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Mild Solutions for the Stochastic Generalized Burgers–Huxley Equation

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Abstract
In this work, we consider the stochastic generalized Burgers–Huxley equation perturbed by space–time white noise and discuss the global solvability results. We show the existence of a unique global mild solution to such equation using a fixed point method and stopping time arguments. The existence of a local mild solution (up to a stopping time) is proved via contraction mapping principle. Then, establishing a uniform bound for the solution, we show the existence and uniqueness of global mild solution to the stochastic generalized Burgers–Huxley equation. Finally, we discuss the inviscid limit of the stochastic Burgers–Huxley equation to the stochastic Burgers as well as Huxley equations.

Keywords Generalized Burgers–Huxley equation · Space–time white noise · Mild solution

Mathematics Subject Classification 60H15 · 35K58 · 35Q35 · 37H10

1 Introduction

We consider the *generalized Burgers–Huxley equation* perturbed by a random forcing, which is a space–time white noise (or Brownian sheet) as (see [27,31] for deterministic model)

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \nu\frac{\partial^2 u(t,x)}{\partial x^2} - \alpha u^\delta(t,x)\frac{\partial u(t,x)}{\partial x}\\ &\quad + \beta u(t,x)(1-u^\delta(t,x))(u^\delta(t,x)-\gamma) + \frac{\partial^2 \widetilde{W}(t,x)}{\partial x\partial t},\end{aligned} \tag{1.1}$$

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where $\alpha, \beta, \nu, \delta$ are parameters such that $\alpha > 0$ is the advection coefficient, $\beta > 0$, $\delta \geq 1$ and $\gamma \in (0, 1)$ are parameters. In the above equation, $\widetilde{W}(t, x)$, $t \geq 0$, $x \in \mathbb{R}$ is a zero mean Gaussian process, whose covariance function is given by

$$\mathbb{E}[\widetilde{W}(t, x)\widetilde{W}(t, y)] = (t \wedge s)(x \wedge y),\ t, s \geq 0,\ x, y \in \mathbb{R}.$$

On the other hand, one can consider a cylindrical Wiener process by setting

$$W(t, x) = \frac{\partial \widetilde{W}(t, x)}{\partial x} = \sum_{k=1}^{\infty} w_k(x)\beta_k(t), \tag{1.2}$$

where $\{w_k(\cdot)\}$ is an orthonormal basis of $L^2(\mathcal{O})$ and $\{\beta_k(\cdot)\}$ is a sequence of mutually independent real-valued Brownian motions in a fixed complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ adapted to the filtration $\{\mathscr{F}_t\}_{t\geq 0}$. Note that the series given in (1.2) does not converge in $L^2(\mathcal{O})$, but it is convergent in any Hilbert space U such that the embedding $L^2(\mathcal{O}) \subset U$ is Hilbert–Schmidt (see Chapter 4, [10]). We rewrite Eq. (1.1) as

$$\mathrm{d}u(t) = \left(\nu \frac{\partial^2 u(t)}{\partial x^2} - \alpha u^\delta(t) \frac{\partial u(t)}{\partial x} + \beta u(t)(1 - u^\delta(t))(u^\delta(t) - \gamma)\right) \mathrm{d}t + \mathrm{d}W(t), \tag{1.3}$$

where $x \in [0, 1]$, $t > 0$, and $W(\cdot)$ is defined by (1.2). We supplement (1.3) with Dirichlet boundary conditions

$$u(0, t) = u(1, t) = 0, \tag{1.4}$$

and the initial condition

$$u(x, 0) = u_0(x),\ x \in \overline{\mathcal{O}}. \tag{1.5}$$

Equation (1.3) describes a prototype model for describing the interaction between reaction mechanisms, convection effects and diffusion transports. In the deterministic setting, the existence and uniqueness of a global weak as well as strong solution for the forced generalized Burgers–Huxley equation are established in [27]. Our goal in this work is to show that problem (1.3) with boundary and initial conditions (1.4) and (1.5) has a unique global mild solution in $\mathrm{C}([0, T]; \mathrm{L}^p(\mathcal{O}))$, for $\delta < p < \infty$.

For $\delta = 1$, $\alpha \neq 0$ and $\beta \neq 0$, Eq. (1.3) becomes

$$\mathrm{d}u(t) = \left(\nu \frac{\partial^2 u(t)}{\partial x^2} - \alpha u(t) \frac{\partial u(t)}{\partial x} + \beta u(t)(1 - u(t))(u(t) - \gamma)\right) \mathrm{d}t + \mathrm{d}W(t), \tag{1.6}$$

which is known as the *stochastic Burgers–Huxley equation* (cf. [25]). The global solvability results as well as asymptotic behavior of solutions of stochastic Burgers–Huxley equation perturbed by multiplicative Gaussian noise are examined in [25].

The log-Harnack inequality for the Markov semigroup associated with the stochastic Burgers–Huxley equation and its applications have been discussed in [26]. For $\alpha = 0$ and $\delta = 1$, Eq. (1.3) takes the form

$$\mathrm{d}u(t) = \left(\nu\frac{\partial^2 u(t)}{\partial x^2} + \beta u(t)(1-u(t))(u(t)-\gamma)\right)\mathrm{d}t + \mathrm{d}W(t), \tag{1.7}$$

which is known as the *stochastic Huxley equation* and it describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals (cf. [35]).

For $\beta = 0$, $\delta = 1$ and $\alpha = 1$, Eq. (1.3) can be reduced to

$$\mathrm{d}u(t) = \left(\nu\frac{\partial^2 u(t)}{\partial x^2} - u(t)\frac{\partial u(t)}{\partial x}\right)\mathrm{d}t + \mathrm{d}W(t), \tag{1.8}$$

which is the well-known *stochastic viscous Burgers equation*. In [8], Burgers studied the deterministic model for modeling the turbulence phenomena (see [3,9] also). The authors in [11] proved the existence and uniqueness of a global mild solution as well as the existence of an invariant measure for the stochastic Burgers equation perturbed by cylindrical Gaussian noise. The existence and uniqueness of the global mild solution for the stochastic Burgers equation perturbed by a multiplicative white noise are established in [12]. Interested readers are referred to see [5,15,16,20,24,36], etc., for more details on mathematical analysis of stochastic Burgers equation. The stochastic generalized Burgers equation

$$\mathrm{d}u(t) = \left(\nu\frac{\partial^2 u(t)}{\partial x^2} - \alpha u^{\delta}(t)\frac{\partial u(t)}{\partial x}\right)\mathrm{d}t + \mathrm{d}W(t) \tag{1.9}$$

with white noise has been considered in [17,22], etc. The stochastic generalized Burgers equation perturbed by different kinds of noises has been considered in the works [18,21,23,34], etc. Various mathematical problems regarding stochastic Burgers equation are available in the literature, and interested readers are referred to see [4,6,7,14,32,33], etc., and the references therein.

