mathrm{d}r\right)^{\frac{1}{p}}\\ &\lesssim\left(\int_0^t\left|(t+h-r)^{\alpha-1}-(t-r)^{\alpha-1}\right|^q(t-r)^{\frac{1-q}{2}}\lambda^{-\frac{q}{p}}(x)\mathrm{d}r\right)^{\frac{1}{q}}\\ &\quad\cdot\left(\int_0^T\left\|J_\alpha(r,\cdot)\right\|^p_{L^p_\lambda(\mathbb{R})}\mathrm{d}r\right)^{\frac{1}{p}}.\end{aligned} \tag{3.57}$$

By (3.54) and (6.8), for any $\gamma\in(0,1)$, if $\alpha<\frac{H}{2}$, then we have

$$\begin{aligned}&\sup_{\substack{t,\, t+h\in[0,T],\\ x\in\mathbb{R}}} \lambda^{\frac{1}{p}}(x)\cdot\left|\mathcal{K}_2(t,h,x)\right|\\ &\lesssim|h|^{\gamma}\cdot\sup_{t\in[0,T]}\left(\int_0^t(t-r)^{q(\alpha-1-\gamma)+\frac{1-q}{2}}\mathrm{d}r\right)^{\frac{1}{q}}\cdot\left(1+\|u_\varepsilon^g\|_{Z^p_{\lambda,T}}\right).\end{aligned}$$

If $\alpha<\frac{H}{2}$ and $q(\alpha-1-\gamma)+\frac{1-q}{2}>-1$, namely if $0<\gamma<\frac{H}{2}-\frac{3}{2p}$, then there exists a positive constant $C_{T,p,H,N,\gamma}$ such that

$$\sup_{\substack{t,\, t+h\in[0,T],\\ x\in\mathbb{R}}} \lambda^{\frac{1}{p}}(x)\cdot\left|\mathcal{K}_2(t,h,x)\right| \le C_{T,p,H,N,\gamma}|h|^{\gamma}\cdot\left(1+\|u_\varepsilon^g\|_{Z^p_{\lambda,T}}\right). \tag{3.58}$$

By Hölder's inequality and Lemma 6.1, we have

$$\begin{aligned}\mathcal{K}_3(t,h,x) &\le \left(\int_t^{t+h}(t+h-r)^{q(\alpha-1)}(t+h-r)^{\frac{1-q}{2}}\lambda^{-\frac{q}{p}}(x)\mathrm{d}r\right)^{\frac{1}{q}}\\ &\quad\cdot\left(\int_0^T\left\|J_\alpha(r,\cdot)\right\|^p_{L^p_\lambda(\mathbb{R})}\mathrm{d}r\right)^{\frac{1}{p}}\\ &\simeq\lambda^{-\frac{1}{p}}(x)\cdot|h|^{\alpha-\frac{3}{2p}}\cdot\left(\int_0^T\left\|J_\alpha(r,\cdot)\right\|^p_{L^p_\lambda(\mathbb{R})}\mathrm{d}r\right)^{\frac{1}{p}}.\end{aligned} \tag{3.59}$$

Let $\gamma = \alpha - \frac{3}{2p}$. If $\frac{3}{2p} < \alpha < \frac{H}{2}$, namely if $0 < \gamma < \frac{H}{2} - \frac{3}{2p}$, then (3.54) yields that

$$\sup_{\substack{t, t+h \in [0,T], \\ x \in \mathbb{R}}} \lambda^{\frac{1}{p}}(x) \cdot \left|\mathcal{K}_3(t,h,x)\right| \leq C_{T,p,H,N,\gamma} |h|^{\gamma} \cdot \left(1 + \|u_\varepsilon^g\|_{Z^p_{\lambda,T}}\right). \tag{3.60}$$

If $p > \frac{3}{H}$ and $0 < \gamma < \frac{H}{2} - \frac{3}{2p}$, then by putting (3.47), (3.56), (3.58) and (3.60) together, we have

$$\sup_{\substack{t, t+h \in [0,T], \\ x \in \mathbb{R}}} \lambda^{\frac{1}{p}}(x) \cdot \left|\Phi(t+h,x) - \Phi(t,x)\right| \leq C_{T,p,H,N,\gamma} |h|^{\gamma} \cdot \left(1 + \|u_\varepsilon^g\|_{Z^p_{\lambda,T}}\right). \tag{3.61}$$

(iii) Notice that

$$\begin{aligned}
&|\Phi(t,x) - \Phi(t,y)| \\
&\simeq \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} \cdot (p_{t-r}(x-z) - p_{t-r}(y-z))\, J_\alpha(r,z) \mathrm{d}z \mathrm{d}r \\
&\lesssim \left(\int_0^t \int_{\mathbb{R}} (t-r)^{q(\alpha-1)} \cdot \left|p_{t-r}(x-z) - p_{t-r}(y-z)\right|^q \lambda^{-\frac{q}{p}}(z) \mathrm{d}z \mathrm{d}r\right)^{\frac{1}{q}} \cdot \\
&\quad \left(\int_0^t \int_{\mathbb{R}} \left|J_\alpha(r,z)\right|^p \lambda(z) \mathrm{d}z \mathrm{d}r\right)^{\frac{1}{p}} \\
&= \left(\int_0^t \int_{\mathbb{R}} (t-r)^{q(\alpha-1)} \cdot \left|p_{t-r}(x-z) - p_{t-r}(y-z)\right|^q \lambda^{-\frac{q}{p}}(z) \mathrm{d}z \mathrm{d}r\right)^{\frac{1}{q}} \cdot \\
&\quad \left(\int_0^t \left\|J_\alpha(r,\cdot)\right\|^p_{L^p_\lambda(\mathbb{R})} \mathrm{d}r\right)^{\frac{1}{p}}.
\end{aligned}$$

For any $\gamma \in (0,1)$, by using the same method as that in the proof of (4.35) in [22] and (3.54), we have

$$\begin{aligned}
&\left|\Phi(t,x) - \Phi(t,y)\right| \\
&\lesssim \left|x-y\right|^{\gamma} \cdot \left[\lambda^{-\frac{1}{p}}(x) + \lambda^{-\frac{1}{p}}(y)\right] \cdot \left(\int_0^t (t-r)^{(\alpha q - \frac{3q}{2} + \frac{1}{2}) - \frac{\gamma q}{2}} \mathrm{d}r\right)^{\frac{1}{q}} \cdot \\
&\quad \left(1 + \|u_\varepsilon^g\|_{Z^p_{\lambda,T}}\right).
\end{aligned} \tag{3.62}$$

If $\alpha < \frac{H}{2}$ and $(\alpha q - \frac{3q}{2} + \frac{1}{2}) - \frac{\gamma q}{2} > -1$, namely if $p > \frac{3}{H}$ and $0 < \gamma < H - \frac{3}{p}$, then we get the desired result (3.36).

The proof is complete. □

Lemma 3.3 *Assume that $u_n \in Z^p_{\lambda,T}$, $n \geq 1$, for some $p \geq 2$. If $u_n \to u$ in $(\mathcal{C}([0,T] \times \mathbb{R}), d_{\mathcal{C}})$ as $n \to \infty$, then we have the following results:*

(i) *u is also in* $Z^p_{\lambda,T}$*;*
(ii) *for any fixed* $t \in [0,T]$,

$$\int_0^t \int_{\mathbb{R}} \|p_{t-s}(x-\cdot)\sigma(s,\cdot,u(s,\cdot))\|^2_{\mathcal{H}} \lambda(x)\mathrm{d}s\mathrm{d}x < \infty. \tag{3.63}$$

Hence, for almost all $x \in \mathbb{R}$,

$$\int_0^t \|p_{t-s}(x-\cdot)\sigma(s,\cdot,u(s,\cdot))\|^2_{\mathcal{H}} \mathrm{d}s < \infty. \tag{3.64}$$

Proof (i) This result is due to [22, Lemma 4.6] by replacing $\|\cdot\|_{L^p_\lambda(\Omega\times\mathbb{R})}$ by $\|\cdot\|_{L^p_\lambda(\mathbb{R})}$, the proof is omitted here.

(ii) Recall $D_t(x,h)$ given by (3.8). By (2.6) and a change of variable, we have

$$\begin{aligned}
&\|p_{t-s}(x-\cdot)\sigma(s,\cdot,u(s,\cdot))\|^2_{\mathcal{H}} \\
&= c_{2,H}\int_{\mathbb{R}^2}\Big[p_{t-s}(x-y-h)\sigma(s,y+h,u(s,y+h)) \\
&\quad - p_{t-s}(x-y)\sigma(s,y,u(s,y))\Big]^2 \cdot |h|^{2H-2}\mathrm{d}h\mathrm{d}y \\
&\lesssim \int_{\mathbb{R}^2} p^2_{t-s}(x-y)\,|\sigma(s,y,u(s,y+h))-\sigma(s,y,u(s,y))|^2\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^2}|D_{t-s}(x-y,h)|^2\,|\sigma(s,y+h,u(s,y))|^2\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^2} p^2_{t-s}(x-y)\,|\sigma(s,y+h,u(s,y))-\sigma(s,y,u(s,y))|^2\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}y \\
&=: F_1(t,s,x)+F_2(t,s,x)+F_3(t,s,x).
\end{aligned} \tag{3.65}$$

By (2.18), (3.18), a change of variable and Lemma 6.1, we have that, for any fixed $t \in [0,T]$,

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}} F_1(t,s,x)\lambda(x)\mathrm{d}x\mathrm{d}s \\
&\lesssim \int_0^t\int_{\mathbb{R}^3} p^2_{t-s}(x-y)\cdot\big|u(s,y+h)-u(s,y)\big|^2\cdot|h|^{2H-2}\lambda(x)\mathrm{d}h\mathrm{d}y\mathrm{d}x\mathrm{d}s \\
&\simeq \int_0^t\int_{\mathbb{R}^2}(t-s)^{-\frac{1}{2}}\cdot\left(\int_{\mathbb{R}} p_{\frac{t-s}{2}}(y-x)\lambda(x)\mathrm{d}x\right) \\
&\qquad \cdot\big|u(s,y+h)-u(s,y)\big|^2\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s \\
&\lesssim \int_0^t\int_{\mathbb{R}^2}(t-s)^{-\frac{1}{2}}\lambda(y)\big|u(s,y+h)-u(s,y)\big|^2\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s
\end{aligned}$$

$$\lesssim \int_0^t (t-s)^{-\frac{1}{2}} \mathrm{d}s \cdot \sup_{s\in[0,T]} \left[\mathcal{N}^*_{\frac{1}{2}-H,2} u(s)\right]^2 < \infty. \tag{3.66}$$

By (2.17), a change of variable and Lemma 6.2, we have that, for any fixed $t \in [0, T]$,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} F_2(t,s,x)\lambda(x)\mathrm{d}x\mathrm{d}s \\
&\lesssim \int_0^t \int_{\mathbb{R}^3} |D_{t-s}(x-y,h)|^2 \cdot \left(1 + \left|u(s,y)\right|^2\right) \cdot |h|^{2H-2}\lambda(x)\mathrm{d}h\mathrm{d}y\mathrm{d}x\mathrm{d}s \\
&= \int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |D_{t-s}(y,h)|^2 \cdot |h|^{2H-2}\lambda(x-y)\mathrm{d}h\mathrm{d}y\right) \cdot \left(1+\left|u(s,x)\right|^2\right)\mathrm{d}x\mathrm{d}s \\
&\lesssim \int_0^t \int_{\mathbb{R}} (t-s)^{H-1}\left(1+\left|u(s,x)\right|^2\right)\lambda(x)\mathrm{d}x\mathrm{d}s \\
&\le \int_0^t (t-s)^{H-1}\mathrm{d}s \cdot \sup_{s\in[0,T]} \left(1 + \|u(s,\cdot)\|^2_{L^2_\lambda(\mathbb{R})}\right) < \infty.
\end{aligned} \tag{3.67}$$

By (3.16), (3.17), (3.18), a change of variable and Lemma 6.1, we have that, for any fixed $t \in [0, T]$,

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}} F_3(t,s,x)\lambda(x)\mathrm{d}x\mathrm{d}s \\
&\lesssim \int_0^t \int_{\mathbb{R}^3} p^2_{t-s}(x-y)\left(1+|u(s,y)|^2\right) \cdot \Big(\mathbf{1}_{\{|h|\le 1\}} \\
&\qquad |h|^2 + \mathbf{1}_{\{|h|>1\}}\Big) |h|^{2H-2}\lambda(x)\mathrm{d}h\mathrm{d}y\mathrm{d}x\mathrm{d}s \\
&\lesssim \int_0^t \int_{\mathbb{R}^2} p^2_{t-s}(x-y)\left(1+|u(s,y)|^2\right)\lambda(x)\mathrm{d}y\mathrm{d}x\mathrm{d}s \\
&\simeq \int_0^t \int_{\mathbb{R}} (t-s)^{-\frac{1}{2}} \cdot \left(\int_{\mathbb{R}} p_{\frac{t-s}{2}}(y-x)\lambda(x)\mathrm{d}x\right) \cdot \left(1+|u(s,y)|^2\right)\mathrm{d}y\mathrm{d}s \\
&\lesssim \int_0^t \int_{\mathbb{R}} (t-s)^{-\frac{1}{2}}\lambda(y)\left(1+|u(s,y)|^2\right)\mathrm{d}y\mathrm{d}s \\
&\le \int_0^t (t-s)^{-\frac{1}{2}}\mathrm{d}s \cdot \sup_{s\in[0,T]}\left(1+\|u(s,\cdot)\|^2_{L^2_\lambda(\mathbb{R})}\right) < \infty.
\end{aligned} \tag{3.68}$$

Putting (3.65)-(3.68) together, we obtain (3.63). In particular, (3.64) holds for almost all $x \in \mathbb{R}$.

