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Precise Local Estimates for Differential Equations driven by Fractional Brownian Motion: Elliptic Case

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Abstract
This article is concerned with stochastic differential equations driven by a d-dimensional fractional Brownian motion with Hurst parameter $H > 1/4$, understood in the rough paths sense. Whenever the coefficients of the equation satisfy a uniform ellipticity condition, we establish a sharp local estimate on the associated control distance function and a sharp local lower estimate on the density of the solution.

Keywords Rough paths · Malliavin calculus · Fractional Brownian motion · Density function

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Contents

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1 Introduction

In this paper, we consider the following stochastic differential equation (SDE)

$$X_t = x + \sum_{i=1}^{d} \int_0^t V_i(X_s) dB_s^i, \quad t \in [0, 1], \tag{1.1}$$

where $x \in \mathbb{R}^N$, $V_1, \cdots, V_d$ are C^∞-bounded vector fields on $\mathbb{R}^N$ and $\{B_t\}_{0\le t\le 1}$ is an d-dimensional fractional Brownian motion. We assume throughout the paper that in (1.1) the fractional Brownian motion has Hurst parameter $H \in (1/4, 1)$ and that the vector fields V_i's satisfy the uniform ellipticity condition. When $H \in (1/2, 1)$, the above equation is understood in Young's sense [19], and when $H \in (1/4, 1/2)$ stochastic integrals in equation (1.1) are interpreted as rough path integrals (see, e.g., [9, 12]) which extends the Young's integral. Existence and uniqueness of solutions to the above equation can be found, for example, in [15]. In particular, when $H = \frac{1}{2}$, this notion of solution coincides with the solution of the corresponding Stratonovitch stochastic differential equation.

It is now well understood that under Hörmander's condition the law of the solution X_t to equation (1.1) admits a smooth probability density $p(t, x, y)$ with respect to the Lebesgue measure on $\mathbb{R}^N$ (cf. [1, 4, 5, 11]). Moreover, it is shown in [2] that, under uniform ellipticity condition, the following global upper bound holds

$$p(t, x, y) \le C \frac{1}{t^{NH}} \exp\left[-\frac{|x-y|^{(2H+1)\wedge 2}}{Ct^{2H}} \right]. \tag{1.2}$$

Clearly, (1.2) is of Gaussian type and sharp when $H \ge 1/2$, while it only gives a sub-Gaussian bound when $H < 1/2$. Whether one should still expect a Gaussian upper bound when $H < 1/2$ remains one of the major open problems in the study of the density function. Another open problem in this direction is to obtain a sharp lower bound for the density $p(t, x, y)$.

On the other hand, the Varadhan type estimate established in [3] shows that

$$\lim_{t\to 0} t^{2H} \log p(t, x, y) = -\frac{1}{2} d(x, y)^2. \tag{1.3}$$

In the above, the control distance function $d(x, y)$ is given by

$$d^2(x, y) = \inf\{\|h\|_{\bar{\mathcal{H}}}^2;\ \Phi_1(x; h) = y\}, \tag{1.4}$$

where $\bar{\mathcal{H}}$ is the Cameron–Martin space of B and $\Phi_t(x; \cdot) : \bar{\mathcal{H}} \to C[0, 1]$ is the deterministic Itô map associated with equation (1.1). Although one cannot directly equate the Varadhan estimate in (1.3) to the upper bound (or a similar lower bound) in (1.2), it naturally motivates the following questions:

Q1. Is the control distance $d(x, y)$ comparable to the Euclidean distance $|x-y|$?
Q2. Can we use techniques developed in proving (1.3) to obtain some information on the bounds of $p(t, x, y)$? [Here we are in particular interested in a lower bound, since progress on the lower bound of the density is limited in the literature.]

Our investigation in the present article shows an effort in answering the above two questions, at least partially. More specifically, our discovery is reported in the following two theorems.

Theorem 1.1 *Let d be the control distance given in (1.4). Under uniform ellipticity conditions (see the forthcoming equation (3.1) for a more explicit version), there exist constants $C, \delta > 0$, such that*

$$\frac{1}{C}|x-y| \leq d(x, y) \leq C|x-y|, \tag{1.5}$$

for all $x, y \in \mathbb{R}^N$ with $|x-y| < \delta$.

Remark 1.2 Theorem (1.1) reflects our attempt in answering Q1. The control distance $d(x, y)$ plays an important role in various analytic properties of X in Eq. (1.1), for example, the large deviations of X_t. Due to the complexity of the Cameron–Martin structure of B, the control distance $d(x, y)$ is far from being a metric (for example, it is not clear whether it satisfies the triangle inequality) and its shape is not clear. Our investigation shows that $d(x, y)$, as a function, is locally comparable to the Euclidean distance. The authors believe that a global equivalence would not hold in general.

Our second result concerns Q2 above and aims at obtaining a lower bound of the density function. It is phrased below in a slightly informal way, and we refer to Theorem 3.4 for a complete statement.

Theorem 1.3 *Let $p(t, x, y)$ be the probability density of X_t. Under uniform ellipticity conditions on the vector fields in V, there exist some constants $C, \tau > 0$ such that*

$$p(t, x, y) \geq \frac{C}{t^{NH}}, \tag{1.6}$$

for all $(t, x, y) \in (0, 1] \times \mathbb{R}^N \times \mathbb{R}^N$ with $|x-y| \leq t^H$, and $t < \tau$.

Remark 1.4 Relation (1.6) presents a local lower bound, both in time and space, for the density function $p_t(x, y)$. It is clearly sharp by a quick examination of the case when X_t is an N-dimensional fractional Brownian motion, i.e., when $N = d$ and $V = \mathrm{Id}$.

In order to summarize the methodology, we have followed for Theorems 1.1 and 1.3; we should highlight two main ingredients:
(i) Some thorough analytic estimates concerning the Cameron–Martin space are related to fractional Brownian motions, which are mostly useful in order to get proper estimates on the distance d defined by (1.4).

(ii) A heavy use of Malliavin calculus, Girsanov's theorem in a fBm context and large deviations techniques are invoked for our local lower bound (1.6).
Our analysis relies thus heavily on the particular fBm setting. Generalizations to a broader class of Gaussian processes seem to be nontrivial and are left for a subsequent publication.

Remark 1.5 As one will see below, our argument for both Theorems 1.1 and 1.3 hinges crucially on uniform ellipticity of the vector fields. The hypoelliptic case is substantially harder and requires a completely different approach, which will be studied in a companion paper [10].

Remark 1.6 For sake of clarity and conciseness, we have restricted most of our analysis to equation (1.1) that is an equation with no drift. However, we shall give some hints at the end of the paper about how to extend our results to more general contexts.

Organization of the present paper. In Sect. 2, we present some basic notions from the analysis of fractional Brownian motion. In particular, we provide substantial detail on the Cameron–Martin space of a fractional Brownian motion. This is needed in order to establish the comparison between control distance and the Euclidean distance and will also be helpful for later references. Our main results Theorems 1.1 and 1.3 are be proved in Sect. 3.

2 Preliminary Results

This section is devoted to some preliminary results on the Cameron–Martin space related to a fractional Brownian motion. We shall also recall some basic facts about rough paths solutions to noisy equations.

2.1 The Cameron–Martin Subspace of Fractional Brownian Motion

Let us start by recalling the definition of fractional Brownian motion.

Definition 2.1 A d-dimensional *fractional Brownian motion* with Hurst parameter $H \in (0, 1)$ is an $\mathbb{R}^d$-valued continuous centered Gaussian process $B_t = (B_t^1, \dots, B_t^d)$ whose covariance structure is given by

$$\mathbb{E}[B_s^i B_t^j] = \frac{1}{2}\left(s^{2H} + t^{2H} - |s-t|^{2H}\right)\delta_{ij} \triangleq R(s,t)\delta_{ij}. \tag{2.1}$$

This process is defined and analyzed in numerous articles (cf. [6, 17, 18] for instance), to which we refer for further details. In this section, we mostly focus on a proper definition of the Cameron–Martin subspace related to B. We also prove two general lemmas about this space which are needed for our analysis of the density $p(t, x, y)$. Notice that we will frequently identify a Hilbert space with its dual in the canonical way without further mentioning.

In order to introduce the Hilbert spaces which will feature in the sequel, consider a one dimensional fractional Brownian motion $\{B_t : 0 \le t \le 1\}$ with Hurst parameter

$H \in (0,1)$. The discussion here can be easily adapted to the multidimensional setting with arbitrary time horizon $[0,T]$. Denote W as the space of continuous paths $w : [0,1] \to \mathbb{R}^1$ with $w_0 = 0$. Let $\mathbb{P}$ be the probability measure over W under which the coordinate process $B_t(w) = w_t$ becomes a fractional Brownian motion. Let $\mathcal{C}_1$ be the associated first order Wiener chaos, i.e., $\mathcal{C}_1 \triangleq \overline{\text{Span}\{B_t : 0 \le t \le 1\}}$ in $L^2(W, \mathbb{P})$.

Definition 2.2 Let B be a one-dimensional fractional Brownian motion as defined in (2.1). Define $\bar{\mathcal{H}}$ to be the space of elements $h \in W$ which can be written as

$$h_t = \mathbb{E}[B_t Z], \quad 0 \le t \le 1, \tag{2.2}$$

where $Z \in \mathcal{C}_1$. We equip $\bar{\mathcal{H}}$ with an inner product structure given by

$$\langle h_1, h_2 \rangle_{\bar{\mathcal{H}}} \triangleq \mathbb{E}[Z_1 Z_2], \quad h_1, h_2 \in \bar{\mathcal{H}},$$

whenever h^1, h^2 are defined by (2.2) for two random variables $Z_1, Z_2 \in \mathcal{C}_1$. The Hilbert space $(\bar{\mathcal{H}}, \langle \cdot, \cdot \rangle_{\bar{\mathcal{H}}})$ is called the *Cameron–Martin subspace* of the fractional Brownian motion.