The rest of the paper is organized as follows. In the next section, we provide the abstract formulation of the problem and provide the necessary function spaces needed to obtain the global solvability results of Eq. (1.3). The existence and uniqueness of a mild solution to the stochastic generalized Burgers–Huxley equation is established in Sect. 3. We first show the existence of a local mild solution up to a stopping time using fixed point arguments (contraction mapping principle, Theorem 3.2). A uniform bound for the solution (for arbitrary deterministic time) is then obtained (Lemma 3.3), and the global existence is established by showing that the stopping time (up to which the local existence has been shown) is same as an arbitrary deterministic time almost surely (Theorem 3.5). In the final section, we discuss the inviscid limit of the stochastic Burgers–Huxley equation to the stochastic Burgers (Proposition 4.1) and Huxley equations (Proposition 4.2).

2 Mathematical Formulation

In this section, we present the necessary function spaces needed to obtain the global solvability results of Eq. (1.3). We provide the properties of linear and nonlinear operators, and the definition of mild solution also.

2.1 Functional Setting

Let $\mathrm{C}_0^\infty(\mathcal{O})$ denote the space of all infinitely differentiable functions with compact support in $\mathcal{O}$. For $p \in [2, \infty)$, the Lebesgue spaces are denoted by $\mathrm{L}^p(\mathcal{O})$ and the Sobolev spaces are denoted by $W^{k,p(\mathcal{O})}$ and $\mathrm{H}^k(\mathcal{O}) := W^{k,2}(\mathcal{O})$. The norm in $\mathrm{L}^p(\mathcal{O})$ is denoted by $\|\cdot\|_{\mathrm{L}^p}$, and for $p = 2$, the inner product in $\mathrm{L}^2(\mathcal{O})$ is denoted by $(\cdot,\cdot)$. Let $\mathrm{H}_0^1(\mathcal{O})$ denote closure of $\mathrm{C}_0^\infty(\mathcal{O})$ in $\mathrm{H}^1(\mathcal{O})$-norm. As $\mathcal{O}$ is a bounded domain, using Poincaré's inequality, one can easily obtain that the norm $\left(\|\cdot\|_{\mathrm{L}^2}^2 + \|\partial_x\cdot\|_{\mathrm{L}^2}^2\right)^{1/2}$ and the seminorm $\|\partial_x\cdot\|_{\mathrm{L}^2}$ are equivalent and $\|\partial_x\cdot\|_{\mathrm{L}^2}$ defines a norm on $\mathrm{H}_0^1(\mathcal{O})$. We also have the continuous embedding $\mathrm{H}_0^1(\mathcal{O}) \subset \mathrm{L}^2(\mathcal{O}) \subset \mathrm{H}^{-1}(\mathcal{O})$, where $\mathrm{H}^{-1}(\mathcal{O})$ is the dual space of $\mathrm{H}_0^1(\mathcal{O})$. Remember that the embedding of $\mathrm{H}_0^1(\mathcal{O}) \subset \mathrm{L}^2(\mathcal{O})$ is compact. The duality paring between $\mathrm{H}_0^1(\mathcal{O})$ and $\mathrm{H}^{-1}(\mathcal{O})$ is denoted by $\langle\cdot,\cdot\rangle$. In one dimension, we have the following continuous embedding: $\mathrm{H}_0^1(\mathcal{O}) \subset \mathrm{L}^\infty(\mathcal{O}) \subset \mathrm{L}^p(\mathcal{O})$, for all $p \in [1, \infty)$. Remember that the embedding of $\mathrm{H}^\sigma(\mathcal{O}) \subset \mathrm{L}^q(\mathcal{O})$ is compact for $\sigma > \frac{1}{2} - \frac{1}{q}$, for $q \geq 2$.

2.2 Linear Operator

Let A denote the self-adjoint and unbounded operator on $\mathrm{L}^2(\mathcal{O})$ defined by[1]

$$Au := -\frac{\partial^2 u}{\partial x^2},$$

with domain $D(A) = \mathrm{H}^2(\mathcal{O}) \cap \mathrm{H}_0^1(\mathcal{O}) = \{u \in \mathrm{H}^2(\mathcal{O}) : u(0) = u(1) = 0\}$. The eigenvalues and the corresponding eigenfunctions of A are given by

$$\lambda_k = k^2\pi^2 \text{ and } w_k(x) = \sqrt{\frac{2}{\pi}}\sin(k\pi x),\ k = 1, 2\ldots.$$

Since $\mathcal{O}$ is a bounded domain, A^{-1} exists and is a compact operator on $\mathrm{L}^2(\mathcal{O})$. Moreover, one can define the fractional powers of A and

$$\|A^{1/2}u\|_{\mathrm{L}^2}^2 = \sum_{j=1}^\infty \lambda_j |(u, w_j)|^2 \geq \lambda_1 \sum_{j=1}^\infty |(u, w_j)|^2 = \lambda_1\|u\|_{\mathrm{L}^2}^2 = \pi^2\|u\|_{\mathrm{L}^2}^2,$$

[1] One has to write $Au = -u''$.

which is the Poincaré inequality. An integration by parts yields

$$(Au, v) = (\partial_x u, \partial_x v) =: a(u, v), \quad \text{for all} \ \ v \in \mathrm{H}_0^1(\mathcal{O}),$$

so that $A : \mathrm{H}_0^1(\mathcal{O}) \to \mathrm{H}^{-1}(\mathcal{O})$. Let us define the operator $A_p = -\frac{\partial^2}{\partial x^2}$ with $D(A_p) = \mathrm{W}_0^{1,p}(\mathcal{O}) \cap \mathrm{W}^{2,p}(\mathcal{O})$, for $1 < p < \infty$ and $D(A_1) = \{u \in \mathrm{W}^{1,1}(\mathcal{O}) : u \in \mathrm{L}^1(\mathcal{O})\}$, for $p = 1$. From Proposition 4.3, Chapter 1 [1], [29], we know that for $1 \leq p < \infty$, A_p generates an analytic semigroup of contractions in $\mathrm{L}^p(\mathcal{O})$.