The proof is complete. ◻

Proof of Proposition 3.1 (**Existence**) By Lemma 3.2 (ii) and (iii), we have that for any $R > 0$ and $\gamma > 0$, there exists $\theta > 0$ such that

$$\max_{\substack{|t-s|+|x-y|\le\theta,\\ 0\le t,s\le T;\ -R\le x,y\le R}} |u^g_\varepsilon(t,x) - u^g_\varepsilon(s,y)| \le \gamma.$$

Combining this with the fact of $u_\varepsilon^g(0,\cdot) \equiv 1$, we know that the family $\{u_\varepsilon^g\}_{\varepsilon>0}$ is relatively compact on the space $(\mathcal{C}([0,T]\times\mathbb{R}), d_{\mathcal{C}})$ by the Arzelà–Ascoli theorem. Thus, there is a subsequence $\varepsilon_n \downarrow 0$ such that $u_{\varepsilon_n}^g$ converges to a function u^g in $(\mathcal{C}([0,T]\times\mathbb{R}), d_{\mathcal{C}})$.

By using the Cauchy–Schwarz inequality, (2.18), Lemma 3.3 (ii) and the dominated convergence theorem, we have

$$\begin{aligned} u_{\varepsilon_n}^g(t,x) = 1 &+ \int_0^t \langle p_{t-s}(x-\cdot)\left(\sigma(s,\cdot,u_{\varepsilon_n}^g(s,\cdot)) - \sigma(s,\cdot,u^g(s,\cdot))\right), g(s,\cdot)\rangle_{\mathcal{H}_{\varepsilon_n}} \mathrm{d}s \\ &+ \int_0^t \langle p_{t-s}(x-\cdot)\sigma(s,\cdot,u^g(s,\cdot)), g(s,\cdot)\rangle_{\mathcal{H}_{\varepsilon_n}} \mathrm{d}s \\ &\longrightarrow 1 + \int_0^t \langle p_{t-s}(x-\cdot)\sigma(s,\cdot,u^g(s,\cdot)), g(s,\cdot)\rangle_{\mathcal{H}} \mathrm{d}s \quad \text{as } \varepsilon_n \downarrow 0. \end{aligned}$$

By the uniqueness of the limit of $\{u_{\varepsilon_n}^g\}_{n\geq 1}$, we know that u^g satisfies Eq. (2.28). By Lemma 3.1 and Lemma 3.3 (i), we have that, for any $p \geq 2$,

$$\sup_{g\in S^N} \|u^g\|_{Z_{\lambda,T}^p} < \infty. \tag{3.69}$$

(**Uniqueness**) Let u^g and v^g be two solutions of (2.28). By using the same technique as that in the proof of Lemma 3.2 (i), we have that, for any $p > \frac{6}{4H-1}$,

$$\sup_{t\in[0,T],\, x\in\mathbb{R}} \lambda^{\frac{1}{p}}(x)\cdot \mathcal{N}_{\frac{1}{2}-H} u^g(t,x) < \infty; \tag{3.70}$$

$$\sup_{t\in[0,T],\, x\in\mathbb{R}} \lambda^{\frac{1}{p}}(x)\cdot \mathcal{N}_{\frac{1}{2}-H} v^g(t,x) < \infty. \tag{3.71}$$

Denote that

$$S_1(t) = \int_{\mathbb{R}} |u^g(t,x) - v^g(t,x)|^2 \lambda(x)\mathrm{d}x,$$

and

$$\begin{aligned} S_2(t) = \int_{\mathbb{R}^2} &\left|u^g(t,x) - v^g(t,x) - u^g(t,x+h) + v^g(t,x+h)\right|^2 \\ &\cdot |h|^{2H-2}\lambda(x)\mathrm{d}h\mathrm{d}x. \end{aligned}$$

According to (3.2), (3.3) and (3.69), we know that

$$\sup_{t\in[0,T]} S_1(t) < \infty, \quad \sup_{t\in[0,T]} S_2(t) < \infty. \tag{3.72}$$

Recall $D_t(x,h)$ defined by (3.8) and denote that

$$\triangle(t,x,y) := \sigma(t,x,u^g(t,y)) - \sigma(t,x,v^g(t,y)).$$

Since $\int_0^T \|g(s,\cdot)\|_{\mathcal{H}}^2 ds < \infty$, by using the Cauchy–Schwarz inequality, (2.6) and a change of variable, we have that, for any $t \in [0,T]$,

$$\begin{aligned}
&\int_{\mathbb{R}} |u^g(t,x) - v^g(t,x)|^2 \lambda(x) dx \\
&= \int_{\mathbb{R}} \left| \int_0^t \langle p_{t-s}(x-\cdot)\triangle(s,\cdot,\cdot), g(s,\cdot)\rangle_{\mathcal{H}} ds \right|^2 \lambda(x) dx \\
&\lesssim \int_{\mathbb{R}} \int_0^t \|g(s,\cdot)\|_{\mathcal{H}}^2 ds \cdot \int_0^t \|p_{t-s}(x-\cdot)\triangle(s,\cdot,\cdot)\|_{\mathcal{H}}^2 ds \lambda(x) dx \\
&\lesssim \int_0^t \int_{\mathbb{R}^3} \left[p_{t-s}(x-y-h)\triangle(s,y+h,y+h) - p_{t-s}(x-y)\triangle(s,y,y)\right]^2 \\
&\qquad \cdot |h|^{2H-2}\lambda(x) dh dy dx ds vz \\
&\lesssim \int_0^t \int_{\mathbb{R}^3} p_{t-s}^2(x-y) \left|\triangle(s,y,y+h) - \triangle(s,y,y)\right|^2 \cdot |h|^{2H-2}\lambda(x) dh dy dx ds \\
&\quad + \int_0^t \int_{\mathbb{R}^3} |D_{t-s}(x-y,h)|^2 \left|\triangle(s,y+h,y)\right|^2 \cdot |h|^{2H-2}\lambda(x) dh dy dx ds \\
&\quad + \int_0^t \int_{\mathbb{R}^3} p_{t-s}^2(x-y) \left|\triangle(s,y+h,y) - \triangle(s,y,y)\right|^2 \cdot |h|^{2H-2}\lambda(x) dh dy dx ds \\
&=: V_1(t) + V_2(t) + V_3(t).
\end{aligned} \tag{3.73}$$

By the integral mean value theorem, we have

$$\begin{aligned}
&|\triangle(s,y,y+h) - \triangle(s,y,y)|^2 \\
=&\Big| \int_0^1 \left[u^g(s,y+h) - v^g(s,y+h)\right] \sigma_\xi'\left(s,y,\theta u^g(s,y+h) + (1-\theta)v^g(s,y)\right) d\theta \\
&- \int_0^1 \left[u^g(s,y) - v^g(s,y)\right] \sigma_\xi'\left(s,y,\theta u^g(s,y) + (1-\theta)v^g(s,y)\right) d\theta \Big|^2.
\end{aligned}$$

By (2.19) and (2.22), we have

$$\begin{aligned}
&|\triangle(s,y,y+h) - \triangle(s,y,y)|^2 \\
\lesssim& \left|u^g(s,y+h) - v^g(s,y+h) - u^g(s,y) + v^g(s,y)\right|^2 \\
&+ \lambda^{\frac{2}{p_0}}(y) |u^g(s,y) - v^g(s,y)|^2 \cdot \\
&\left[|u^g(s,y+h) - u^g(s,y)|^2 + |v^g(s,y+h) - v^g(s,y)|^2\right].
\end{aligned} \tag{3.74}$$

where $p_0 \in \left(\frac{6}{4H-1}, \infty\right)$ is the constant appeared in (2.22).

By (3.18) and Lemma 6.1, we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} p_{t-s}^2(x-y) \left|u^g(s,y+h) - v^g(s,y+h) - u^g(s,y) + v^g(s,y)\right|^2 \cdot \\
&\quad |h|^{2H-2}\lambda(x)\mathrm{d}h\mathrm{d}y\mathrm{d}x\mathrm{d}s \\
&\simeq \int_0^t (t-s)^{-\frac{1}{2}} \int_{\mathbb{R}^2} \cdot \left(\int_{\mathbb{R}} p_{\frac{t-s}{2}(y-x)}\lambda(x)\mathrm{d}x\right) \\
&\qquad \cdot \left|u^g(s,y+h) - v^g(s,y+h) - u^g(s,y) + v^g(s,y)\right|^2 \cdot |h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s \\
&\lesssim \int_0^t (t-s)^{-\frac{1}{2}} \int_{\mathbb{R}^2} \lambda(y) \left|u^g(s,y+h) - v^g(s,y+h) - u^g(s,y) + v^g(s,y)\right|^2 \cdot \\
&\quad |h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s = \int_0^t (t-s)^{-\frac{1}{2}} \cdot S_2(s)\mathrm{d}s. \qquad (3.75)
\end{aligned}$$

By (3.18), (3.70), (3.71) and Lemma 6.1, we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}^3} p_{t-s}^2(x-y)\lambda^{\frac{2}{p_0}}(y)|u^g(s,y) - v^g(s,y)|^2 \\
&\quad \cdot \left[|u^g(s,y+h) - u^g(s,y)|^2 + |v^g(s,y+h) - v^g(s,y)|^2\right] \cdot \\
&\quad |h|^{2H-2}\lambda(x)\mathrm{d}h\mathrm{d}y\mathrm{d}x\mathrm{d}s \\
&\lesssim \int_0^t \int_{\mathbb{R}^2} p_{t-s}^2(x-y)|u^g(s,y) - v^g(s,y)|^2\lambda(x)\mathrm{d}y\mathrm{d}x\mathrm{d}s \\
&\lesssim \int_0^t (t-s)^{-\frac{1}{2}} \int_{\mathbb{R}} \cdot \left(\int_{\mathbb{R}} p_{\frac{t-s}{2}}(y-x)\lambda(x)\mathrm{d}x\right) |u^g(s,y) - v^g(s,y)|^2\mathrm{d}y\mathrm{d}s \\
&\lesssim \int_0^t (t-s)^{-\frac{1}{2}} \int_{\mathbb{R}} \lambda(y)|u^g(s,y) - v^g(s,y)|^2\mathrm{d}y\mathrm{d}s \\
&= \int_0^t (t-s)^{-\frac{1}{2}} \cdot S_1(s)\mathrm{d}s. \qquad (3.76)
\end{aligned}$$

Combining (3.74), (3.75) with (3.76), we obtain that

$$V_1(t) \lesssim \int_0^t (t-s)^{-\frac{1}{2}} \cdot (S_1(s) + S_2(s))\,\mathrm{d}s. \qquad (3.77)$$

A change of variable, Lemma 6.5 and (2.18) yield that

$$\begin{aligned}
V_2(t) &\lesssim \int_0^t \int_{\mathbb{R}^3} |D_{t-s}(x-y,h)|^2 \cdot |u^g(s,y) - v^g(s,y)|^2 \cdot |h|^{2H-2}\lambda(x)\mathrm{d}h\mathrm{d}y\mathrm{d}x\mathrm{d}s \\
&= \int_0^t \int_{\mathbb{R}} \cdot \left(\int_{\mathbb{R}^2} |D_{t-s}(x,h)|^2 \cdot |h|^{2H-2}\lambda(y-x)\mathrm{d}h\mathrm{d}x\right) \cdot |u^g(s,y) - v^g(s,y)|^2 \cdot \\
&\quad \mathrm{d}y\mathrm{d}s
\end{aligned}$$

$$\lesssim \int_0^t (t-s)^{H-1} \int_{\mathbb{R}} \lambda(y)|u^g(s,y)-v^g(s,y)|^2 \mathrm{d}y\mathrm{d}s$$
$$= \int_0^t (t-s)^{H-1} \cdot S_1(s)\mathrm{d}s. \tag{3.78}$$

If $h > 1$, then we have that by (2.19),

$$(\triangle(s,y+h,y)-\triangle(s,y,y))^2 = \left| \int_{u^g}^{v^g} \left(\sigma'_\xi(s,y+h,\xi) - \sigma'_\xi(s,y,\xi) \right) \mathrm{d}\xi \right|^2$$
$$\lesssim \left| u^g(s,y) - v^g(s,y) \right|^2. \tag{3.79}$$

If $h \leq 1$, then we have that by (2.21),

$$(\triangle(s,y+h,y)-\triangle(s,y,y))^2 = \left| \int_{u^g}^{v^g} \left(\sigma'_\xi(s,y+h,\xi) - \sigma'_\xi(s,y,\xi) \right) \mathrm{d}\xi \right|^2$$
$$\lesssim \left| u^g(s,y) - v^g(s,y) \right|^2 \cdot |h|^2. \tag{3.80}$$

