



Exact Uniform Modulus of Continuity and Chung’s LIL for the Generalized Fractional Brownian Motion

Ran Wang¹ · Yimin Xiao²

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Abstract

The generalized fractional Brownian motion (GFBM) $X := \{X(t)\}_{t \geq 0}$ with parameters $\gamma \in [0, 1)$ and $\alpha \in (-\frac{1}{2} + \frac{\gamma}{2}, \frac{1}{2} + \frac{\gamma}{2})$ is a centered Gaussian H -self-similar process introduced by Pang and Taqqu (2019) as the scaling limit of power-law shot noise processes, where $H = \alpha - \frac{\gamma}{2} + \frac{1}{2} \in (0, 1)$. When $\gamma = 0$, X is the ordinary fractional Brownian motion. When $\gamma \in (0, 1)$, GFBM X does not have stationary increments, and its sample path properties such as Hölder continuity, path differentiability/non-differentiability, and the functional law of the iterated logarithm (LIL) have been investigated recently by Ichiba et al. (J Theoret Probab 10.1007/s10959-020-01066-1, 2021). They mainly focused on sample path properties that are described in terms of the self-similarity index H (e.g., LILs at infinity or at the origin). In this paper, we further study the sample path properties of GFBM X and establish the exact uniform modulus of continuity, small ball probabilities, and Chung’s laws of iterated logarithm at any fixed point $t > 0$. Our results show that the local regularity properties away from the origin and fractal properties of GFBM X are determined by the index $\alpha + \frac{1}{2}$ instead of the self-similarity index H . This is in contrast with the properties of ordinary fractional Brownian motion whose local and asymptotic properties are determined by the single index H .

Keywords Gaussian self-similar process · Generalized fractional Brownian motion · Exact uniform modulus of continuity · Small ball probability · Chung’s LIL · Tangent process · Lamperti’s transformation

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✉ Yimin Xiao
xiaoy@msu.edu

Ran Wang
rwang@whu.edu.cn

¹ School of Mathematics and Statistics, Wuhan University, Wuhan 430072, China

² Department of Statistics and Probability, Michigan State University, East Lansing, MI 48824, USA

1 Introduction and Main Results

The generalized fractional Brownian motion (GFBM, for short) $X := \{X(t)\}_{t \geq 0}$ is a centered Gaussian self-similar process introduced by Pang and Taqqu [28] as the scaling limit of power-law shot noise processes. It has the following stochastic integral representation:

$$\{X(t)\}_{t \geq 0} \stackrel{d}{=} \left\{ \int_{\mathbb{R}} ((t-u)_+^\alpha - (-u)_+^\alpha) |u|^{-\gamma/2} B(du) \right\}_{t \geq 0}, \tag{1.1}$$

where the parameters γ and α satisfy

$$\gamma \in [0, 1), \quad \alpha \in \left(-\frac{1}{2} + \frac{\gamma}{2}, \frac{1}{2} + \frac{\gamma}{2} \right), \tag{1.2}$$

and where $B(du)$ is a Gaussian random measure in \mathbb{R} with the Lebesgue control measure du . It follows that the Gaussian process X is self-similar with index H given by

$$H = \alpha - \frac{\gamma}{2} + \frac{1}{2} \in (0, 1). \tag{1.3}$$

When $\gamma = 0$, X becomes an ordinary fractional Brownian motion (FBM) B^H which can be represented as:

$$\{B^H(t)\}_{t \geq 0} \stackrel{d}{=} \left\{ \int_{\mathbb{R}} \left((t-u)_+^{H-\frac{1}{2}} - (-u)_+^{H-\frac{1}{2}} \right) B(du) \right\}_{t \geq 0}. \tag{1.4}$$

However, when $\gamma \neq 0$, X does not have the property of stationary increments.

Fractional Brownian motion B^H has been studied extensively in the literature. It is well known that B^H arises naturally as the scaling limit of many interesting stochastic systems. For example, [17] or [30, Chapter 3.4] showed that the scaled power-law Poisson shot noise process with stationary increments converges to B^H . Pang and Taqqu [28] studied a class of integrated shot-noise processes with power-law non-stationary conditional variance functions and proved in their Theorem 3.1 that the corresponding scaled process converges weakly to GFBM X .

As shown by Pang and Taqqu [28], GFBM X is a natural generalization of the ordinary FBM. It preserves the self-similarity property while the factor $|u|^{-\gamma/2}$ introduces non-stationarity of increments, which is useful for reflecting the non-stationarity in physical stochastic systems. Ichiba, Pang and Taqqu [11] raised the interesting question: ‘‘How does the parameter γ affect the sample path properties of GFBM?’’. They proved in [11] that, for any $T > 0$ and $\varepsilon > 0$, the sample paths of X are H\"older continuous in $[0, T]$ of order $H - \varepsilon$ and the functional and local laws of the iterated logarithm of X are determined by the self-similarity index H . More recently, Ichiba, Pang and Taqqu [12] studied the semimartingale properties of GFBM X and its mixtures and applied them to model the volatility processes in finance.

In this paper, we study precise local sample path properties of GFBM X , including the exact uniform modulus of continuity, small ball probabilities, Chung’s law of the iterated logarithm at any fixed point $t > 0$, and the tangent processes. Our main results are Theorems 1.1–1.6 below. They show that the local regularity properties of GFBM X away from the origin are determined by the index $\alpha + \frac{1}{2}$, instead of the self-similarity index $H = \alpha - \frac{\gamma}{2} + \frac{1}{2}$. Our results also imply that the fractal properties of GFBM X are determined by $\alpha + \frac{1}{2}$, see Remarks 3.1 (ii) and 7.1 below. This is in contrast with the ordinary fractional Brownian motion whose local, fractal, and asymptotic properties are determined by the single index H . We remark that our results are also useful for studying other fine sample path properties of GFBM X . For example, one can determine the exact Hausdorff measure functions for various fractals generated by the sample paths and prove sharp Hölder conditions and tail probability estimates for the local times of GFBM X as in, e.g., [32,34,38–40].

The first result is related to Theorems 3.1 and 4.1 of Ichiba, Pang and Taqqu [11] and provides the exact uniform modulus of continuity for X and its derivative X' (when it exists) in $[a, b]$, where $0 < a < b < \infty$ are constants.

Theorem 1.1 *Let $X := \{X(t)\}_{t \geq 0}$ be the GFBM defined in (1.1) and let $0 < a < b < \infty$ be constants.*

(a) *If $\alpha \in (-1/2 + \gamma/2, 1/2)$, then there exists a constant $\kappa_1 \in (0, \infty)$ such that*

$$\lim_{r \rightarrow 0^+} \sup_{a \leq t \leq b} \sup_{0 \leq h \leq r} \frac{|X(t+h) - X(t)|}{h^{\alpha + \frac{1}{2}} \sqrt{\ln h^{-1}}} = \kappa_1, \quad \text{a.s.} \tag{1.5}$$

(b) *If $\alpha = 1/2$, then there exists a constant $\kappa_2 \in (0, \infty)$ such that*

$$\lim_{r \rightarrow 0^+} \sup_{a \leq t \leq b} \sup_{0 \leq h \leq r} \frac{|X(t+h) - X(t)|}{h \ln h^{-1}} \leq \kappa_2, \quad \text{a.s.} \tag{1.6}$$

(c) *If $\alpha \in (1/2, 1/2 + \gamma)$, then X has a modification that is continuously differentiable on $[a, b]$ and its derivative X' satisfies that*

$$\lim_{r \rightarrow 0^+} \sup_{a \leq t \leq b} \sup_{0 \leq h \leq r} \frac{|X'(t+h) - X'(t)|}{h^{\alpha - \frac{1}{2}} \sqrt{\ln h^{-1}}} = \kappa_3, \quad \text{a.s.}, \tag{1.7}$$

where $\kappa_3 \in (0, \infty)$ is a constant.

Remark 1.1 (i) Theorem 3.1 in [11] states that, for all $\varepsilon > 0$, X has a modification that satisfies the uniform Hölder condition in $[0, T]$ of order $\alpha - \gamma/2 + 1/2 - \varepsilon$. Our Theorem 1.1 shows that the sample path of X on any interval $[a, b]$ with $a > 0$ is smoother than its behavior at $t = 0$, which is determined by the self-similarity index $H = \alpha - \gamma/2 + 1/2$ as suggested by $\mathbb{E}[X(t)^2] = c(\alpha, \gamma)t^{2H}$, with

$$c(\alpha, \gamma) = \mathcal{B}(2\alpha + 1, 1 - \gamma) + \int_0^\infty ((1 + u)^\alpha - u^\alpha)^2 u^{-\gamma} du. \tag{1.8}$$

Here and below, $\mathcal{B}(\cdot, \cdot)$ denotes the Beta function.

- (ii) We believe that the equality in (1.6) holds. However, we have not been able to prove this. The reason is that, when $\alpha = 1/2$, the lower bounds in Lemma 3.1 and Proposition 3.2 are different. Similarly, the case of $\alpha = 1/2$ is excluded in Theorems 1.2–1.4 below.

The next two results are on the small ball probabilities of X . They show a clear difference for the two cases of $s \in [0, r]$ and $s \in [t-r, t+r]$ with $t > r > 0$. The small ball probabilities are not only useful for proving Chung’s law of the iterated logarithm (Chung’s LIL, for short) in Theorem 1.4 but also have many other applications. We refer to Li and Shao [21] for more information.

Theorem 1.2 *Assume $\alpha \in (-1/2 + \gamma/2, 1/2)$. Then, there exist constants $\kappa_4, \kappa_5 \in (0, \infty)$ such that for all $r > 0$ and $0 < \varepsilon < 1$,*

$$\exp\left(-\kappa_4\left(\frac{r^H}{\varepsilon}\right)^{\frac{1}{\alpha+1/2}}\right) \leq \mathbb{P}\left\{\sup_{s \in [0,r]} |X(s)| \leq \varepsilon\right\} \leq \exp\left(-\kappa_5\left(\frac{r^H}{\varepsilon}\right)^{\frac{1}{\alpha+1/2}}\right), \tag{1.9}$$

where $H = \alpha - \gamma/2 + 1/2$.

Theorem 1.3 (a) *Assume $\alpha \in (-1/2 + \gamma/2, 1/2)$. Then, there exist constants $\kappa_6, \kappa_7 \in (0, \infty)$ such that for all $t > 0, r \in (0, t/2)$ and $\varepsilon \in (0, r^{\alpha+1/2})$,*

$$\begin{aligned} \exp\left(-\kappa_6 r c_1(t)\left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1/2}}\right) &\leq \mathbb{P}\left\{\sup_{|h| \leq r} |X(t+h) - X(t)| \leq \varepsilon\right\} \\ &\leq \exp\left(-\kappa_7 r c_2(t)\left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1/2}}\right), \end{aligned} \tag{1.10}$$

where $c_1(t) = \max\{t^{\alpha-\gamma/2-1/2}, t^{-\gamma/(2\alpha+1)}\}$ and $c_2(t) = t^{-\gamma/(2\alpha+1)}$.

- (b) *Assume $\alpha \in (1/2, 1/2 + \gamma/2)$. Then, there exist constants $\kappa_8, \kappa_9 \in (0, \infty)$ such that for all $t > 0, r \in (0, t/2)$ and $\varepsilon \in (0, r^{\alpha-1/2})$,*

$$\begin{aligned} \exp\left(-\kappa_8 r c_3(t)\left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha-1/2}}\right) &\leq \mathbb{P}\left\{\sup_{|h| \leq r} |X'(t+h) - X'(t)| \leq \varepsilon\right\} \\ &\leq \exp\left(-\kappa_9 r c_4(t)\left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha-1/2}}\right), \end{aligned} \tag{1.11}$$

where $c_3(t) = \max\{t^{\alpha-\gamma/2-3/2}, t^{-\gamma/(2\alpha-1)}\}$ and $c_4(t) = t^{-\gamma/(2\alpha-1)}$.

The following are Chung’s laws of the iterated logarithm for X and X' . It is interesting to notice that the parameters γ and α play different roles. Since X and X' do not have stationary increments when $\gamma > 0$, the limits in their Chung’s LILs depend on the location of $t > 0$. (1.12) and (1.13) show that the oscillations decrease at the rate $t^{-\gamma/2}$ as t increases. This provides an explicit answer in the context of Chung’s LIL

to the aforementioned question of Ichiba, Pang and Taqqu [11] regarding the effect of the parameter γ .

Theorem 1.4 (a) *If $\alpha \in (-1/2 + \gamma/2, 1/2)$, then there exists a constant $\kappa_{10} \in (0, \infty)$ such that for every $t > 0$,*

$$\liminf_{r \rightarrow 0+} \sup_{|h| \leq r} \frac{|X(t+h) - X(t)|}{r^{\alpha+1/2}/(\ln \ln 1/r)^{\alpha+1/2}} = \kappa_{10} t^{-\gamma/2}, \quad a.s. \tag{1.12}$$

(b) *If $\alpha \in (1/2, 1/2 + \gamma/2)$, then there exists a constant $\kappa_{11} \in (0, \infty)$ such that for every $t > 0$,*

$$\liminf_{r \rightarrow 0+} \sup_{|h| \leq r} \frac{|X'(t+h) - X'(t)|}{r^{\alpha-1/2}/(\ln \ln 1/r)^{\alpha-1/2}} = \kappa_{11} t^{-\gamma/2}, \quad a.s. \tag{1.13}$$

Remark 1.2 From the proofs of Theorems 1.3 and 1.4, we know that if $\sup_{|h| \leq r}$ is replaced by $\sup_{0 \leq h \leq r}$ in (1.10–1.13), then the corresponding results also hold.

For completeness, we also include the following law of the iterated logarithm for GFBM X at any fixed point $t > 0$. Part (a) of our Theorem 1.5 supplements Theorem 6.1 in [11] where the case of $t = 0$ was considered. See also Proposition 7.1 at the end of the present paper for a slight improvement of [11, Theorem 6.1] using the time inversion property of GFBM. Theorems 1.4 and 1.5 together describe precisely the large and small oscillations of X in the neighborhood of every fixed point $t > 0$. As such they are useful for studying fine fractal properties (such as exact Hausdorff measure function and multifractal structure) of the sample path of X .