2.3 Nonlinear Operators

Let us now define $b : \mathrm{H}_0^1(\mathcal{O}) \times \mathrm{H}_0^1(\mathcal{O}) \times \mathrm{H}_0^1(\mathcal{O}) \to \mathbb{R}$ as

$$b(u, v, w) = \int_0^1 (u(x))^\delta \frac{\partial v(x)}{\partial x} w(x)\mathrm{d}x.$$

Using an integration by parts and boundary conditions, it can be easily seen that

$$\begin{aligned} b(u, u, u) = (u^\delta \partial_x u, u) &= \int_0^1 (u(x))^\delta \frac{\partial u(x)}{\partial x} u(x) dx \\ &= \frac{1}{\delta + 2} \int_0^1 \frac{\partial}{\partial x} (u(x))^{\delta+2} dx = 0, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} b(u, u, v) &= \int_0^1 (u(x))^\delta \frac{\partial u(x)}{\partial x} v(x) dx = \frac{1}{\delta + 1} \int_0^1 \frac{\partial (u(x))^{\delta+1}}{\partial x} v(x)\mathrm{d}x \\ &= -\frac{1}{\delta + 1} \int_0^1 (u(x))^\delta \frac{\partial v(x)}{\partial x} u(x)\mathrm{d}x = -\frac{1}{\delta + 1} b(u, v, u), \end{aligned} \tag{2.2}$$

for all $u, v \in \mathrm{H}_0^1(\mathcal{O})$. In general, for all $p > 2$ and $u \in \mathrm{H}_0^1(\mathcal{O})$, we consider

$$\begin{aligned} b(u, u, |u|^{p-2}u) = (u^\delta \partial_x u, |u|^{p-2}u) &= \frac{1}{\delta + 2} \int_0^1 \frac{\partial}{\partial x} (u(x))^{\delta+2} |u(x)|^{p-2} dx \\ &= -\frac{1}{\delta + 2} \int_0^1 (u(x))^{\delta+2} \frac{\partial}{\partial x} |u(x)|^{p-2} dx \\ &= -\frac{p-2}{\delta + 2} \int_0^1 (u(x))^{\delta+2} |u(x)|^{p-4} u(x) \frac{\partial u(x)}{\partial x} dx \\ &= -\frac{p-2}{\delta + 2} (u^\delta \partial_x u, |u|^{p-2}u) = -\frac{p-2}{\delta + 2} b(u, u, |u|^{p-2}u), \end{aligned} \tag{2.3}$$

which implies

$$b(u, u, |u|^{p-2}u) = (u^\delta \partial_x u, |u|^{p-2}u) = 0, \tag{2.4}$$

for all $p \geq 2$ and $u \in \mathrm{H}_0^1(\mathcal{O})$.

For $w \in \mathrm{L}^2(\mathcal{O})$, we can define an operator $B(\cdot,\cdot) : \mathrm{H}_0^1(\mathcal{O}) \times \mathrm{H}_0^1(\mathcal{O}) \to \mathrm{L}^2(\mathcal{O})$ by

$$(B(u, v), w) = b(u, v, w) \leq \|u\|_{\mathrm{L}^\infty}^\delta \|\partial_x v\|_{\mathrm{L}^2} \|w\|_{\mathrm{L}^2} \leq \|u\|_{\mathrm{H}_0^1}^\delta \|v\|_{\mathrm{H}_0^1} \|w\|_{\mathrm{L}^2},$$

so that $\|B(u, v)\|_{\mathrm{L}^2} \leq \|u\|_{\mathrm{H}_0^1}^\delta \|v\|_{\mathrm{H}_0^1}$. We denote $B(u) = B(u, u)$, so that one can easily obtain $\|B(u)\|_{\mathrm{L}^2} \leq \|u\|_{\mathrm{H}_0^1}^{\delta+1}$.

Let us define $c(u) = u(1 - u^\delta)(u^\delta - \gamma)$. It should be noted that

$$\begin{aligned}(c(u), u) &= (u(1 - u^\delta)(u^\delta - \gamma), u) = ((1 + \gamma)u^{\delta+1} - \gamma u - u^{2\delta+1}, u) \\ &= (1 + \gamma)(u^{\delta+1}, u) - \gamma\|u\|_{L^2}^2 - \|u\|_{L^{2(\delta+1)}}^{2(\delta+1)},\end{aligned} \tag{2.5}$$

for all $u \in \mathrm{L}^{2(\delta+1)}(\mathcal{O})$.

3 Mild Solution

In this section, we show the existence and uniqueness of global mild solution to Eq. (1.3) with Dirichlet boundary conditions. First, we establish the existence of a local mild solution (up to a stopping time) using the contraction mapping principle. Then, we show a uniform bound for the mild solution and deduce the existence and uniqueness of global mild solution to the stochastic generalized Burgers–Huxley equation.

3.1 Linear Problem

We know that the solution to the linear problem (cf. [10]):

$$\begin{cases} \mathrm{d}u(t) = -\nu A u(t)\mathrm{d}t + \mathrm{d}W(t), \ t \in (0, T), \\ u(0) = 0, \end{cases} \tag{3.1}$$

is unique and is given by the stochastic convolution

$$W_A(t) = \int_0^t R(t - s)\mathrm{d}W(s), \tag{3.2}$$

where $R(t) = e^{-t\nu A}$. Note that W_A is a Gaussian process and it is mean square continuous with values in $L^2(\mathcal{O})$ and W_A has a modification, which has $\mathbb{P}$-a.s., α-Hölder continuous paths with respect to $(t, x) \in [0, T] \times [0, 1]$, for any $\alpha \in [0, 1/4)$ (see Theorem 5.22 and Example 5.24, [10] for more details).

3.2 Local Existence and Uniqueness

With the notations given in Sect. 2, one can write down the abstract formulation of the problem (1.3)–(1.5) as

$$\begin{cases} \mathrm{d}u(t) = [-\nu Au(t) - \alpha B(u(t)) + \beta c(u(t))]\mathrm{d}t + \mathrm{d}W(t), \ t \in (0,T), \\ u(0) = u_0. \end{cases} \tag{3.3}$$

Let us now provide the definition of mild solution to Eq. (3.3). Let the initial data u_0 be $\mathscr{F}_0$-measurable and belong to $L^p(\mathcal{O})$, for $\delta < p < \infty$, $\mathbb{P}$-a.s.