Thus, by (3.39), (3.79), (3.80) and Lemma 6.1, we have

$$\begin{aligned}
V_3(t) &= \int_0^t \int_{\mathbb{R}^2} \int_{|h|>1} p^2_{t-s}(x-y) \cdot |u^g(s,y)-v^g(s,y)|^2 \cdot |h|^{2H-2} \lambda(x) \mathrm{d}h\mathrm{d}y\mathrm{d}x\mathrm{d}s \\
&\quad + \int_0^t \int_{\mathbb{R}^2} \int_{|h|\leq 1} p^2_{t-s}(x-y-h) \cdot |u^g(s,y)-v^g(s,y)|^2 \cdot \\
&\quad |h|^{2H} \lambda(x)\mathrm{d}h\mathrm{d}y\mathrm{d}x\mathrm{d}s \\
&\lesssim \int_0^t \int_{\mathbb{R}^2} p^2_{t-s}(x-y) \cdot |u^g(s,y)-v^g(s,y)|^2 \lambda(x)\mathrm{d}y\mathrm{d}x\mathrm{d}s \\
&\simeq \int_0^t (t-s)^{\frac{1}{2}} \int_{\mathbb{R}} \cdot \left(\int_{\mathbb{R}} p_{\frac{t-s}{2}}(y-x)\lambda(x)\mathrm{d}x \right) \cdot |u^g(s,y)-v^g(s,y)|^2 \mathrm{d}y\mathrm{d}s \\
&\lesssim \int_0^t (t-s)^{\frac{1}{2}} \int_{\mathbb{R}} \lambda(y)|u^g(s,y)-v^g(s,y)|^2\mathrm{d}y\mathrm{d}s \\
&= \int_0^t (t-s)^{-\frac{1}{2}} \cdot S_1(s)\mathrm{d}s.
\end{aligned} \tag{3.81}$$

By (3.73), (3.77), (3.78) and (3.81), we have

$$S_1(t) \lesssim \int_0^t (t-s)^{H-1} \cdot [S_1(s)+S_2(s)]\mathrm{d}s. \tag{3.82}$$

Using the similar procedure as in the proof of (3.82), we obtain that

$$S_2(t) \lesssim \int_0^t (t-s)^{2H-\frac{3}{2}} \cdot [S_1(s) + S_2(s)] \mathrm{d}s. \tag{3.83}$$

Therefore, by (3.82) and (3.83), we have

$$S_1(t) + S_2(t) \lesssim \int_0^t \left((t-s)^{H-1} + (t-s)^{2H-\frac{3}{2}} \right) \cdot [S_1(s) + S_2(s)] \mathrm{d}s.$$

Notice that $S_1(t)$ and $S_2(t)$ are uniformly bounded on $[0, T]$ as in (3.72). By the fractional Grönwall lemma ([15, Lemma 7.1.1]), we have

$$S_1(t) + S_2(t) = 0 \quad \text{for all } t \in [0, T].$$

This, together with the continuities of u^g and v^g, implies that $u^g = v^g$.

The proof is complete. □

4 Verification of Condition 2.2 (*a*)

We now verify Condition 2.2 (a). Recall that $\Gamma^0 \left(\int_0^{\cdot} g(s) \mathrm{d}s \right) = u^g$ for $g \in \mathbb{S}$, where u^g is the solution of Eq. (2.28).

Proposition 4.1 *Assume that σ satisfies the hypothesis* $(\mathbf{H})$. *For any $N \geq 1$, let $g_n, g \in S^N$ be such that $g_n \to g$ weakly as $n \to \infty$. Let u^{g_n} denote the solution to Eq.* (2.28) *replacing g by g_n. Then, as $n \to \infty$,*

$$u^{g_n} \longrightarrow u^g \quad \text{in } \mathcal{C}([0, T] \times \mathbb{R}).$$

Proof The proof is divided into two steps. In Step 1, we prove that the family $\{u^{g_n}\}_{n \geq 1}$ is relatively compact in $\mathcal{C}([0, T] \times \mathbb{R})$, which implies that there exists a subsequence of $\{u^{g_n}\}_{n \geq 1}$ (still denoted by $\{u^{g_n}\}_{n \geq 1}$) such that $u^{g_n} \to u$ as $n \to \infty$ in $\mathcal{C}([0, T] \times \mathbb{R})$ for some function $u \in \mathcal{C}([0, T] \times \mathbb{R})$. In Step 2, we show that $u = u^g$.

Step 1 Recall that

$$u^{g_n}(t, x) = 1 + \int_0^t \langle p_{t-s}(x - \cdot) \sigma(s, \cdot, u^{g_n}(s, \cdot)), g_n(s, \cdot) \rangle_{\mathcal{H}} \mathrm{d}s. \tag{4.1}$$

Since $\{g_n\}_{n \geq 1} \subset S^N$, by Proposition 3.1, we know that for any $p \geq 2$,

$$\sup_{n \geq 1} \left\| u^{g_n} \right\|_{Z^p_{\lambda, T}} < \infty. \tag{4.2}$$

As in (3.37), for any $\alpha \in (0, 1)$, we have

$$\int_0^t \langle p_{t-s}(x - \cdot) \sigma(s, \cdot, u^{g_n}(s, \cdot)), g_n(s, \cdot) \rangle_{\mathcal{H}} \mathrm{d}s$$

$$\simeq \int_0^t \int_{\mathbb{R}} (t-r)^{\alpha-1} \cdot p_{t-r}(x-z) \cdot I_\alpha(r,z)\mathrm{d}z\mathrm{d}r,$$

where

$$I_\alpha(r,z) := \int_0^r \langle (r-s)^{-\alpha} p_{r-s}(z-\cdot)\sigma(s,\cdot,u^{g_n}(s,\cdot)), g_n(s,\cdot)\rangle_{\mathcal{H}}\mathrm{d}s.$$

If $\alpha < \frac{H}{2}$, then by using the same technique as that in the proof of (3.54), we have

$$\left\| I_\alpha(r,\cdot)\right\|^p_{L^p_\lambda(\mathbb{R})} \lesssim 1 + \|u^{g_n}\|^p_{Z^p_{\lambda,T}}. \tag{4.3}$$

Fix $\gamma \in (0,1)$. By using the same method as that in the proof of (3.61), if $\alpha < \frac{H}{2}$, then we have

$$\begin{aligned} &\sup_{\substack{t,t+h\in[0,T],\\ x\in\mathbb{R}}} \lambda^{\frac{1}{p}}(x) \cdot \left|u^{g_n}(t+h,x) - u^{g_n}(t,x)\right| \\ &\lesssim |h|^\gamma \cdot \left(\int_0^t (t-r)^{q(\alpha-1-\gamma)+\frac{1-q}{2}}\mathrm{d}r\right)^{\frac{1}{q}} \cdot \left(1 + \|u^{g_n}\|_{Z^p_{\lambda,T}}\right). \end{aligned} \tag{4.4}$$

If $\alpha < \frac{H}{2}$ and $q(\alpha-1-\gamma) + \frac{1-q}{2} > -1$, namely if $p > \frac{3}{H}$ and $0 < \gamma < \frac{H}{2} - \frac{3}{2p}$, then there exists a positive constant $C_{T,p,H,N,\gamma}$ such that

$$\begin{aligned} &\sup_{\substack{t,t+h\in[0,T],\\ x\in\mathbb{R}}} \lambda^{\frac{1}{p}}(x) \cdot \left|u^{g_n}(t+h,x) - u^{g_n}(t,x)\right| \\ &\quad \le C_{T,p,H,N,\gamma} \cdot |h|^\gamma \cdot \left(1 + \|u^{g_n}\|_{Z^p_{\lambda,T}}\right). \end{aligned} \tag{4.5}$$

By using the same method as that in the proof of (3.62), if $\alpha < \frac{H}{2}$, then we have

$$\begin{aligned} &\left|u^{g_n}(t,x) - u^{g_n}(t,y)\right| \\ &\lesssim \left|x-y\right|^\gamma \cdot \left[\lambda^{-\frac{1}{p}}(x) + \lambda^{-\frac{1}{p}}(y)\right] \cdot \left(\int_0^t (t-r)^{(\alpha q - \frac{3q}{2} + \frac{1}{2}) - \frac{\gamma q}{2}}\mathrm{d}r\right)^{\frac{1}{q}} \cdot \\ &\quad \left(1 + \|u^{g_n}\|_{Z^p_{\lambda,T}}\right). \end{aligned} \tag{4.6}$$

If $\alpha < \frac{H}{2}$ and $(\alpha q - \frac{3q}{2} + \frac{1}{2}) - \frac{\gamma q}{2}$, namely if $p > \frac{3}{H}$ and $0 < \gamma < H - \frac{3}{p}$, then there exists a positive constant $C_{T,p,H,N,\gamma}$ such that

$$\sup_{\substack{t\in[0,T],\\ x,h\in\mathbb{R}}} \frac{|u^{g_n}(t,x) - u^{g_n}(t,y)|}{\lambda^{-\frac{1}{p}}(x) + \lambda^{-\frac{1}{p}}(y)} \le C_{T,p,H,N,\gamma} \cdot |x-y|^\gamma \cdot \left(1 + \|u^{g_n}\|_{Z^p_{\lambda,T}}\right). \tag{4.7}$$

By (4.5) and (4.7), we have that, for any $R > 0$ and $\gamma > 0$, there exists $\theta > 0$ such that

$$\max_{\substack{|t-s|+|x-y|\le\theta,\\ 0\le t,s\le T;\,-R\le x,y\le R}} |u^{g_n}(t,x) - u^{g_n}(s,y)| \le \gamma.$$

Combining this with the initial value $u^{g_n}(0,\cdot) \equiv 1$, we know that $\{u^{g_n}\}_{n\ge1}$ is relatively compact on the space $(\mathcal{C}([0,T]\times\mathbb{R}), d_{\mathcal{C}})$ by the Arzelà–Ascoli theorem. Thus, there exists a subsequence of $\{u^{g_n}\}_{n\ge1}$ (still denoted by $\{u^{g_n}\}_{n\ge1}$) and $u \in \mathcal{C}([0,T]\times\mathbb{R})$ such that $u^{g_n} \to u$ as $n \to \infty$. By Lemma 3.3 (i) and (4.2), we have that, for any $p \ge 2$,

$$\|u\|_{Z^p_{\lambda,T}} < \infty. \tag{4.8}$$

Step 2 In this step, we prove that $u = u^g$. Denote that

$$D_1(t) = \int_{\mathbb{R}} |u^{g_n}(t,x) - u^g(t,x)|^2 \lambda(x)\mathrm{d}x,$$

and

$$D_2(t) = \int_{\mathbb{R}^2} \left|u^{g_n}(t,x) - u^g(t,x) - u^{g_n}(t,x+h) + u^g(t,x+h)\right|^2 \cdot \\ |h|^{2H-2}\lambda(x)\mathrm{d}h\mathrm{d}x.$$

According to (3.2), (3.3) and (3.69), we know that $\sup_{t\in[0,T]} D_1(t) < \infty$ and $\sup_{t\in[0,T]} D_2(t) < \infty$. Recall $D_t(x,h)$ defined in (3.8) and denote that

$$\triangle_n(t,x,y) := \sigma(t,x,u^{g_n}(t,y)) - \sigma(t,x,u^g(t,y)).$$

By (2.28) and (4.1), we have

$$\begin{aligned} &D_1(t) + D_2(t) \\ \le\, &2\int_{\mathbb{R}} \left|\int_0^t \langle p_{t-s}(x-\cdot)\triangle_n(s,\cdot,\cdot), g_n(s,\cdot)\rangle_{\mathcal{H}}\mathrm{d}s\right|^2 \lambda(x)\mathrm{d}x \\ &+ 2\int_{\mathbb{R}} \left|\int_0^t \langle p_{t-s}(x-\cdot)\sigma(s,\cdot,u^g(s,\cdot)), g_n(s,\cdot) - g(s,\cdot)\rangle_{\mathcal{H}}\mathrm{d}s\right|^2 \lambda(x)\mathrm{d}x \\ &+ 2\int_{\mathbb{R}} \left|\int_0^t \langle D_{t-s}(x-\cdot,h)\triangle_n(s,\cdot,\cdot), g_n(s,\cdot)\rangle_{\mathcal{H}}\mathrm{d}s\right|^2 |h|^{2H-2}\lambda(x)\mathrm{d}h\mathrm{d}x \\ &+ 2\int_{\mathbb{R}} \left|\int_0^t \langle D_{t-s}(x-\cdot)\sigma(s,\cdot,u^g(s,\cdot)), g_n(s,\cdot) - g(s,\cdot)\rangle_{\mathcal{H}}\mathrm{d}s\right|^2 \\ &|h|^{2H-2}\lambda(x)\mathrm{d}h\mathrm{d}x \end{aligned}$$

$$=: \ 2\left(E_1(t)+E_2(t)+E_3(t)+E_4(t)\right). \tag{4.9}$$

By the similar technique as that in the uniqueness of the proof of Proposition 3.1, we have

$$E_1(t)+E_3(t) \lesssim \int_0^t \left((t-s)^{H-1}+(t-s)^{2H-\frac{3}{2}}\right)\cdot[D_1(s)+D_2(s)]\,\mathrm{d}s. \tag{4.10}$$

On the other hand, denote that

$$G_n(t,x) := \left|\int_0^t \langle p_{t-s}(x-\cdot)\sigma(s,\cdot,u^g(s,\cdot)), g_n(s,\cdot)-g(s,\cdot)\rangle_{\mathcal{H}}\mathrm{d}s\right|^2.$$