One of the advantages of working with fractional Brownian motion is that a convenient analytic description of $\bar{\mathcal{H}}$ in terms of fractional calculus is available (cf. [6]). Namely recall that given a function f defined on $[a,b]$, the right and left *fractional integrals* of f of order $\alpha > 0$ are, respectively, defined by

$$(I^{\alpha}_{a+} f)(t) \triangleq \frac{1}{\Gamma(\alpha)} \int_a^t f(s)(t-s)^{\alpha-1} ds, \quad \text{and} \quad (I^{\alpha}_{b-} f)(t) \triangleq \frac{1}{\Gamma(\alpha)} \int_t^b f(s)(s-t)^{\alpha-1} ds. \tag{2.3}$$

In the same way the right and left *fractional derivatives* of f of order $\alpha > 0$ are, respectively, defined by

$$(D^{\alpha}_{a+} f)(t) \triangleq \left(\frac{d}{dt}\right)^{[\alpha]+1} (I^{1-\{\alpha\}}_{a+} f)(t), \quad \text{and} \quad (D^{\alpha}_{b-} f)(t) \triangleq \left(-\frac{d}{dt}\right)^{[\alpha]+1} (I^{1-\{\alpha\}}_{b-} f)(t), \tag{2.4}$$

where $[\alpha]$ is the integer part of α and $\{\alpha\} \triangleq \alpha - [\alpha]$ is the fractional part of α. The following formula for D^{α}_{a+}, valid for $\alpha \in (0,1)$, will be useful for us:

$$(D^{\alpha}_{a+} f)(t) = \frac{1}{\Gamma(1-\alpha)} \left(\frac{f(t)}{(t-a)^{\alpha}} + \alpha \int_a^t \frac{f(t) - f(s)}{(t-s)^{\alpha+1}} ds \right), \quad t \in [a,b]. \tag{2.5}$$

The fractional integral and derivative operators are inverse to each other. For this and other properties of fractional derivatives, the reader is referred to [13].

Let us now go back to the construction of the Cameron–Martin space for B, and proceed as in [6]. Namely, define an isomorphism K between $L^2([0,1])$ and $I_{0+}^{H+1/2}(L^2([0,1]))$ in the following way:

$$K\varphi \triangleq \begin{cases} C_H \cdot I_{0+}^1\left(t^{H-\frac{1}{2}} \cdot I_{0+}^{H-\frac{1}{2}}\left(s^{\frac{1}{2}-H}\varphi(s)\right)(t)\right), & H > \frac{1}{2}; \\ C_H \cdot I_{0+}^{2H}\left(t^{\frac{1}{2}-H} \cdot I_{0+}^{\frac{1}{2}-H}\left(s^{H-\frac{1}{2}}\varphi(s)\right)(t)\right), & H \leq \frac{1}{2}, \end{cases} \tag{2.6}$$

where C_H is a universal constant depending only on H. One can easily compute K^{-1} from the definition of K in terms of fractional derivatives. Moreover, the operator K admits a kernel representation, i.e., there exits a function $K(t,s)$ such that

$$(K\varphi)(t) = \int_0^t K(t,s)\varphi(s)ds, \quad \varphi \in L^2([0,1]).$$

The kernel $K(t,s)$ is defined for $s < t$ (taking zero value otherwise). One can write down $K(t,s)$ explicitly thanks to the definitions (2.3) and (2.4), but this expression is not included here since it will not be used later in our analysis. A crucial property for $K(t,s)$ is that

$$R(t,s) = \int_0^{t\wedge s} K(t,r)K(s,r)dr, \tag{2.7}$$

where $R(t,s)$ is the fractional Brownian motion covariance function introduced in (2.1). This essential fact enables the following analytic characterization of the Cameron–Martin space in [6, Theorem 3.1].

Theorem 2.3 *Let $\bar{\mathcal{H}}$ be the space given in Definition 2.2. As a vector space, we have $\bar{\mathcal{H}} = I_{0+}^{H+1/2}(L^2([0,1]))$, and the Cameron–Martin norm is given by*

$$\|h\|_{\bar{\mathcal{H}}} = \|K^{-1}h\|_{L^2([0,1])}. \tag{2.8}$$

In order to define Wiener integrals with respect to B, it is also convenient to look at the Cameron–Martin subspace in terms of the covariance structure. Specifically, we define another space $\mathcal{H}$ as the completion of the space of simple step functions with inner product induced by

$$\langle \mathbf{1}_{[0,s]}, \mathbf{1}_{[0,t]} \rangle_{\mathcal{H}} \triangleq R(s,t). \tag{2.9}$$

The space $\mathcal{H}$ is easily related to $\bar{\mathcal{H}}$. Namely, define the following operator

$$\mathcal{K}^* : \mathcal{H} \to L^2([0,1]), \quad \text{such that} \quad \mathbf{1}_{[0,t]} \mapsto K(t,\cdot). \tag{2.10}$$

We also set

$$\mathcal{R} \triangleq K \circ \mathcal{K}^* : \mathcal{H} \to \bar{\mathcal{H}}, \tag{2.11}$$

where the operator K is introduced in (2.6). Then, it can be proved that $\mathcal{R}$ is an isometric isomorphism (cf. Lemma 2.7 for the surjectivity of $\mathcal{K}^*$). In addition, under this identification, $\mathcal{K}^*$ is the adjoint of K, i.e., $\mathcal{K}^* = K^* \circ \mathcal{R}$. This can be seen by acting on indicator functions and then taking limits. Another explicit description of $\mathcal{R}$ is the following:

$$\mathcal{R}(\mathbf{1}_{[0,t]})(s) = \mathbb{E}[B_t B_s]$$

and for any $h \in \bar{\mathcal{H}}$, $\mathcal{R}^{-1}(h)$ is the unique element $g \in \mathcal{H}$ such that

$$h(t) = \int_0^t K(t,s)(\mathcal{R}^{-1}h)(s)ds.$$

As mentioned above, one advantage of the space $\mathcal{H}$ is that the fractional Wiener integral operator $I : \mathcal{H} \to \mathcal{C}_1$ induced by $\mathbf{1}_{[0,t]} \mapsto B_t$ is an isometric isomorphism; more explicitly, we have

$$I(f) \sim N(0, \|f\|_{\mathcal{H}}), \ \langle f, g\rangle_{\mathcal{H}} = \mathbb{E}[I(f)I(g)] \ \ \forall f, g \in \mathcal{H}.$$

According to relation (2.7), B_t admits a Wiener integral representation with respect to an underlying Wiener process W:

$$B_t = \int_0^t K(t,s)dW_s. \tag{2.12}$$

Moreover, the process W in (2.12) can be expressed as a Wiener integral with respect to B, that is, $W_s = I((\mathcal{K}^*)^{-1}\mathbf{1}_{[0,s]})$ (cf. [17, relation(5.15)]).

Let us also mention the following useful formula for the natural pairing between $\mathcal{H}$ and $\bar{\mathcal{H}}$. Denote by $C^{H^-}([0,1];\mathbb{R}^d)$ the space of α-Hölder continuous paths for all $\alpha < H$.

Lemma 2.4 *Let $\mathcal{H}$ be the space defined as the completion of the indicator functions with respect to the inner product (2.9). Also recall that $\bar{\mathcal{H}}$ is introduced in Definition 2.2. Then, through the isometric isomorphism $\mathcal{R}$ defined by (2.11), the natural pairing between $\mathcal{H}$ and $\bar{\mathcal{H}}$ is given by*

$$_{\mathcal{H}}\langle f, h\rangle_{\bar{\mathcal{H}}} = \int_0^1 f_s \, dh_s, \tag{2.13}$$

for all $f \in C^{H^-}([0,1];\mathbb{R}^d)$ and $h \in \bar{\mathcal{H}}$. In the above, the integral on the right-hand side is understood in Young's sense, thanks to Proposition 2.6.

Proof Let $f \in C^{H-}([0,1];\mathbb{R}^d)$ and $h \in \bar{\mathcal{H}}$. Let Z be the random variable in the first chaos $\mathcal{C}_1$ such that $h_t = \mathbb{E}[ZB_t]$ (cf. Definition 2.2). The natural pairing between f and h is given by

$$_{\mathcal{H}}\langle f, h\rangle_{\bar{\mathcal{H}}} = \mathbb{E}[ZI(f)],$$

where we recall that $I : \mathcal{H} \to \mathcal{C}_1$ is the fractional Wiener integral operator. According to Young's integration theory (cf. [19]),

$$\int_0^1 f_s dh_s = \lim_{\text{mesh}(\mathcal{P})\to 0} \sum_{t_i \in \mathcal{P}} f_{t_{i-1}}(h_{t_i} - h_{t_{i-1}}),$$

where $\mathcal{P}$ denotes an arbitrary finite partition of [0, 1]. On the other hand, for each partition $\mathcal{P}$, we have

$$\begin{aligned}\sum_{t_i \in \mathcal{P}} f_{t_{i-1}}(h_{t_i} - h_{t_{i-1}}) &= \sum_{t_i \in \mathcal{P}} f_{t_{i-1}} \mathbb{E}[Z(B_{t_i} - B_{t_{i-1}})] \\ &= \sum_{t_i \in \mathcal{P}} f_{t_{i-1}} \mathbb{E}\big[ZI(\mathbf{1}_{(t_{i-1},t_i]})\big] = \mathbb{E}[ZI(f^{\mathcal{P}})],\end{aligned}$$

where

$$f_t^{\mathcal{P}} \triangleq \sum_{t_i \in \mathcal{P}} f_{t_{i-1}} \mathbf{1}_{(t_{i-1},t_i]}(t), \quad 0 \le t \le 1.$$

Since $f^{\mathcal{P}} \to f$ in $\mathcal{H}$ as mesh$(\mathcal{P}) \to 0$, we know that $I(f^{\mathcal{P}}) \to I(f)$ in L^2. Therefore,

$$\mathbb{E}[ZI(f^{\mathcal{P}})] \to \mathbb{E}[ZI(f)]$$

as mesh$(\mathcal{P}) \to 0$, which implies the desired relation (2.13). □

The space $\mathcal{H}$ can also be described in terms of fractional calculus, since the operator $\mathcal{K}^*$ defined by (2.10) can be expressed as

$$(\mathcal{K}^* f)(t) = \begin{cases} C_H \cdot t^{\frac{1}{2}-H} \cdot \left(I_{1^-}^{H-\frac{1}{2}}\left(s^{H-\frac{1}{2}} f(s)\right)\right)(t), & H > \frac{1}{2}; \\ C_H \cdot t^{\frac{1}{2}-H} \cdot \left(D_{1^-}^{\frac{1}{2}-H}\left(s^{H-\frac{1}{2}} f(s)\right)\right)(t), & H \le \frac{1}{2}. \end{cases} \tag{2.14}$$

Starting from this expression, it is readily checked that when $H > 1/2$ the space $\mathcal{H}$ coincides with the following subspace of the Schwartz distributions $\mathcal{S}'$:

$$\mathcal{H} = \left\{ f \in \mathcal{S}';\ t^{1/2-H} \cdot (I_{1^-}^{H-1/2}(s^{H-1/2} f(s)))(t) \text{ is an element of } L^2([0,1]) \right\}. \tag{2.15}$$

In the case $H \le 1/2$, we simply have

$$\mathcal{H} = I_{1^-}^{1/2-H}(L^2([0,1])). \tag{2.16}$$

These characterizations can be found in [18].