Definition 3.1 An $L^p(\mathcal{O})$-valued and $\mathscr{F}_t$-adapted stochastic process $u : [0,\infty) \times [0,1] \times \Omega \to \mathbb{R}$ with $\mathbb{P}$-a.s. continuous trajectories on $t \in [0,T]$, is a *mild solution* to Eq. (3.3), if for any $T > 0$, $u(t) := u(t,\cdot,\cdot)$ satisfies the following integral equation:

$$u(t) = R(t)u_0 - \alpha \int_0^t R(t-s)B(u(s))\mathrm{d}s + \beta \int_0^t R(t-s)c(u(s))\mathrm{d}s + \int_0^t R(t-s)\mathrm{d}W(s), \tag{3.4}$$

$\mathbb{P}$-a.s., for each $t \in [0,T]$.

Let us set

$$v(t) := u(t) - W_A(t), \ t \geq 0. \tag{3.5}$$

Then, $u(\cdot)$ is a solution to (3.3) if and only if $v(\cdot)$ is a solution of

$$\begin{cases} \dfrac{dv(t)}{\mathrm{d}t} = -\nu Av(t) - \dfrac{\alpha}{\delta+1}\partial_x((v(t)+W_A(t))^{\delta+1}) \\ \qquad + \beta(v(t)+W_A(t))(1-(v(t)+W_A(t))^{\delta})((v(t)+W_A(t))^{\delta}-\gamma), \ t \in (0,T), \\ v(0) = u_0. \end{cases} \tag{3.6}$$

We rewrite (3.6) as

$$v(t) = R(t)u_0 - \frac{\alpha}{\delta+1}\int_0^t R(t-s)\partial_x((v(s)+W_A(s))^{\delta+1})\mathrm{d}s + \beta \int_0^t R(t-s)(v(s)+W_A(s))(1-(v(s)+W_A(s))^{\delta})((v(s)+W_A(s))^{\delta}-\gamma)\mathrm{d}s; \tag{3.7}$$

then, if $v(\cdot)$ satisfies (3.7), we say that it is a *mild solution* of (3.6). By using fixed point arguments, we show the existence of mild solution to Eq. (3.6) in the space $\mathrm{C}([0,T^*];\mathrm{L}^p(\mathcal{O}))$, for $p > \delta$, $\mathbb{P}$-a.s., and for some $T^* > 0$ (random time). Let us set

$$\Sigma(m,T^*) = \left\{ v \in \mathrm{C}([0,T^*];\mathrm{L}^p(\mathcal{O})) : \|v(t)\|_{L^p(\mathcal{O})} \leq m, \ \text{ for all } \ t \in [0,T^*] \right\}. \tag{3.8}$$

Let us now show that (3.7) has a meaning as an equality in $\mathrm{L}^p(\mathcal{O})$ and establish the existence of a mild solution to the problem (3.6).

Theorem 3.2 *(Local existence) For $\|u_0\|_{\mathrm{L}^p} < m$, there exists a stopping time T^* such that (3.7) has a unique solution in $\Sigma(m, T^*)$.*

Proof Let us take any $v \in \Sigma(m, T^*)$ and define $z = Gv$ by

$$
\begin{aligned}
z(t) &= R(t)u_0 - \frac{\alpha}{\delta+1}\int_0^t R(t-s)\partial_x((v(s)+W_A(s))^{\delta+1})\mathrm{d}s \\
&\quad + \beta\int_0^t R(t-s)(v(s)+W_A(s))(1-(v(s)+W_A(s))^{\delta})((v(s)+W_A(s))^{\delta}-\gamma)\mathrm{d}s.
\end{aligned} \tag{3.9}
$$

Step 1. $G : \Sigma(m, T^*) \to \Sigma(m, T^*)$. From (3.9), we have

$$
\begin{aligned}
\|z(t)\|_{\mathrm{L}^p} &\leq \|R(t)u_0\|_{\mathrm{L}^p} + \frac{\alpha}{\delta+1}\int_0^t \left\|R(t-s)\partial_x((v(s)+W_A(s))^{\delta+1})\right\|_{L^p}\mathrm{d}s \\
&\quad + \beta\int_0^t \big\|R(t-s)(v(s)+W_A(s))(1-(v(s)+W_A(s))^{\delta}) \\
&\quad \times ((v(s)+W_A(s))^{\delta}-\gamma)\big\|_{L^p}\mathrm{d}s.
\end{aligned} \tag{3.10}
$$

Remember that $e^{-t\nu A}$ is a contraction semigroup on $\mathrm{L}^p(\mathcal{O})$. In order to estimate the terms on the right-hand side of the inequality (3.10), the following Sobolev embedding is needed:

$$
\|u\|_{L^{q_1}} \leq C\|u\|_{\mathrm{W}^{k,q_2}}, \quad \text{whenever } k < \frac{1}{q_2}, \tag{3.11}
$$

where $\frac{1}{q_1} = \frac{1}{q_2} - k$ (Theorem 6, page 284, [13]). We also need a smoothing property of the heat semigroup, that is, for any $r_1 \leq r_2$ in $\mathbb{R}$, and $\theta \geq 1$, $R(t)$ maps $\mathrm{W}^{r_1,\theta}(\mathcal{O})$ into $\mathrm{W}^{r_2,\theta}(\mathcal{O})$, for all $t > 0$. Furthermore, the following estimate holds (see Lemma 3, Part I, [30], [11])

$$
\|R(t)u\|_{\mathrm{W}^{r_2,\theta}} \leq C(t^{\frac{r_1-r_2}{2}}+1)\|u\|_{\mathrm{W}^{r_1,\theta}}, \tag{3.12}
$$

for all $u \in \mathrm{W}^{r_1,\theta}(\mathcal{O})$, where $C = C(r_1, r_2, \theta)$ is a positive constant. Applying (3.11) with $q_1 = p$, $q_2 = \frac{p}{\delta+1}$ and $k = \frac{\delta}{p}$, and then using the smoothing property (3.12) with $r_1 = -1$, $r_2 = \frac{\delta}{p}$ and $\theta = \frac{p}{\delta+1}$, we evaluate

$$
\begin{aligned}
&\|R(t-s)\partial_x((v(s)+W_A(s))^{\delta+1})\|_{L^p} \\
&\leq C\|R(t-s)\partial_x((v(s)+W_A(s))^{\delta+1})\|_{\mathrm{W}^{\frac{\delta}{p},\frac{p}{\delta+1}}} \\
&\leq C(1+(t-s)^{\frac{-p-\delta}{2p}})\|\partial_x((v(s)+W_A(s))^{\delta+1})\|_{\mathrm{W}^{-1,\frac{p}{\delta+1}}}
\end{aligned}
$$

$$
\begin{aligned}
&= C(1+(t-s)^{\frac{-p-\delta}{2p}})\|((v(s)+W_A(s))^{\delta+1})\|_{\mathrm{L}^{\frac{p}{\delta+1}}} \\
&= C(1+(t-s)^{\frac{-p-\delta}{2p}})\|v(s)+W_A(s)\|_{\mathrm{L}^p}^{\delta+1}. \qquad (3.13)
\end{aligned}
$$