By Lemma 3.3 (ii), we know that for almost all $x\in\mathbb{R}$,

$$\int_0^t \left\|p_{t-s}(x-\cdot)\sigma(s,\cdot,u^g(s,\cdot))\right\|_{\mathcal{H}}^2\mathrm{d}s < \infty. \tag{4.11}$$

This, together with the weak convergence of g_n to g, implies that almost all $x\in\mathbb{R}$,

$$G_n(t,x)\to 0, \quad \text{as } n\to\infty. \tag{4.12}$$

Since $g_n, g\in S^N$, by using the Cauchy–Schwarz inequality, (2.6), and a change of variable, we have that, for any $p\geq 2$,

$$\begin{aligned}
&\int_{\mathbb{R}} G_n(t,x)^{\frac{p}{2}}\lambda(x)\mathrm{d}x\\
&\lesssim \int_{\mathbb{R}}\left(\int_0^t \left\|p_{t-s}(x-\cdot)\sigma(s,\cdot,u^g(s,\cdot))\right\|_{\mathcal{H}}^2\mathrm{d}s\right)^{\frac{p}{2}}\lambda(x)\mathrm{d}x\\
&\simeq \int_{\mathbb{R}}\left(\int_0^t\int_{\mathbb{R}^2}\Big|p_{t-s}(x-y-h)\sigma(s,y+h,u^g(s,y+h))\right.\\
&\qquad \left.-\,p_{t-s}(x-y)\sigma(s,y,u^g(s,y))\Big|^2\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s\right)^{\frac{p}{2}}\lambda(x)\mathrm{d}x.
\end{aligned} \tag{4.13}$$

By the similar technique as that in Step 2 in the proof of Proposition 3.1, we have

$$\sup_{n\geq 1}\int_{\mathbb{R}} G_n(t,x)^{\frac{p}{2}}\lambda(x)\mathrm{d}x \lesssim \|u^g\|_{Z^p_{\lambda,T}}^p < \infty.$$

It follows from [8, p. 105, Exercise 8] that $\{G_n(t,x)\}_{n\geq 1}$ is L^1-uniformly integrable in $(\mathbb{R},\lambda(x)\mathrm{d}x)$, namely

$$\lim_{M\to\infty}\sup_{n\geq 1}\int_{G_n(t,x)>M} G_n(t,x)\lambda(x)\mathrm{d}x = 0. \tag{4.14}$$

By (4.12) and the dominated convergence theorem, we have

$$\lim_{n\to\infty}\int_{G_n(t,x)\le M} G_n(t,x)\lambda(x)\mathrm{d}x = 0. \tag{4.15}$$

By (4.14) and (4.15), we have

$$\lim_{n\to\infty} E_2(t) = 0. \tag{4.16}$$

Using the same technique as that in the proof of (4.16), we can prove that

$$\lim_{n\to\infty} E_4(t) = 0. \tag{4.17}$$

Since $D_1(t)$ and $D_2(t)$ are uniformly bounded on $[0, T]$, they are integrable on $[0, T]$. Putting (4.9), (4.10), (4.16) and (4.17) together, by the fractional Grönwall lemma ([15, Lemma 7.1.1]), we have

$$\lim_{n\to\infty} [D_1(t) + D_2(t)] = 0, \text{ for all } t \in [0, T].$$

In particular, $u^{g_n}(t,\cdot) \to u^g(t,\cdot)$ as $n \to \infty$ in the space $L^2_\lambda(\mathbb{R})$ for all $t \in [0, T]$. Since u^{g_n} also converges to u as $n \to \infty$ in the space $(\mathcal{C}([0, T] \times \mathbb{R}), d_{\mathcal{C}})$, the uniqueness of the limit of u^{g_n} implies that $u = u^g$.

The proof is complete. □

5 Verification of Condition 2.2 (*b*)

For any $\varepsilon > 0$, define the solution functional $\Gamma^\varepsilon : C([0, T]; \mathbb{R}^\infty) \to \mathcal{C}([0, T] \times \mathbb{R})$ by

$$\Gamma^\varepsilon (W(\cdot)) := u^\varepsilon, \tag{5.1}$$

where u^ε stands for the solution of Eq. (1.1) and W can be regarded as a cylindrical Brownian motion on $\mathcal{H}$ by (2.7).

Let $\{g^\varepsilon\}_{\varepsilon>0} \subset \mathcal{U}^N$ be a given family of stochastic processes. By the Girsanov theorem, it is easily to see that $\tilde{u}^\varepsilon := \Gamma^\varepsilon \left(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^\cdot g^\varepsilon(s)\mathrm{d}s\right)$ is the unique solution of the equation

$$\begin{aligned}\frac{\partial \tilde{u}^\varepsilon(t,x)}{\partial t} &= \frac{\partial^2 \tilde{u}^\varepsilon(t,x)}{\partial x^2} + \sqrt{\varepsilon}\sigma(t,x,\tilde{u}^\varepsilon(t,x))\dot{W}(t,x) \\ &\quad + \left\langle \sigma(t,\cdot,\tilde{u}^\varepsilon(t,\cdot)), g^\varepsilon(t,\cdot)\right\rangle_{\mathcal{H}}, \quad t>0,\ x\in\mathbb{R}\end{aligned} \tag{5.2}$$

with the initial value $\tilde{u}^\varepsilon(0,\cdot) \equiv 1$.

Recall the map Γ^0 defined by (2.29). Then $\bar{u}^\varepsilon := \Gamma^0\left(\int_0^\cdot g^\varepsilon(s)\mathrm{d}s\right)$ solves the equation

$$\frac{\partial \bar{u}^\varepsilon(t,x)}{\partial t} = \frac{\partial^2 \bar{u}^\varepsilon(t,x)}{\partial x^2} + \left\langle \sigma(t,\cdot,\bar{u}^\varepsilon(t,\cdot)), g^\varepsilon(t,\cdot)\right\rangle_{\mathcal{H}}, \ t>0, \ x\in\mathbb{R} \tag{5.3}$$

with the initial value $\bar{u}^\varepsilon(0,\cdot)\equiv 1$.

Equivalently, we have

$$\begin{aligned}\tilde{u}^\varepsilon(t,x) = 1 &+ \sqrt{\varepsilon}\int_0^t\int_{\mathbb{R}} p_{t-s}(x-y)\sigma(s,y,\tilde{u}^\varepsilon(s,y))W(\mathrm{d}s,\mathrm{d}y)\\ &+\int_0^t \left\langle p_{t-s}(x-\cdot)\sigma(s,\cdot,\tilde{u}^\varepsilon(s,\cdot)), g^\varepsilon(s,\cdot)\right\rangle_{\mathcal{H}}\mathrm{d}s,\end{aligned} \tag{5.4}$$

and

$$\bar{u}^\varepsilon(t,x) = 1 + \int_0^t \left\langle p_{t-s}(x-\cdot)\sigma(s,\cdot,\bar{u}^\varepsilon(s,\cdot)), g^\varepsilon(s,\cdot)\right\rangle_{\mathcal{H}}\mathrm{d}s. \tag{5.5}$$

Proposition 5.1 *Assume that σ satisfies the hypothesis* (**H**) *for some constant $p_0 > \frac{6}{4H-1}$. For every $N<+\infty$ and $\{g^\varepsilon\}_{\varepsilon>0}\subset\mathcal{U}^N$, it holds that for any $\delta>0$,*

$$\lim_{\varepsilon\to 0}\mathbb{P}\left(d_{\mathcal{C}}(\tilde{u}^\varepsilon,\bar{u}^\varepsilon)>\delta\right)=0.$$

Before proving Proposition 5.1, we give the following lemmas.

Lemma 5.1 *Assume that σ satisfies the hypothesis* (**H**) *for some constant $p_0>\frac{6}{4H-1}$. Then, it holds that for any $p>p_0$,*

$$\sup_{0<\varepsilon<1}\|\tilde{u}^\varepsilon\|_{\mathcal{Z}^p_{\lambda,T}}<\infty, \quad \sup_{0<\varepsilon<1}\|\bar{u}^\varepsilon\|_{\mathcal{Z}^p_{\lambda,T}}<\infty. \tag{5.6}$$

Proof We give the details of the proof for $\tilde{u}^\varepsilon$, while the proof for $\bar{u}^\varepsilon$ is similar but simpler which is omitted. Here, the proof is inspired by the proof of [22, Lemma 4.5].

Step 1 As in (4.38) of [22], let

$$\frac{\partial}{\partial x}W_\eta(t,x) := \int_{\mathbb{R}} p_\eta(x-y)W(t,\mathrm{d}y), \quad \text{for any } \eta>0.$$

Consider

$$\begin{aligned}\tilde{u}^\varepsilon_\eta(t,x) = 1 &+ \sqrt{\varepsilon}\int_0^t\int_{\mathbb{R}} p_{t-s}(x-y)\sigma(s,y,\tilde{u}^\varepsilon_\eta(s,y))W_\eta(\mathrm{d}s,\mathrm{d}y)\\ &+\int_0^t \left\langle p_{t-s}(x-\cdot)\sigma(s,\cdot,\tilde{u}^\varepsilon_\eta(s,\cdot)), g^\varepsilon(s,\cdot)\right\rangle_{\mathcal{H}_\eta}\mathrm{d}s\\ =:&\, 1+\sqrt{\varepsilon}\Phi^\varepsilon_{1,\eta}(t,x)+\Phi^\varepsilon_{2,\eta}(t,x).\end{aligned} \tag{5.7}$$

Let

$$\tilde{u}_\eta^{\varepsilon,0}(t,x)=1,$$

and recursively for $n=0,1,2,\ldots,$

$$\begin{aligned}\tilde{u}_\eta^{\varepsilon,n+1}(t,x)=&1+\sqrt{\varepsilon}\int_0^t\int_{\mathbb{R}}p_{t-s}(x-y)\sigma(s,y,\tilde{u}_\eta^{\varepsilon,n}(s,y))W_\eta(\mathrm{d}s,\mathrm{d}y)\\&+\int_0^t\left\langle p_{t-s}(x-\cdot)\sigma(s,\cdot,\tilde{u}_\eta^{\varepsilon,n}(s,\cdot)),g^\varepsilon(s,\cdot)\right\rangle_{\mathcal{H}_\eta}\mathrm{d}s\\=:&1+\sqrt{\varepsilon}\Phi_{1,\eta}^{\varepsilon,n}(t,x)+\Phi_{2,\eta}^{\varepsilon,n}(t,x).\end{aligned}\tag{5.8}$$

By using [19, Lemma 4.15] and the similar argument as that in Step 1 in the proof of Lemma 3.1, we know that for any fixed $t\in[0,T]$ and $\eta>0$, when n goes to infinity, $\tilde{u}_\eta^{\varepsilon,n}(t,\cdot)$ converges to $\tilde{u}_\eta^{\varepsilon}(t,\cdot)$ in $L_\lambda^p(\Omega\times\mathbb{R})$.

By using the same method as that in the proof of Lemma 3.1, we obtain that

$$\begin{aligned}&\|\Phi_{2,\eta}^{\varepsilon,n+1}(t,\cdot)\|^2_{L_\lambda^p(\Omega\times\mathbb{R})}+\left[\mathcal{N}^*_{\frac12-H,p}\Phi_{2,\eta}^{\varepsilon,n+1}(t)\right]^2\\&\lesssim 1+\int_0^t\left((t-s)^{2H-\frac32}+(t-s)^{H-1}+(t-s)^{-\frac12}\right)\cdot\|\tilde{u}_\eta^{\varepsilon,n}(s,\cdot)\|^2_{L_\lambda^p(\Omega\times\mathbb{R})}\mathrm{d}s\\&\quad+\int_0^t\left((t-s)^{H-1}+(t-s)^{-\frac12}\right)\cdot\left[\mathcal{N}^*_{\frac12-H,p}\tilde{u}_\eta^{\varepsilon,n}(s)\right]^2\mathrm{d}s.\end{aligned}\tag{5.9}$$

Next, we will give some estimates for $\|\Phi_{1,\eta}^{\varepsilon,n}(t,\cdot)\|_{L_\lambda^p(\Omega\times\mathbb{R})}$ and $\mathcal{N}^*_{\frac12-H,p}\Phi_{1,\eta}^{\varepsilon,n}(t)$.

Step 2 In this step, we estimate $\|\Phi_{1,\eta}^{\varepsilon,n}(t,\cdot)\|_{L_\lambda^p(\Omega\times\mathbb{R})}$. By the Burkholder–Davis–Gundy inequality, we have that

$$\begin{aligned}&\mathbb{E}\left[|\Phi_{1,\eta}^{\varepsilon,n+1}(t,x)|^p\right]\\&\lesssim\mathbb{E}\Bigg(\int_0^t\int_{\mathbb{R}^2}\big|p_{t-s}(x-y-h)\sigma(s,y+h,\tilde{u}_\eta^{\varepsilon,n}(s,y+h))\\&\quad-p_{t-s}(x-y)\sigma(s,y,\tilde{u}_\eta^{\varepsilon,n}(s,y))\big|^2\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s\Bigg)^{\frac p2}\\&\lesssim\mathbb{E}\Bigg(\int_0^t\int_{\mathbb{R}^2}p^2_{t-s}(x-y-h)\big|\sigma(s,y+h,\tilde{u}_\eta^{\varepsilon,n}(s,y+h))\\&\quad-\sigma(s,y,\tilde{u}_\eta^{\varepsilon,n}(s,y+h))\big|^2\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s\Bigg)^{\frac p2}\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{E}\bigg(\int_0^t\int_{\mathbb{R}^2} p_{t-s}^2(x-y-h)\left|\sigma(s,y,\tilde{u}_\eta^{\varepsilon,n}(s,y+h))-\sigma(s,y,\tilde{u}_\eta^{\varepsilon,n}(s,y))\right|^2 \\
&\quad\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s\bigg)^{\frac{p}{2}} \\
&+\mathbb{E}\bigg(\int_0^t\int_{\mathbb{R}^2}|D_{t-s}(x-y,h)|^2\cdot\left|\sigma(s,y,\tilde{u}_\eta^{\varepsilon,n}(s,y))\right|^2\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s\bigg)^{\frac{p}{2}} \\
&=:\mathcal{L}_1(t,x)+\mathcal{L}_2(t,x)+\mathcal{L}_3(t,x),
\end{aligned}
\tag{5.10}
$$

where $D_{t-s}(x-y,h)$ is defined by (3.8).