The topic of LILs for Gaussian processes has been studied extensively by many authors, see for example, Arcones [3], Marcus and Rosen [24], Meerschaert et al. [26]. In particular, Chapter 7 of [24] provides explicit information about the constant in LIL for Gaussian processes with stationary increments under extra regularity conditions on the variance of the increments or the spectral density functions. However, the results in [24] cannot be applied to GFBM X directly. Instead, we will make use of Theorem 5.1 in [26] and the stationary Gaussian process U in Sect. 4 to prove the following theorem. As in Theorem 1.4, our result below describes explicitly the roles played by the parameters γ and α and the location $t > 0$.

Theorem 1.5 (a). *If $\alpha \in (-1/2 + \gamma/2, 1/2)$, then there exists a constant $\kappa_{12} \in (0, \infty)$ such that for every $t > 0$,*

$$\limsup_{r \rightarrow 0+} \sup_{|h| \leq r} \frac{|X(t+h) - X(t)|}{r^{\alpha+1/2} \sqrt{\ln \ln 1/r}} = \kappa_{12} t^{-\gamma/2}, \quad a.s. \tag{1.14}$$

(b). *If $\alpha = 1/2$, then there exists a constant $\kappa_{13} \in (0, \infty)$ such that for every $t > 0$,*

$$\limsup_{r \rightarrow 0+} \sup_{|h| \leq r} \frac{|X(t+h) - X(t)|}{r \sqrt{\ln(1/r) \ln \ln 1/r}} = \kappa_{13} t^{-\gamma/2}, \quad a.s. \tag{1.15}$$

(c). If $\alpha \in (1/2, 1/2 + \gamma/2)$, then there exists a constant $\kappa_{14} \in (0, \infty)$ such that for every $t > 0$,

$$\limsup_{r \rightarrow 0^+} \sup_{|h| \leq r} \frac{|X'(t+h) - X'(t)|}{r^{\alpha-1/2} \sqrt{\ln \ln 1/r}} = \kappa_{14} t^{-\gamma/2}, \quad \text{a.s.} \tag{1.16}$$

In order to prove the theorems stated above, we consider the following decomposition of $X(t)$ for all $t \geq 0$:

$$\begin{aligned} X(t) &= \int_{-\infty}^0 ((t-u)^\alpha - (-u)^\alpha) (-u)^{-\gamma/2} B(du) + \int_0^t (t-u)^\alpha u^{-\gamma/2} B(du) \\ &=: Y(t) + Z(t). \end{aligned} \tag{1.17}$$

Then, the two processes $Y = \{Y(t)\}_{t \geq 0}$ and $Z = \{Z(t)\}_{t \geq 0}$ are independent. The process Z in (1.17) is well defined for $\alpha > -1/2$ and $\gamma < 1$ and is called a *generalized Riemann–Liouville FBM*, following the terminology of Ichiba, Pang and Taqqu [11]. Notice that the ranges of the parameters α and γ for Z are wider than that in (1.2). As in [28, Remark 5.1], one can verify that Z is a self-similar Gaussian process with Hurst index $H = \alpha - \frac{\gamma}{2} + \frac{1}{2}$ which is negative if $\alpha \in (-1/2, -1/2 + \gamma/2)$. It follows from Lemma 3.1 below that Z has a modification whose sample function is continuous on $(0, \infty)$ a.s. In Sect. 2, we will prove that Y has a modification that is continuously differentiable in $(0, \infty)$. Hence, in order to study the regularity properties of X , we only need to study in detail the regularity properties of the sample path of Z when the parameters $\alpha > -1/2$ and $\gamma \in [0, 1)$.

Intuitively, if $u \in [a, b] \subset (0, \infty)$, the perturbation of $u^{-\gamma/2}$ is bounded and it does not deeply affect the sample path properties of $Z(t)$. Consequently, the process Z shares many regularity properties of the following process:

$$Z^{\alpha,0}(t) := \int_0^t (t-u)^\alpha B(du), \quad \alpha > -1/2 \tag{1.18}$$

which is the *Riemann–Liouville FBM* introduced by Lévy [20], see also Mandelbrot and Van Ness [23], Marinucci and Robinson [25] for further information. When $\alpha \geq 1$ is a positive integer, then $Z^{\alpha,0}$ is, up to a constant factor, an α -fold primitive of Brownian motion and its precise local asymptotic properties were studied by Lachal [18].

Theorems 1.4–1.5 demonstrate that, since GFBM X has non-stationary increments, the local oscillation properties of X near a point $t \in (0, \infty)$ are location dependent. As suggested by one of the referees,¹ another way to study the local structure of X near t is to determine the limit (in the sense of all the finite dimensional distributions or in the sense of weak convergence) of the following sequence of scaled enlargements of X around t :

¹ We thank the referee for raising this interesting question.

$$\left\{ \frac{X(t + r_n \tau) - X(t)}{c_n} \right\}_{\tau \geq 0}, \tag{1.19}$$

where $\{r_n\}$ and $\{c_n\}$ are sequences of real numbers such that $r_n \searrow 0$ and $c_n \searrow 0$. The (small scale) limiting process of (1.19), when it exists, is called a tangent process of X at t by Falconer [8,9]. If the limit in (1.19) exists for $c_n = r_n^\chi$ for some constant $\chi \in (0, 1]$ which may depend on t , one also says that X is weakly locally asymptotically self-similar of order χ at t (cf. [6]). Tangent processes of stochastic processes and random fields are useful in both theoretical and applications. We refer to [8,9] for a general framework of tangent processes and to [4,5] for some statistical applications. Recently, Skorniakov [31] provided sufficient and necessary conditions for a class of self-similar Gaussian processes to admit a unique tangent process at any fixed point $t > 0$. The theorems in [31] are applicable to the Riemann–Liouville FBM, sub-fractional Brownian motion, and bi-fractional Brownian motion. However, it is not obvious to verify that GFBM X satisfies the conditions of Theorem 2.1 of Skorniakov [31]. In the following, we provide some results on the tangent processes of GFBM X by using direct arguments.

For any fixed $t > 0$, $u \geq 0$ and a constant $\chi \in (0, 1]$, define the scaled process $\{V^{t,u}(\tau)\}_{\tau \geq 0}$ around t by

$$V^{t,u}(\tau) := \frac{X(t + u\tau) - X(t)}{u^\chi}, \quad \tau \geq 0. \tag{1.20}$$

- (a) If $t = 0$, then, by the self-similarity of X , we take $\chi = H$ and see that the corresponding tangent process is X itself.
- (b) If $t > 0$, then the choice of χ depends on the parameter α .
 - (b1) If $\alpha \in (1/2, 1/2 + \gamma/2)$, then by the differentiability of X (see also [8, Example 2]), we can take $\chi = 1$ and derive that, for any $t > 0$,

$$\{V^{t,u}(\tau)\}_{\tau \geq 0} \text{ converges in distribution to } \{c_{1,1} t^{H-1} B^1(\tau)\}_{\tau \geq 0}$$

in $C(\mathbb{R}_+, \mathbb{R})$ (the space of continuous functions from \mathbb{R}_+ to \mathbb{R}), as $u \rightarrow 0+$, where $B^1(\tau) = \tau \mathcal{N}$ with \mathcal{N} being a standard Gaussian random variable and

$$c_{1,1} = \alpha \left(\int_{-\infty}^t (t - u)^{2\alpha-2} |u|^{-\gamma} du \right)^{1/2}.$$

- (b2) If $\alpha \in (-1/2 + \gamma/2, 1/2)$, then it follows from the decomposition (1.17) and the differentiability of Y (see Proposition 2.1 below) that the weak limit of $\{V^{t,u}(\tau)\}_{\tau \geq 0}$ as $u \rightarrow 0+$ is the same as the analogous scaled process for the generalized Riemann–Liouville FBM Z (See Proposition 3.1 below). More precisely, we obtain the result in Theorem 1.6 below.

Theorem 1.6 Assume $\alpha \in (-1/2 + \gamma/2, 1/2)$, $t > 0$ and $\chi = \alpha + \frac{1}{2}$ in (1.20). Then, the process $\{V^{t,u}(\tau)\}_{\tau \geq 0}$ defined by (1.20) converges in distribution to

$\{\kappa_{15}t^{-\gamma/2}B^{\alpha+1/2}(\tau)\}_{\tau \geq 0}$ in $C(\mathbb{R}_+, \mathbb{R})$, as $u \rightarrow 0+$. Here, $B^{\alpha+1/2}$ is a FBM with index $\alpha + 1/2$, and

$$\kappa_{15} = \left(\frac{1}{2\alpha + 1} + \int_0^\infty [(1 + v)^\alpha - v^\alpha]^2 dv \right)^{1/2}.$$

The rest of the paper is organized as follows. In Sect. 2, we prove that the sample paths of the process Y are differentiable in $(0, \infty)$ almost surely. From Sects. 3–6, we focus on the generalized Riemann–Liouville FBM Z . More precisely, we give estimates on the moment of increments, prove the existence of the tangent process and establish the one-sided strong local nondeterminism of Z in Sect. 3; study the Lamperti transformation of Z and give some spectral density estimates in Sect. 4; determine the small ball probabilities for Z in Sect. 5; and prove a Chung’s LIL for Z in Sect. 6. In Sect. 7, we prove the main theorems for GFBM X .

2 Sample Path Properties of Y

In this section, we consider the process $Y = \{Y(t)\}_{t \geq 0}$ defined in (1.17), namely,

$$Y(t) = \int_{-\infty}^0 ((t - u)^\alpha - (-u)^\alpha) (-u)^{-\gamma/2} B(du) \tag{2.1}$$

and show that its sample function is smooth away from the origin.

Lemma 2.1 *Assume $-1/2 + \gamma/2 < \alpha < 1/2 + \gamma/2$. Then, for all $0 < s < t < \infty$,*

$$c_{2,1} \frac{|t - s|^2}{t^{2-2H}} \leq \mathbb{E} \left[(Y(t) - Y(s))^2 \right] \leq c_{2,1} \frac{|t - s|^2}{s^{2-2H}} \tag{2.2}$$

where $c_{2,1} = \alpha^2 \int_0^\infty (1 + u)^{2\alpha-2} u^{-\gamma} du$.

Proof For any $0 < s < t < \infty$ and $u > 0$, by the mean-value theorem, there exists a number $\theta(u) \in (0, 1)$ (which also depends on s, t) such that

$$(t + u)^\alpha - (s + u)^\alpha = \alpha [s + u + \theta(u)(t - s)]^{\alpha-1} (t - s). \tag{2.3}$$

In the proofs of this lemma and Proposition 2.1 below, we will use (2.3) to derive lower and upper bounds for $|(t + u)^\alpha - (s + u)^\alpha|$. Hence, we have

$$\begin{aligned} \mathbb{E} \left[(Y(t) - Y(s))^2 \right] &= \int_0^\infty [(t + u)^\alpha - (s + u)^\alpha]^2 u^{-\gamma} du \\ &\geq \alpha^2 (t - s)^2 \int_0^\infty (t + u)^{2(\alpha-1)} u^{-\gamma} du \\ &= \alpha^2 (t - s)^2 t^{2\alpha-\gamma-1} \int_0^\infty (1 + v)^{2\alpha-2} v^{-\gamma} dv \\ &= c_{2,1} \frac{|t - s|^2}{t^{2-2H}}. \end{aligned}$$

Similarly, we can prove that

$$\mathbb{E} \left[(Y(t) - Y(s))^2 \right] \leq c_{2,1} \frac{|t - s|^2}{s^{2-2H}}.$$

This proves (2.2). □

By Lemma 2.1, the Gaussian property of Y , and the Kolmogorov continuity theorem (see, e.g., [16, Theorem C.6]), we know that, for any $\varepsilon > 0$, Y has a modification that is Hölder continuous with index $1 - \varepsilon$ on any interval $[a, b]$ with $0 < a < b$. We will apply this fact in Sect. 7 to prove Theorems 1.1, 1.4–1.6, from the results on the generalized Riemann–Liouville FBM Z .

In the following, we prove the differentiability of Y by using the argument in the proof of Lemma 3.6 in [16].

Proposition 2.1 *Assume $-1/2 + \gamma/2 < \alpha < 1/2 + \gamma/2$. For any integer $n \geq 1$, the Gaussian process $Y = \{Y(t)\}_{t \geq 0}$ has a modification (still denoted by Y) that is continuously differentiable of order n in $(0, \infty)$.*

Proof For any $t > 0$, define

$$Y'(t) := \alpha \int_{-\infty}^0 (t - u)^{\alpha-1} (-u)^{-\frac{\gamma}{2}} B(du). \tag{2.4}$$

The integrand is in $L^2((-\infty, 0); \mathbb{R})$, and hence, $\{Y'(t)\}_{t > 0}$ is a well-defined mean-zero Gaussian process. For every $s, t \in [a, b] \subset (0, \infty)$ with $s < t$, applying (2.3), we have

$$\begin{aligned} &\mathbb{E} \left[|Y'(t) - Y'(s)|^2 \right] \\ &= \alpha^2 \int_0^\infty \left| (t + u)^{\alpha-1} - (s + u)^{\alpha-1} \right|^2 u^{-\gamma} du \\ &\leq \alpha^2 (\alpha - 1)^2 \left[\int_0^1 s^{2\alpha-4} u^{-\gamma} du + \int_1^\infty u^{2\alpha-4-\gamma} du \right] \cdot |t - s|^2 \\ &\leq \alpha^2 (\alpha - 1)^2 \left(\frac{a^{2\alpha-4}}{1 - \gamma} - \frac{1}{2\alpha - 3 - \gamma} \right) |t - s|^2. \end{aligned} \tag{2.5}$$

This, together with the Kolmogorov continuity theorem and the arbitrariness of a and b , implies that Y' is continuous in $(0, \infty)$ up to a modification.