Using $q_1 = p, q_2 = \frac{p}{\delta}$ and $k = \frac{p}{\delta+1}$ in (3.11), and $r_1 = 0$, $r_2 = \frac{\delta}{p}$ and $\theta = \frac{p}{\delta+1}$, we obtain

$$
\begin{aligned}
&\|R(t-s)(v(s)+W_A(s))^{\delta+1}\|_{L^p} \\
&\leq C\|R(t-s)(v(s)+W_A(s))^{\delta+1}\|_{\mathrm{W}^{\frac{\delta}{p},\frac{p}{\delta+1}}} \\
&\leq C(1+(t-s)^{\frac{-\delta}{2p}})\|(v(s)+W_A(s))^{\delta+1}\|_{L^{\frac{p}{\delta+1}}} \\
&\leq C(1+(t-s)^{\frac{-\delta}{2p}})\|v(s)+W_A(s)\|_{L^p}^{\delta+1}. \qquad (3.14)
\end{aligned}
$$

Taking $q_1 = p, q_2 = \frac{p}{2\delta+1}$ and $k = \frac{2\delta}{p}$ in (3.11), and $r_1 = 0$, $r_2 = \frac{2\delta}{p}$ and $\theta = \frac{p}{2\delta+1}$, we estimate the term $\|R(t-s)(u(s))^{2\delta+1}\|_{\mathrm{L}^p}$ as

$$
\begin{aligned}
&\|R(t-s)(v(s)+W_A(s))^{2\delta+1}\|_{L^p} \\
&\leq C\|R(t-s)(v(s)+W_A(s))^{2\delta+1}\|_{\mathrm{W}^{\frac{2\delta}{p},\frac{p}{2\delta+1}}} \\
&\leq C(1+(t-s)^{-\frac{\delta}{p}})\|(v(s)+W_A(s))^{2\delta+1}\|_{\mathrm{L}^{\frac{p}{2\delta+1}}} \\
&\leq C(1+(t-s)^{-\frac{\delta}{p}})\|v(s)+W_A(s)\|_{\mathrm{L}^p}^{2\delta+1}. \qquad (3.15)
\end{aligned}
$$

Combining (3.13)–(3.15) and substituting it in (3.10), we find

$$
\begin{aligned}
&\|z(t)\|_{L^p} \\
&\leq \|u_0\|_{L^p} + \frac{C\alpha}{\delta+1}\int_0^t (1+(t-s)^{\frac{-p-\delta}{2p}})\|v(s)+W_A(s)\|_{\mathrm{L}^p}^{\delta+1}\mathrm{d}s \\
&\quad + \beta\gamma\int_0^t \|v(s)+W_A(s)\|_{L^p}\mathrm{d}s \\
&\quad + C\beta(1+\gamma)\int_0^t (1+(t-s)^{\frac{-\delta}{2p}})\|v(s)+W_A(s)\|_{L^p}^{\delta+1}\mathrm{d}s \\
&\quad + C\beta\int_0^t (1+(t-s)^{\frac{-\delta}{p}})\|v(s)+W_A(s)\|_{\mathrm{L}^p}^{2\delta+1}\mathrm{d}s \\
&\leq \|u_0\|_{L^p} + \frac{C\alpha}{\delta+1}\left(\sup_{s\in[0,t]}\|v(s)\|_{L^p} + \sup_{s\in[0,t]}\|W_A(s)\|_{\mathrm{L}^p}\right)^{\delta+1}\left(t+\frac{2p}{p-\delta}t^{\frac{p-\delta}{2p}}\right) \\
&\quad + \beta\gamma t\left(\sup_{s\in[0,t]}\|v(s)\|_{L^p} + \sup_{s\in[0,t]}\|W_A(s)\|_{L^p}\right) \\
&\quad + C\beta(1+\gamma)\left(\sup_{s\in[0,t]}\|v(s)\|_{L^p} + \sup_{s\in[0,t]}\|W_A(s)\|_{L^p}\right)^{\delta+1}\left(t+\frac{2p}{2p-\delta}t^{\frac{2p-\delta}{2p}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ C\beta \left(\sup_{s\in[0,t]} \|v(s)\|_{L^p} + \sup_{s\in[0,t]} \|W_A(s)\|_{L^p} \right)^{2\delta+1} \left(t + \frac{p}{p-\delta} t^{\frac{p-\delta}{p}} \right) \\
&\le \|u_0\|_{L^p} + \frac{C\alpha}{\delta+1} \left(t + \frac{2p}{p-\delta} t^{\frac{p-\delta}{2p}} \right) \left(m+\mu_p\right)^{\delta+1} + \beta\gamma t (m+\mu_p) \\
&+ C\beta(1+\gamma) \left(m+\mu_p\right)^{\delta+1} \left(t + \frac{2p}{2p-\delta} t^{\frac{2p-\delta}{2p}} \right) \\
&+ C\beta \left(m+\mu_p\right)^{2\delta+1} \left(t + \frac{p}{p-\delta} t^{\frac{p-\delta}{p}} \right),
\end{aligned} \tag{3.16}
$$

provided $p > \delta$, where

$$
\mu_p = \sup_{t\in[0,T]} \|W_A(t)\|_{L^p}.
$$

Thus, $\|z(t)\|_{L^p} \le m$, for all $t \in [0, T^*]$, provided

$$
\begin{aligned}
&\|u_0\|_{L^p} + \frac{C\alpha}{\delta+1} \left(T^* + \frac{2p}{p-\delta} T^{*\frac{p-\delta}{2p}} \right) \left(m+\mu_p\right)^{\delta+1} + \beta\gamma T^* (m+\mu_p) \\
&+ C\beta(1+\gamma) \left(m+\mu_p\right)^{\delta+1} \left(T^* + \frac{2p}{2p-\delta} T^{*\frac{2p-\delta}{2p}} \right) \\
&+ C\beta \left(m+\mu_p\right)^{2\delta+1} \left(T^* + \frac{p}{p-\delta} T^{*\frac{p-\delta}{p}} \right) \le m.
\end{aligned} \tag{3.17}
$$