By using the same arguments as to that in Step 3 of the proof of Lemma 3.1, we have

$$
\begin{aligned}
&\left(\int_{\mathbb{R}}\mathcal{L}_1(t,x)\lambda(x)\mathrm{d}x\right)^{\frac{2}{p}} \\
&\lesssim\int_0^t\int_{\mathbb{R}}p_{t-s}^2(y)\left(\int_{\mathbb{R}}\left(1+\|\tilde{u}_\eta^{\varepsilon,n}(s,x+y)\|_{L^p(\Omega)}^p\right)\lambda(x)\mathrm{d}x\right)^{\frac{2}{p}}\mathrm{d}y\mathrm{d}s \\
&\lesssim\int_0^t(t-s)^{-\frac{1}{2}}\cdot\left(\int_{\mathbb{R}^2}p_{\frac{t-s}{2}}(y)\left(1+\|\tilde{u}_\eta^{\varepsilon,n}(s,x)\|_{L^p(\Omega)}^p\right)\lambda(x-y)\mathrm{d}x\mathrm{d}y\right)^{\frac{2}{p}}\mathrm{d}s \\
&\lesssim\int_0^t(t-s)^{-\frac{1}{2}}\cdot\left(1+\|\tilde{u}_\eta^{\varepsilon,n}(s,\cdot)\|_{L_\lambda^p(\Omega\times\mathbb{R})}^2\right)\mathrm{d}s,
\end{aligned}
\tag{5.11}
$$

$$
\begin{aligned}
&\left(\int_{\mathbb{R}}\mathcal{L}_2(t,x)\lambda(x)\mathrm{d}x\right)^{\frac{2}{p}} \\
&\lesssim\int_0^t\int_{\mathbb{R}^2}p_{t-s}^2(y)\left(\int_{\mathbb{R}}\|\tilde{u}_\eta^{\varepsilon,n}(s,x+y+h)-\tilde{u}_\eta^{\varepsilon,n}(s,x+y)\|_{L^p(\Omega)}^p\lambda(x)\mathrm{d}x\right)^{\frac{2}{p}}\cdot \\
&\quad|h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s \\
&\lesssim\int_0^t\int_{\mathbb{R}}(t-s)^{-\frac{1}{2}}\cdot\bigg(\int_{\mathbb{R}^2}p_{\frac{t-s}{2}}(y)\|\tilde{u}_\eta^{\varepsilon,n}(s,x+h)-\tilde{u}_\eta^{\varepsilon,n}(s,x)\|_{L^p(\Omega)}^p \\
&\quad\lambda(x-y)\mathrm{d}x\mathrm{d}y\bigg)^{\frac{2}{p}}\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}s \\
&\lesssim\int_0^t(t-s)^{-\frac{1}{2}}\cdot\left[\mathcal{N}_{\frac{1}{2}-H,p}^{*}\tilde{u}_\eta^{\varepsilon,n}(s)\right]^2\mathrm{d}s,
\end{aligned}
\tag{5.12}
$$

and

$$
\begin{aligned}
&\left(\int_{\mathbb{R}}\mathcal{L}_3(t,x)\lambda(x)\mathrm{d}x\right)^{\frac{2}{p}} \\
&\lesssim\int_0^t\int_{\mathbb{R}^2}|D_{t-s}(y,h)|^2\cdot\left(\int_{\mathbb{R}}\left(1+\|\tilde{u}_\eta^{\varepsilon,n}(s,x+y)\|_{L^p(\Omega)}^p\right)\lambda(x)\mathrm{d}x\right)^{\frac{2}{p}}\cdot|h|^{2H-2}\mathrm{d}h\mathrm{d}y\mathrm{d}s
\end{aligned}
$$

$$\lesssim \int_0^t (t-s)^{H-1} \cdot \left(\int_{\mathbb{R}^3} (t-s)^{1-H} \cdot |D_{t-s}(y,h)|^2 \cdot \left(1 + \|\tilde{u}_\eta^{\varepsilon,n}(s,x)\|_{L^p(\Omega)}^p\right) \lambda(x-y) \right.$$

$$\left. \cdot |h|^{2H-2} \mathrm{d}h\mathrm{d}x\mathrm{d}y \right)^{\frac{2}{p}} \mathrm{d}s$$

$$\lesssim \int_0^t (t-s)^{H-1} \cdot \left(1 + \|\tilde{u}_\eta^{\varepsilon,n}(s,\cdot)\|_{L_\lambda^p(\Omega\times\mathbb{R})}^2\right) \mathrm{d}s. \tag{5.13}$$

Therefore, by (5.10), (5.11), (5.12) and (5.13), we have

$$\begin{aligned}\|\Phi_{1,\eta}^{\varepsilon,n+1}(t,\cdot)\|_{L_\lambda^p(\Omega\times\mathbb{R})}^2 &= \left(\int_{\mathbb{R}} \mathbb{E}\left[\left|\Phi_{1,\eta}^{\varepsilon,n+1}(t,x)\right|^p \right] \lambda(x)\mathrm{d}x \right)^{\frac{2}{p}} \\ &\lesssim 1 + \int_0^t \left((t-s)^{-\frac{1}{2}} + (t-s)^{H-1} \right) \cdot \|\tilde{u}_\eta^{\varepsilon,n}(s,\cdot)\|_{L_\lambda^p(\Omega\times\mathbb{R})}^2 \mathrm{d}s \\ &\quad + \int_0^t (t-s)^{-\frac{1}{2}} \cdot \left[\mathcal{N}_{\frac{1}{2}-H,\,p}^* \tilde{u}_\eta^{\varepsilon,n}(s) \right]^2 \mathrm{d}s. \end{aligned} \tag{5.14}$$

Step 3 In this step, we deal with $\mathcal{N}_{\frac{1}{2}-H,\,p}^* \Phi_{1,\eta}^{\varepsilon,n}(t)$. By the Burkholder–Davis–Gundy inequality and the hypothesis (**H**), the similar calculation as that in Step 2 of the proof of Lemma 3.1 implies that

$$\begin{aligned}&\mathbb{E}\Big[\left|\Phi_{1,\eta}^{\varepsilon,n+1}(t,x) - \Phi_{1,\eta}^{\varepsilon,n+1}(t,x+h)\right|^p \Big] \\ &\simeq \mathbb{E}\bigg(\int_0^t \int_{\mathbb{R}^2} \Big| D_{t-s}(x-y-z,h)\sigma(s,y+z,\tilde{u}_\eta^{\varepsilon,n}(s,y+z)) \\ &\qquad - D_{t-s}(x-z,h)\sigma(s,z,\tilde{u}_\eta^{\varepsilon,n}(s,z)) \Big|^2 \cdot |y|^{2H-2} \mathrm{d}z\mathrm{d}y\mathrm{d}s \bigg)^{\frac{p}{2}} \\ &\lesssim \mathbb{E}\bigg(\int_0^t \int_{\mathbb{R}} |D_{t-s}(x-z,h)|^2 \cdot \left(1 + |\tilde{u}_\eta^{\varepsilon,n}(s,z)|^2\right) \mathrm{d}z\mathrm{d}s \bigg)^{\frac{p}{2}} \\ &\quad + \mathbb{E}\bigg(\int_0^t \int_{\mathbb{R}^2} |D_{t-s}(x-y-z,h)|^2 \cdot \left| \tilde{u}_\eta^{\varepsilon,n}(s,y+z) - \tilde{u}_\eta^{\varepsilon,n}(s,z) \right|^2 \cdot \\ &\qquad |y|^{2H-2} \mathrm{d}z\mathrm{d}y\mathrm{d}s \bigg)^{\frac{p}{2}} \\ &\quad + \mathbb{E}\left(\int_0^t \int_{\mathbb{R}^2} |\Box_{t-s}(x-z,y,h)|^2 \cdot \left(1 + |\tilde{u}_\eta^{\varepsilon,n}(s,z)|^2\right) \cdot |y|^{2H-2} \mathrm{d}z\mathrm{d}y\mathrm{d}s \right)^{\frac{p}{2}} \\ &=: \mathcal{M}_1(t,x,h) + \mathcal{M}_2(t,x,h) + \mathcal{M}_3(t,x,h),\end{aligned}$$

where $\Box_{t-s}(x-z,y,h)$ is defined by (3.9).

By (2.15) and (3.3), we have

$$\left[\mathcal{N}_{\frac{1}{2}-H,\,p}^* \Phi_{1,\eta}^{\varepsilon,n+1}(t) \right]^2 \lesssim \sum_{i=1}^3 \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathcal{M}_i(t,x,h)\lambda(x)\mathrm{d}x \right)^{\frac{2}{p}} \cdot |h|^{2H-2}\mathrm{d}h. \tag{5.15}$$

Applying a change of variable, Minkowski's inequality, Jensen's inequality and Lemma 6.5, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathcal{M}_1(t,x,h)\lambda(x)\mathrm{d}x \right)^{\frac{2}{p}} |h|^{2H-2}\mathrm{d}h \\
&\lesssim \int_0^t (t-s)^{H-1} \cdot \left(1 + \|\tilde{u}_\eta^{\varepsilon,n}(s,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}\right) \mathrm{d}s; \qquad (5.16)\\
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathcal{M}_2(t,x,h)\lambda(x)\mathrm{d}x \right)^{\frac{2}{p}} |h|^{2H-2}\mathrm{d}h \\
&\lesssim \int_0^t (t-s)^{H-1} \cdot \left[\mathcal{N}^*_{\frac{1}{2}-H,\,p}\tilde{u}_\eta^{\varepsilon,n}(s)\right]^2 \mathrm{d}s; \qquad (5.17)\\
&\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \mathcal{M}_3(t,x,h)\lambda(x)\mathrm{d}x \right|^{\frac{2}{p}} \cdot |h|^{2H-2}\mathrm{d}h \\
\lesssim &\int_0^t (t-s)^{2H-\frac{3}{2}} \Bigg((t-s)^{\frac{3}{2}-2H} \int_{\mathbb{R}^4} \left|\Box_{t-s}(z,y,h)\right|^2 \cdot |y|^{2H-2} \cdot |h|^{2H-2} \\
&\quad \cdot \left(1 + \mathbb{E}\left[\left|\tilde{u}_\eta^{\varepsilon,n}(s,x)\right|^p\right]\right) \lambda(x-z)\mathrm{d}x\mathrm{d}y\mathrm{d}h\mathrm{d}z \Bigg)^{\frac{2}{p}} \mathrm{d}s \\
\lesssim &\int_0^t (t-s)^{2H-\frac{3}{2}} \cdot \left(1 + \|\tilde{u}_\eta^{\varepsilon,n}(s,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}\right) \mathrm{d}s. \qquad (5.18)
\end{aligned}$$

Therefore, by (5.15), (5.16), (5.17) and (5.18), we have

$$\begin{aligned}
&\left[\mathcal{N}^*_{\frac{1}{2}-H,\,p}\Phi^{\varepsilon,n+1}_{1,\eta}(t)\right]^2 \\
&\quad \lesssim 1 + \int_0^t \left((t-s)^{H-1} + (t-s)^{2H-\frac{3}{2}}\right) \cdot \|\tilde{u}_\eta^{\varepsilon,n}(s,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}\mathrm{d}s \qquad (5.19)\\
&+ \int_0^t (t-s)^{H-1} \cdot \left[\mathcal{N}^*_{\frac{1}{2}-H,\,p}\tilde{u}_\eta^{\varepsilon,n}(s)\right]^2 \mathrm{d}s.
\end{aligned}$$

Step 4 By (5.14) and (5.19), we obtain that

$$\begin{aligned}
&\|\Phi^{\varepsilon,n+1}_{1,\eta}(t,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})} + \left[\mathcal{N}^*_{\frac{1}{2}-H,\,p}\Phi^{\varepsilon,n+1}_{1,\eta}(t)\right]^2 \\
&\lesssim 1 + \int_0^t \left((t-s)^{H-1} + (t-s)^{-\frac{1}{2}}\right) \cdot \|\tilde{u}_\eta^{\varepsilon,n}(s,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})}\mathrm{d}s \qquad (5.20)\\
&\quad + \int_0^t \left((t-s)^{H-1} + (t-s)^{-\frac{1}{2}}\right) \cdot \left[\mathcal{N}^*_{\frac{1}{2}-H,\,p}\tilde{u}_\eta^{\varepsilon,n}(s)\right]^2 \mathrm{d}s.
\end{aligned}$$

For any $t \geq 0$, let

$$\tilde{\Psi}_\eta^{\varepsilon,n}(t) := \|\tilde{u}_\eta^{\varepsilon,n}(t,\cdot)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})} + \left[\mathcal{N}^*_{\frac{1}{2}-H,\,p}\tilde{u}_\eta^{\varepsilon,n}(t)\right]^2.$$

By (5.8), (5.9) and (5.20), there exists a constant $C_{T,p,H,N} > 0$ such that

$$\tilde{\Psi}_{\eta}^{\varepsilon,n+1}(t) \leq C_{T,p,H,N}\left(1+\int_0^t (t-s)^{2H-\frac{3}{2}} \cdot \tilde{\Psi}_{\eta}^{\varepsilon,n}(s)\mathrm{d}s\right).$$

By the extension of Grönwall's lemma [11, Lemma 15], we have

$$\sup_{n\geq 1}\sup_{t\in[0,T]} \tilde{\Psi}_{\eta}^{\varepsilon,n}(t) \leq C,$$

where C is a constant independent of $\eta \in (0,\infty)$ and $\varepsilon \in (0,1)$.