Remark 2.5 As the Hurst parameter H increases, $\mathcal{H}$ gets larger (and contains distributions when $H > 1/2$) while $\bar{\mathcal{H}}$ gets smaller. This fact is apparent from Theorem 2.3 and relations (2.15)-(2.16). When $H = 1/2$, the process B_t coincides with the usual Brownian motion. In this case, we have $\mathcal{H} = L^2([0,1])$ and $\bar{\mathcal{H}} = W_0^{1,2}$, the space of absolutely continuous paths starting at the origin with square integrable derivative.

Next we mention a variational embedding theorem for the Cameron–Martin subspace $\bar{\mathcal{H}}$ which will be used in a crucial way. The case when $H > 1/2$ is a simple exercise starting from the definition (2.2) of $\bar{\mathcal{H}}$ and invoking the Cauchy–Schwarz inequality. The case when $H \leq 1/2$ was treated in [8]. From a pathwise point of view, this allows us to integrate a fractional Brownian path against a Cameron–Martin path or vice versa (cf. [19]) and to make sense of ordinary differential equations driven by a Cameron–Martin path (cf. [14]).

Proposition 2.6 *If $H > \frac{1}{2}$, then $\bar{\mathcal{H}} \subseteq C_0^H([0,1];\mathbb{R}^d)$, the space of H-Hölder continuous paths. If $H \leq \frac{1}{2}$, then for any $q > (H+1/2)^{-1}$, we have $\bar{\mathcal{H}} \subseteq C_0^{q\text{-var}}([0,1];\mathbb{R}^d)$, the space of continuous paths with finite q-variation. In addition, the above inclusions are continuous embeddings.*

Finally, we prove two general lemmas on the Cameron–Martin subspace that are needed later on. These properties do not seem to be contained in the literature and they require some care based on fractional calculus. The first one claims the surjectivity of $\mathcal{K}^*$ on properly defined spaces.

Lemma 2.7 *Let $H \in (0,1)$, and consider the operator $\mathcal{K}^* : \mathcal{H} \to L^2([0,1])$ defined by (2.10). Then, $\mathcal{K}^*$ is surjective.*

Proof If $H > 1/2$, we know that the image of $\mathcal{K}^*$ contains all indicator functions (cf. [17, Equation (5.14)]). Therefore, $\mathcal{K}^*$ is surjective.

If $H < 1/2$, we first claim that the image of $\mathcal{K}^*$ contains functions of the form $t^{1/2-H}p(1-t)$ where $p(t)$ is a polynomial. Indeed, given an arbitrary $\beta \geq 0$, consider the function

$$f_\beta(t) \triangleq t^{\frac{1}{2}-H}(1-t)^{\beta+\frac{1}{2}-H}.$$

It is readily checked that $D_{1-}^{\frac{1}{2}-H} f_\beta \in L^2([0,1])$, and hence, $f_\beta \in I_{1-}^{\frac{1}{2}-H}(L^2([0,1])) = \mathcal{H}$. Using the analytic expression (2.14) for $\mathcal{K}^*$, we can compute $\mathcal{K}^* f_\beta$ explicitly (cf. [13, Chapter 2, Equation (2.45)]) as

$$(\mathcal{K}^* f_\beta)(t) = C_H \frac{\Gamma\left(\beta+\frac{3}{2}-H\right)}{\Gamma(\beta+1)} t^{\frac{1}{2}-H}(1-t)^\beta.$$

Since β is arbitrary and $\mathcal{K}^*$ is linear, the claim follows.

Now it remains to show (with a change of variable) that the space of functions of the form $(1-t)^{\frac{1}{2}-H}p(t)$ with $p(t)$ being a polynomial is dense in $L^2([0,1])$. To this end, let $\varphi \in C_c^\infty((0,1))$. Then, $\psi(t) \triangleq (1-t)^{-(1/2-H)}\varphi(t) \in C_c^\infty((0,1))$. According to Bernstein's approximation theorem, for any $\varepsilon > 0$, there exists a polynomial $p(t)$ such that

$$\|\psi - p\|_\infty < \varepsilon,$$

and thus

$$\sup_{0\le t\le 1} |\varphi(t) - (1-t)^{\frac{1}{2}-H} p(t)| < \varepsilon.$$

Therefore, functions in $C_c^\infty((0,1))$ (and thus in $L^2([0,1])$) can be approximated by functions of the desired form. □

Our second lemma gives some continuous embedding properties for $\mathcal{H}$ and $\bar{\mathcal{H}}$ in the irregular case $H < 1/2$, whose proof relies on Lemma 2.7.

Lemma 2.8 *For $H < 1/2$, the inclusions $\mathcal{H} \subseteq L^2([0,1])$ and $W_0^{1,2} \subseteq \bar{\mathcal{H}}$ are continuous embeddings.*

Proof For the first assertion, let $f \in \mathcal{H}$. We wish to prove that

$$\|f\|_{L^2([0,1])} \le C_H \|f\|_{\mathcal{H}}. \tag{2.17}$$

Toward this aim, define $\varphi \triangleq \mathcal{K}^* f$, where $\mathcal{K}^*$ is defined by (2.10). Observe that $\mathcal{K}^* : \mathcal{H} \to L^2([0,1])$, and thus, $f \in L^2([0,1])$. By solving f in terms of φ using the analytic expression (2.14) for $\mathcal{K}^*$, we have

$$f(t) = C_H t^{\frac{1}{2}-H} \left(I_{1-}^{\frac{1}{2}-H} \left(s^{H-\frac{1}{2}} \varphi(s) \right) \right)(t). \tag{2.18}$$

We now bound the right-hand side of (2.18). Our first step in this direction is to notice that according to the definition (2.3) of fractional integral we have

$$\begin{aligned}
\left| \left(I_{1-}^{\frac{1}{2}-H} (s^{H-\frac{1}{2}} \varphi(s)) \right)(t) \right| &= C_H \left| \int_t^1 (s-t)^{-\frac{1}{2}-H} s^{H-\frac{1}{2}} \varphi(s) ds \right| \\
&\le C_H \int_t^1 (s-t)^{-\frac{1}{2}-H} s^{H-\frac{1}{2}} |\varphi(s)| ds \\
&= C_H \int_t^1 (s-t)^{-\frac{1}{4}-\frac{H}{2}} \left((s-t)^{-\frac{1}{4}-\frac{H}{2}} s^{H-\frac{1}{2}} |\varphi(s)| \right) ds.
\end{aligned}$$

Hence, a direct application of Cauchy–Schwarz inequality gives

$$\left|\left(I_{1^-}^{\frac{1}{2}-H}(s^{H-\frac{1}{2}}\varphi(s))\right)(t)\right| \leq C_H\left(\int_t^1 (s-t)^{-\frac{1}{2}-H}ds\right)^{\frac{1}{2}} \left(\int_t^1 (s-t)^{-\frac{1}{2}-H}s^{2H-1}|\varphi(s)|^2 ds\right)^{\frac{1}{2}}$$
$$= C_H(1-t)^{\frac{1}{2}\left(\frac{1}{2}-H\right)}\left(\int_t^1 (s-t)^{-\frac{1}{2}-H}s^{2H-1}|\varphi(s)|^2 ds\right)^{\frac{1}{2}}, \tag{2.19}$$

where we recall that C_H is a positive constant which can change from line to line. Therefore, plugging (2.19) into (2.18) we obtain

$$\|f\|^2_{L^2([0,1])} \leq C_H\int_0^1\left(t^{1-2H}(1-t)^{\frac{1}{2}-H}\int_t^1 (s-t)^{-\frac{1}{2}-H}s^{2H-1}|\varphi(s)|^2 ds\right)dt.$$

We now bound all the terms of the form s^β with $\beta > 0$ by 1. This gives

$$\begin{aligned}\|f\|^2_{L^2([0,1])} &\leq C_H\int_0^1 dt\int_t^1 (s-t)^{-\frac{1}{2}-H}|\varphi(s)|^2 ds\\ &= C_H\int_0^1 |\varphi(s)|^2 ds\int_0^s (s-t)^{-\frac{1}{2}-H}dt\\ &= C_H\int_0^1 s^{\frac{1}{2}-H}|\varphi(s)|^2 ds \leq C_H\|\varphi\|^2_{L^2([0,1])} = C_H\|f\|^2_{\mathcal{H}},\end{aligned}$$

which is our claim (2.17).