Since $\|u_0\|_{\mathrm{L}^p} < m$, then there exists a $T^* > 0$ satisfying (3.17).
Step 2. *G is a contraction on* $\Sigma(m, T^*)$. Let us now consider $v_1, v_2 \in \Sigma(m, T^*)$ and set $z_i = Gv_i$, for $i = 1, 2$ and $z = z_1 - z_2$. Then, $z(t)$ satisfies

$$
\begin{aligned}
z(t) &= \frac{\alpha}{\delta+1} \int_0^t R(t-s)\partial_x((v_1(s)+W_A(s))^{\delta+1} - (v_2(s)+W_A(s))^{\delta+1}) \mathrm{d}s \\
&+ \beta \int_0^t R(t-s) \Big\{ \big[(v_1(s)+W_A(s))(1-(v_1(s)+W_A(s))^{\delta})((v_1(s)+W_A(s))^{\delta}-\gamma)\big] \\
&- \big[(v_2(s)+W_A(s))(1-(v_2(s)+W_A(s))^{\delta})((v_2(s)+W_A(s))^{\delta}-\gamma)\big] \Big\} \mathrm{d}s.
\end{aligned} \tag{3.18}
$$

Using Taylor's formula, for some $0 < \theta_1 < 1$, we have

$$
(v_1+W_A)^{\delta+1} - (v_2+W_A)^{\delta+1} = (\delta+1)(v_1-v_2)(\theta_1(v_1+W_A) + (1-\theta_1)(v_2+W_A))^{\delta}. \tag{3.19}
$$

A calculation similar to (3.13) yields

$$
\begin{aligned}
&\|R(t-s)\partial_x((v_1(s)+W_A(s))^{\delta+1} - (v_2(s)+W_A(s))^{\delta+1})\|_{L^p} \\
&= (\delta+1)\|R(t-s)\partial_x \\
&\quad \big[((v_1-v_2)(s))((\theta_1(v_1(s)+W_A(s)) + (1-\theta_1)(v_2(s)+W_A(s)))^{\delta})\big]\|_{L^p}
\end{aligned}
$$

$$
\begin{aligned}
&\le C(\delta+1)(1+(t-s)^{\frac{-p-\delta}{2p}})\|(v_1-v_2)(s)\|_{L^p}\|\theta_1(v_1(s)\\
&\quad+W_A(s))+(1-\theta_1)(v_2(s)+W_A(s))\|_{L^p}^{\delta}\\
&\le C(\delta+1)(1+(t-s)^{\frac{-p-\delta}{2p}})\|(v_1-v_2)(s)\|_{L^p}(\|v_1(s)\|_{\mathrm{L}^p}\\
&\quad+\|v_2(s)\|_{L^p}+2\|W_A(s)\|_{L^p})^{\delta}\\
&\le C(\delta+1)(m+\mu_p)^{\delta}(1+(t-s)^{\frac{-p-\delta}{2p}})\|(v_1-v_2)(s)\|_{\mathrm{L}^p}. \qquad (3.20)
\end{aligned}
$$

Similar to (3.19), for some $0<\theta_2,\theta_3<1$, we get

$$
\begin{aligned}
&\left[(v_1+W_A)(1-(v_1+W_A)^{\delta})((v_1+W_A)^{\delta}-\gamma)\right]\\
&\quad-\left[(v_2+W_A)(1-(v_2+W_A)^{\delta})((v_2+W_A)^{\delta}-\gamma)\right]\\
&=-\gamma(v_1-v_2)+(\delta+1)(1+\gamma)(v_1-v_2)[\theta_2(v_1+W_A)+(1-\theta_2)(v_2+W_A)]^{\delta}\\
&\quad-(2\delta+1)(v_1-v_2)[\theta_3(v_1+W_A)+(1-\theta_3)(v_2+W_A)]^{2\delta}.
\end{aligned}
$$

Estimates similar to (3.14) and (3.15) yield

$$
\begin{aligned}
&\Big\|R(t-s)\Big\{\Big[(v_1(s)+W_A(s))(1-(v_1(s)+W_A(s))^{\delta})((v_1(s)+W_A(s))^{\delta}-\gamma)\Big]\\
&\quad-\Big[(v_2(s)+W_A(s))(1-(v_2(s)+W_A(s))^{\delta})((v_2(s)+W_A(s))^{\delta}-\gamma)\Big]\Big\}\Big\|_{L^p}\\
&\le\gamma\|(v_1-v_2)(s)\|_{\mathrm{L}^p}\\
&\quad+C(\delta+1)(1+\gamma)\left(1+(t-s)^{-\frac{\delta}{2p}}\right)\|(v_1-v_2)(s)\|_{L^p}\\
&\qquad\times(\|v_1(s)\|_{L^p}+\|v_2(s)\|_{\mathrm{L}^p}+2\|W_A(s)\|_{\mathrm{L}^p})^{\delta}\\
&\quad+C(2\delta+1)\left(1+(t-s)^{-\frac{\delta}{p}}\right)\|(v_1-v_2)(s)\|_{L^p}\\
&\qquad\times(\|v_1(s)\|_{L^p}+\|v_2(s)\|_{L^p}+2\|W_A(s)\|_{L^p})^{2\delta}\\
&\le\gamma\|(v_1-v_2)(s)\|_{L^p}+C(\delta+1)(1+\gamma)\left(1+(t-s)^{-\frac{\delta}{2p}}\right)\|(v_1-v_2)(s)\|_{L^p}\left(m+\mu_p\right)^{\delta}\\
&\quad+C(2\delta+1)\left(1+(t-s)^{-\frac{\delta}{p}}\right)\\
&\quad\|(v_1-v_2)(s)\|_{\mathrm{L}^p}\left(m+\mu_p\right)^{2\delta}. \qquad (3.21)
\end{aligned}
$$