Assume that $\phi \in C_c^\infty((0, \infty))$ (the space of all infinitely differentiable functions with compact supports). By the stochastic Fubini theorem [16, Corollary 2.9] and the formula of integration by parts, we know a.s.,

$$\begin{aligned} \int_0^\infty Y'(t)\phi(t)dt &= \int_{-\infty}^0 B(du) \int_0^\infty \frac{\partial}{\partial t} \left[((t-u)^\alpha - (-u)^\alpha)u^{-\frac{\gamma}{2}} \right] \phi(t)dt \\ &= - \int_{-\infty}^0 B(du) \int_0^\infty \left[((t-u)^\alpha - (-u)^\alpha)u^{-\frac{\gamma}{2}} \right] \frac{d}{dt}\phi(t)dt. \end{aligned} \tag{2.6}$$

Applying the stochastic Fubini theorem again, we have

$$\int_0^\infty Y'(t)\phi(t)dt = - \int_0^\infty Y(t)\frac{d}{dt}\phi(t)dt, \quad \text{a.s.} \tag{2.7}$$

This means that $Y'(t)$ is the weak derivative of $Y(t)$ for all $t > 0$. Since Y' is continuous in $(0, \infty)$, (2.7) shows that Y' is in fact almost surely the ordinary derivative of Y in $(0, \infty)$. By induction, we use the same argument as above to show that, for every integer $n \geq 1$, Y has a modification that is continuously differentiable of order n in $(0, \infty)$. □

3 Moment Estimates for the Increments and One-Sided SLND of Z

Consider the generalized Riemann–Liouville FBM $Z = \{Z(t)\}_{t \geq 0}$ with indices α and γ defined by

$$Z(t) = \int_0^t (t-u)^\alpha u^{-\gamma/2} B(du). \tag{3.1}$$

This Gaussian process is well defined if the constants α and γ satisfy $\alpha > -\frac{1}{2}$ and $\gamma < 1$ and is self-similar with index $H = \alpha - \frac{\gamma}{2} + \frac{1}{2}$. Notice that $H \leq 0$ if $-\frac{1}{2} < \alpha \leq -\frac{1}{2} + \frac{\gamma}{2}$ and $H > 0$ if $\alpha > -\frac{1}{2} + \frac{\gamma}{2}$.

In this section, we derive optimal estimates on the moment of the increments, prove the existence of the tangent process and establish the one-sided strong local nondeterminism for Z . These properties are useful for studying the sample properties of Z .

3.1 Moment Estimates

In the following, Lemmas 3.1, 3.2 and 3.3 provide optimal estimates on $\mathbb{E} [(Z(t) - Z(s))^2]$. These estimates are essential for establishing sharp sample path properties of Z . Notice that the upper bounds in (i) and (ii) in Lemma 3.1 below are the

same (up to a constant factor) when $\alpha < 1/2$. We will use these bounds for estimating the small ball probability and the uniform modulus of continuity in Sects. 4 and 7.

Lemma 3.1 *Assume $\alpha \in (-1/2, 1/2]$ and $\gamma \in [0, 1)$. The following statements hold:*

(i). *If $0 < s < t < \infty$ and $0 < s \leq 2(t - s)$, then*

$$c_{3,1} \frac{|t - s|^{2\alpha+1}}{t^\gamma} \leq \mathbb{E} \left[(Z(t) - Z(s))^2 \right] \leq c_{3,2} \frac{|t - s|^{2\alpha+1}}{s^\gamma}, \tag{3.2}$$

here $c_{3,1} = 1/(2\alpha+1)$ and $c_{3,2} = 2/(1-\gamma) + 1/(2\alpha+1) + \mathcal{B}(2\alpha+1, 1-\gamma)2^{2\alpha+1}$.

(ii). *If $0 < s < t < \infty$ and $s > 2(t - s)$, then*

$$\mathbb{E} \left[(Z(t) - Z(s))^2 \right] \asymp \begin{cases} \frac{|t-s|^{2\alpha+1}}{s^\gamma}, & \text{if } \alpha < 1/2, \\ \frac{(t-s)^2}{s^\gamma} \left(1 + \ln \left| \frac{s}{t-s} \right| \right), & \text{if } \alpha = 1/2. \end{cases} \tag{3.3}$$

Here and below, for two real-valued functions f and g defined on a set I , the notation $f \asymp g$ means that

$$c \leq f(x)/g(x) \leq c' \text{ for all } x \in I,$$

for some positive and finite constants c and c' which may depend on f, g and I .

Proof For any $0 < s < t < \infty$, we have

$$\begin{aligned} & \mathbb{E} \left[(Z(t) - Z(s))^2 \right] \\ &= \int_0^s ((t - u)^\alpha - (s - u)^\alpha)^2 u^{-\gamma} du + \int_s^t (t - u)^{2\alpha} u^{-\gamma} du \tag{3.4} \\ &=: I_1 + I_2. \end{aligned}$$

To bound the integral I_1 , we make a change of variable with $u = s - (t - s)v$ to obtain

$$I_1 = (t - s)^{2\alpha-\gamma+1} \int_0^{\frac{s}{t-s}} [(1 + v)^\alpha - v^\alpha]^2 \left(\frac{s}{t - s} - v \right)^{-\gamma} dv. \tag{3.5}$$

In order to estimate I_1 in Case (i), we distinguish the two cases $\alpha \in [0, 1/2]$ and $\alpha \in (-1/2, 0)$.

When $\alpha \in [0, 1/2]$, we use the fact that $3^\alpha - 2^\alpha \leq (1 + v)^\alpha - v^\alpha \leq 1$ for all $v \in [0, 2]$ to derive

$$\begin{aligned}
 I_1 &\leq (t - s)^{2\alpha - \gamma + 1} \int_0^{\frac{s}{t-s}} \left(\frac{s}{t-s} - v\right)^{-\gamma} dv \\
 &= \frac{s^{1-\gamma}}{1-\gamma} (t - s)^{2\alpha} \\
 &\leq \frac{2}{1-\gamma} \frac{(t - s)^{2\alpha + 1}}{s^\gamma}.
 \end{aligned}
 \tag{3.6}$$

When $\alpha \in (-1/2, 0)$, we use the fact that $0 < v^\alpha - (1 + v)^\alpha < v^\alpha$ for all $v \in [0, 2]$ to derive

$$\begin{aligned}
 I_1 &\leq (t - s)^{2\alpha - \gamma + 1} \int_0^{\frac{s}{t-s}} v^{2\alpha} \left(\frac{s}{t-s} - v\right)^{-\gamma} dv \\
 &= s^{2\alpha - \gamma + 1} \int_0^1 v^{2\alpha} (1 - v)^{-\gamma} dv \\
 &\leq \mathcal{B}(2\alpha + 1, 1 - \gamma) 2^{2\alpha + 1} \frac{(t - s)^{2\alpha + 1}}{s^\gamma}.
 \end{aligned}
 \tag{3.7}$$

In the above, we made a change of variable and the assumption that $s \leq 2(t - s)$.

For the integral I_2 in (3.4), by the change of variable $u = s + (t - s)v$, we have

$$\begin{aligned}
 I_2 &= (t - s)^{2\alpha - \gamma + 1} \int_0^1 (1 - v)^{2\alpha} \left(\frac{s}{t-s} + v\right)^{-\gamma} dv \\
 &\leq (t - s)^{2\alpha - \gamma + 1} \left(\frac{s}{t-s}\right)^{-\gamma} \int_0^1 (1 - v)^{2\alpha} dv \\
 &= \frac{1}{2\alpha + 1} \frac{(t - s)^{2\alpha + 1}}{s^\gamma}.
 \end{aligned}
 \tag{3.8}$$

On the other hand, in Case (i), $\left(\frac{s}{t-s} + v\right)^{-\gamma} \geq \left(\frac{t}{t-s}\right)^{-\gamma}$ for all $v \in [0, 1]$. Hence,

$$I_2 \geq (t - s)^{2\alpha - \gamma + 1} \left(\frac{t}{t-s}\right)^{-\gamma} \int_0^1 (1 - v)^{2\alpha} dv = \frac{1}{2\alpha + 1} \frac{(t - s)^{2\alpha + 1}}{t^\gamma}.
 \tag{3.9}$$

Consequently, the lower bound in (3.2) follows from (3.4) and (3.9), and the upper bound in (3.2) follows from (3.4), (3.6), (3.7), and (3.8).

Now, we consider Case (ii). Since $s > 2(t - s)$, we write

$$\begin{aligned}
 I_1 &= (t - s)^{2\alpha - \gamma + 1} \left(\int_0^1 [(1 + v)^\alpha - v^\alpha]^2 \left(\frac{s}{t - s} - v\right)^{-\gamma} dv \right. \\
 &\quad \left. + \int_1^{\frac{s}{t - s}} [(1 + v)^\alpha - v^\alpha]^2 \left(\frac{s}{t - s} - v\right)^{-\gamma} dv \right) \\
 &=: (t - s)^{2\alpha - \gamma + 1} (I_{1,1} + I_{1,2}).
 \end{aligned}
 \tag{3.10}$$

We will see that the main term is the integral $I_{1,2}$. For estimating the integral $I_{1,1}$, again we distinguish the two cases $\alpha \in [0, 1/2]$ and $\alpha \in (-1/2, 0)$.

When $\alpha \in [0, 1/2]$, we use the facts that $2^\alpha - 1 \leq (1 + v)^\alpha - v^\alpha \leq 1$ for all $v \in [0, 1]$ and

$$\frac{s}{2(t - s)} \leq \frac{s}{t - s} - v \leq \frac{s}{t - s}, \quad \forall v \in [0, 1],
 \tag{3.11}$$

to derive that

$$(2^\alpha - 1)^2 \left(\frac{t - s}{s}\right)^\gamma \leq I_{1,1} \leq 2^\gamma \left(\frac{t - s}{s}\right)^\gamma.
 \tag{3.12}$$

When $\alpha \in (-1/2, 0)$, we use the fact that $(1 - 2^\alpha)v^\alpha \leq v^\alpha - (1 + v)^\alpha \leq v^\alpha$ for $v \in [0, 1]$ and (3.11) to get

$$\frac{(1 - 2^\alpha)^2}{2\alpha + 1} \left(\frac{t - s}{s}\right)^\gamma \leq I_{1,1} \leq \frac{2^\gamma}{2\alpha + 1} \left(\frac{t - s}{s}\right)^\gamma.
 \tag{3.13}$$

Next, we estimate the integral $I_{1,2}$. For $\alpha \in (-1/2, 1/2]$, we use the inequality $|(1 + v)^\alpha - v^\alpha| \asymp v^{\alpha - 1}$ for all $v \in [1, \infty)$ to derive

$$\begin{aligned}
 I_{1,2} &\asymp \int_1^{\frac{s}{t - s}} v^{2(\alpha - 1)} \left(\frac{s}{t - s} - v\right)^{-\gamma} dv \\
 &= \left(\frac{s}{t - s}\right)^{2\alpha - 1 - \gamma} \int_{\frac{t - s}{s}}^1 w^{2(\alpha - 1)} (1 - w)^{-\gamma} dw,
 \end{aligned}
 \tag{3.14}$$

where the above equality is obtained by the change of variable $v = \frac{s}{t - s}w$. By splitting the last integral over the intervals $[\frac{t - s}{s}, \frac{3}{4}]$ and $[\frac{3}{4}, 1]$, we have

$$\begin{aligned}
 I_{1,2} &\asymp \left(\frac{s}{t - s}\right)^{2\alpha - 1 - \gamma} \left(\int_{\frac{t - s}{s}}^{\frac{3}{4}} w^{2(\alpha - 1)} dw + \int_{\frac{3}{4}}^1 w^{2(\alpha - 1)} (1 - w)^{-\gamma} dw \right) \\
 &\asymp \begin{cases} \left(\frac{t - s}{s}\right)^\gamma, & \text{if } -1/2 < \alpha < 1/2, \\ \left(\frac{t - s}{s}\right)^\gamma \left(1 + \ln \left|\frac{s}{t - s}\right|\right), & \text{if } \alpha = 1/2. \end{cases}
 \end{aligned}
 \tag{3.15}$$

Combining (3.10)-(3.15) yields that in Case (ii),

$$I_1 \asymp \begin{cases} \frac{(t-s)^{2\alpha+1}}{s^\gamma}, & \text{if } -1/2 < \alpha < 1/2, \\ \frac{(t-s)^2}{s^\gamma} \left(1 + \ln \left| \frac{s}{t-s} \right| \right), & \text{if } \alpha = 1/2. \end{cases} \tag{3.16}$$

It follows from this and (3.4) that

$$\mathbb{E} \left[(Z(t) - Z(s))^2 \right] \geq c_{3,3} \begin{cases} \frac{|t-s|^{2\alpha+1}}{s^\gamma}, & \text{if } -1/2 < \alpha < 1/2, \\ \frac{(t-s)^2}{s^\gamma} \left(1 + \ln \left| \frac{s}{t-s} \right| \right), & \text{if } \alpha = 1/2, \end{cases} \tag{3.17}$$

where $c_{3,3} > 0$ is a finite constant.

On the other hand, it follows from (3.4), (3.8) and (3.16) that

$$\mathbb{E} \left[(Z(t) - Z(s))^2 \right] \leq c_{3,4} \begin{cases} \frac{|t-s|^{2\alpha+1}}{s^\gamma}, & \text{if } -1/2 < \alpha < 1/2, \\ \frac{|t-s|^2}{s^\gamma} \left(1 + \ln \left| \frac{s}{t-s} \right| \right), & \text{if } \alpha = 1/2, \end{cases} \tag{3.18}$$

where $c_{3,4} > 0$ is a finite constant. This finishes the proof of (3.3). □

Remark 3.1 The following are some remarks about Lemma 3.1 and some of its consequences.

- (i). It follows from Lemma 3.1 that for any $0 < a < b < \infty$ there exist constants $c_{3,5}, \dots, c_{3,8} \in (0, \infty)$ such that for $s, t \in [a, b]$, we have that for $\alpha \in (-1/2, 1/2)$,

$$c_{3,5}|t - s|^{2\alpha+1} \leq \mathbb{E} \left[(Z(t) - Z(s))^2 \right] \leq c_{3,6}|t - s|^{2\alpha+1}; \tag{3.19}$$

for $\alpha = 1/2$,

$$\begin{aligned} c_{3,7}|t - s|^2 (1 + |\ln |t - s||) &\leq \mathbb{E} \left[(Z(t) - Z(s))^2 \right] \\ &\leq c_{3,8}|t - s|^2 (1 + |\ln |t - s||). \end{aligned} \tag{3.20}$$

Consequently, the process Z has a modification that is uniformly Hölder continuous on $[a, b]$ of order $\alpha + 1/2 - \varepsilon$ for all $\varepsilon > 0$. In the proof of Theorem 1.1 below, we will establish an exact uniform modulus of continuity of Z on any interval $[a, b]$ for $0 < a < b < \infty$.