By using the same argument as that in Step 3 of the proof of [22, Lemma 4.5] (or Step 4 in the proof of Lemma 3.1), we have

$$\sup_{\eta>0} \|\tilde{u}_{\eta}^{\varepsilon}\|_{\mathcal{Z}_{\lambda,T}^{p}} \leq C \quad \text{for any } p \geq p_0, \tag{5.21}$$

where C is a constant independent of $\varepsilon \in (0,1)$.

By using the same methods as that in the proof of [22, Lemma 4.7 (ii), (iii)] and Lemma 3.2 (ii), (iii), we have that, for any $R > 0$ and $\gamma > 0$, there exists $\theta > 0$ such that for each $i = 1, 2$,

$$\lim_{\theta\downarrow 0}\mathbb{P}\left(\left\{\Phi_{i,\eta}^{\varepsilon} \in \mathcal{C}([0,T]\times\mathbb{R}) : m^{T,R}\left(\Phi_{i,\eta}^{\varepsilon},\theta\right) > \gamma\right\}\right) = 0,$$

where

$$m^{T,R}\left(\Phi_{i,\eta}^{\varepsilon},\theta\right) := \max_{\substack{|t-s|+|x-y|\leq\theta,\\ 0\leq t,s\leq T,\ -R\leq x,y\leq R}} |\Phi_{i,\eta}^{\varepsilon}(t,x) - \Phi_{i,\eta}^{\varepsilon}(s,y)|.$$

By Lemma 6.7, the family $\{\tilde{u}_{\eta}^{\varepsilon}\}_{\eta>0}$ is tight on the space $\mathcal{C}([0,T]\times\mathbb{R})$. Thus, $\tilde{u}_{\eta}^{\varepsilon} \to \tilde{u}^{\varepsilon}$ almost surely in the space $(\mathcal{C}([0,T]\times\mathbb{R}), d_{\mathcal{C}})$ as $\eta \to 0$. By the same method as that in the proofs of [22, Theorem 1.5] and proposition 3.1, we know that $\tilde{u}^{\varepsilon}$ is the solution of Eq.(5.4). By (5.21) and [22, Lemma 4.6], we have

$$\sup_{\varepsilon\in(0,1)} \|\tilde{u}^{\varepsilon}\|_{\mathcal{Z}_{\lambda,T}^{p}} < \infty, \quad \text{for any } p \geq p_0.$$

The proof is complete. □

For any $u \in \mathcal{Z}_{\lambda,T}^{p}$ and $g \in \mathcal{U}^N$, let

$$Y(t,x) := \int_0^t \langle p_{t-s}(x-\cdot)\sigma(s,\cdot,u(s,\cdot)), g(s,\cdot)\rangle_{\mathcal{H}}\,\mathrm{d}s. \tag{5.22}$$

By using Lemma 3.2 and Minkowski's inequality, we have the following results.

Lemma 5.2 *Assume that σ satisfies the hypothesis* (**H**) *for some constant $p_0 > \frac{6}{4H-1}$. Then we have the following results:*

(i) *For any $p > p_0$, there exists a constant $C_{T,p,H,N} > 0$ such that*

$$\left\| \sup_{t\in[0,T],\, x\in\mathbb{R}} \lambda^{\frac{1}{p}}(x)\mathcal{N}_{\frac{1}{2}-H}Y(t,x) \right\|_{L^p(\Omega)} \le C_{T,p,H,N}\left(1+\|u\|_{\mathcal{Z}^p_{\lambda,T}}\right); \tag{5.23}$$

(ii) *If $p > \frac{3}{H}$ and $0 < \gamma < \frac{H}{2} - \frac{3}{2p}$, then there exists a positive constant $C_{T,p,H,N,\gamma}$ such that*

$$\begin{aligned} &\left\| \sup_{\substack{t,\, t+h\in[0,T],\\ x\in\mathbb{R}}} \lambda^{\frac{1}{p}}(x)\big[Y(t+h,x)-Y(t,x)\big] \right\|_{L^p(\Omega)} \\ &\quad \le C_{T,p,H,N,\gamma}|h|^{\gamma}\cdot\left(1+\|u\|_{\mathcal{Z}^p_{\lambda,T}}\right); \end{aligned} \tag{5.24}$$

(iii) *If $p > \frac{3}{H}$ and $0 < \gamma < H - \frac{3}{p}$, then there exists a positive constant $C_{T,p,H,N,\gamma}$ such that*

$$\begin{aligned} &\left\| \sup_{\substack{t\in[0,T],\\ x,y\in\mathbb{R}}} \frac{Y(t,x)-Y(t,y)}{\lambda^{-\frac{1}{p}}(x)+\lambda^{-\frac{1}{p}}(y)} \right\|_{L^p(\Omega)} \\ &\quad \le C_{T,p,H,N,\gamma}|x-y|^{\gamma}\cdot\left(1+\|u\|_{\mathcal{Z}^p_{\lambda,T}}\right). \end{aligned} \tag{5.25}$$

Recall $\tilde{u}^{\varepsilon}$ and $\bar{u}^{\varepsilon}$ defined by (5.4) and (5.5), respectively. For any $k \ge 1$ and any $p \ge p_0$, where p_0 is defined by (2.22), define the stopping time

$$\begin{aligned} \tau_k := \inf\Bigg\{ r\ge 0 : &\sup_{0\le s\le r,\, x\in\mathbb{R}} \lambda^{\frac{1}{p}}(x)\mathcal{N}_{\frac{1}{2}-H}\tilde{u}^{\varepsilon}(s,x) \ge k, \\ &\text{or} \sup_{0\le s\le r,\, x\in\mathbb{R}} \lambda^{\frac{1}{p}}(x)\mathcal{N}_{\frac{1}{2}-H}\bar{u}^{\varepsilon}(s,x) \ge k \Bigg\}. \end{aligned} \tag{5.26}$$

By [22, Lemma 4.7], Lemmas 5.1 and 5.2, we know that

$$\sup_{\varepsilon\in(0,1)} \left\| \sup_{t\in[0,T],\, x\in\mathbb{R}} \lambda^{\frac{1}{p}}(x)\mathcal{N}_{\frac{1}{2}-H}\tilde{u}^{\varepsilon}(t,x) \right\|_{L^p(\Omega)} < \infty,$$

$$\sup_{\varepsilon\in(0,1)} \left\| \sup_{t\in[0,T],\, x\in\mathbb{R}} \lambda^{\frac{1}{p}}(x)\mathcal{N}_{\frac{1}{2}-H}\bar{u}^{\varepsilon}(t,x) \right\|_{L^p(\Omega)} < \infty.$$

Those, together with Chebychev's inequality, imply that

$$\tau_k \uparrow \infty, \text{ a.s., as } k \to \infty. \tag{5.27}$$

For any $t \in [0, T]$, let

$$\tilde{u}_k^{\varepsilon}(t, \cdot) := \tilde{u}^{\varepsilon}(t \wedge \tau_k, \cdot), \qquad \bar{u}_k^{\varepsilon}(t, \cdot) := \bar{u}^{\varepsilon}(t \wedge \tau_k, \cdot). \tag{5.28}$$

Obviously, when $\tau_k > T$, $\tilde{u}_k^{\varepsilon}(t, \cdot) = \tilde{u}^{\varepsilon}(t, \cdot)$ and $\bar{u}_k^{\varepsilon}(t, \cdot) = \bar{u}^{\varepsilon}(t, \cdot)$ for any $t \in [0, T]$.

Lemma 5.3 *Assume that σ satisfies the hypothesis* (**H**) *for some constant $p_0 > \frac{6}{4H-1}$. Then for any $p \geq p_0$,*

$$\lim_{\varepsilon \to 0} \left\| \tilde{u}_k^{\varepsilon} - \bar{u}_k^{\varepsilon} \right\|_{\mathcal{Z}_{\lambda,T}^p} = 0. \tag{5.29}$$

Proof Let

$$\Phi_1^{\varepsilon}(t, x) := \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y)\sigma(s, y, \tilde{u}_k^{\varepsilon}(s, y)) W(\mathrm{d}s, \mathrm{d}y), \tag{5.30}$$

and

$$\Phi_2^{\varepsilon}(t, x) := \int_0^t \left\langle p_{t-s}(x - \cdot)\triangle(s, \cdot, \cdot).g^{\varepsilon}(s, \cdot)\right\rangle_{\mathcal{H}} \mathrm{d}s, \tag{5.31}$$

where $\triangle(t, x, y) := \sigma(t, x, \tilde{u}_k^{\varepsilon}(t, y)) - \sigma(t, x, \bar{u}_k^{\varepsilon}(t, y))$. Then, by (5.4) and (5.5), we have

$$\left\| \tilde{u}_k^{\varepsilon} - \bar{u}_k^{\varepsilon} \right\|_{\mathcal{Z}_{\lambda,T}^p} \leq \sqrt{\varepsilon} \left\| \Phi_1^{\varepsilon} \right\|_{\mathcal{Z}_{\lambda,T}^p} + \left\| \Phi_2^{\varepsilon} \right\|_{\mathcal{Z}_{\lambda,T}^p}. \tag{5.32}$$

According to Lemmas 4.5 and 4.6 in [22], we have $\left\| \Phi_1^{\varepsilon} \right\|_{\mathcal{Z}_{\lambda,T}^p} < \infty$ for any $p \geq p_0$. Now, it remains to give an estimate for $\left\| \Phi_2^{\varepsilon} \right\|_{\mathcal{Z}_{\lambda,T}^p}$.

By using the same technique as that in the proof of (5.14), we have

$$\begin{aligned} &\left\| \Phi_2^{\varepsilon}(t, \cdot) \right\|_{L_\lambda^p(\Omega \times \mathbb{R})}^2 \\ \lesssim &\int_0^t \left((t-s)^{H-1} + (t-s)^{-\frac{1}{2}} \right) \cdot \left\| \tilde{u}_k^{\varepsilon}(s, \cdot) - \bar{u}_k^{\varepsilon}(s, \cdot) \right\|_{L_\lambda^p(\Omega \times \mathbb{R})}^2 \mathrm{d}s \\ &+ \int_0^t (t-s)^{-\frac{1}{2}} \cdot \left[\mathcal{N}_{\frac{1}{2}-H, p}^* \left(\tilde{u}_k^{\varepsilon}(s) - \bar{u}_k^{\varepsilon}(s) \right) \right]^2 \mathrm{d}s. \end{aligned} \tag{5.33}$$

Next, we deal with the term $\mathcal{N}_{\frac{1}{2}-H, p}^* \Phi_2^{\varepsilon}(t)$. Since $g^{\varepsilon} \in \mathcal{U}^N$, by the Cauchy–Schwarz inequality and (2.6), we have

$$\begin{aligned} &\mathbb{E}\left[\left| \Phi_2^{\varepsilon}(t, x) - \Phi_2^{\varepsilon}(t, x+h) \right|^p \right] \\ = &\mathbb{E}\left[\left| \int_0^t \left\langle D_{t-s}(x - \cdot, h)\triangle(s, \cdot, \cdot), g^{\varepsilon}(s, \cdot) \right\rangle_{\mathcal{H}} \mathrm{d}s \right|^p \right] \end{aligned}$$

$$
\begin{aligned}
&\lesssim \mathbb{E}\left[\int_0^t \left\| D_{t-s}(x-\cdot,h)\triangle(s,\cdot,\cdot)\right\|_{\mathcal{H}}^2 \mathrm{d}s\right]^{\frac{p}{2}} \\
&\simeq \mathbb{E}\Bigg[\int_0^t \int_{\mathbb{R}^2} \big|D_{t-s}(x-y-l,h)\triangle(s,y+l,y+l) \\
&\qquad - D_{t-s}(x-y,h)\triangle(s,y,y)\big|^2 \cdot |l|^{2H-2}\mathrm{d}l\mathrm{d}y\mathrm{d}s\Bigg]^{\frac{p}{2}} \\
&\lesssim \mathcal{R}_1^{\varepsilon}(t,x,h)+\mathcal{R}_2^{\varepsilon}(t,x,h)+\mathcal{R}_3^{\varepsilon}(t,x,h),
\end{aligned}
\tag{5.34}
$$

where

$$
\begin{aligned}
\mathcal{R}_1^{\varepsilon}(t,x,h) &:= \mathbb{E}\Bigg[\int_0^t \int_{\mathbb{R}^2} \big|D_{t-s}(x-y-l,h)\big|^2 \\
&\qquad \cdot \big|\triangle(s,y+l,y+l)-\triangle(s,y+l,y)\big|^2 \cdot |l|^{2H-2}\mathrm{d}l\mathrm{d}y\mathrm{d}s\Bigg]^{\frac{p}{2}}; \\
\mathcal{R}_2^{\varepsilon}(t,x,h) &:= \mathbb{E}\Bigg[\int_0^t \int_{\mathbb{R}^2} \big|\square_{t-s}(x-y,-l,h)\big|^2 \cdot \big|\triangle(s,y+l,y)\big|^2 \cdot \\
&\qquad |l|^{2H-2}\mathrm{d}l\mathrm{d}y\mathrm{d}s\Bigg]^{\frac{p}{2}}; \\
\mathcal{R}_3^{\varepsilon}(t,x,h) &:= \mathbb{E}\Bigg[\int_0^t \int_{\mathbb{R}^2} \big|D_{t-s}(x-y,h)\big|^2 \cdot \big|\triangle(s,y+l,y)-\triangle(s,y,y)\big|^2 \cdot \\
&\qquad |l|^{2H-2}\mathrm{d}l\mathrm{d}y\mathrm{d}s\Bigg]^{\frac{p}{2}}.
\end{aligned}
$$