Combining (3.20)–(3.21) and substituting it in (3.18), we obtain

$$
\begin{aligned}
&\|G(v_1-v_2)(t)\|_{\mathrm{L}^p}\\
&\le C\alpha(m+\mu_p)\int_0^t(1+(t-s)^{\frac{-p-\delta}{2p}})\|(v_1-v_2)(s)\|_{L^p}\mathrm{d}s+\beta\gamma\int_0^t\|(v_1-v_2)(s)\|_{L^p}\mathrm{d}s\\
&\quad+C\beta(1+\gamma)(1+\delta)(m+\mu_p)^{\delta}\int_0^t(1+(t-s)^{\frac{-\delta}{2p}})\|(v_1-v_2)(s)\|_{L^p}\mathrm{d}s\\
&\quad+C\beta(2\delta+1)(m+\mu_p)^{2\delta}\int_0^t(1+(t-s)^{\frac{-\delta}{p}})\|(v_1-v_2)(s)\|_{L^p}\mathrm{d}s.
\end{aligned}
$$

Therefore, we deduce that

$$\sup_{s\in[0,t]} \|G(v_1-v_2)(s)\|_{L^p} \leq \left\{C\alpha\left(t+\frac{2p}{p-\delta}t^{\frac{p-\delta}{2p}}\right)\left(m+\mu_p\right)^{\delta}+\beta\gamma t(m+\mu_p)\right.$$
$$+C\beta(1+\gamma)(1+\delta)\left(m+\mu_p\right)^{\delta}\left(t+\frac{2p}{2p-\delta}t^{\frac{2p-\delta}{2p}}\right)$$
$$\left.+C\beta(2\delta+1)\left(m+\mu_p\right)^{2\delta}\left(t+\frac{p}{p-\delta}t^{\frac{p-\delta}{p}}\right)\right\}\sup_{s\in[0,t]}\|(v_1-v_2)(s)\|_{L^p}, \quad (3.22)$$

provided $p>\delta$. We can choose a $T^*>0$ such that

$$C\alpha\left(T^*+\frac{2p}{p-\delta}T^{*\frac{p-\delta}{2p}}\right)\left(m+\mu_p\right)^{\delta}+\beta\gamma T^*(m+\mu_p)$$
$$+C\beta(1+\gamma)(1+\delta)\left(m+\mu_p\right)^{\delta}\left(T^*+\frac{2p}{2p-\delta}T^{*\frac{2p-\delta}{2p}}\right)$$
$$+C\beta(2\delta+1)\left(m+\mu_p\right)^{2\delta}\left(T^*+\frac{p}{p-\delta}T^{*\frac{p-\delta}{p}}\right)<1, \quad (3.23)$$

and (3.17) holds true for all $t\in[0,T^*]$. Hence, G is a strict contraction on $\Sigma(m,T^*)$ and it proves the existence of a mild solution to (3.6). Uniqueness follows from the representation (3.7). □

3.3 Global Existence and Uniqueness

Let us now show the global existence of generalized Burgers–Huxley equation (3.3). The result obtained in Theorem 3.2 is valid $\mathbb{P}$-a.s., and up to a random time as μ_p and T^* depend on $\omega\in\Omega$. In this subsection, we show that $T^*=T$, $\mathbb{P}$-a.s. (arbitrary time T), and hence, one can remove the dependence on ω for the time interval on which the solution exists. In order to prove our main result, we need the following lemma.

Lemma 3.3 *If $v\in \mathrm{C}([0,T];L^p(\mathcal{O}))$, $p>\delta$, satisfies (3.7), then*

$$\|v(t)\|_{L^p}^p+\frac{\nu p(p-1)}{2}\int_0^t\||v(s)|^{\frac{p-2}{2}}\partial_x v(s)\|_{\mathrm{L}^2}^2\mathrm{d}s+\frac{p\beta}{8}\int_0^t\|v(s)\|_{\mathrm{L}^{2\delta+p}}^{2\delta+p}\mathrm{d}s$$
$$\leq\|u_0\|_{L^p}^p+\left(\frac{2\delta T}{2\delta+p}\right)\left(\frac{8p}{(2\delta+p)\beta}\right)^{\frac{p}{2\delta}}\left\{\frac{(p-1)\alpha^2 2^{\delta-2}}{\nu}+\beta(1+\gamma)^2 2^{2\delta}\right.$$
$$+\beta(1+\gamma)2^{\delta}+\beta\gamma+\beta(2\delta+1)^{2\delta}2^{2\delta(2\delta-1)}\mu_\infty^{2\delta}+\beta(2\delta+1)2^{2\delta-1}$$
$$\left.+\left(\frac{(p-1)\alpha^2 2^{(\delta-1)}}{2\nu}\right)^{\delta}\frac{1}{\delta}\left(\frac{4(\delta-1)}{\beta\delta}\right)^{\delta-1}\mu_\infty^{2\delta}\right\}^{\frac{2\delta+p}{2\delta}}$$
$$+T\left\{\frac{(p-1)^2\alpha^2 2^{2\delta-1}}{p\nu}\left(\frac{p-2}{p}\right)^{\frac{2}{p-2}}+\frac{\beta(1+\gamma)2^{\delta}}{p}\left(\frac{p-1}{p}\right)^{p-1}\right\}\mu_\infty^{p(\delta+1)}$$

$$+\frac{\beta(1+\gamma)2^{\delta}T}{p}\left(\frac{p-1}{p}\right)^{p-1}\mu_{\infty}^{p(\delta+1)}+\frac{\beta(2\delta+1)2^{2\delta-1}T}{p}\left(\frac{p-1}{p}\right)^{p-1}\mu_{\infty}^{p(2\delta+1)}, \tag{3.24}$$

for all $t\in[0,T]$*, where* $\mu_{\infty}=\sup_{t\in[0,T]}\|W_A(t)\|_{\mathrm{L}^{\infty}}$.