For the second assertion about the embedding of $W_0^{1,2}$ in $\bar{\mathcal{H}}$, let $h \in W_0^{1,2}$. We thus also have $h \in \bar{\mathcal{H}}$ and we can write $h = K\varphi$ for some $\varphi \in L^2([0,1])$. We first claim that

$$\int_0^1 f(s)dh(s) = \int_0^1 \mathcal{K}^* f(s)\varphi(s)ds \tag{2.20}$$

for all $f \in \mathcal{H}$. This assertion can be reduced in the following way: since $\mathcal{H} \hookrightarrow L^2([0,1])$ continuously and $\mathcal{K}^* : \mathcal{H} \to L^2([0,1])$ is continuous, one can take limits along indicator functions in (2.20). Thus, it is sufficient to consider $f = \mathbf{1}_{[0,t]}$ in (2.20). In addition, relation (2.20) can be checked easily for $f = \mathbf{1}_{[0,t]}$. Namely, we have

$$\int_0^1 \mathbf{1}_{[0,t]}(s)dh(s) = h(t) = \int_0^t K(t,s)\varphi(s)ds = \int_0^1 \left(\mathcal{K}^*\mathbf{1}_{[0,t]}\right)(s)\varphi(s)ds.$$

Therefore, our claim (2.20) holds true. Now from Lemma 2.7, if $\varphi \in L^2([0,1])$ there exists $f \in \mathcal{H}$ such that $\varphi = \mathcal{K}^* f$. For this particular f, invoking relation (2.20) we get

$$\int_0^1 f(s)dh(s) = \|\varphi\|^2_{L^2([0,1])}. \tag{2.21}$$

But we also know that

$$\|\varphi\|_{L^2([0,1])} = \|h\|_{\bar{\mathcal{H}}} = \|f\|_{\mathcal{H}}, \quad \text{and thus} \quad \|\varphi\|^2_{L^2([0,1])} = \|h\|_{\bar{\mathcal{H}}}\|f\|_{\mathcal{H}}. \tag{2.22}$$

In addition, recall that the $W^{1,2}$ norm can be written as

$$\|h\|_{W^{1,2}} = \sup_{\psi\in L^2([0,1])} \frac{\left|\int_0^1 \psi(s)dh(s)\right|}{\|\psi\|_{L^2([0,1])}}$$

Owing to (2.21) and (2.22), we thus get

$$\|h\|_{W^{1,2}} \geq \frac{\int_0^1 f(s)dh(s)}{\|f\|_{L^2([0,1])}} = \frac{\|h\|_{\bar{\mathcal{H}}}\|f\|_{\mathcal{H}}}{\|f\|_{L^2([0,1])}} \geq C_H\|h\|_{\bar{\mathcal{H}}},$$

where the last step stems from (2.17). The continuous embedding $W_0^{1,2} \subseteq \bar{\mathcal{H}}$ follows. □

2.2 Malliavin Calculus for Fractional Brownian Motion

In this section, we review some basic aspects of Malliavin calculus and set up corresponding notations. The reader is referred to [17] for further details.

We consider the fractional Brownian motion $B = (B^1, \ldots, B^d)$ as in Definition (2.1), defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For sake of simplicity, we assume that $\mathcal{F}$ is generated by $\{B_t;\ t \in [0, T]\}$. An $\mathcal{F}$-measurable real-valued random variable F is said to be *cylindrical* if it can be written, with some $m \geq 1$, as

$$F = f\left(B_{t_1}, \ldots, B_{t_m}\right), \quad \text{for} \quad 0 \leq t_1 < \cdots < t_m \leq 1,$$

where $f : \mathbb{R}^m \to \mathbb{R}$ is a C_b^∞ function. The set of cylindrical random variables is denoted by $\mathcal{S}$.

The Malliavin derivative is defined as follows: for $F \in \mathcal{S}$, the derivative of F in the direction $h \in \mathcal{H}$ is given by

$$\mathbf{D}_h F = \sum_{i=1}^m \frac{\partial f}{\partial x_i}\left(B_{t_1}, \ldots, B_{t_m}\right) h_{t_i}.$$

More generally, we can introduce iterated derivatives. Namely, if $F \in \mathcal{S}$, we set

$$\mathbf{D}^k_{h_1,\ldots,h_k} F = \mathbf{D}_{h_1} \ldots \mathbf{D}_{h_k} F.$$

For any $p \geq 1$, it can be checked that the operator $\mathbf{D}^k$ is closable from $\mathcal{S}$ into $\mathbf{L}^p(\Omega; \mathcal{H}^{\otimes k})$. We denote by $\mathbb{D}^{k,p}(\mathcal{H})$ the closure of the class of cylindrical random variables with respect to the norm

$$\|F\|_{k,p} = \left(\mathbb{E}\left[|F|^p\right] + \sum_{j=1}^{k} \mathbb{E}\left[\left\| \mathbf{D}^j F \right\|_{\mathcal{H}^{\otimes j}}^p \right] \right)^{\frac{1}{p}}, \tag{2.23}$$

and we also set $\mathbb{D}^\infty(\mathcal{H}) = \cap_{p\geq 1} \cap_{k\geq 1} \mathbb{D}^{k,p}(\mathcal{H})$.

Estimates of Malliavin derivatives are crucial in order to get information about densities of random variables, and Malliavin covariance matrices as well as non-degenerate random variables will feature importantly in the sequel.

Definition 2.9 Let $F = (F^1, \ldots, F^n)$ be a random vector whose components are in $\mathbb{D}^\infty(\mathcal{H})$. Define the *Malliavin covariance matrix* of F by

$$\gamma_F = (\langle \mathbf{D}F^i, \mathbf{D}F^j \rangle_{\mathcal{H}})_{1\leq i,j\leq n}. \tag{2.24}$$

Then, F is called *non-degenerate* if γ_F is invertible $a.s.$ and

$$(\det \gamma_F)^{-1} \in \cap_{p\geq 1} L^p(\Omega).$$

It is a classical result that the law of a non-degenerate random vector $F = (F^1, \ldots, F^n)$ admits a smooth density with respect to the Lebesgue measure on $\mathbb{R}^n$.

The following integration by parts formula is borrowed from [17, Proposition 2.1.4].

Proposition 2.10 *Let $F = (F^1, \ldots, F^n)$ be a non-degenerate random vector as in Definition 2.9. Let $G \in \mathbb{D}^\infty$ and φ be a function in the space $C_p^\infty(\mathbb{R}^n)$. Then, for any multi-index $\alpha \in \{1, 2, \ldots, n\}^k$, $k \geq 1$, there exists an element $H_\alpha(F, G) \in \mathbb{D}^\infty$ such that*

$$\mathbb{E}[\partial_\alpha \varphi(F) G] = \mathbb{E}[\varphi(F) H_\alpha(F, G)],$$

Moreover, the elements $H_\alpha(F, G)$ are recursively given by

$$\begin{aligned} H_{(i)}(F, G) &= \textstyle\sum_{j=1}^n \delta\left(G(\gamma_F^{-1})^{ij}\, \mathbf{D}F^j\right) \quad \textit{and} \\ H_\alpha(F, G) &= H_{\alpha_k}(F, H_{(\alpha_1,\ldots,\alpha_{k-1})}(F, G)), \end{aligned} \tag{2.25}$$

and for $1 \leq p < q < \infty$ we have

$$\|H_\alpha(F, G)\|_p \leq c_{p,q} \|\gamma_F^{-1}\, \mathbf{D}F\|_{k,2^{k-1}r}^k \|G\|_{k,q}^k, \tag{2.26}$$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

Remark 2.11 Through an approximation procedure, the above integration by parts formula can be extended to the case when $\varphi = \delta_x$, the Dirac delta function. We refer the readers to the proof of [17, Theorem 2.1.4] or [17, Section 2.1.5] for more details.

3 Proof of Main Results

In this section, we prove Theorems 1.1 and 1.3. We emphasize that our analysis relies crucially on the uniform ellipticity of the vector fields V_i's in equation (1.1), which is spelled out explicitly below.

Uniform Ellipticity Assumption The $\mathcal{C}_b^\infty$ vector fields $V = \{V_1, \dots, V_d\}$ are such that

$$\Lambda_1 |\xi|^2 \leq \xi^* V(x) V(x)^* \xi \leq \Lambda_2 |\xi|^2, \quad \forall x, \xi \in \mathbb{R}^N, \tag{3.1}$$

with some constants $\Lambda_1, \Lambda_2 > 0$, where $(\cdot)^*$ denotes matrix transpose.

We now split our proofs in two subsections, corresponding, respectively, to Theorems 1.1 and 1.3.

3.1 Proof of the Distance Comparison

In order to prove Theorem 1.1, recall first that $\Phi_t(x; \cdot) : \bar{\mathcal{H}} \to C[0, 1]$ is the deterministic Itô map associated with Eq. (1.1). For $x, y \in \mathbb{R}^N$, set

$$\Pi_{x,y} \triangleq \left\{ h \in \bar{\mathcal{H}} : \Phi_1(x; h) = y \right\} . \tag{3.2}$$

Otherwise stated, $\Pi_{x,y}$ is the set of Cameron–Martin paths that joining x to y though the Itô map. Under our assumption (3.1), it is easy to construct an $h \in \bar{\mathcal{H}} \in \Pi_{x,y}$ explicitly, which will ease our computations later on.

Lemma 3.1 *Let $V = \{V_1, \dots, V_d\}$ be vector fields satisfying the uniform elliptic assumption (3.1). Given $x, y \in \mathbb{R}^N$, define*

$$h_t \triangleq \int_0^t V^*(z_s) \cdot (V(z_s) V^*(z_s))^{-1} \cdot (y - x) ds, \tag{3.3}$$

where $z_t \triangleq (1-t)x + ty$ is the line segment from x to y. Then, $h \in \Pi_{x,y}$, where $\Pi_{x,y}$ is defined by relation (3.2).