- (ii). By Lemma 2.1, we see that for $\alpha \in (-1/2 + \gamma/2, 1/2]$, the inequalities (3.19) and (3.20) hold for GFBM X . These inequalities can be applied to determine the fractal dimensions of random sets (e.g., range, graph, level set, etc) generated by the sample path of X . For example, we can derive by using standard covering and

capacity methods (cf. [10,14,40]) the following Hausdorff dimension results: for any $T > 0$,

$$\dim_{\text{H}} \text{Gr}X([0, T]) = 2 - \left(\frac{1}{2} + \frac{\alpha}{2}\right) = \frac{3 - \alpha}{2}, \quad \text{a.s.}, \tag{3.21}$$

where $\text{Gr}X([0, T]) = \{(t, X(t)) : t \in [0, T]\}$ is the graph set of X , and for every $x \in \mathbb{R}$ we have

$$\dim_{\text{H}} X^{-1}(x) = 1 - \left(\frac{1}{2} + \frac{\alpha}{2}\right) = \frac{1 - \alpha}{2} \tag{3.22}$$

with positive probability, where $X^{-1}(x) = \{t \geq 0 : X(t) = x\}$ is the level set of X . We remark that, due to the σ -stability of Hausdorff dimension \dim_{H} ([10]), the asymptotic behavior of X at $t = 0$ (hence the self-similarity index H) has no effect on (3.21) and (3.22). Later on, we will indicate how more precise results than (3.21) and (3.22) can be established for X ; see Remark 7.1 below.

- (iii). Let $\xi = \{\xi(t)\}_{t \geq 0}$ be a centered Gaussian process. If there exists an even, non-negative, and non-decreasing function $\varphi(h)$ satisfying $\lim_{h \downarrow 0} h/\varphi(h) = 0$ and

$$\mathbb{E}[(\xi(t+h) - \xi(t))^2] \geq \varphi(h)^2, \quad t \geq 0, h \in (0, 1),$$

then by using the argument in Yeh [41], one can prove that the sample functions of ξ are nowhere differentiable with probability one. See also [15]. Thus, if $-1/2 < \alpha \leq 1/2$, then (3.19) and (3.20) imply that the sample paths of the generalized Riemann–Liouville FBM Z are nowhere differentiable in $(0, \infty)$ with probability one.

The next lemma will be used for studying the tangent processes of Z in Proposition 3.1.

Lemma 3.2 *Assume $\alpha \in (-1/2, 1/2)$ and $\gamma \in [0, 1)$. Then, for every fixed $t \in (0, \infty)$,*

$$\lim_{s \rightarrow t} \frac{\mathbb{E}[(Z(t) - Z(s))^2]}{|t - s|^{2\alpha+1}} = c_{3,9}t^{-\gamma}, \tag{3.23}$$

where $c_{3,9} = \frac{1}{2\alpha+1} + \int_0^\infty [(1+v)^\alpha - v^\alpha]^2 dv$.

Proof For simplicity, we consider the limit in (3.23) as $s \uparrow t$ only. For $0 < s < t < \infty$, recall from (3.4) the decomposition of $\mathbb{E}[(Z(t) - Z(s))^2]$.

The term I_2 is easier, and we handle it first. Since $u^{-\gamma} \in [t^{-\gamma}, s^{-\gamma}]$ for any $u \in [s, t]$, we obtain that

$$\frac{1}{2\alpha + 1} \frac{|t - s|^{2\alpha+1}}{t^\gamma} \leq I_2 \leq \frac{1}{2\alpha + 1} \frac{|t - s|^{2\alpha+1}}{s^\gamma}. \tag{3.24}$$

This implies that

$$\lim_{s \uparrow t} \frac{I_2}{|t - s|^{2\alpha+1}} = \frac{1}{2\alpha + 1} t^{-\gamma}. \tag{3.25}$$

For the integral I_1 , we use (3.5) and write it as

$$\begin{aligned} I_1 &= \int_0^s [(t - u)^\alpha - (s - u)^\alpha]^2 u^{-\gamma} du \\ &= (t - s)^{2\alpha+1} \int_0^{\frac{s}{t-s}} [(1 + v)^\alpha - v^\alpha]^2 (s - v(t - s))^{-\gamma} dv \\ &= (t - s)^{2\alpha+1} \int_0^{\frac{s}{2(t-s)}} [(1 + v)^\alpha - v^\alpha]^2 (s - v(t - s))^{-\gamma} dv \\ &\quad + (t - s)^{2\alpha+1} \int_{\frac{s}{2(t-s)}}^{\frac{s}{t-s}} [(1 + v)^\alpha - v^\alpha]^2 (s - v(t - s))^{-\gamma} dv \\ &=: J_{1,1} + J_{1,2}. \end{aligned} \tag{3.26}$$

Notice that, for every $v \in (0, s/[2(t - s)])$, we have $s^{-\gamma} \leq (s - v(t - s))^{-\gamma} \leq 2^\gamma s^{-\gamma}$. Thus, the dominated convergence theorem gives

$$\lim_{s \uparrow t} \frac{J_{1,1}}{|t - s|^{2\alpha+1}} = t^{-\gamma} \int_0^\infty [(1 + v)^\alpha - v^\alpha]^2 dv. \tag{3.27}$$

On the other hand, we use the inequality $|(1 + v)^\alpha - v^\alpha| \leq |\alpha|v^{\alpha-1}$ for all $v \geq 1$ to obtain

$$\begin{aligned} J_{1,2} &\leq (t - s)^{2\alpha+1} \int_{\frac{s}{2(t-s)}}^{\frac{s}{t-s}} \alpha^2 v^{2\alpha-2} (s - v(t - s))^{-\gamma} dv \\ &\leq \alpha^2 (t - s)^{2\alpha+1} \left(\frac{s}{2(t - s)}\right)^{2\alpha-2} \int_{\frac{s}{2(t-s)}}^{\frac{s}{t-s}} (s - v(t - s))^{-\gamma} dv \\ &= \frac{\alpha^2}{1 - \gamma} \left(\frac{s}{2}\right)^{2\alpha-1-\gamma} (t - s)^2. \end{aligned}$$

It follows that

$$\lim_{s \uparrow t} \frac{J_{1,2}}{|t - s|^{2\alpha+1}} = 0. \tag{3.28}$$

Therefore, (3.23) follows from (3.25), (3.26), (3.27) and (3.28). The proof is complete. □

The following lemma deals with the case when $\alpha > 1/2$ and provides estimates on the second moments of the increments of $Z(t)$ and its mean-square derivative $Z'(t)$.

The latter estimate allows us to show that $\{Z(t)\}_{t \geq 0}$ has a modification whose sample functions are continuously differentiable in $(0, \infty)$.

For simplicity, we only consider the case when $\alpha \in (1/2, 3/2)$ and s, t stay away from the origin. This is sufficient for our study of the sample path properties of GFBM X .

Lemma 3.3 *Assume $\alpha \in (1/2, 3/2)$ and $\gamma \in [0, 1)$. For any $0 < a < b < \infty$, it holds that for any $s, t \in [a, b]$,*

$$c_{3,10}|t - s|^2 \leq \mathbb{E} \left[(Z(t) - Z(s))^2 \right] \leq c_{3,11}|t - s|^2, \tag{3.29}$$

here, $c_{3,10} = \frac{\alpha^2}{1-\gamma} b^{2\alpha-2} a^{1-\gamma}$ and $c_{3,11} = \alpha^2 a^{2\alpha-1-\gamma} \mathcal{B}(2\alpha - 1, 1 - \gamma)$.

The process $\{Z(t)\}_{t \in [a,b]}$ has a modification, which is still denoted by Z , such that its derivative process $\{Z'(t)\}_{t \in [a,b]}$ is continuous almost surely. Furthermore, there exist constants $c_{3,12}, c_{3,13} \in (0, \infty)$ such that for any $s, t \in [a, b]$,

$$c_{3,12}|t - s|^{2\alpha-1} \leq \mathbb{E} \left[(Z'(t) - Z'(s))^2 \right] \leq c_{3,13}|t - s|^{2\alpha-1}. \tag{3.30}$$

Proof The proof of (3.29) is similar to that of Lemma 3.1. Here, we only prove the lower bound in (3.29) and the existence of a modification of Z whose sample functions are continuously differentiable on $(0, \infty)$ almost surely.

For any $s, t \in [a, b]$ with $s < t$, by (2.3) and (3.4), we have

$$\begin{aligned} \mathbb{E} \left[(Z(t) - Z(s))^2 \right] &\geq \int_0^s ((t - u)^\alpha - (s - u)^\alpha)^2 u^{-\gamma} du \\ &\geq \alpha^2 (t - s)^2 \int_0^s (t - u)^{2\alpha-2} u^{-\gamma} du \\ &\geq \alpha^2 (t - s)^2 t^{2\alpha-2} \frac{s^{1-\gamma}}{1 - \gamma} \\ &\geq \frac{\alpha^2}{1 - \gamma} b^{2\alpha-2} a^{1-\gamma} (t - s)^2. \end{aligned}$$

Thus, the lower bound in (3.29) holds.

For any $t \geq 0$, define

$$Z'(t) := \alpha \int_0^t (t - u)^{\alpha-1} u^{-\frac{\gamma}{2}} B(du), \tag{3.31}$$

with $Z'(0) = 0$. Notice that, since $\alpha \in (1/2, 3/2)$, the process $\{Z'(t)\}_{t \geq 0}$ is a generalized Riemann–Liouville FBM with indices $\alpha - 1$ and γ . It is self-similar with index $\tilde{H} = \alpha - 1/2 - \gamma/2$. Hence, by (3.19), we see that (3.30) holds. By the Kolmogorov continuity theorem (see, e.g., [16, Theorem C.6]), the Gaussian property of Z' and the arbitrariness of a and b , we know that Z' has a modification (still denoted by Z') that is Hölder continuous in $(0, \infty)$ with index $\alpha - 1/2 - \varepsilon$ for any $\varepsilon > 0$. With

this modification, we define a Gaussian process $\tilde{Z} = \{\tilde{Z}(t)\}_{t \geq 0}$ by

$$\tilde{Z}(t) = \int_0^t Z'(s)ds, \quad \forall t \geq 0.$$

Then, by the stochastic Fubini theorem [16, Corollary 2.9], we derive that for every $t \geq 0$,

$$\tilde{Z}(t) = \alpha \int_0^t \left(\int_u^t (s - u)^{\alpha-1} ds \right) u^{-\frac{\gamma}{2}} B(du) = Z(t), \quad \text{a.s.}$$

Hence, \tilde{Z} is modification of the generalized Riemann–Liouville FBM $Z = \{Z(t)\}_{t \geq 0}$ and the sample function of \tilde{Z} is a.s. continuously differentiable in $(0, \infty)$. The proof is complete. \square

3.2 The Tangent Process of Z

Let $C(\mathbb{R}_+, \mathbb{R})$ be the space of all continuous functions from \mathbb{R}_+ to \mathbb{R} , endowed with the locally uniform convergence topology. We say that a family of stochastic processes $V_n = \{V_n(\tau)\}_{\tau \geq 0}$ converges weakly (or in distribution) to $V = \{V(\tau)\}_{\tau \geq 0}$ in $C(\mathbb{R}_+, \mathbb{R})$, if $\mathbb{E}[f(V_n)] \rightarrow \mathbb{E}[f(V)]$ for every bounded, continuous function $f : C(\mathbb{R}_+, \mathbb{R}) \rightarrow \mathbb{R}$ (cf. [7]).

Proposition 3.1 *Assume $\alpha \in (-1/2, 1/2)$ and $\gamma \in [0, 1)$. For any $t, u > 0$, let*

$$V_Z^{t,u}(\tau) := \frac{Z(t + u\tau) - Z(t)}{u^{\alpha+1/2}}, \quad \tau \geq 0.$$

Then, as $u \rightarrow 0+$, $\{V_Z^{t,u}(\tau)\}_{\tau \geq 0}$ converges in distribution to $\{c_{3,9}^{1/2} t^{-\gamma/2} B^{\alpha+1/2}(\tau)\}_{\tau \geq 0}$ in $C(\mathbb{R}_+, \mathbb{R})$, where $c_{3,9} = \frac{1}{2\alpha+1} + \int_0^\infty [(1+v)^\alpha - v^\alpha]^2 dv$ is the constant in Lemma 3.2.

Proof By the self-similarity of Z and (3.23), we know that for any $t, s > 0$,

$$\begin{aligned} & \mathbb{E}[Z(t)Z(s)] \\ &= \frac{1}{2} \mathbb{E}[Z(t)^2 + Z(s)^2 - (Z(t) - Z(s))^2] \\ &= \frac{1}{2} \left(\mathcal{B}(2\alpha + 1, 1 - \gamma) \left(t^{2H} + s^{2H} \right) - c_{3,9} t^{-\gamma} |t - s|^{2\alpha+1} \right) + o(t^{-\gamma} |t - s|^{2\alpha+1}). \end{aligned} \tag{3.32}$$

For any $t > 0, \tau_1, \tau_2 \geq 0$, by (3.32), we have

$$\lim_{u \rightarrow 0+} \mathbb{E}[V_Z^{t,u}(\tau_1)V_Z^{t,u}(\tau_2)] = \frac{c_{3,9}}{2} t^{-\gamma} \left(\tau_1^{2\alpha+1} + \tau_2^{2\alpha+1} - |\tau_1 - \tau_2|^{2\alpha+1} \right). \tag{3.33}$$

Hence, the Gaussian process $\{V_Z^{t,u}(\tau)\}_{\tau \geq 0}$ converges to $c_{3,9}^{1/2} t^{-\gamma/2} B^{\alpha+1/2}$ in finite-dimensional distributions, as $u \rightarrow 0+$, where $B^{\alpha+1/2}$ is a FBM with index $\alpha + 1/2$. By Lemma 3.1, there exists a constant $c_{3,14} \in (0, \infty)$ such that for all $t > 0, u > 0, \tau_1, \tau_2 \geq 0$,

$$\mathbb{E} \left[\left(V_Z^{t,u}(\tau_1) - V_Z^{t,u}(\tau_2) \right)^2 \right] \leq c_{3,14} t^{-\gamma} |\tau_1 - \tau_2|^{2\alpha+1}. \tag{3.34}$$

Hence, by [8, Proposition 4.1], we know the family $\{V_Z^{t,u}\}_{u>0}$ is tight in $C(\mathbb{R}_+, \mathbb{R})$ and then it converges in distribution in $C(\mathbb{R}_+, \mathbb{R})$, as $u \rightarrow 0+$. The proof is complete. \square

3.3 One-Sided Strong Local Nondeterminism

We establish the following one-sided strong local nondeterminism (SLND, for short) for Z . This property is essential for dealing with problems that involve joint distributions of random variables $Z(t_1), \dots, Z(t_n)$. From the proof, it is clear that GFBM X has the same SLND property.