By a change of variable, we have

$$
\begin{aligned}
&\mathcal{R}_1^{\varepsilon}(t,x,h) \\
&\quad \lesssim \mathbb{E}\Bigg[\int_0^t \int_{\mathbb{R}^2} \big|D_{t-s}(x-y,h)\big|^2 \\
&\quad \cdot \big|\tilde{u}_k^{\varepsilon}(s,y+l)-\bar{u}_k^{\varepsilon}(s,y+l)-\tilde{u}_k^{\varepsilon}(s,y)+\bar{u}_k^{\varepsilon}(s,y)\big|^2 \cdot |l|^{2H-2}\mathrm{d}l\mathrm{d}y\mathrm{d}s\Bigg]^{\frac{p}{2}} \\
&\quad + \mathbb{E}\Bigg[\int_0^t \int_{\mathbb{R}} \big|D_{t-s}(x-y,h)\big|^2 \cdot \big|\tilde{u}_k^{\varepsilon}(s,y)-\bar{u}_k^{\varepsilon}(s,y)\big|^2 \\
&\quad \cdot \left(\int_{\mathbb{R}} \lambda^{\frac{2}{p_0}}(y)\big|\tilde{u}_k^{\varepsilon}(s,y+l)-\tilde{u}_k^{\varepsilon}(s,y)\big|^2 \cdot |l|^{2H-2}\mathrm{d}l\right)\mathrm{d}y\mathrm{d}s\Bigg]^{\frac{p}{2}} \\
&\quad + \mathbb{E}\Bigg[\int_0^t \int_{\mathbb{R}} \big|D_{t-s}(x-y,h)\big|^2 \cdot \big|\tilde{u}_k^{\varepsilon}(s,y)-\bar{u}_k^{\varepsilon}(s,y)\big|^2
\end{aligned}
$$

$$\cdot \left(\int_{\mathbb{R}} \lambda^{\frac{2}{p_0}}(y) \left| \bar{u}_k^{\varepsilon}(s, y+l) - \bar{u}_k^{\varepsilon}(s, y) \right|^2 \cdot |l|^{2H-2} \mathrm{d}l \right) \mathrm{d}y \mathrm{d}s \Bigg]^{\frac{p}{2}}$$

$$=: \mathcal{R}_{11}^{\varepsilon}(t, x, h) + \mathcal{R}_{12}^{\varepsilon}(t, x, h) + \mathcal{R}_{13}^{\varepsilon}(t, x, h),$$

where p_0 is the constant given by (2.22).

By using a change of variable, the Minkowski inequality, Jensen's inequality and Lemma 6.5, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathcal{R}_{11}^{\varepsilon}(t, x, h) \lambda(x) \mathrm{d}x \right)^{\frac{2}{p}} \cdot |h|^{2H-2} \mathrm{d}h \\
&\lesssim \int_0^t \int_{\mathbb{R}} (t-s)^{H-1} \cdot \Bigg[\int_{\mathbb{R}^3} (t-s)^{1-H} \left| D_{t-s}(y, h) \right|^2 \cdot |h|^{2H-2} \\
&\quad \cdot \mathbb{E}\left[\left| \tilde{u}_k^{\varepsilon}(s, x+l) - \bar{u}_k^{\varepsilon}(s, x+l) - \tilde{u}_k^{\varepsilon}(s, x) + \bar{u}_k^{\varepsilon}(s, x) \right|^p \right] \cdot \lambda(x-y) \mathrm{d}x \mathrm{d}y \mathrm{d}h \Bigg]^{\frac{2}{p}} \cdot \\
&\quad |l|^{2H-2} \mathrm{d}l \mathrm{d}s \\
&\lesssim \int_0^t (t-s)^{H-1} \cdot \left[\mathcal{N}_{\frac{1}{2}-H, p}^{*} \left(\tilde{u}_k^{\varepsilon}(s) - \bar{u}_k^{\varepsilon}(s) \right) \right]^2 \mathrm{d}s,
\end{aligned} \tag{5.35}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left(\mathcal{R}_{12}^{\varepsilon}(t, x, h) + \mathcal{R}_{13}^{\varepsilon}(t, x, h) \right) \lambda(x) \mathrm{d}x \right)^{\frac{2}{p}} \cdot |h|^{2H-2} \mathrm{d}h \\
&\lesssim k^2 \int_0^t (t-s)^{H-1} \cdot \Bigg[\int_{\mathbb{R}^3} (t-s)^{1-H} \cdot \left| D_{t-s}(y, h) \right|^2 \cdot |h|^{2H-2} \\
&\quad \cdot \mathbb{E}\left[\left| \tilde{u}_k^{\varepsilon}(s, x) - \bar{u}_k^{\varepsilon}(s, x) \right|^p \right] \cdot \lambda(x-y) \mathrm{d}x \mathrm{d}y \mathrm{d}h \Bigg]^{\frac{2}{p}} \mathrm{d}s \\
&\lesssim k^2 \int_0^t (t-s)^{H-1} \cdot \left\| \tilde{u}_k^{\varepsilon}(s, \cdot) - \bar{u}_k^{\varepsilon}(s, \cdot) \right\|_{L_{\lambda}^p(\Omega \times \mathbb{R})}^2 \mathrm{d}s.
\end{aligned} \tag{5.36}$$

By (2.18), we have

$$\begin{aligned}
\mathcal{R}_2^{\varepsilon}(t, x, h) \lesssim \mathbb{E}\Bigg[\int_0^t \int_{\mathbb{R}^2} &\left| \square_{t-s}(x-y, -l, h) \right|^2 \cdot \\
&\left| \tilde{u}_k^{\varepsilon}(s, y) - \bar{u}_k^{\varepsilon}(s, y) \right|^2 \cdot |l|^{2H-2} \mathrm{d}l \mathrm{d}y \mathrm{d}s \Bigg]^{\frac{p}{2}}.
\end{aligned}$$

Hence, by a change of variable, Minkowski's inequality, Jensen's inequality and Lemma 6.5, we have

$$\int_{\mathbb{R}} \left| \int_{\mathbb{R}} \mathcal{R}_2^{\varepsilon}(t, x, h) \lambda(x) \mathrm{d}x \right|^{\frac{2}{p}} \cdot |h|^{2H-2} \mathrm{d}h$$

$$\lesssim \int_0^t (t-s)^{2H-\frac{3}{2}} \cdot \Big((t-s)^{\frac{3}{2}-2H} \cdot \int_{\mathbb{R}^4} \big|\Box_{t-s}(y,l,h)\big|^2 \cdot |l|^{2H-2}|h|^{2H-2}$$
$$\cdot\, \mathbb{E}\big[\big|\tilde{u}_k^\varepsilon(s,x) - \bar{u}_k^\varepsilon(s,x)\big|^p\big]\lambda(x-y)\mathrm{d}x\mathrm{d}l\mathrm{d}y\mathrm{d}h\Big)^{\frac{2}{p}} \mathrm{d}s$$
$$\lesssim \int_0^t (t-s)^{2H-\frac{3}{2}} \cdot \big\|\tilde{u}_k^\varepsilon(s,\cdot) - \bar{u}_k^\varepsilon(s,\cdot)\big\|^2_{L^p_\lambda(\Omega\times\mathbb{R})} \mathrm{d}s. \tag{5.37}$$

By Lemma 6.5, we have

$$\begin{aligned} &\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \mathcal{R}_3^\varepsilon(t,x,h)\lambda(x)\mathrm{d}x \Big)^{\frac{2}{p}} \cdot |h|^{2H-2}\mathrm{d}h \\ &\lesssim \int_0^t (t-s)^{H-1} \cdot \big\|\tilde{u}_k^\varepsilon(s,\cdot) - \bar{u}_k^\varepsilon(s,\cdot)\big\|^2_{L^p_\lambda(\Omega\times\mathbb{R})} \mathrm{d}s. \end{aligned} \tag{5.38}$$

Putting (5.34), (5.35), (5.36), (5.37) and (5.38) together, we have

$$\begin{aligned} &\Big[\mathcal{N}^*_{\frac{1}{2}-H,p}\Phi_2^\varepsilon(t)\Big]^2 \\ &:= \int_{\mathbb{R}} \Big[\int_{\mathbb{R}} \mathbb{E}\big[\big|\Phi_2^\varepsilon(t,x) - \Phi_2^\varepsilon(t,x+h)\big|^p\big]\lambda(x)\mathrm{d}x \Big]^{\frac{2}{p}} \cdot |h|^{2H-2}\mathrm{d}h \\ &\lesssim k^2 \int_0^t \Big((t-s)^{H-1} + (t-s)^{2H-\frac{3}{2}} \Big) \cdot \big\|\tilde{u}_k^\varepsilon(s,\cdot) - \bar{u}_k^\varepsilon(s,\cdot)\big\|^2_{L^p_\lambda(\Omega\times\mathbb{R})} \mathrm{d}s \\ &\quad + \int_0^t (t-s)^{H-1} \cdot \Big[\mathcal{N}^*_{\frac{1}{2}-H,\,p}\big(\tilde{u}_k^\varepsilon(s) - \bar{u}_k^\varepsilon(s)\big)\Big]^2 \mathrm{d}s. \end{aligned} \tag{5.39}$$

It follows from Lemma 5.1 that $\big\|\tilde{u}_k^\varepsilon - \bar{u}_k^\varepsilon\big\|_{\mathcal{Z}^p_{\lambda,T}}$ is finite. By (5.32), (5.33), (5.39) and the fractional Grönwall lemma ([15, Lemma 7.1.1]), we have that, for any fixed $k \geq 1$ and $p > p_0$,

$$\lim_{\varepsilon\to 0} \big\|\tilde{u}_k^\varepsilon - \bar{u}_k^\varepsilon\big\|_{\mathcal{Z}^p_{\lambda,T}} = 0.$$

The proof is complete. □

We now give the proof of Proposition 5.1.

Proof of Proposition 5.1 By [22, Lemma 4.7], Lemmas 5.1, 5.2 and 6.7, we know that the probability measures on the space $(\mathcal{C}([0,T]\times\mathbb{R}), \mathcal{B}(\mathcal{C}([0,T]\times\mathbb{R})), d_{\mathcal{C}})$ corresponding to the processes $\{\tilde{u}^\varepsilon - \bar{u}^\varepsilon\}_{\varepsilon>0}$ are tight. Thus, there is a subsequence $\varepsilon_n \downarrow 0$ such that $\tilde{u}^{\varepsilon_n} - \bar{u}^{\varepsilon_n}$ converges weakly to some stochastic process $Z = \{Z(t,x), t\in[0,T], x\in\mathbb{R}\}$ in $(\mathcal{C}([0,T]\times\mathbb{R}), \mathcal{B}(\mathcal{C}([0,T]\times\mathbb{R})), d_{\mathcal{C}})$.