Proof Since $\bar{\mathcal{H}} = I_{0+}^{H+1/2}(L^2([0, 1]))$ contains smooth paths, it is obvious that $h \in \bar{\mathcal{H}}$. As far as z_t is concerned, the definition $z_t = (1-t)x + ty$ clearly implies that $z_0 = x$, $z_1 = y$ and $\dot{z}_t = y - x$. In addition, since $VV^*(\xi)$ is invertible for all $\xi \in \mathbb{R}^N$ under our condition (3.1), we get

$$\dot{z}_t = y - x = \left(VV^*(VV^*)^{-1} \right)(z_t) \cdot (y - x) = V(z_t) \dot{h}_t,$$

where the last identity stems from the definition (3.3) of h. Therefore, $h \in \Pi_{x,y}$ according to our definition (3.2). □

Remark 3.2 The intuition behind Lemma 3.1 is very simple. Indeed, given any smooth path x_t with $x_0 = x, x_1 = y$, since the vector fields are elliptic, there exist smooth functions $\lambda^1(t), \ldots, \lambda^d(t)$, such that

$$\dot{x}_t = \sum_{\alpha=1}^{d} \lambda^\alpha(t) V_\alpha(x_t), \quad 0 \le t \le 1.$$

In matrix notation, $\dot{x}_t = V(x_t) \cdot \lambda(t)$. A canonical way to construct $\lambda(t)$ is writing it as $\lambda(t) = V^*(x_t)\eta(t)$ so that from ellipticity we can solve for $\eta(t)$ as

$$\eta(t) = (V(x_t)V^*(x_t))^{-1}\dot{x}_t.$$

It follows that the path $h_t \triangleq \int_0^t \lambda(s)ds$ belongs to $\Pi_{x,y}$.

Now we can prove the following result which asserts that the control distance function is locally comparable with the Euclidean metric that is Theorem 1.3 under elliptic assumptions.

Theorem 3.3 *Let $V = \{V_1, \ldots, V_d\}$ be vector fields satisfying the uniform elliptic assumption (3.1). Consider the control distance d given in (1.4) for a given $H > \frac{1}{4}$. Then, there exist constants $C_1, C_2 > 0$ depending only on H and the vector fields, such that*

$$C_1|x - y| \le d(x, y) \le C_2|x - y| \tag{3.4}$$

for all $x, y \in \mathbb{R}^N$ with $|x - y| \le 1$.

Proof We first consider the case when $H \le 1/2$, which is simpler due to Lemma 2.8. Given $x, y \in \mathbb{R}^N$, define $h \in \Pi_{x,y}$ as in Lemma 3.1. According to Lemma 2.8 and (1.4), we have

$$d(x, y)^2 \le \|h\|^2_{\bar{\mathcal{H}}} \le C_H \|h\|^2_{W^{1,2}}.$$

Therefore, according to the definition (3.3) of h, we get

$$d(x, y)^2 \le C_H \int_0^1 |V^*(z_s)(V(z_s)V^*(z_s))^{-1} \cdot (y - x)|^2 ds \le C_{H,V}|y - x|^2,$$

where the last inequality stems from the uniform ellipticity assumption (3.1) and the fact that V^* is bounded. This proves the upper bound in (3.4).

We now turn to the lower bound in (3.4). To this aim, consider an arbitrary $h \in \Pi_{x,y}$. We want to show that $|y - x| \le C\|h\|_{\bar{\mathcal{H}}}$ with some constant C. Since $|y - x| \le 1$, we may assume without loss of generality that $\|h\|_{\bar{\mathcal{H}}} \le 1$ as the claim holds trivially otherwise. Recalling the definition (3.2) of $\Pi_{x,y}$, we have

$$y - x = \int_0^1 V(\Phi_t(x; h))dh_t.$$

According to Proposition 2.6 (specifically the embedding $\bar{\mathcal{H}} \subseteq C_0^{q-\text{var}}([0,1];\mathbb{R}^d)$ for $q > (H+1/2)^{-1}$) and the pathwise variational estimate given by [9, Theorem 10.14], we have

$$|y-x| \le C_{H,V}\left(\|h\|_{q-\text{var}} \vee \|h\|_{q-\text{var}}^q\right) \le C_{H,V}\left(\|h\|_{\bar{\mathcal{H}}} \vee \|h\|_{\bar{\mathcal{H}}}^q\right). \tag{3.5}$$

Since $q \ge 1$, we conclude that

$$|y-x| \le C_{H,V}\|h\|_{\bar{\mathcal{H}}}.$$

The desired lower bound follows from the arbitrariness of h.

Next we consider the case when $H > 1/2$. The lower bound in (3.4) can be proved with the same argument as in the case $H \le 1/2$, the only difference being that in (3.5) we replace $\bar{\mathcal{H}} \subseteq C_0^{q-\text{var}}([0,1];\mathbb{R}^d)$ by $\bar{\mathcal{H}} \subseteq C_0^H([0,1];\mathbb{R}^d)$ and the pathwise variational estimate of [9, Theorem 10.14] by a Hölder estimate borrowed from [7, Proposition 8.1].

For the upper bound in (3.4), we again take $h \in \Pi_{x,y}$ as given by Lemma 3.1 and estimate its Cameron–Martin norm. Note that due to our uniform ellipticity assumption (3.1), one can define the function

$$\gamma_t \equiv \int_0^t (V^*(VV^*)^{-1})(z_s)ds = \int_0^t g((1-s)x+sy)ds, \tag{3.6}$$

where g is a matrix-valued C_b^∞ function. We will now prove that γ can be written as $\gamma = K\varphi$ for $\varphi \in L^2([0,1])$. Indeed, one can solve for φ in the analytic expression (2.6) for $H > 1/2$ and get

$$\varphi(t) = C_H t^{H-\frac{1}{2}}\left(D_{0+}^{H-\frac{1}{2}}\left(s^{\frac{1}{2}-H}\dot{\gamma}_s\right)\right)(t).$$

We now use the expression (2.4) for $D_{0+}^{H-1/2}$, which yield (after an elementary change of variable)

$$\begin{aligned}
\varphi(t) &= C_H t^{H-\frac{1}{2}}\frac{d}{dt}\int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}g((1-s)x+sy)ds \\
&= C_H t^{H-\frac{1}{2}}\frac{d}{dt}\left(t^{2-2H}\int_0^1 (u(1-u))^{\frac{1}{2}-H}g((1-tu)x+tuy)du\right) \\
&= C_H t^{\frac{1}{2}-H}\int_0^1 (u(1-u))^{\frac{1}{2}-H}g((1-tu)x+tuy)du \\
&\quad + C_H t^{\frac{3}{2}-H}\int_0^1 (u(1-u))^{\frac{1}{2}-H}u\nabla g((1-tu)x+tuy)\cdot(y-x)du.
\end{aligned}$$

Hence, thanks to the fact that g and ∇g are bounded plus the fact that $t \le 1$, we get

$$|\varphi(t)| \le C_{H,V}(t^{\frac{1}{2}-H} + |y-x|),$$

from which φ is clearly an element of $L^2([0,1])$. Since $|y-x| \leq 1$, we conclude that

$$\|\gamma\|_{\bar{\mathcal{H}}} = \|\varphi\|_{L^2([0,1])} \leq C_{H,V}.$$

Therefore, recalling that h is given by (3.3) and γ is defined by (3.6), we end up with

$$\begin{aligned} d(x,y) \leq \|h\|_{\bar{\mathcal{H}}} &= \left\| \left(\int_0^{\cdot} (V^*(VV^*)^{-1})(z_s) ds \right) \cdot (y-x) \right\|_{\bar{\mathcal{H}}} \\ &= \|\gamma\|_{\bar{\mathcal{H}}} |y-x| \leq C_{H,V} |y-x|. \end{aligned}$$

This concludes the proof. □

3.2 Lower Bound for the Density

With Theorem 3.3 in hand, we are now ready to state Theorem 1.3 rigorously and prove it. Specifically, our main local bound on the density of X_t takes the following form.

Theorem 3.4 *Let $p(t,x,y)$ be the density of the solution X_t to Eq. (1.1). Under the uniform ellipticity assumption (3.1), there exist constants $C_1, C_2, \tau > 0$ depending only on H and the vector fields V, such that*

$$p(t,x,y) \geq \frac{C_1}{t^{NH}} \tag{3.7}$$

for all $(t,x,y) \in (0,1] \times \mathbb{R}^N \times \mathbb{R}^N$ satisfying $|x-y| \leq C_2 t^H$ and $t < \tau$.

Remark 3.5 From Theorem 3.3, we know that $|B_d(x,t^H)| \asymp t^{NH}$ when t is small. Therefore, Theorem 1.3 becomes the following result, which is consistent with the intuition that the density $p(t,x,y)$ of the solution to Eq. (1.1) should behave like the Gaussian kernel:

$$p(t,x,y) \asymp \frac{C_1}{t^{NH}} \exp\left(-\frac{C_2 |y-x|^2}{t^{2H}} \right).$$

The main idea behind the proof of Theorem 3.4 is to translate the small time estimate in (3.7) into a large deviation estimate. To this aim, we will first recall some preliminary notions taken from [3]. Denote by $\Phi(\cdot;\cdot) : \mathbb{R}^N \times \Omega \to \mathbb{R}^N$ the Itô-Lyons map associated with (1.1), that is, $\Phi_t(x;\omega) := X_t(\omega)$ for any initial point x and $\omega \in \Omega$. By the scaling invariance of fractional Brownian motion, we have

$$\Phi_t(x;B) \stackrel{\text{law}}{=} \Phi_1(x;\varepsilon B), \tag{3.8}$$

where $\varepsilon \triangleq t^H$. Therefore, since the random variable $\Phi_t(x;B)$ is nondegenerate under our standing assumption (3.1), the density $p(t,x,y)$ can be written as

$$p(t,x,y) = \mathbb{E}\left[\delta_y \left(\Phi_1(x;\varepsilon B) \right) \right]. \tag{3.9}$$

With expression (3.9), we focus on $\Phi_1(x, \varepsilon B)$ in the rest of the proof. We first label a proposition which gives a lower bound on $p(t, x, y)$ in terms of some conveniently chosen shifts on the Wiener space.