Proposition 3.2 (a) *Assume $\alpha \in (-1/2, 1/2]$ and $\gamma \in [0, 1)$. For any constant $b > 0$, it holds that for any $s, t \in [0, b]$ with $s < t$,*

$$\text{Var}(Z(t)|Z(r) : r \leq s) \geq \frac{1}{(2\alpha + 1)b^\gamma} |t - s|^{2\alpha+1}, \tag{3.35}$$

where $\text{Var}(Z(t)|Z(r) : r \leq s)$ denotes the conditional variance of $Z(t)$ given the σ -algebra $\sigma(Z(r) : r \leq s)$.

(b) *Assume $\alpha \in (1/2, 3/2)$ and $\gamma \in [0, 1)$. For any $b > 0$, it holds that for any $s, t \in [0, b]$ with $s < t$,*

$$\text{Var}(Z'(t)|Z'(r) : r \leq s) \geq \frac{1}{(2\alpha - 1)b^\gamma} |t - s|^{2\alpha-1}. \tag{3.36}$$

Proof Assume that $0 \leq s < t \leq b$. We write $Z(t)$ as

$$Z(t) = \int_0^s (t - u)^\alpha u^{-\frac{\gamma}{2}} B(du) + \int_s^t (t - u)^\alpha u^{-\frac{\gamma}{2}} B(du).$$

The first term is measurable with respect to $\sigma(B(r) : r \leq s)$, and the second term is independent of $\sigma(B(r) : r \leq s)$. Since $\sigma(Z(r) : r \leq s) \subseteq \sigma(B(r) : r \leq s)$, we have

$$\text{Var}(Z(t)|Z(r) : r \leq s) \geq \text{Var}(Z(t)|B(r) : r \leq s) = \int_s^t (t - u)^{2\alpha} u^{-\gamma} du.$$

This implies (3.35).

The proof of (3.36) is similar. The details are omitted. The proof is complete. \square

4 Lamperti’s Transformation of Z

Inspired by [35], we consider the centered stationary Gaussian process $U = \{U(t)\}_{t \in \mathbb{R}}$ defined through Lamperti’s transformation of Z :

$$U(t) := e^{-tH} Z(e^t) \quad \text{for all } t \in \mathbb{R}. \tag{4.1}$$

Let $r_U(t) := \mathbb{E}[U(0)U(t)]$ be the covariance function of U . By Bochner’s theorem, r_U is the Fourier transform of a finite measure F_U which is called the spectral measure of U . Notice that $r_U(t)$ is an even function and

$$r_U(t) = e^{-tH} \int_0^{1 \wedge e^t} (e^t - u)^\alpha (1 - u)^\alpha u^{-\gamma} du \quad \text{for all } t \in \mathbb{R}. \tag{4.2}$$

We can verify that $r_U(t) = O(e^{-t(1-\gamma)/2})$ as $t \rightarrow \infty$. It follows that $r_U(\cdot) \in L^1(\mathbb{R})$. Hence, the spectral measure F_U has a continuous spectral density function f_U which can be represented as the inverse Fourier transform of $r_U(\cdot)$:

$$f_U(\lambda) = \frac{1}{\pi} \int_0^\infty r_U(t) \cos(t\lambda) dt \quad \text{for all } \lambda \in \mathbb{R}.$$

It is known that U has the stochastic integral representation:

$$U(t) = \int_{\mathbb{R}} e^{i\lambda t} W(d\lambda) \quad \text{for all } t \in \mathbb{R}, \tag{4.3}$$

where W is a complex Gaussian measure with control measure F_U . Then, for any $s, t \in \mathbb{R}$,

$$\begin{aligned} \mathbb{E}[(U(s) - U(t))^2] &= 2(r_U(0) - r_U(t - s)) \\ &= 2 \int_{\mathbb{R}} [1 - \cos((s - t)\lambda)] f_U(\lambda) d\lambda. \end{aligned} \tag{4.4}$$

The following lemma provides bounds for $\mathbb{E}[(U(s) - U(t))^2]$ when Z has rough (fractal) sample paths.

Lemma 4.1 *Assume $\alpha \in (-1/2, 1/2]$. Then, for any $b > 0$, there exist positive constants $\varepsilon_0, c_{4,1}$ and $c_{4,2}$ such that for all $s, t \in [0, b]$ with $|s - t| \leq \varepsilon_0$,*

$$\begin{aligned} c_{4,1}|s - t|^{2\alpha+1} (1 + |\ln |t - s||)^\eta &\leq \mathbb{E}[(U(s) - U(t))^2] \\ &\leq c_{4,2}|s - t|^{2\alpha+1} (1 + |\ln |t - s||)^\eta, \end{aligned} \tag{4.5}$$

where $\eta = 0$ if $\alpha \in (-1/2, 1/2)$ and $\eta = 1$ if $\alpha = 1/2$.

Proof Since U is stationary, it is sufficient to consider $\mathbb{E}[(U(t) - U(0))^2]$ for $t > 0$. It follows from (4.1) and the elementary inequality $(x + y)^2 \leq 2(x^2 + y^2)$ that

$$\begin{aligned} \mathbb{E}[(U(t) - U(0))^2] &= \mathbb{E}\left[\left(Z(e^t) - Z(1) + (e^{-tH} - 1)Z(e^t)\right)^2\right] \\ &\leq 2\left(\mathbb{E}\left[(Z(e^t) - Z(1))^2\right] + (e^{-tH} - 1)^2 \mathbb{E}\left[Z(e^t)^2\right]\right). \end{aligned} \tag{4.6}$$

This, together with Lemma 3.1, implies that the upper bound in (4.5) holds for all $s, t \in [0, b]$ with $|s - t| \leq \varepsilon_0$ for some positive constant ε_0 .

On the other hand, the first equation in (4.6) and the inequality $(x + y)^2 \geq \frac{1}{2}x^2 - y^2$ imply

$$\mathbb{E}[(U(t) - U(0))^2] \geq \frac{1}{2}\mathbb{E}\left[(Z(e^t) - Z(1))^2\right] - (e^{-tH} - 1)^2 \mathbb{E}\left[Z(e^t)^2\right]. \tag{4.7}$$

It follows from Lemma 3.1 that the lower bounds in (4.5) hold if $t > 0$ is small enough, say, $0 < t \leq \varepsilon_0$. This completes the proof of Lemma 4.1. \square

The following are truncation inequalities in Loève [22, Page 209] that are expressed in terms of the spectral density f_U : for any $u > 0$, we have

$$\begin{aligned} \int_{|\lambda| < u} \lambda^2 f_U(\lambda) d\lambda &\leq Ku^2 \int_{\mathbb{R}} (1 - \cos(\lambda/u)) f_U(\lambda) d\lambda, \\ \int_{|\lambda| \geq u} f_U(\lambda) d\lambda &\leq Ku \int_0^{1/u} dv \int_{\mathbb{R}} (1 - \cos(v\lambda)) f_U(\lambda) d\lambda. \end{aligned}$$

By these inequalities, (4.4) and the upper bound in Lemma 4.1, we have the following properties of the spectral density $f_U(\lambda)$ at the origin and infinity, respectively.

Lemma 4.2 *Assume $\alpha \in (-1/2, 1/2]$. There exist positive constants $u_0, c_{4,3}$ and $c_{4,4}$ such that for any $u > u_0$,*

$$\int_{|\lambda| < u} \lambda^2 f_U(\lambda) d\lambda \leq c_{4,3} u^{1-2\alpha} (1 + |\ln u|)^\eta \tag{4.8}$$

and

$$\int_{|\lambda| \geq u} f_U(\lambda) d\lambda \leq c_{4,4} u^{-(2\alpha+1)} (1 + |\ln u|)^\eta. \tag{4.9}$$

In the above, $\eta = 0$ if $\alpha \in (-1/2, 1/2)$ and $\eta = 1$ if $\alpha = 1/2$.

5 Small Ball Probabilities of Z

By Lemmas 3.1, 3.3 and Proposition 3.2, we prove the following estimates on the small ball probabilities of the Gaussian process Z and its derivative Z' when it exists.

Proposition 5.1 (a) *Assume $\gamma \in [0, 1)$ and $\alpha \in (-1/2 + \gamma/2, 1/2)$. There exist constants $c_{5,1}, c_{5,2} \in (0, \infty)$ such that for all $r > 0, 0 < \varepsilon < 1$,*

$$\exp\left(-c_{5,1}\left(\frac{r^H}{\varepsilon}\right)^{\frac{2}{2\alpha+1}}\right) \leq \mathbb{P}\left\{\sup_{s \in [0, r]} |Z(s)| \leq \varepsilon\right\} \leq \exp\left(-c_{5,2}\left(\frac{r^H}{\varepsilon}\right)^{\frac{2}{2\alpha+1}}\right). \tag{5.1}$$

(b) *Assume $\gamma \in [0, 1)$ and $\alpha \in (1/2 + \gamma/2, 3/2)$. There exist constants $c_{5,3}, c_{5,4} \in (0, \infty)$ such that for all $r > 0, 0 < \varepsilon < 1$,*

$$\exp\left(-c_{5,3}\left(\frac{r^{\tilde{H}}}{\varepsilon}\right)^{\frac{2}{2\alpha-1}}\right) \leq \mathbb{P}\left\{\sup_{s \in [0, r]} |Z'(s)| \leq \varepsilon\right\} \leq \exp\left(-c_{5,4}\left(\frac{r^{\tilde{H}}}{\varepsilon}\right)^{\frac{2}{2\alpha-1}}\right), \tag{5.2}$$

where $\tilde{H} = \alpha - \gamma/2 - 1/2$.

Remark 5.1 The following are two remarks about Proposition 5.1.

- Notice that the case of $\alpha = 1/2$ is excluded in Proposition 5.1. The reason is that, the bounds of the one-sided SLND in Proposition 3.2 and the optimal bounds in (3.3) do not coincide when $\alpha = 1/2$. The method for proving (5.1) will not be able to prove optimal upper and lower bounds in the case of $\alpha = 1/2$.
- In Part (b), we assume that the self-similarity index $\tilde{H} = \alpha - \gamma/2 - 1/2$ of Z' is positive. When $\tilde{H} \leq 0$, (5.2) does not hold. In fact, by using $\mathbb{E}[Z'(t)^2] = \mathbb{E}[Z'(1)^2]t^{2\tilde{H}}$, one sees that for any $0 < \tau \leq r$,

$$\mathbb{P}\left\{\sup_{s \in [0, r]} |Z'(s)| \leq \varepsilon\right\} \leq \mathbb{P}\{|Z'(\tau)| \leq \varepsilon\} = \mathbb{P}\{|Z'(1)| \leq \tau^{-\tilde{H}}\varepsilon\}.$$

If $\tilde{H} < 0$, then the last probability goes to 0 as $\tau \rightarrow 0$. This implies that for all $r > 0$ and $\varepsilon > 0$ we have $\mathbb{P}\{\sup_{s \in [0, r]} |Z'(s)| \leq \varepsilon\} = 0$.

When $\tilde{H} = 0$, the self-similarity implies that the small ball probability in (5.2) does not depend on r . We will let $r \rightarrow \infty$ to show that this probability is in fact 0 for all $\varepsilon > 0$. To this end, we consider the centered stationary Gaussian process $V = \{V(s)\}_{s \in \mathbb{R}}$ defined by $V(s) = Z'(e^s)$, which is the Lamperti transform of Z' , and apply Theorem 5.2 of Pickands [29]. Notice that the covariance function of V ,

$$r_V(s) = \int_0^1 (1-u)^{\alpha-1}(e^s-u)^{\alpha-1}u^{-\gamma} du, \text{ for } s > 0.$$

Then, $r_V(s) \leq 2^{1-\alpha} e^{(\alpha-1)s}$ for all s large enough. Since $\alpha = \frac{1+\gamma}{2} < 1$, we have $\lim_{s \rightarrow \infty} r_V(s) \ln s = 0$, so the condition of [29, Theorem 5.2] is satisfied. Hence,

$$\liminf_{t \rightarrow \infty} \left(\sup_{s \in [0, t]} V(s) - \sqrt{2 \ln t} \right) \geq 0, \quad \text{a.s.}$$

This implies that for all $\varepsilon > 0$,

$$\mathbb{P} \left\{ \sup_{s \in [0, r]} |Z'(s)| \leq \varepsilon \right\} = \lim_{t \rightarrow \infty} \mathbb{P} \left\{ \sup_{s \in [0, t]} |Z'(s)| \leq \varepsilon \right\} = 0.$$

For proving the lower bound in (5.1), we apply the general lower bound on the small ball probability of Gaussian processes due to Talagrand (cf. Lemma 2.2 of [32]). We will make use of the following reformulation of Talagrand’s lower bound given by Ledoux [19, (7.11)-(7.13) on Page 257].