On the other hand, for any $k \geq 1$, $p \geq p_0$, $\gamma > 0$,

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\leq t\leq T}\int_{\mathbb{R}}\left|\tilde{u}^{\varepsilon}(t,x)-\bar{u}^{\varepsilon}(t,x)\right|^p\lambda(x)\mathrm{d}x>\gamma\right)\\
&\leq \mathbb{P}\left(\sup_{0\leq t\leq T}\int_{\mathbb{R}}\left|\tilde{u}^{\varepsilon}(t,x)-\bar{u}^{\varepsilon}(t,x)\right|^p\lambda(x)\mathrm{d}x>\gamma,\tau_k>T\right)+\mathbb{P}(\tau_k\leq T) \qquad (5.40)\\
&\leq \mathbb{P}\left(\sup_{0\leq t\leq T}\int_{\mathbb{R}}\left|\tilde{u}_k^{\varepsilon}(t,x)-\bar{u}_k^{\varepsilon}(t,x)\right|^p\lambda(x)\mathrm{d}x>\gamma\right)+\mathbb{P}(\tau_k\leq T)\,.
\end{aligned}$$

First letting $\varepsilon \to 0$ and then letting $k \to \infty$, by Lemma 5.3 and (5.27), we have

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}}\left|\tilde{u}^{\varepsilon}(t,x)-\bar{u}^{\varepsilon}(t,x)\right|^p\lambda(x)\mathrm{d}x\to 0 \text{ in probability, as } \varepsilon\to 0.$$

Thus, for any fixed $t \in [0, T]$, the processes $\{\tilde{u}^{\varepsilon}(t,x)(\omega)-\bar{u}^{\varepsilon}(t,x)(\omega);\,(\omega,x)\in\Omega\times\mathbb{R}\}$ converges in probability in the product probability space $(\Omega\otimes\mathbb{R}, \mathbb{P}\otimes\lambda(x)\mathrm{d}x)$. This, together with the weak convergence of $\{\tilde{u}^{\varepsilon_n}-\bar{u}^{\varepsilon_n}\}_{n\geq 1}$ and the uniqueness of the limit distribution of $\tilde{u}^{\varepsilon_n}-\bar{u}^{\varepsilon_n}$, implies that $Z(t,x)\equiv 0$, $t\in[0,T]$, $x\in\mathbb{R}$, almost surely. Thus, as $\varepsilon\to 0$, $\tilde{u}^{\varepsilon}-\bar{u}^{\varepsilon}$ converges weakly to 0 in $(\mathcal{C}([0,T]\times\mathbb{R}), \mathcal{B}(\mathcal{C}([0,T]\times\mathbb{R})), d_{\mathcal{C}})$. Equivalently, the sequence of real-valued random variables $d_{\mathcal{C}}(\tilde{u}^{\varepsilon},\bar{u}^{\varepsilon})$ converges to 0 in distribution as ε goes to 0. This implies that as $\varepsilon\to 0$,

$$d_{\mathcal{C}}(\tilde{u}^{\varepsilon},\bar{u}^{\varepsilon})\longrightarrow 0 \text{ in probability.}$$

See, e.g., [8, p. 98, Exercise 4]. The proof is complete. □

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Appendix

In this section, we give some lemmas related to the heat kernel $p_t(x)$. Recall $\lambda(x)$ defined by (2.13), $D_t(x,h)$, $\square_t(x,y,h)$ defined by (3.8) and (3.9), respectively.

Lemma 6.1 ([22, Lemma 2.5]) *For any* $T > 0$,

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\frac{1}{\lambda(x)}\int_{\mathbb{R}} p_t(x-y)\lambda(y)\mathrm{d}y < \infty. \tag{6.1}$$

Lemma 6.2 ([22, Lemma 2.8]) *For any* $H \in (\frac{1}{4}, \frac{1}{2})$*, there exists some constant* C_H *such that*

$$\int_{\mathbb{R}^2} |D_t(x,h)|^2 \cdot |h|^{2H-2}\mathrm{d}h\mathrm{d}x = C_H t^{H-1}, \tag{6.2}$$

and

$$\int_{\mathbb{R}^3} |\Box_t(x,y,h)|^2 \cdot |h|^{2H-2} \cdot |y|^{2H-2}\mathrm{d}y\mathrm{d}h\mathrm{d}x = C_H t^{2H-\frac{3}{2}}. \tag{6.3}$$

Lemma 6.3 ([22, Lemma 2.10]) *For any* $t > 0$*, there exists some constant* C_H *such that*

$$\int_{\mathbb{R}} |D_t(x,h)|^2 \cdot |h|^{2H-2}\mathrm{d}h \le C_H\left(t^{H-\frac{3}{2}} \wedge \frac{|x|^{2H-2}}{\sqrt{t}}\right), \tag{6.4}$$

where $0 < H < \frac{1}{2}$.

Lemma 6.4 ([22, Lemma 2.11]) *For any* $t > 0$*, there exists some constant* C_H *such that*

$$\int_{\mathbb{R}^2} |\Box_t(x,y,h)|^2 \cdot |h|^{2H-2} \cdot |y|^{2H-2}\mathrm{d}y\mathrm{d}h \le C_H\left(t^{2H-2} \wedge \frac{|x|^{2H-2}}{t^{1-H}}\right). \tag{6.5}$$

Lemma 6.5 ([22, Lemma 2.12]) *For any* $t > 0$*, there exists some constant* $C_{T,H}$ *such that*

$$\int_{\mathbb{R}^2} |D_t(x,h)|^2 \cdot |h|^{2H-2}\lambda(z-x)\mathrm{d}x\mathrm{d}h \le C_{T,H}t^{H-1}\lambda(z), \tag{6.6}$$

and

$$\int_{\mathbb{R}^3} |\Box_t(x,y,h)|^2 \cdot |h|^{2H-2} \cdot |y|^{2H-2}\lambda(z-x)\mathrm{d}x\mathrm{d}y\mathrm{d}h \le C_{T,H}t^{2H-\frac{3}{2}}\lambda(z). \tag{6.7}$$

Lemma 6.6 ([22, (4.29)], [22, (4.32)]) *For some fixed* $\gamma \in (0,1)$ *and* $\alpha \in (0,1)$*, the following two inequalities hold:*

$$|(t+h)^{\alpha-1} - t^{\alpha-1}| \lesssim |t|^{\alpha-1-\gamma}h^{\gamma}, \tag{6.8}$$

and

$$|p_{t+h}(x) - p_t(x)| \lesssim h^{\gamma}t^{-\gamma}\left[p_{\frac{2}{\gamma}(t+h)}(x) + p_{\frac{2t}{\gamma}}(x)\right]. \tag{6.9}$$

Lemma 6.7 ([22, Theorem 4.4]) *A sequence* $\{\mathbb{P}_n\}_{n=1}^{\infty}$ *of probability measures on* $(\mathcal{C}([0,T]\times\mathbb{R}), \mathcal{B}(\mathcal{C}([0,T]\times\mathbb{R})))$ *is tight if and only if the following conditions hold:*

(i). $\lim_{\lambda\uparrow\infty}\sup_{n\geq 1}\mathbb{P}_n\left(\{\omega\in\mathcal{C}([0,T]\times\mathbb{R}) : |\omega(0,0)|>\lambda\}\right)=0.$

(ii). *For any* $R>0$ *and* $\gamma>0$,

$$\lim_{\delta\downarrow 0}\sup_{n\geq 1}\mathbb{P}_n\left(\left\{\omega\in\mathcal{C}([0,T]\times\mathbb{R}) : m^{T,R}(\omega,\theta)>\gamma\right\}\right)=0,$$

where

$$m^{T,R}(\omega,\theta) := \max_{\substack{|t-s|+|x-y|\leq\theta,\\ 0\leq t,s\leq T;\ -R\leq x,y\leq R}} |\omega(t,x)-\omega(s,y)|$$

is the modulus of continuity on $[0,T]\times[-R,R]$.

References

1. Balan, R., Jolis, M., Quer-Sardanyons, L.: SPDEs with affine multiplicative fractional noise in space with index $\frac{1}{4}<H<\frac{1}{2}$. Electron. J. Probab. **2**(54), 1–36 (2015)
2. Balan, R., Jolis, M., Quer-Sardanyons, L.: SPDEs with rough noise in space: Hölder continuity of the solution. Stat. Probab. Lett. **119**, 310–316 (2016)
3. Budhiraja, A., Chen, J., Dupuis, P.: Large deviations for stochastic partial differential equations driven by a Poisson random measure. Stoch. Process. Appl. **123**(2), 523–560 (2013)
4. Budhiraja, A., Dupuis, P.: A variational representation for positive functionals of infinite dimensional Brownian motion. Probab. Math. Statist. **20**(1), 39–61 (2000)
5. Budhiraja, A., Dupuis, P.: Analysis and Approximation of Rare Events: Representations and Weak Convergence Methods. Springer, New York (2019)
6. Budhiraja, A., Dupuis, P., Maroulas, V.: Large deviations for infinite dimensional stochastic dynamical systems. Ann. Probab. **36**(4), 1390–1420 (2008)
7. Budhiraja, A., Dupuis, P., Maroulas, V.: Variational representations for continuous time processes. Ann. Inst. Henri Poincaré Probab. Stat. **47**(3), 725–747 (2011)
8. Chung, K.L.: A Course in Probability Theory, 3rd edn. Academic Press Inc, San Diego (2001)
9. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions, 2nd edn. Cambridge University Press, Cambridge (2014)
10. Dai, Y., Li, R.: Transportation inequality for stochastic heat equation with rough dependence in space. *Acta Math. Sin* (Engl. Ser.) **38**(11), 2019–2038 (2022)
11. Dalang, R.C.: Extending the martingale measure stochastic integral with applications to spatially homogeneous spde's. Electron. J. Probab. **4**(6), 1–29 (1999)
12. Dalang, R.C., Quer-Sardanyons, L.: Stochastic integrals for spde's: a comparison. Expo. Math. **29**(1), 67–109 (2011)
13. Dong, Z., Wu, J., Zhang, R., Zhang, T.: Large deviation principles for first-order scalar conservation laws with stochastic forcing. Ann. Appl. Probab. **30**(1), 324–367 (2020)
14. Dupuis, P., Ellis, R.: A Weak Convergence Approach to the Theory of Large Deviations. Wiley, New York (1997)
15. Henry, D.: Geometric theory of Semilinear Parabolic Equations. Lecture Notes in Mathematics, vol. 840. Springer-Verlag, Berlin, New York (1981)
16. Hong, W., Hu S., Liu, W.: McKean-Vlasov SDEs and SPDEs with locally monotone coefficients. arXiv:2205.04043, (2022)
17. Hong, W., Li, S., Liu, W.: Large deviation principle for McKean–Vlasov quasilinear stochastic evolution equation. Appl. Math. Optim. **84**, S1119–S1147 (2021)
18. Hu, Y.: Some recent progress on stochastic heat equations. Acta Math. Sci. Ser. B (Engl. Ed.) **39**(3), 874–914 (2019)

19. Hu, Y., Huang, J., Lê, K., Nualart, D., Tindel, S.: Stochastic heat equation with rough dependence in space. Ann. Probab. **45**(6), 4561–4616 (2017)
20. Hu, Y., Huang, J., Lê, K., Nualart, D., Tindel, S.: Parabolic Anderson model with rough dependence in space. Computation and combinatorics in dynamics, stochastics and control, 477-498. Abel Symp., 13, Springer, Cham (2018)
21. Hu, Y., Nualart, D., Zhang, T.: Large deviations for stochastic heat equation with rough dependence in space. Bernoulli **24**(1), 354–385 (2018)
22. Hu, Y., Wang, X.: Stochastic heat equation with general rough noise. Ann. Inst. Henri Poincaré Probab. Stat. **58**(1), 379–423 (2022)
23. Liu, J.: Moderate deviations for stochastic heat equation with rough dependence in space. Acta Math. Sin. (Engl. Ser.) **35**(9), 1491–1510 (2019)
24. Liu, S., Hu, Y., Wang, X.: Nonlinear stochastic wave equation driven by rough noise. J. Differ. Equ. **331**, 99–161 (2022)
25. Liu, W.: Large deviations for stochastic evolution equations with small multiplicative noise. Appl. Math. Optim. **61**(1), 27–56 (2010)
26. Liu, W., Song, Y., Zhai, J., Zhang, T.: Large and moderate deviation principles for McKean–Vlasov with jumps. *Potential Anal.*, in press
27. Liu, W., Tao, C., Zhu, J.: Large deviation principle for a class of SPDE with locally monotone coefficients. Sci. China Math. **63**(6), 1181–1202 (2020)
28. Márquez-Carreras, D., Sarrà, M.: Large deviation principle for a stochastic heat equation with spatially correlated noise. Electron. J. Probab. **8**(12), 1–39 (2003)
29. Matoussi, A., Sabbagh, W., Zhang, T.: Large deviation principles of obstacle problems for quasilinear stochastic PDEs. Appl. Math. Optim. **83**(2), 849–879 (2021)
30. Peszat, S., Zabczyk, J.: Stochastic evolution equations with a spatially homogeneous Wiener process. Stoch. Process. Appl. **72**, 187–204 (1997)
31. Peszat, S., Zabczyk, J.: Nonlinear stochastic wave and heat equations. Probab. Theory Relat. Fields **116**, 421–443 (2000)
32. Pipiras, V., Taqqu, M.S.: Integration questions related to fractional Brownian motion. Probab. Theory Relat. Fields **118**(2), 251–91 (2000)
33. Ren, J., Zhang, X.: Freidlin–Wentzell's large deviations for stochastic evolution equations. J. Funct. Anal. **254**, 3148–3172 (2008)
34. Song, J.: SPDEs with colored Gaussian noise: a survey. Commun. Math. Stat. **6**(4), 481–492 (2018)
35. Song, J., Song, X., Xu, F.: Fractional stochastic wave equation driven by a Gaussian noise rough in space. Bernoulli **26**(4), 2699–2726 (2020)
36. Wang, R., Zhang, S., Zhai, J.: Large deviation principle for stochastic Burgers type equation with reflection. Commun. Pure Appl. Anal. **21**(1), 213–238 (2022)
37. Wu, W., Zhai, J.: Large deviations for stochastic porous media equation on general measure space. J. Differ. Equ. **269**, 10002–10036 (2020)
38. Xiong, J., Zhai, J.: Large deviations for locally monotone stochastic partial differential equations driven by Lévy noise. Bernoulli **24**, 2842–2874 (2018)
39. Xu, T., Zhang, T.: White noise driven SPDEs with reflection: existence, uniqueness and large deviation principles. Stoch. Process. Appl. **119**, 3453–3470 (2009)