Proposition 3.6 *Assume that the vector fields* $\{V_1, \ldots, V_d\}$ *satisfy the uniform elliptic assumption (3.1). Then, the following holds true.*
(i) Let Φ_t *be the solution map of Eq. (1.1),* $h \in \bar{\mathcal{H}}$*, and let*

$$X^\varepsilon(h) \triangleq \frac{\Phi_1(x; \varepsilon B + h) - \Phi_1(x; h)}{\varepsilon}. \tag{3.10}$$

Then, $X^\varepsilon(h)$ *converges in* $\mathbb{D}^\infty$ *to X(h) uniformly in* $h \in \bar{\mathcal{H}}$ *with* $\|h\|_{\bar{\mathcal{H}}} \leq M$ *(for any fixed* $M > 0$*). Moreover,* $X(h)$ *is a* $\mathbb{R}^N$*-valued centered Gaussian random variable whose covariance matrix will be specified later.*
(ii) Let $\varepsilon > 0$ *and consider* $x, y \in \mathbb{R}^N$ *such that* $d(x, y) \leq \varepsilon$*, where* $d(\cdot, \cdot)$ *is the distance considered in Theorem 3.3. Choose* $h \in \Pi_{x,y}$ *(cf. (3.2)) so that*

$$\|h\|_{\bar{\mathcal{H}}} \leq d(x, y) + \varepsilon. \tag{3.11}$$

Then, there exists a constant $C > 0$ *not depending on* ε *such that*

$$p(t, x, y) = \mathbb{E}\left[\delta_y\left(\Phi_1(x; \varepsilon B)\right)\right] \geq C\varepsilon^{-N} \cdot \mathbb{E}\left[\delta_0\left(X^\varepsilon(h)\right) \mathrm{e}^{-I\left(\frac{h}{\varepsilon}\right)}\right]. \tag{3.12}$$

Proof The first statement is proved in [3, Proposition 2.15]. We therefore focus on the second statement. Recall that $I : \mathcal{H} \to \mathcal{C}_1$ is the Wiener integral operator and that $\mathcal{R} : \mathcal{H} \to \bar{\mathcal{H}}$ is the isomorphism between $\mathcal{H}$ and $\bar{\mathcal{H}}$ introduced in Section 2.1. Throughout the proof and the rest of the paper, we will abuse the notation slightly and use $I(h)$ for the more accurate $I(\mathcal{R}^{-1}h)$ whenever $h \in \bar{\mathcal{H}}$. First note that by Cameron–Martin theorem we have

$$\mathbb{E}\left[\delta_y\left(\Phi_1(x; \varepsilon B)\right)\right] = \mathrm{e}^{-\frac{\|h\|^2_{\bar{\mathcal{H}}}}{2\varepsilon^2}} \mathbb{E}\left[\delta_y\left(\Phi_1(x; \varepsilon B + h)\right) \mathrm{e}^{-I\left(\frac{h}{\varepsilon}\right)}\right]. \tag{3.13}$$

Thanks to inequality (3.11) and the fact that $d(x, y) \leq \varepsilon$, we have $\|h\|_{\bar{\mathcal{H}}} \leq d(x, y) + \varepsilon \leq 2\varepsilon$. Hence, the factor $\exp(-\|h\|^2_{\bar{\mathcal{H}}}/2\varepsilon^2)$ in (3.13) is lower bounded by a positive constant independent of ε. Plugging this information into the above equation, we therefore obtain

$$p(t, x, y) = \mathbb{E}\left[\delta_y\left(\Phi_1(x; \varepsilon B)\right)\right] \geq C \cdot \mathbb{E}\left[\delta_y\left(\Phi_1(x; \varepsilon B + h)\right) \mathrm{e}^{-I\left(\frac{h}{\varepsilon}\right)}\right]. \tag{3.14}$$

In addition, we have chosen $h \in \Pi_{x,y}$, which means that $\Phi_1(x; h) = y$. Therefore, Eq. (3.14) becomes

$$p(t, x, y) \geq C \cdot \mathbb{E}\left[\delta_0\left(\Phi_1(x; \varepsilon B + h) - \Phi_1(x; h)\right) \mathrm{e}^{-I\left(\frac{h}{\varepsilon}\right)}\right]. \tag{3.15}$$

Note that we have the following scaling property for the Dirac delta function in $\mathbb{R}^N$: $\delta_0(cx) = c^{-N}\delta_0(x)$ for any $c > 0$. Indeed, suppose $\rho_\varepsilon(x) = \varepsilon^{-N}\rho(\varepsilon^{-1}x)$ is an approximation of δ_0. On the one hand,

$$\rho_\varepsilon(cx) \to \delta_0(cx).$$

On the other hand, after a change of variable $c^{-1}\varepsilon = \varepsilon'$, we also have

$$\rho_\varepsilon(cx) = c^{-N}\rho_{\varepsilon'}(x) \to c^{-N}\delta_0(x).$$

The desired scaling property of δ_0 is thus obtained. Plugging this scaling property into (3.15), we end up with

$$p(t, x, y) \geq C\varepsilon^{-N} \cdot \mathbb{E}\left[\delta_0\left(\frac{\Phi_1(x; \varepsilon B + h) - \Phi_1(x; h)}{\varepsilon}\right) \mathrm{e}^{-I\left(\frac{h}{\varepsilon}\right)}\right].$$

Our claim (3.12) thus follows from the definition (3.10) of $X^\varepsilon(h)$. □

Let us now describe the covariance matrix of $X(h)$ introduced in Proposition 3.6. To this aim, we first note that the Itô-Lyons map Φ_t can be restricted to $\bar{\mathcal{H}}$, that is, $\Phi_t(\cdot; \cdot) : \mathbb{R}^N \times \bar{\mathcal{H}} \to \mathbb{R}^N$ which we call the deterministic Itô-Lyons map associated with (1.1). For any fixed $h \in \bar{\mathcal{H}}$, the Jacobian of $\Phi_t(\cdot\,; h) : \mathbb{R}^N \to \mathbb{R}^N$ is denoted by $J_t(\cdot\,; h)$. According to [7, Sec. 11.3.2], it satisfies the following linear ODE,

$$J_t(x, h) = \mathrm{Id} + \sum_{i=1}^{d} \int_0^t \partial V_i(\Phi_s(x; h)) J_s(x, h) dh_s^i, \tag{3.16}$$

where Id is the $N \times N$ identity matrix. Let D be the Malliavin derivative operator. Similarly, the l-directional Malliavin differential $D^l\Phi_t := \langle D\Phi_t(x, h), l\rangle_{\bar{\mathcal{H}}}$ of Φ satisfies

$$D^l\Phi_t = \sum_{i=1}^{d} \int_0^t \partial V_i(\Phi_s(x; h)) D^l\Phi_s dh_s^i + \sum_{i=1}^{d} \int_0^t V_i(\Phi_s(x; h)) dl_s^i, \quad \text{for all } l \in \bar{\mathcal{H}}. \tag{3.17}$$

By using (3.16) and (3.17) to write down the ODE for $J_t^{-1}(x; h)D^l\Phi_t$, it is easily seen that

$$\langle D\Phi_t(x; h), l\rangle_{\bar{\mathcal{H}}} = J_t(x; h) \cdot \int_0^t J_s^{-1}(x; h) \cdot V(\Phi_s(x; h)) dl_s. \tag{3.18}$$

This is a standard application of Duhamel's formula when h is a smooth driving paths. One can then take limits in Young integrals in order to obtain the result (3.18) for a general $h \in \bar{\mathcal{H}}$. See [7, Sec. 11.3.2, Equation (11.17)] for more details. According to the pairing (2.13), when viewed as an $\mathcal{H}$-valued functional, we have

$$D_s\Phi_t^i(x;h) = \Big(J_t(x;h)J_s^{-1}(x;h)V(\Phi_s(x;h))\Big)^i \mathbf{1}_{[0,t]}(s), \quad 1 \le i \le N. \tag{3.19}$$

Now recall that $X_t(h) = \lim_{\varepsilon\downarrow 0}(\Phi(x;\varepsilon B + h) - \Phi_t(x;h))/\varepsilon$. Since the Itô-Lyons map is smooth with respect to both the initial condition and the driving path (cf. [9, Proposition 11.5]), it is seen that for any $l \in \bar{\mathcal{H}}$ the l-directional Malliavin differential of $X_t(h)$ satisfies the same equation as (3.17). Note again that this equation is deterministic. This implies that $X_t(h)$ is a Gaussian random variable and the $N \times N$ covariance matrix of $X_t(h)$ admits the following representation

$$\mathrm{Cov}(X_t(h)) := \Gamma_{\Phi_t(x;h)} = \langle D\Phi_t(x;h), D\Phi_t(x;h)\rangle_{\mathcal{H}}. \tag{3.20}$$

With (3.20) in hand, a crucial point for proving Theorem 3.4 is the fact that $\Gamma_{\Phi_1(x;h)}$ is uniformly non-degenerate with respect to all h. This is the content of the following result which is another special feature of ellipticity that fails in the hypoelliptic case. Its proof is an adaptation of the argument in [3] to the deterministic context.

Lemma 3.7 *Let $M > 0$ be a localizing constant. Consider the Malliavin covariance matrix $\Gamma_{\Phi_1(x;h)}$ defined by (3.20). Under the uniform ellipticity assumption (3.1), there exist $C_1, C_2 > 0$ depending only on H, M and the vector fields, such that*

$$C_1 \le \det \Gamma_{\Phi_1(x;h)} \le C_2 \tag{3.21}$$

for all $x \in \mathbb{R}^N$ and $h \in \bar{\mathcal{H}}$ with $\|h\|_{\bar{\mathcal{H}}} \le M$.