Lemma 5.1 *Let $\{Z(t)\}_{t \in S}$ be a separable, real-valued, centered Gaussian process indexed by a bounded set S with the canonical metric $d_Z(s, t) = (\mathbb{E}|Z(s) - Z(t)|^2)^{1/2}$. Let $N_\varepsilon(S)$ denote the smallest number of d_Z -balls of radius ε needed to cover S . If there is a decreasing function $\psi : (0, \delta] \rightarrow (0, \infty)$ such that $N_\varepsilon(S) \leq \psi(\varepsilon)$ for all $\varepsilon \in (0, \delta]$ and there are constants $K_2 \geq K_1 > 1$ such that*

$$K_1 \psi(\varepsilon) \leq \psi(\varepsilon/2) \leq K_2 \psi(\varepsilon) \tag{5.3}$$

for all $\varepsilon \in (0, \delta]$, then there is a constant $K \in (0, \infty)$ depending only on K_1, K_2 and d_Z such that for all $u \in (0, \delta)$,

$$\mathbb{P} \left\{ \sup_{s, t \in S} |Z(s) - Z(t)| \leq u \right\} \geq \exp(-K \psi(u)). \tag{5.4}$$

We are ready to prove Proposition 5.1.

Proof We will only prove (5.1). The proof of (5.2) is the same because Z' is also a generalized GFBM with indices $\gamma \in [0, 1)$ and $\alpha - 1 \in (-1/2 + \gamma/2, 1/2)$.

By the self-similarity property of Z , we know that (5.1) is equivalent to the following statement: there exist constants $c_{5,1}, c_{5,2} \in (0, \infty)$ such that for all $0 < \varepsilon < 1$,

$$\exp \left(-c_{5,1} \left(\frac{1}{\varepsilon} \right)^{\frac{2}{2\alpha+1}} \right) \leq \mathbb{P} \left\{ \sup_{s \in [0, 1]} |Z(s)| \leq \varepsilon \right\} \leq \exp \left(-c_{5,2} \left(\frac{1}{\varepsilon} \right)^{\frac{2}{2\alpha+1}} \right). \tag{5.5}$$

In order to prove the lower bound in (5.5), we take $S = [0, 1]$ and apply Lemma 5.1. For any $\varepsilon \in (0, 1)$, we construct a covering of $[0, 1]$ by sub-intervals of d_Z -radius ε , which will give an upper bound for $N_\varepsilon([0, 1])$.

Recall that $\mathbb{E}[Z(t)^2] = c t^{2H}$ for all $t \geq 0$, where $H = \alpha - \frac{\gamma}{2} + \frac{1}{2} > 0$. Since constants c here and those in Lemma 3.1 can be absorbed by the constants $c_{5,1}$ and $c_{5,2}$

in (5.5), without loss of generality we will take these constants to be 1 (otherwise, we consider the processes obtained by dividing Z by the maximum and minimum of these constants, respectively, and prove the upper and lower bounds in (5.5) separately.)

Let $t_0 = 0, t_1 = \varepsilon^{1/H}$. For any $n \geq 2$, if t_{n-1} has been defined, we define

$$t_n = t_{n-1} + t_{n-1}^{\frac{\gamma}{2\alpha+1}} \varepsilon^{\frac{2}{2\alpha+1}}. \tag{5.6}$$

It follows from Lemma 3.1 that

$$\mathbb{E} \left[(Z(t_n) - Z(t_{n-1}))^2 \right] \leq \frac{c}{t_{n-1}^\gamma} |t_n - t_{n-1}|^{2\alpha+1} \leq \varepsilon^2. \tag{5.7}$$

Hence, $d_Z(t_n, t_{n-1}) \leq \varepsilon$ for all $n \geq 1$.

Since $[0, 1]$ can be covered by the intervals $[t_{n-1}, t_n]$ for $n = 1, 2, \dots, L_\varepsilon$, where L_ε is the largest integer n such that $t_n \leq 1$, we have $N_\varepsilon([0, 1]) \leq L_\varepsilon + 1 \leq 2L_\varepsilon$.

In order to estimate L_ε , we write $t_n = a_n \varepsilon^{1/H}$ for all $n \geq 1$. Then, by (5.6), we have $a_1 = 1$,

$$a_n = a_{n-1} + a_{n-1}^{\frac{\gamma}{2\alpha+1}}, \quad \forall n \geq 2. \tag{5.8}$$

Denote by $\beta = 1 - \frac{\gamma}{2\alpha+1} = \frac{2H}{2\alpha+1}$. We claim that there exist positive and finite constants $c_{5,5} \leq 2^{-\gamma/(2H\beta)} \beta^{1/\beta}$ and $c_{5,6} \geq 1$ such that

$$c_{5,5} n^{1/\beta} \leq a_n \leq c_{5,6} n^{1/\beta}, \quad \forall n \geq 1. \tag{5.9}$$

We postpone the proof of (5.9). Let us estimate L_ε and prove the lower bound in (5.5) first.

By (5.9), we have

$$L_\varepsilon = \max \left\{ n : a_n \leq \varepsilon^{-\frac{1}{H}} \right\} \leq c_{5,5}^{-\beta} \varepsilon^{-\frac{\beta}{H}} = c_{5,5}^{-\beta} \varepsilon^{-\frac{2}{2\alpha+1}}.$$

This implies that for all $\varepsilon \in (0, 1)$,

$$N_\varepsilon([0, 1]) \leq 2c_{5,5}^{-\beta} \varepsilon^{-\frac{2}{2\alpha+1}} =: \psi(\varepsilon). \tag{5.10}$$

Since the function $\psi(\varepsilon)$ satisfies (5.3) with $K_1 = K_2 = 2^{\frac{2}{2\alpha+1}} > 1$, we see that the lower bound in (5.5) follows from (5.10) and (5.4) in Lemma 5.1.

Now, we prove (5.9) by using induction. Clearly (5.9) holds for $n = 1$. Assume that it holds for $n = k$. Then for $n = k + 1$, it follows from (5.8) and (5.9) that

$$a_{n+1} \leq c_{5,6} n^{1/\beta} + (c_{5,6} n^{1/\beta})^{\frac{\gamma}{2\alpha+1}} \leq c_{5,6} (n + 1)^{1/\beta},$$

where the last inequality can be checked by using the mean-value theorem and the facts that $c_{5,6} \geq 1$ and $0 < \beta < 1$. This verifies the upper bound in (5.9). The desired

lower bound for a_{n+1} is derived similarly using the mean-value theorem and the fact that $c_{5,5} \leq 2^{-\gamma/(2H\beta)} \beta^{1/\beta}$. Hence, the claim (5.9) holds.

Next, we prove the upper bound in (5.5). Lemma 3.1 and Proposition 3.2 show that the conditions of Theorem 2.1 of Monrad and Rootzén [27] are satisfied. Hence, the upper bound in (5.1) follows from Theorem 2.1 of [27]. The proof is complete. \square

Similarly to the proof of Proposition 5.1, we can prove the following estimates on the small ball probabilities for the increments of Z and Z' at points away from the origin. We will use these estimates to prove Chung’s LILs for Z and Z' . Also, notice that no extra condition of $\alpha - \gamma/2 - 1/2 > 0$ is needed for (b).

Proposition 5.2 (a) *Assume $\alpha \in (-1/2, 1/2)$. Then, there exist constants $c_{5,7}, c_{5,8} \in (0, \infty)$ such that for all $t > 0, r \in (0, t/2)$ and $\varepsilon \in (0, r^{\alpha+1/2})$,*

$$\begin{aligned} \exp\left(-c_{5,7} r t^{-\frac{\gamma}{2\alpha+1}} \varepsilon^{-\frac{2}{2\alpha+1}}\right) &\leq \mathbb{P}\left\{\sup_{|s|\leq r} |Z(t+s) - Z(t)| \leq \varepsilon\right\} \\ &\leq \exp\left(-c_{5,8} r t^{-\frac{\gamma}{2\alpha+1}} \varepsilon^{-\frac{2}{2\alpha+1}}\right). \end{aligned} \tag{5.11}$$

(b) *Assume $\alpha \in (1/2, 3/2)$. Then, there exist constants $c_{5,9}, c_{5,10} \in (0, \infty)$ such that for all $t > 0, r \in (0, t/2)$ and $\varepsilon \in (0, r^{\alpha-1/2})$,*

$$\begin{aligned} \exp\left(-c_{5,9} r t^{-\frac{\gamma}{2\alpha-1}} \varepsilon^{-\frac{2}{2\alpha-1}}\right) &\leq \mathbb{P}\left\{\sup_{|s|\leq r} |Z'(t+s) - Z'(t)| \leq \varepsilon\right\} \\ &\leq \exp\left(-c_{5,10} r t^{-\frac{\gamma}{2\alpha-1}} \varepsilon^{-\frac{2}{2\alpha-1}}\right). \end{aligned} \tag{5.12}$$

Proof (a). Assume $\alpha \in (-1/2, 1/2), 0 < r < t/2$. Let $I(t, r) = [t - r, t + r]$. It follows from Lemma 3.1 that

$$\mathbb{E}[(Z(s) - Z(s'))^2] \asymp t^{-\gamma} |s - s'|^{2\alpha+1} \quad \text{for all } s, s' \in I(t, r).$$

Hence, there exists a constant $c_{5,11} \in (0, \infty)$ satisfying that for all $0 < \varepsilon < r^{\alpha+1/2}$,

$$N_\varepsilon(I(t, r)) \leq c_{5,11} t^{-\frac{\gamma}{2\alpha+1}} \varepsilon^{-\frac{2}{2\alpha+1}} r =: \psi(\varepsilon).$$

Then, the function $\psi(\varepsilon)$ satisfies (5.3) with $K_1 = K_2 = 2^{\frac{2}{2\alpha+1}} > 1$. Hence, the lower bound in (5.11) follows from (5.4) in Lemma 5.1.

Next, Lemma 3.1 and Proposition 3.2 show that the conditions of Theorem 2.1 of Monrad and Rootzén [27] are satisfied. Hence, the upper bound in (5.11) follows from Theorem 2.1 of [27].

(b). As noted earlier, when $\alpha \in (1/2, 3/2)$ the Gaussian process $\{Z'(t)\}_{t \geq 0}$ is a generalized Riemann–Liouville FBM. Hence, (5.12) follows from (5.11). This finishes the proof. \square

6 Chung’s Law of the Iterated Logarithm for Z

As applications of small ball probability estimates, Monrad and Rootzén [27], Xiao [38], and Li and Shao [21] established Chung’s LILs for fractional Brownian motions and other strongly locally nondeterministic Gaussian processes with stationary increments. Notice that the generalized Riemann–Liouville FBM Z does not have stationary increments. Here, we will use the small ball probability estimates and the Lamperti transformation in the last two sections to establish Chung’s LIL for Z at points away from the origin when $\alpha \in (-1/2, 1/2)$. The case of $\alpha = 1/2$ is open for Z as well as GFBM X .

Proposition 6.1 *Assume $\alpha \in (-1/2, 1/2)$. There exists a constant $c_{6,1} \in (0, \infty)$ such that for every $t > 0$,*

$$\liminf_{r \rightarrow 0^+} \sup_{|s| \leq r} \frac{|Z(t+s) - Z(t)|}{r^{\alpha+1/2} / (\ln \ln 1/r)^{\alpha+1/2}} = c_{6,1} t^{-\gamma/2}, \quad a.s. \tag{6.1}$$

For proving Proposition 6.1, we will make use of the following zero-one law, which implies the existence of the limit in the left hand side of (6.1). Notice that the constant $c'_{6,1}$ in (6.2) can be 0 or ∞ .

Lemma 6.1 *Assume $\alpha \in (-1/2, 1/2)$. There exists a constant $c'_{6,1} \in [0, \infty]$ such that for every $t > 0$,*

$$\liminf_{r \rightarrow 0^+} \sup_{|s| \leq r} \frac{|Z(t+s) - Z(t)|}{r^{\alpha+1/2} / (\ln \ln 1/r)^{\alpha+1/2}} = c'_{6,1} t^{-\gamma/2}, \quad a.s. \tag{6.2}$$

Proof We start with the stationary Gaussian process $U = \{U(s)\}_{s \in \mathbb{R}}$ in (4.3). As in the proof of Lemma 2.1 in [37], we write for every $n \geq 1$,

$$U_n(s) = \int_{n-1 \leq |\lambda| < n} e^{i\lambda s} W(d\lambda).$$

Then, the Gaussian processes $U_n = \{U_n(s)\}_{s \in \mathbb{R}}$, $n \geq 1$, are independent and

$$U(s) = \sum_{n=1}^{\infty} U_n(s),$$

where the series is a.s. uniformly convergent on every compact interval in \mathbb{R} . As in the proof of Proposition 2.1, we can verify that every $U_n(s)$ is a.s. continuously differentiable (or as in [37], we show that $U_n(s)$ is almost Lipschitz on compact intervals). Therefore, for every $x \in \mathbb{R}$ and $N \in \mathbb{N}$,

$$\liminf_{r \rightarrow 0^+} \sup_{|s| \leq r} \frac{|\sum_{n=1}^N (U_n(x+s) - U_n(s))|}{r^{\alpha+1/2} / (\ln \ln 1/r)^{\alpha+1/2}} = 0.$$

Hence, for every $x \in \mathbb{R}$ and $c \geq 0$, the event

$$E_c = \left\{ \liminf_{r \rightarrow 0^+} \sup_{|s| \leq r} \frac{|U(x+s) - U(x)|}{r^{\alpha+1/2}/(\ln \ln 1/r)^{\alpha+1/2}} \leq c \right\}$$

is a tail event with respect to the sequence $\{U_n\}_{n \geq 1}$ of independent processes. By Kolmogorov’s zero-one law, we have $\mathbb{P}(E_c) = 0$ or 1 . Let $c'_{6,1} = \inf\{c \geq 0 : \mathbb{P}(E_c) = 1\}$, with the convention that $\inf \emptyset = \infty$. Then, $c'_{6,1} \in [0, \infty]$ and

$$\liminf_{r \rightarrow 0^+} \sup_{|s| \leq r} \frac{|U(x+s) - U(x)|}{r^{\alpha+1/2}/(\ln \ln 1/r)^{\alpha+1/2}} = c'_{6,1}, \quad \text{a.s.} \tag{6.3}$$

Moreover, $c'_{6,1}$ does not depend on x because of the stationarity of U .