Proof We consider the cases of $H > 1/2$ and $H \le 1/2$ separately. We only study the lower bound of $\Gamma_{\Phi_1(x;h)}$ since the upper bound is standard from pathwise estimates by (3.19) and (3.20), plus the fact that $\|h\|_{\mathcal{H}} \le M$.
(i) Proof of the lower bound when $H > 1/2$. According to relation (3.20) and the expression for the inner product in $\mathcal{H}$ given by [17, equation (5.6)], we have

$$\Gamma_{\Phi_1(x;h)} = C_H \sum_{\alpha=1}^{d} \int_{[0,1]^2} J_1 J_s^{-1} V_\alpha(\Phi_s) V_\alpha^*(\Phi_t)(J_t^{-1})^* J_1^* |t-s|^{2H-2} ds dt,$$

where we have omitted the dependence on x and h for Φ and J inside the integral for notational simplicity. It follows that for any $z \in \mathbb{R}^N$, we have

$$z^*\Gamma_{\Phi_1(x;h)} z = C_H \int_{[0,1]^2} \langle \xi_s, \xi_t\rangle_{\mathbb{R}^d} |t-s|^{2H-2} ds dt, \tag{3.22}$$

where ξ is the function in $\mathcal{H}$ defined by

$$\xi_t \triangleq V^*(\Phi_t)(J_t^{-1})^* J_1^* z. \tag{3.23}$$

According to an interpolation inequality proved by Baudoin-Hairer (cf. [1, Proof of Lemma 4.4]), given $\gamma > H - 1/2$, we have

$$\int_{[0,1]^2} \langle f_s, f_t\rangle_{\mathbb{R}^d} |t-s|^{2H-2} ds dt \geq C_\gamma \frac{\left(\int_0^1 v^\gamma (1-v)^\gamma |f_v|^2 dv\right)^2}{\|f\|_\gamma^2} \tag{3.24}$$

for all $f \in C^\gamma([0,1];\mathbb{R}^d)$, where the γ-Hölder variational semi-norm in (3.24) is defined by

$$\|f\|_\gamma \triangleq \sup_{s,t\in[0,1]} \frac{|f_t - f_s|}{|t-s|^\gamma}.$$

Observe that, due to our uniform ellipticity assumption (3.1) and the non-degeneracy of J_t, we have

$$\inf_{0\leq t\leq 1} |\xi_t|^2 \geq C_{H,V,M} |z|^2. \tag{3.25}$$

Furthermore, recall that Φ_t is driven by $h \in \bar{\mathcal{H}}$. We have also seen that $\bar{\mathcal{H}} \hookrightarrow C_0^H$ whenever $H > 1/2$. Thus, for $H - 1/2 < \gamma < H$, we get $\|\Phi_t\|_\gamma \leq C_{H,V} \|h\|_\gamma$, and the same inequality holds true for the Jacobian J in (3.23). Therefore, going back to Eq. (3.23) again, we have

$$\|\xi\|_\gamma^2 \leq C_{H,V,M} \|h\|_{\bar{\mathcal{H}}} |z|^2 \leq C_{H,V,M} |z|^2, \tag{3.26}$$

where the last inequality stems from our assumption $\|h\|_{\bar{\mathcal{H}}} \leq M$. Therefore, taking $f_t = \xi_t$ in (3.24), plugging inequalities (3.25) and (3.26), and recalling relation (3.22), we conclude that

$$z^* \Gamma_{\Phi_1(x;h)} z \geq C_{H,V,M} |z|^2$$

uniformly for $\|h\|_{\bar{\mathcal{H}}} \leq M$. Hence, our result (3.21) follows when $H > \frac{1}{2}$.
(ii) Proof of the lower bound when $H \leq 1/2$. Recall again that (3.20) yields

$$z^* \Gamma_{\Phi_1(x;h)} z = \|z^* D\Phi_1(x;h)\|_{\mathcal{H}}^2.$$

Then, owing to the continuous embedding $\mathcal{H} \subseteq L^2([0,1])$ proved in Lemma 2.8, and expression (3.19) for $D\Phi_t$, we have for any $z \in \mathbb{R}^N$,

$$\begin{aligned} z^* \Gamma_{\Phi_1(x;h)} z &\geq C_H \|z^* D\Phi_1(x;h)\|_{L^2([0,1])}^2 \\ &= C_H \int_0^1 z^* J_1 J_t^{-1} V(\Phi_t) V^*(\Phi_t) (J_t^{-1})^* J_1^* z dt = C_H \int_0^1 |\xi_t|^2 dt, \end{aligned}$$

where we have used the definition (3.23) for the last step. Resorting to (3.25), we thus discover that

$$z^*\Gamma_{\Phi_1(x;h)}z \geq C_{H,V,M}|z|^2\,,$$

uniformly for $\|h\|_{\bar{\mathcal{H}}} \leq M$. Our claim (3.21) now follows as in the case $H > 1/2$. □

With the preliminary results of Proposition 3.6 and Lemma 3.7 in hand, we are now able to complete the proof of Theorem 3.4.

Proof of Theorem 3.4 Recall that $X^\varepsilon(h)$ is defined by (3.10). According to our preliminary bound (3.12), it remains to show that there exits a constant $C_{H,V} > 0$ such that when ε is small enough,

$$\mathbb{E}\left[\delta_0\left(X^\varepsilon(h)\right)\mathrm{e}^{-I\left(\frac{h}{\varepsilon}\right)}\right] \geq C_{H,V}, \tag{3.27}$$

uniformly for all h with $\|h\|_{\bar{\mathcal{H}}} \leq 2\varepsilon$. The proof of this fact consists of the following two steps:
(i) Prove that there exists a constant $C_{H,V} > 0$ such that $\mathbb{E}[\delta_0(X(h))\mathrm{e}^{-I(h/\varepsilon)}] \geq C_{H,V}$ for all $\varepsilon > 0$ and $h \in \bar{\mathcal{H}}$ with $\|h\|_{\bar{\mathcal{H}}} \leq 2\varepsilon$;
(ii) Upper bound the difference

$$\mathbb{E}\left[\delta_0\left(X^\varepsilon(h)\right)\mathrm{e}^{-I\left(\frac{h}{\varepsilon}\right)}\right] - \mathbb{E}\left[\delta_0(X(h))\mathrm{e}^{-I\left(\frac{h}{\varepsilon}\right)}\right],$$

and show that when ε is small the above difference is small uniformly for all h with $\|h\|_{\bar{\mathcal{H}}} \leq 2\varepsilon$. We now treat the above two parts separately.
Proof of item (i): Recall that the first chaos $\mathcal{C}_1$ has been defined in Section 2.1. Then, observe that the centered Gaussian random variable random variable $X(h) = (X^1(h), \ldots, X^N(h))$ introduced in Proposition 3.6 sits in $\mathcal{C}_1$. We decompose the Wiener integral $I(h/\varepsilon)$ as

$$I\,(h/\varepsilon) = G_1^\varepsilon + G_2^\varepsilon,$$

where G_1^ε and G_2^ε satisfy

$$G_1^\varepsilon \in \mathrm{Span}\{X^i(h);\, 1 \leq i \leq N\}, \quad G_2^\varepsilon \in \mathrm{Span}\{X^i(h);\, 1 \leq i \leq N\}^\perp$$

where the orthogonal complement is considered in $\mathcal{C}_1$. With this decomposition in hand, we get

$$\mathbb{E}\left[\delta_0(X(h))\mathrm{e}^{-I\left(\frac{h}{\varepsilon}\right)}\right] = \mathbb{E}\left[\delta_0(X(h))\mathrm{e}^{-G_1^\varepsilon}\right]\cdot\mathbb{E}\left[\mathrm{e}^{-G_2^\varepsilon}\right].$$

Furthermore, $\mathbb{E}[\mathrm{e}^G] \geq 1$ for any centered Gaussian random variable G. Thus,

$$\mathbb{E}\left[\delta_0(X(h))\mathrm{e}^{-I\left(\frac{h}{\varepsilon}\right)}\right] \geq \mathbb{E}\left[\delta_0(X(h))\mathrm{e}^{-G_1^\varepsilon}\right]. \tag{3.28}$$

We claim that

$$\mathbb{E}\left[\delta_0(X(h))\mathrm{e}^{-G_1^\varepsilon}\right] = \mathbb{E}\left[\delta_0(X(h))\right]. \tag{3.29}$$

Indeed, by the definition of G_1^ε, it can be expressed as a linear combination of components of $X(h)$, that is, $G_1^\varepsilon = a_1 X^1(h)+\cdots+a_N X^N(h)$ for some constants $a_1, \ldots, a_N$. If we approximate δ_0 above by a sequence of smooth functions $\{\psi_n; n \geq 1\}$ compactly supported in $B(0, 1/n) \subset \mathbb{R}^N$, it follows as $n \to \infty$

$$\begin{aligned}\mathbb{E}\left[\psi_n(X(h))\mathrm{e}^{-G_1^\varepsilon}\right] &= \int_{\mathbb{R}^N} p_{X(h)}(x)\psi_n(x)e^{a_1x^1+\cdots+a^Nx^N}dx \\ &= \int_{B(0,1/n)} p_{X(h)}(x)\psi_n(x)e^{a_1x^1+\cdots+a^Nx^N}dx \\ &\to p_{X(h)}(0) = \mathbb{E}\left[\delta_0(X(h))\right],\end{aligned}$$

where we have denoted by $p_{X(h)}$ the probability density function of the Gaussian random variable $X(h)$. Since the right hand-side of the above equation converges to $\mathbb{E}\left[\delta_0(X(h))\mathrm{e}^{-G_1^\varepsilon}\right]$ as $n \to \infty$, our claim in (3.29) follows. Combining (3.28) and (3.29), we get

$$\mathbb{E}\left[\delta_0(X(h))\mathrm{e}^{-I\left(\frac{h}{\varepsilon}\right)}\right] \geq \mathbb{E}[\delta_0(X(h))] = p_{X(h)}(0).$$

We now resort to the fact that $X(h)$ is a Gaussian random variable with covariance matrix $\Gamma_{\Phi_1(x;h)}$ by (3.20), which satisfies relation (3.21). This yields

$$\mathbb{E}\left[\delta_0(X(h))\mathrm{e}^{-I\left(\frac{h}{\varepsilon}\right)}\right] \geq \frac{1}{(2\pi)^{\frac{N}{2}}\sqrt{\det\Gamma_{\Phi_1(x;h)}}} \geq C_{H,V},$$

uniformly for all h with $\|h\|_{\bar{\mathcal{H}}} \leq 2\varepsilon$ (indeed, the proof shows that the above lower bound holds for all h with $\|h\|_{\bar{\mathcal{H}}} \lesssim 1$) . This ends the proof of item (i).
Proof of item (ii): By using the integration by parts formula in Malliavin's calculus (see Proposition 2.10), we have

$$\mathbb{E}[\delta_0(X(h))\mathrm{e}^{-I(h/\varepsilon)}] = \mathbb{E}\left[1_{\{X(h)\geq 0\}}H(X(h), I(h/\varepsilon))\right],$$

where $X(h) \geq 0$ is interpreted component-wise, and $H(X(h), I(h/\varepsilon))$ is a random variable which can be expressed explicitly in terms of the Malliavin derivatives of $I(h/\varepsilon)$, $X(h)$ and the inverse of the Malliavin covariance matrix $\gamma_{X(h)}$ of $X(h)$ (see (2.25) for details). Similarly, we have

$$\mathbb{E}[\delta_0(X^\varepsilon(h))\mathrm{e}^{-I(h/\varepsilon)}] = \mathbb{E}\left[1_{\{X^\varepsilon(h)\geq 0\}}H(X^\varepsilon(h), I(h/\varepsilon))\right].$$

Therefore,

$$\begin{aligned}&\left|\mathbb{E}[\delta_0(X(h))\mathrm{e}^{-I(h/\varepsilon)}] - \mathbb{E}[\delta_0(X^\varepsilon(h))\mathrm{e}^{-I(h/\varepsilon)}]\right| \\ &\quad \le \left|\mathbb{E}\left[\left(\mathbf{1}_{\{X^\varepsilon(h)\ge 0\}} - \mathbf{1}_{\{X(h)\ge 0]\}}\right) H(X(h), I(h/\varepsilon))\right]\right| \\ &\qquad + \left|\mathbb{E}\left[\mathbf{1}_{\{X^\varepsilon(h)\ge 0\}}\left(H(X^\varepsilon(h), I(h/\varepsilon)) - H(X(h), I(h/\varepsilon))\right)\right]\right|. \end{aligned} \tag{3.30}$$

Note that since $\|h\|_{\bar{\mathcal{H}}} \le 2\varepsilon$, $\|h/2\varepsilon\|_{\bar{\mathcal{H}}}$ remains bounded and hence the random variable $H(X(h), I(h/\varepsilon))$ has bounded p-th moment (uniform in ε) for all $p \ge 1$. It is thus clear from Proposition 3.6-(i) and an easy application of Hölder's inequality that the first term in the right-hand side of (3.30) can be made small when ε is small.

As for the second term in the right-hand side of (3.30), first of all thanks to Lemma 3.8, $\det \gamma_{X^\varepsilon(h)}$ has negative moments of all orders uniformly for all $\varepsilon \in (0, 1)$ and bounded $h \in \bar{\mathcal{H}}$. Together with the convergence in Proposition 3.6-(i), it follows that

$$\det \gamma^{-1}_{X^\varepsilon(h)} \xrightarrow{L^p} \det \gamma^{-1}_{X(h)}, \qquad \text{as } \varepsilon \to 0, \tag{3.31}$$

uniformly for $\|h\|_{\bar{\mathcal{H}}} \lesssim 1$ for each $p \ge 1$. Now recall from (2.25) that $H(X(h), I(h/\varepsilon))$ is a random variable which can be expressed explicitly in terms of the Malliavin derivatives of $I(h/\varepsilon)$, $X(h)$ and the inverse Malliavin covariance matrix $M_{X(h)}$ of $X(h)$. The convergence in (3.31) and Proposition 3.6-(i) is sufficient to conclude that $H(X^\varepsilon(h), I(h/\varepsilon)) - H(X(h), I(h/\varepsilon)) \to 0$ in $L^p(\Omega)$ as $\varepsilon \downarrow 0$ for all $p \ge 1$. Thus, the second term in the right-hand side of (3.30) can be made small when ε is small. Therefore, the assertion of item (ii) holds.

Once item (i) and (ii) are proved, it is easy to obtain (3.27) and the details are omitted. This finishes the proof of Theorem 3.4. □

Lemma 3.8 *Let $\gamma_{X^\varepsilon(h)}$ be the Malliavin matrix of $X^\epsilon(h)$ as defined in (2.24). We have*

$$\left\|\det \gamma^{-1}_{X^\varepsilon(h)}\right\|_p < \infty,$$

for all $p \ge 1$ and $\varepsilon \in (0, 1]$.

Proof We first recall that Φ is the Itô-Lyons map associated with (1.1) and J is the Jacobian (associated with the deterministic Φ) introduced in (3.16). Note that although J is introduced in (3.16) as a map on $\mathbb{R}^N \times \bar{\mathcal{H}}$, it can be extended to $\mathbb{R}^N \times \Omega$ by general rough path theory. Therefore, the notation $J(x; \varepsilon B + h)$ makes perfect sense. Indeed, it is simply the solution to Eq. (3.16) driven by $\varepsilon B + h$ (instead of h itself).

To lighten the notation, in what follows we will write Φ^ε and J^ε for $\Phi(x; \varepsilon B + h)$ and $J(x; \varepsilon B + h)$, respectively.

By the same argument as before (cf. (3.19)), the Malliavin derivative of the random variable Φ^ε_t can be represented in terms of the Jacobian J^ε as

$$\mathbf{D}_s \Phi^{\varepsilon,i}_t = \varepsilon \left(J^\varepsilon_t (J^\varepsilon_s)^{-1} V(\Phi^\varepsilon_s)\right)^i \mathbf{1}_{[0,t]}(s), \quad 1 \le i \le N.$$

In the above, the factor ε on the right hand-side of the equation is inherited from εB.

Recall the definition of $X^{\varepsilon}(h)$ in (3.10), we thus have

$$\mathbf{D}_s X^{\varepsilon,i}(h) = \frac{\mathbf{D}_s \Phi_t^{\varepsilon,i}}{\varepsilon} = \left(J_t^{\varepsilon} (J_s^{\varepsilon})^{-1} V(\Phi_s^{\varepsilon}) \right)^i \mathbf{1}_{[0,t]}(s), \quad 1 \le i \le N.$$

On the other hand, the Jacobian J^{ε} has an inverse which satisfies a similar SDE as (3.16)

$$(J_t^{\varepsilon})^{-1} = \mathrm{Id} - \sum_{i=1}^{d} \int_0^t (J_s^{\varepsilon})^{-1} \partial V_i(\Phi_s^{\varepsilon}) d(\varepsilon B^i + h^i)_s. \tag{3.32}$$

Therefore, general rough path estimates ensure that, for any $\gamma < H$, the γ-Hölder norm of components of both J^{ε} and $(J^{\varepsilon})^{-1}$ have finite p-th moments for all $p \ge 1$.

The rest of the proof now follows the same lines as in the proof of Lemma 3.3 of [3]. □

We conclude our discussion by a remark regarding SDE with a drift.

Remark 3.9 One can also consider the SDE in (1.1) but with a smooth and bounded drift

$$Z_t = x + \int_0^t V_0(Z_s) ds + \sum_{i=1}^{d} \int_0^t V_i(Z_s) dB_s^i, \quad t \in [0, 1]. \tag{3.33}$$

It turns our the control distance of the system (3.33) (in terms of large deviation, etc.) is the same as the one without a drift; that is, the same as being defined in (1.4). Hence, the corresponding local lower bound for the density function of Z_t is the same as stated in Theorem 1.6. In order to see this, recall that $\Phi_t(x; \cdot) : \bar{\mathcal{H}} \to C[0, 1]$ is the deterministic Itô map associated with Eq. (1.1). For each $\epsilon > 0$ we further define $\Phi_t^{\epsilon}(x; \cdot)$ to be the solution map of the equation

$$Z_t^{\epsilon} = x + \epsilon^{\frac{1}{H}} \int_0^t V_0(Z_s^{\epsilon}) ds + \sum_{i=1}^{d} \int_0^t V_i(Z_s^{\epsilon}) dB_s^i, \quad t \in [0, 1].$$

That is, $Z_t^{\epsilon} = \Phi_t^{\epsilon}(x; B)$. Similar to (3.8), we have for $\epsilon = t^H$,

$$Z_t = \Phi_t^1(x; B) \overset{\text{law}}{=} \Phi_1^{\epsilon}(x; \varepsilon B).$$

Now we proceed as in the proof of Theorem 1.6, and denote by $p(t, x, z)$ the density function of Z_t. Eq. (3.9) becomes

$$p(t, x, z) = \mathbb{E}\left[\delta_z \left(\Phi_1^{\epsilon}(x; \varepsilon B)\right)\right] = \mathbb{E}\left[\delta_z \left(\Phi_1(x; \epsilon B) + (\Phi_1^{\epsilon}(x; \varepsilon B) - \Phi_1(x; \epsilon B))\right)\right].$$

As a result, if we still pick $h \in \Pi_{x,z}$ as before (that is, $\Phi_1(x, h) = z$), the expectation on the right hand-side of (3.12) becomes

$$\mathbb{E}\left[\delta_0\left(X^\varepsilon(h) + \frac{\Phi_1^\epsilon(x; \varepsilon B + h) - \Phi_1(x, \varepsilon B + h)}{\epsilon}\right) e^{-I\left(\frac{h}{\varepsilon}\right)}\right].$$

The observation is that rough differential equations are Lipschitz continuous with respect to the vector fields. Hence, the extra term

$$\frac{\Phi_1^\epsilon(x; \varepsilon B + h) - \Phi_1(x, \varepsilon B + h)}{\epsilon}$$

is of order $\epsilon^{\frac{1}{H}-1}$, and can be considered negligible since $0 < H < 1$. Therefore, all the previous argument goes through as if there was no drift. We leave it to the enterprising readers to fill in the details.

Data Availability Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest All authors declare that they have no conflicts of interest.

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