Next, we use (6.3) and Lamperti’s transformation to show (6.2). For any $t > 0$ fixed, $0 < r < t$ and $|s| \leq r$, we write

$$\begin{aligned} Z(t+s) - Z(t) &= [(t+s)^H - t^H]U(\ln(t+s)) + t^H[U(\ln(t+s)) - U(\ln t)] \\ &= [(t+s)^H - t^H]U(\ln(t+s)) + t^H[U(\ln t + \ln(1+s/t)) - U(\ln t)]. \end{aligned} \tag{6.4}$$

Since the first term is Lipschitz in $s \in [-r, r]$ and $\ln(1 + \frac{s}{t}) \sim s/t$ as $s \rightarrow 0$, it can be verified that (6.2) follows from (6.4) and (6.3) with $x = \ln t$. \square

It follows from Lemma 6.1 that Proposition 6.1 will be established if we show $c'_{6,1} \in (0, \infty)$. This is where Propositions 3.2, 5.2, Lemma 4.2 and the following version of Fernique’s lemma from [13, Lemma 1.1, p.138] are needed.

Lemma 6.2 *Let $\{\xi(t)\}_{t \geq 0}$ be a separable, centered, real-valued Gaussian process. Assume that*

$$\mathbb{E} \left[(\xi(t+h) - \xi(t))^2 \right] \leq \varphi(h)^2, \quad t > 0, \quad h > 0,$$

for some continuous nondecreasing function φ with $\varphi(0) = 0$. For any positive integer $k > 1$ and any positive constants t, x and $\theta(p)$, $p \in \mathbb{N}$, we have

$$\mathbb{P} \left\{ \sup_{0 \leq s \leq t} |\xi(s) - \xi(0)| > x\varphi(t) + \sum_{p=1}^{\infty} \theta(p)\varphi(tk^{-2p}) \right\} \leq k^2 e^{-x^2/2} + \sum_{p=1}^{\infty} k^{2p+1} e^{-\theta(p)^2/2}.$$

We now give the proof of Proposition 6.1.

Proof of Proposition 6.1 Without loss of generality, we may assume $t > 1$. We first prove the lower bound. For any integer $n \geq 1$, let $r_n = e^{-n}$. Let $0 < \delta < c_{5,8}$ be a constant and consider the event

$$A_n = \left\{ \sup_{|s| \leq r_n} |Z(t+s) - Z(t)| \leq \delta^{\alpha+1/2} t^{-\gamma/2} r_n^{\alpha+1/2} / (\ln \ln(1/r_n))^{\alpha+1/2} \right\}.$$

Proposition 5.2 implies that for any $n \in \mathbb{N}$,

$$\mathbb{P}\{A_n\} \leq \exp\left(-\frac{c_{5,8}}{\delta} \ln n\right) = n^{-\frac{c_{5,8}}{\delta}}. \tag{6.5}$$

Since $\sum_{n=1}^\infty \mathbb{P}\{A_n\} < \infty$, the Borel–Cantelli lemma implies

$$\liminf_{n \rightarrow \infty} \sup_{|s| \leq r_n} \frac{|Z(t+s) - Z(t)|}{r_n^{\alpha+1/2} / (\ln \ln(1/r_n))^{\alpha+1/2}} \geq \delta^{\alpha+1/2} t^{-\gamma/2}, \quad \text{a.s.} \tag{6.6}$$

It follows from (6.6) and a standard monotonicity argument that

$$\liminf_{r \rightarrow 0+} \sup_{|s| \leq r} \frac{|Z(t+s) - Z(t)|}{r^{\alpha+1/2} / (\ln \ln(1/r))^{\alpha+1/2}} \geq c_{6,2} t^{-\gamma/2}, \quad \text{a.s.}, \tag{6.7}$$

for some constant $c_{6,2} \in (0, \infty)$ which is independent of $t > 0$.

The upper bound is a little more difficult to prove due to the dependence structure of Z . In order to create independence, as in Tudor and Xiao [35], we will make use of the following stochastic integral representation of Z :

$$Z(t) = t^H \int_{\mathbb{R}} e^{i\lambda \ln t} W(d\lambda), \quad t > 0 \tag{6.8}$$

where $H = \alpha - \gamma/2 + 1/2$. This follows from the spectral representation (4.3) of U .

For every integer $n \geq 1$, we take

$$t_n = n^{-n} \quad \text{and} \quad d_n = n^{n+1/2-\alpha}. \tag{6.9}$$

It is sufficient to prove that there exists a finite constant $c_{6,3}$ such that

$$\liminf_{n \rightarrow \infty} \sup_{|s| \leq t_n} \frac{|Z(t+s) - Z(t)|}{t_n^{\alpha+1/2} / (\ln \ln(1/t_n))^{\alpha+1/2}} \leq c_{6,3} t^{-\gamma/2} \quad \text{a.s.}, \tag{6.10}$$

Let us define two Gaussian processes Z_n and \tilde{Z}_n by

$$Z_n(t) = t^H \int_{|\lambda| \in (d_{n-1}, d_n]} e^{i\lambda \ln t} W(d\lambda) \tag{6.11}$$

and

$$\tilde{Z}_n(t) = t^H \int_{|\lambda| \notin (d_{n-1}, d_n]} e^{i\lambda \ln t} W(d\lambda), \tag{6.12}$$

respectively. Clearly, $Z(t) = Z_n(t) + \tilde{Z}_n(t)$ for all $t > 0$. It is important to note that the Gaussian processes Z_n ($n = 1, 2, \dots$) are independent and, moreover, for every $n \geq 1$, Z_n and \tilde{Z}_n are independent as well.

Denote $h(r) = r^{\alpha+1/2} (\ln \ln(1/r))^{-(\alpha+1/2)}$. We make the following two claims:

(i) There is a constant $\delta > 0$ such that

$$\sum_{n=1}^{\infty} \mathbb{P} \left\{ \sup_{|s| \leq t_n} |Z_n(t+s) - Z_n(t)| \leq \delta^{\alpha+1/2} t^{-\gamma/2} h(t_n) \right\} = \infty. \tag{6.13}$$

(ii) For every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P} \left\{ \sup_{|s| \leq t_n} |\tilde{Z}_n(t+s) - \tilde{Z}_n(t)| > \varepsilon t^{-\gamma/2} h(t_n) \right\} < \infty. \tag{6.14}$$

Since the events in (6.13) are independent, we see that (6.10) follows from (6.13), (6.14) and a standard Borel–Cantelli argument.

It remains to verify the claims (i) and (ii) above. By Proposition 5.2 and Anderson’s inequality [2], we have

$$\begin{aligned} & \mathbb{P} \left\{ \sup_{|s| \leq t_n} |Z_n(t+s) - Z_n(s)| \leq \delta^{\alpha+1/2} t^{-\gamma/2} h(t_n) \right\} \\ & \geq \mathbb{P} \left\{ \sup_{|s| \leq t_n} |Z(t+s) - Z(s)| \leq \delta^{\alpha+1/2} t^{-\gamma/2} h(t_n) \right\} \\ & \geq \exp \left(-\frac{c_{5,7}}{\delta} \ln(n \ln n) \right) \\ & = (n \ln n)^{-\frac{c_{5,7}}{\delta}}. \end{aligned} \tag{6.15}$$

Hence, (i) holds for $\delta \geq c_{5,7}$.

In order to prove (ii), we divide it into two terms: For any $|s| < t_n$,

$$\begin{aligned} \mathbb{E} \left((\tilde{Z}_n(t+s) - \tilde{Z}_n(t))^2 \right) &= \int_{|\lambda| \leq d_{n-1}} |(t+s)^H e^{i\lambda \ln(t+s)} - t^H e^{i\lambda \ln t}|^2 f_U(\lambda) d\lambda \\ &\quad + \int_{|\lambda| > d_n} |(t+s)^H e^{i\lambda \ln(t+s)} - t^H e^{i\lambda \ln t}|^2 f_U(\lambda) d\lambda \\ &=: \mathcal{J}_1 + \mathcal{J}_2. \end{aligned} \tag{6.16}$$

The second term is easy to estimate: For any $|s| \leq t_n$, there exists $c_{6,4} \in (0, \infty)$ satisfying that

$$\mathcal{J}_2 \leq 2 \left(t^{2H} + (t+s)^{2H} \right) \int_{|\lambda| > d_n} f_U(\lambda) d\lambda \leq c_{6,4} n^{-(2\alpha+1)(n+1/2-\alpha)}, \tag{6.17}$$

where the last inequality follows from (4.9).

For the first term \mathcal{J}_1 , we use the following elementary inequalities:

$$\left| (x + y)^H - x^H \right| \leq |H| \max \left\{ x^{H-1}, (x + y)^{H-1} \right\} |y|, \quad \text{for all } 0 \leq |y| < x,$$

$1 - \cos x \leq x^2$ for all $x \in \mathbb{R}$ and $\ln(1 + x) \leq x$ for all $x \geq 0$ to derive that for any $|s| \leq t_n$,

$$\begin{aligned} \mathcal{J}_1 &= \int_{|\lambda| \leq d_{n-1}} \left[\left((t + s)^H - t^H \right)^2 + 2(t + s)^H t^H \left(1 - \cos \left(\lambda \ln \frac{t + s}{t} \right) \right) \right] f_U(\lambda) d\lambda \\ &\leq \frac{1}{H^2} \max \left\{ t^{2(H-1)}, (t + s)^{2(H-1)} \right\} s^2 \int_{\mathbb{R}} f_U(\lambda) d\lambda \\ &\quad + 2(t + s)^H t^H \ln^2 \left(1 + \frac{s}{t} \right) \int_{|\lambda| \leq d_{n-1}} \lambda^2 f_U(\lambda) d\lambda \\ &\leq c_{6,5} \left(\int_{\mathbb{R}} f_U(\lambda) d\lambda + \int_{|\lambda| \leq d_{n-1}} \lambda^2 f_U(\lambda) d\lambda \right) n^{-2n}. \end{aligned}$$

Here, $c_{6,5}$ is in $(0, \infty)$. By (4.8) and (4.9), and the fact that $2n \geq (2\alpha + 1)(n + 1/2 - \alpha)$ for any $\alpha \in (-1/2 + \gamma/2, 1/2)$. we have

$$\mathcal{J}_1 \leq c_{6,6} n^{-(2\alpha+1)(n+1/2-\alpha)}, \tag{6.18}$$

for some constant $c_{6,6} \in (0, \infty)$.

Put $\delta = (2\alpha + 1)(1/2 - \alpha)$. By Lemma 3.3, (6.16), (6.17) and (6.18), there exists a constant $K > 0$ such that for $0 \leq h \leq t_n$,

$$\varphi_n(h)^2 := K \min \left\{ h^{2\alpha+1}, n^{-(2\alpha+1)n-\delta} \right\} \geq \mathbb{E} \left[\left(\tilde{Z}_n(t + h) - \tilde{Z}_n(t) \right)^2 \right].$$

Put $x_n = (8 \ln n)^{1/2}$. Given $\varepsilon > 0$, define

$$\theta_n(p) = \varepsilon(p + 1)^{-2} h(t_n) / \varphi_n \left(t_n n^{-2p} \right) \quad \text{for all } p \geq 1.$$

For large enough n , we have

$$\theta_n(p) > 4(\ln n)^{1/2} 2^{p/2} \quad \text{for all } p \geq 1$$

and

$$x_n \varphi_n(t_n) + \sum_{p=1}^{\infty} \theta_n(p) \varphi_n \left(t_n n^{-2p} \right) < \varepsilon h(t_n).$$

Since

$$\sum_{n=1}^{\infty} n^2 e^{-x_n^2/2} + \sum_{n=1}^{\infty} \sum_{p=1}^{\infty} n^{2p+1} e^{-8(\ln n)2^p} < \infty,$$

by applying Lemma 6.2 with $\xi(s) = \tilde{Z}_n(t \pm s) - \tilde{Z}_n(t)$ for $0 \leq s \leq t_n$, we obtain that

$$\sum_{n=1}^{\infty} \mathbb{P} \left\{ \sup_{|s| \leq t_n} |\tilde{Z}_n(t+s) - \tilde{Z}_n(t)| > \varepsilon t^{-\gamma/2} h(t_n) \right\} < \infty.$$

This proves (6.14) and hence Proposition 6.1. □

Remark 6.1 In light of Proposition 6.1, it is natural to study Chung’s LIL of Z at the origin. While doing so, we found that there is an error in the proof of Theorem 3.1 in Tudor and Xiao [35], which gives Chung’s LIL for bifractional Brownian motion at the origin. More precisely, the inequality (3.30) in [35] does not hold. Hence, Theorem 3.1 in [35] should be modified as a Chung’s LIL at $t > 0$, which is similar to Theorem 1.4 and Proposition 6.1 in the present paper.

It turns out that Chung’s LIL at the origin for self-similar Gaussian processes that do not have stationary increments such as GFBM X , the generalized Riemann–Liouville FBM Z , and bifractional Brownian motion is quite subtle. A different method than that in the proof of Proposition 6.1 or that in [35] is needed for proving the desired upper bound. Recently, this problem has been studied in the subsequent paper [36] for GFBM X by modifying the arguments of Talagrand [33].

Similarly, for $\alpha \in (1/2, 3/2)$, we have the following Chung’s LIL for Z' .

Proposition 6.2 *Assume $\alpha \in (1/2, 3/2)$. There exists a constant $c_{6,7} \in (0, \infty)$ such that for any $t > 0$,*

$$\liminf_{r \rightarrow 0^+} \sup_{0 \leq s \leq r} \frac{|Z'(t+s) - Z'(t)|}{r^{\alpha-1/2}/(\ln \ln 1/r)^{\alpha-1/2}} = c_{6,7} t^{-\gamma/2}, \quad a.s. \tag{6.19}$$

Proof Recall from (3.31) that, for $\alpha \in (1/2, 3/2)$, the derivative process $\{Z'(t)\}_{t \geq 0}$ is a generalized Riemann–Liouville FBM with indices $\alpha' = \alpha - 1 \in (-1/2, 1/2)$ and $\gamma \in (0, 1)$. Hence, the proof of (6.19) follows the same line as in the proof of Proposition 6.1. We omit the details. □

7 Proofs of the Main Theorems

In this section, we prove our main results for GFBM X stated in Section 1.

7.1 Proof of Theorem 1.1

(a) Assume that $\alpha \in (-1/2 + \gamma/2, 1/2)$. By Proposition 2.1, we know that for any $0 < a < b < \infty$, $Y(t)$ is continuously differentiable on $[a, b]$. Then,

$$\lim_{\varepsilon \rightarrow 0} \sup_{a \leq t \leq b} \sup_{0 \leq h \leq \varepsilon} \frac{|Y(t+h) - Y(t)|}{h^{\alpha + \frac{1}{2}} \sqrt{\ln h^{-1}}} = 0.$$

By (1.17), to prove (1.5), it is sufficient to prove that for any $0 < a < b < \infty$,

$$\lim_{\varepsilon \rightarrow 0} \sup_{a \leq t \leq b} \sup_{0 \leq h \leq \varepsilon} \frac{|Z(t+h) - Z(t)|}{h^{\alpha + \frac{1}{2}} \sqrt{\ln h^{-1}}} = c_{7,1}, \tag{7.1}$$

where $c_{7,1}$ is a positive constant satisfying

$$c_{7,2} := \sqrt{\frac{2}{(\alpha + 1/2)^2 b^\gamma c_{3,6}}} \leq c_{7,1} \leq \sqrt{\frac{2c_{3,6}}{(\alpha + 1/2)c_{3,5}}} =: c_{7,3}. \tag{7.2}$$

Here, $c_{3,5}, c_{3,6}$ are constants given in (3.19).

For any $\varepsilon > 0$, let

$$J(\varepsilon) := \sup_{a \leq t \leq b} \sup_{0 \leq h \leq \varepsilon} \frac{|Z(t+h) - Z(t)|}{h^{\alpha + \frac{1}{2}} \sqrt{\ln h^{-1}}}.$$

Since $\varepsilon \mapsto J(\varepsilon)$ is non-decreasing, the limit in the left-hand side of (7.1) exists almost surely. Moreover, the zero-one law for the uniform modulus of continuity in [24, Lemma 7.1.1] implies that the limit in (7.1) is a constant almost surely. Hence, it remains to prove that with probability one,

$$\lim_{\varepsilon \rightarrow 0+} J(\varepsilon) \leq c_{7,3} \tag{7.3}$$

and

$$\lim_{\varepsilon \rightarrow 0+} J(\varepsilon) \geq c_{7,2}. \tag{7.4}$$

The proof of (7.3) is standard. It follows from (3.19) and the metric-entropy bound (cf. Theorem 1.3.5 in [1]), or one may prove this directly by applying Fernique’s inequality in Lemma 6.2 and the Borel–Cantelli lemma. The lower bound in (7.4) follows from the one-sided SLND (3.35) and Theorem 4.1 in [26]. This proves (a).

(b) When $\alpha = 1/2$, we see that (3.20) holds for all $s, t \in [a, b]$. This implies that the canonical metric of Z satisfies

$$d_Z(s, t) \asymp |s - t| \sqrt{1 + |\ln |s - t||}$$

on $[a, b]$. Hence, for any $\varepsilon > 0$ small, the covering number $N_\varepsilon([a, b])$ of $[a, b]$ under d_Z satisfies

$$N_\varepsilon([a, b]) \leq c_{7,4} \varepsilon^{-1} (\ln \varepsilon^{-1})^{1/2}, \tag{7.5}$$

where $c_{7,4} > 0$ is a finite constant. It follows from Theorem 1.3.5 in [1] that almost surely for all $\delta > 0$ small enough,

$$\sup_{a \leq t \leq b} \sup_{d_Z(t, t+h) \leq \delta} |Z(t+h) - Z(t)| \leq c \int_0^\delta \sqrt{\ln N_\varepsilon([a, b])} d\varepsilon \leq c\delta \sqrt{\ln \delta^{-1}}. \tag{7.6}$$

Notice that, by (3.20), $d_Z(t, t+h) \leq \delta$ is compatible to $|h| \sqrt{|\ln |h|^{-1}} \leq \delta$, up to a constant factor. Hence, (7.6) implies that

$$\sup_{a \leq t \leq b} \sup_{0 \leq h \leq r} |Z(t+h) - Z(t)| \leq c r \ln r^{-1}. \tag{7.7}$$

Now, it can be seen that (1.6) follows from (7.7).

(c). The proof of (1.7) is similar: the uniform modulus of continuity of X' on $[a, b]$ is the same as that of Z' on $[a, b]$, and the latter can be derived from (3.30) in Lemma 3.3 and the one-sided SLND (3.36) in Proposition 3.2. We omit the details here. The proof is complete. □

7.2 Proofs of Theorems 1.2 and 1.3

Similarly to the proof of Proposition 5.1, by using Lemmas 2.1 and 5.1, we obtain the following lower bounds for the small ball probabilities of Y .

Lemma 7.1 (a) *Assume $\alpha \in (-1/2 + \gamma/2, 1/2 + \gamma/2)$. Then, there exists a constant $c_{7,5} \in (0, \infty)$ such that for all $r > 0, \varepsilon > 0$,*

$$\mathbb{P} \left\{ \sup_{0 \leq s \leq r} |Y(s)| \leq \varepsilon \right\} \geq \exp \left(- c_{7,5} r^H \varepsilon^{-1} \right). \tag{7.8}$$

(b) *Assume $\alpha \in (-1/2 + \gamma/2, 1/2 + \gamma/2)$. Then, there exists a constant $c_{7,6} \in (0, \infty)$ such that for all $t > 0, 0 < \varepsilon < r < t/2$,*

$$\mathbb{P} \left\{ \sup_{|s| \leq r} |Y(t+s) - Y(t)| \leq \varepsilon \right\} \geq \exp \left(- c_{7,6} r t^{H-1} \varepsilon^{-1} \right). \tag{7.9}$$

Proof of Theorem 1.2 By the self-similarity of X , we know that (1.9) is equivalent to the following: For any $0 < \varepsilon < 1$,

$$\exp\left(-\kappa_4\left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1/2}}\right) \leq \mathbb{P}\left\{\sup_{s \in [0,1]} |X(s)| \leq \varepsilon\right\} \leq \exp\left(-\kappa_5\left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1/2}}\right). \tag{7.10}$$

By (5.1), (7.8) and the independence of Y and Z , we have

$$\begin{aligned} \mathbb{P}\left\{\sup_{s \in [0,1]} |X(s)| \leq \varepsilon\right\} &\geq \mathbb{P}\left\{\sup_{s \in [0,1]} |Y(s)| \leq \varepsilon/2\right\} \cdot \mathbb{P}\left\{\sup_{s \in [0,1]} |Z(s)| \leq \varepsilon/2\right\} \\ &\geq \exp\left(-c_{7,5}\frac{2}{\varepsilon}\right) \cdot \exp\left(-c_{5,1}\left(\frac{2}{\varepsilon}\right)^{\frac{1}{\alpha+1/2}}\right) \\ &\geq \exp\left(-\left(2c_{7,5} + 2^{\frac{1}{\alpha+1/2}}c_{5,1}\right)\left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1/2}}\right), \end{aligned}$$

where we have used the facts that $\frac{1}{\varepsilon} \leq \left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1/2}}$ as $\varepsilon \in (0, 1)$ and $\alpha < 1/2$. This proves the lower bound in (7.10).

On the other hand, by using the Anderson inequality [2] and (5.1), we have

$$\begin{aligned} \mathbb{P}\left\{\sup_{s \in [0,1]} |X(s)| \leq \varepsilon\right\} &= \mathbb{E}\left[\mathbb{P}\left\{\sup_{s \in [0,1]} |Y(s) + Z(s)| \leq \varepsilon \mid Y(s), s \in [0, 1]\right\}\right] \\ &\leq \mathbb{P}\left\{\sup_{s \in [0,1]} |Z(s)| \leq \varepsilon\right\} \leq \exp\left(-c_{5,2}\left(\frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1/2}}\right). \end{aligned}$$

This proves the upper bound in (7.10). The proof is complete. □

Similarly to the proof of Theorem 1.2, by using Proposition 5.2 and Lemma 7.1, we can prove Theorem 1.3. The detail is omitted.

7.3 Proof of Theorem 1.4

By Proposition 2.1, $Y(t)$ is continuously differentiable on $[a, b]$ for any $0 < a < b < \infty$. Hence, Chung’s LILs of X (or X' when it exists) at a fixed point $t > 0$ is the same as that of Z (or Z') at t . Therefore, (1.12) and (1.13) follows from (6.1) and (6.19), respectively. □

Remark 7.1 The small ball probabilities in Theorems 1.2 and 1.3 and the one-sided SLND in Proposition 3.2 can be applied as in [38] to refine the results in (3.21) and (3.22) by determining the exact Hausdorff measure functions for the graph and level set of GFBM X . We refer to Talagrand [32,34] for further ideas on investigating other related fractal properties of X .

7.4 Proof of Theorem 1.5 and the time inversion of X

Proof of Theorem 1.5 As in the proof of Theorem 1.4, it is sufficient to prove (1.14)-(1.16) for Z and Z' , respectively.

First, we consider the case when $\alpha \in (-1/2 + \gamma/2, 1/2)$. It follows from (6.4) that for any fixed $t > 0$,

$$\limsup_{r \rightarrow 0+} \sup_{|h| \leq r} \frac{|Z(t+h) - Z(t)|}{r^{\alpha+1/2} \sqrt{\ln \ln 1/r}} = t^{-\gamma/2} \limsup_{r \rightarrow 0+} \sup_{|h| \leq r} \frac{|U(x+h) - U(x)|}{r^{\alpha+1/2} \sqrt{\ln \ln 1/r}}, \tag{7.11}$$

where $x = \ln t$. By Lemma 4.1, we see that the Gaussian process $\{U(s) - U(0)\}_{s \in \mathbb{R}}$ satisfies the conditions of Theorem 5.1 in [26] with $N = 1$ and $\sigma(s, t) \asymp |s - t|^{\alpha+1/2}$ on any compact interval of $(0, \infty)$. Consequently, there exists a constant $\kappa_{12} \in (0, \infty)$ which does not depend on x such that

$$\limsup_{r \rightarrow 0+} \sup_{|h| \leq r} \frac{|U(x+h) - U(x)|}{r^{\alpha+1/2} \sqrt{\ln \ln 1/r}} = \kappa_{12}, \quad \text{a.s.} \tag{7.12}$$

Combining (7.11) and (7.12) yields (1.14) for Z .

When $\alpha = 1/2$, we see from Lemma 4.1 that Condition (A1) in [26] is satisfied with $N = 1$ and $\sigma(s, t) \asymp |s - t| \sqrt{1 + |\ln |s - t||}$ on any compact interval of $(0, \infty)$. This case is not explicitly covered by Theorem 5.1 in [26]. However, an examination of its proof shows that we have

$$\limsup_{r \rightarrow 0+} \sup_{|h| \leq r} \frac{|U(x+h) - U(x)|}{r \sqrt{\ln(1/r) \ln \ln 1/r}} = \kappa_{13}, \quad \text{a.s.} \tag{7.13}$$

for a constant $\kappa_{13} \in (0, \infty)$ that does not depend on x . Therefore, by combining (7.11) and (7.13) we obtain (1.15) for Z .

Finally, when $\alpha \in (1/2, 1/2 + \gamma/2)$, the proof of (1.16) for Z' is similar to that of (1.14) for Z given above. There is no need to repeat it. The proof of Theorem 1.5 is complete. □

The following result is the *time inversion* property of GFBM X . It allows us to improve slightly Theorem 6.1 in [11], where the law of the iterated logarithm of X at the origin was proved under the extra condition of $\alpha > 0$.

Proposition 7.1 *Let $X = \{X(t)\}_{t \geq 0}$ be a GFBM with parameters $\gamma \in (0, 1)$ and $\alpha \in (-1/2 + \gamma/2, 1/2 + \gamma)$. Define the Gaussian process $\tilde{X} = \{\tilde{X}(t)\}_{t \geq 0}$ by*

$$\tilde{X}(0) = 0, \quad \tilde{X}(t) = t^{2H} X(1/t), \quad t > 0.$$

Then, X and \tilde{X} have the same distribution. Consequently, there exists a constant $\kappa_{15} \in (0, \infty)$ such that

$$\limsup_{r \rightarrow 0^+} \frac{|X(r)|}{r^H \sqrt{\ln \ln r^{-1}}} = \kappa_{15}, \quad a.s. \tag{7.14}$$

Proof Since X and \tilde{X} are centered Gaussian processes, it is sufficient to check their covariance functions. For any $0 < s \leq t$,

$$\begin{aligned} \mathbb{E}[\tilde{X}(s)\tilde{X}(t)] &= (st)^{2H} \int_{\mathbb{R}} \left(\left(\frac{1}{s} - u \right)_+^\alpha - (-u)_+^\alpha \right) \left(\left(\frac{1}{t} - u \right)_+^\alpha - (-u)_+^\alpha \right) |u|^{-\gamma} du \\ &= \int_{\mathbb{R}} ((t-v)_+^\alpha - (-v)_+^\alpha) ((s-v)_+^\alpha - (-v)_+^\alpha) |v|^{-\gamma} dv \\ &= \mathbb{E}[X(s)X(t)], \end{aligned}$$

where a change of variable $u = v/(st)$ is used in the second step. This proves the first part of Proposition 7.1.

The time inversion property, together with the LIL of X at infinity in [11, Theorem 5.1], implies (7.14). The proof is complete. \square

7.5 Proof of Theorem 1.6

By Proposition 2.1, $Y(t)$ is continuously differentiable in $(0, \infty)$. Hence, for every $\alpha \in (-1/2 + \gamma/2, 1/2)$, $t > 0$ and for every compact set $\mathcal{K} \subset \mathbb{R}_+$,

$$\lim_{u \rightarrow 0^+} \sup_{\tau \in \mathcal{K}} \left| \frac{X(t + u\tau) - X(t)}{u^{\alpha+1/2}} - \frac{Z(t + u\tau) - Z(t)}{u^{\alpha+1/2}} \right| = 0, \quad a.s. \tag{7.15}$$

Therefore, X has the same tangent process as that of Z at $t \in (0, \infty)$, and Theorem 1.6 follows from Proposition 3.1. \square

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