Proof Let u_0^n be a sequence in $\mathrm{C}^{\infty}(\mathcal{O})$ such that

$$u_0^n\to u_0,\ \text{ in }\ \mathrm{L}^p(\mathcal{O}),$$

and let $W^n(t)=\sum_{j=1}^n w_j\beta_j(t)$ be the finite-dimensional approximation of $W(t)$ defined in (1.2). Then, using Itô's isometry, it can be easily seen that

$$\begin{aligned}\mathbb{E}\left[|W_A(t)-W_A^n(t)|^2\right]&=\mathbb{E}\left[\left|\sum_{j=n+1}^{\infty}\int_0^t e^{-\nu\lambda_j(t-s)}w_j(x)d\beta_j(t)\right|^2\right]\\&=\sum_{j=n+1}^{\infty}\mathbb{E}\left[\int_0^t e^{-2\nu\lambda_j(t-s)}|w_j(x)|^2\mathrm{d}s\right]\\&\le\frac{2}{\pi}\sum_{j=n+1}^{\infty}\left(\frac{1-e^{-2\nu\lambda_j t}}{2\nu\lambda_j}\right)\\&\le\frac{1}{\pi\nu}\sum_{j=n+1}^{\infty}\frac{1}{\lambda_j}=\frac{1}{\pi^3\nu}\sum_{j=n+1}^{\infty}\frac{1}{j^2}\to 0\ \text{ as }\ n\to\infty,\end{aligned}$$

for every $(t,x)\in[0,T]\times[0,1]$. Thus, along a subsequence, we obtain $W_A^n(t)\to W_A(t)$, $\mathbb{P}$-a.s., for all for every $(t,x)\in[0,T]\times[0,1]$. Since $W_A^n(\cdot)$ and $W_A(\cdot)$ are unique solutions, the whole sequence converges. Furthermore, for every $\varepsilon>0$, there exist a $\delta=\varepsilon$ and $N\in\mathbb{N}$, so that $n\ge N$ and $\mathbb{E}\left[|W_A(t)-W_A^n(t)|^2\right]\le\delta$ implies

$$\mathbb{E}\left[|W_A(t)-W_A^{n+k}(t)|^2\right]\le\frac{1}{\pi^3\nu}\sum_{j=n+k+1}^{\infty}\frac{1}{j^2}\le\frac{1}{\pi^3\nu}\sum_{j=n+1}^{\infty}\frac{1}{j^2}\le\varepsilon,$$

for all $(t,x)\in[0,T]\times[0,1]$ and $n\in\mathbb{N}$. The above estimate implies the uniform convergence $\mathbb{P}$-a.s. along a further subsequence (cf. Theorem 1, [19]), and the continuity of the processes $W_A^n(\cdot)$ and $W_A(\cdot)$ (uniqueness also) implies that

$$W_A^n(t)=\int_0^t R(t-s)\mathrm{d}W^n(s)\to W_A(t),$$

in $\mathrm{C}([0,T]\times[0,1])$, $\mathbb{P}$-a.s. Note that

$$\mu_{n,p}:=\sup_{t\in[0,T]}\|W_A^n(t)\|_{\mathrm{L}^p}\le\sup_{t\in[0,T]}\|W_A(t)\|_{\mathrm{L}^p}=:\mu_p, \tag{3.25}$$

for all $p \in [2, \infty]$.

Let $v^n(\cdot)$ be a solution of

$$\begin{aligned} v^n(t) &= R(t)u_0^n - \frac{\alpha}{\delta+1}\int_0^t R(t-s)\partial_x((v^n(s)+W_A^n(s))^\delta)\mathrm{d}s \\ &\quad + \beta\int_0^t R(t-s)(v^n(s)+W_A^n(s))(1-(v^n(s)+W_A^n(s))^\delta)((v^n(s)+W_A^n(s))^\delta-\gamma)\mathrm{d}s. \end{aligned} \tag{3.26}$$

Making use of Theorem 3.2, we know that v^n exists on an interval $[0, T_n]$ such that $T_n \to T^*$, $\mathbb{P}$-a.s., and that v^n converges to v in $\mathrm{C}([0, T^*]; L^p(\mathcal{O}))$, $\mathbb{P}$-a.s., for $\delta < p < \infty$. This result can be obtained in the following way: Let us consider

$$\begin{aligned} &\|v^n(t)-v(t)\|_{L^p} \\ &\le \|R(t)(u_0^n-u_0)\|_{L^p} \\ &\quad + \frac{\alpha}{\delta+1}\int_0^t \|R(t-s)\partial_x(((v^n(s)+W_A^n(s))^\delta-(v(s)+W_A(s))^\delta))\|_{\mathrm{L}^p}ds \\ &\quad + \beta\int_0^t \|R(t-s)[(v^n(s)+W_A^n(s))(1-(v^n(s)+W_A^n(s))^\delta)((v^n(s)+W_A^n(s))^\delta-\gamma) \\ &\qquad -(v^n(s)+W_A^n(s))(1-(v^n(s)+W_A^n(s))^\delta)((v^n(s)+W_A^n(s))^\delta-\gamma)]\|_{\mathrm{L}^p}\mathrm{d}s \\ &\le \|u_0^n-u_0\|_{\mathrm{L}^p} + C\alpha(m+\mu_p)^\delta\int_0^t (1+(t-s)^{\frac{-p-\delta}{2p}})(\|v^n(s)-v(s)\|_{\mathrm{L}^p} \\ &\quad + \|W_A^n(s)-W_A(s)\|_{L^p})\mathrm{d}s \\ &\quad + C\beta\gamma\int_0^t (\|v^n(s)-v(s)\|_{L^p}+\|W_A^n(s)-W_A(s)\|_{L^p})\mathrm{d}s \\ &\quad + C\beta(\delta+1)(1+\gamma)(m+\mu_p)^\delta \\ &\qquad \times\int_0^t (1+(t-s)^{\frac{-\delta}{2p}})(\|v^n(s)-v(s)\|_{L^p}+\|W_A^n(s)-W_A(s)\|_{L^p})\mathrm{d}s \\ &\quad + C\beta(2\delta+1)(m+\mu_p)^{2\delta} \\ &\qquad \times\int_0^t (1+(t-s)^{\frac{-\delta}{2p}})(\|v^n(s)-v(s)\|_{\mathrm{L}^p}+\|W_A^n(s)-W_A(s)\|_{L^p})\mathrm{d}s, \end{aligned}$$

where we have used calculations similar to (3.20)–(3.21). Thus, from the above estimate it is immediate that

$$\begin{aligned} &\|v^n(t)-v(t)\|_{L^p} \\ &\le \left\{\|u_0^n-u_0\|_{\mathrm{L}^p} + C(\alpha,\beta,\gamma,\delta,m,\mu_p,T^*)\sup_{s\in[0,T^*]}\|W_A^n(s)-W_A(s)\|_{L^p}\right\} \\ &\quad + C(\alpha,\beta,\gamma,\delta,m,\mu_p) \\ &\quad \times\int_0^t (1+(t-s)^{\frac{-p-\delta}{2p}}+(t-s)^{\frac{-\delta}{2p}}+(t-s)^{\frac{-\delta}{p}})\|v^n(s)-v(s)\|_{L^p}\mathrm{d}s, \end{aligned} \tag{3.27}$$

for all $t \in [0, T^n]$. An application of Gronwall's inequality in (3.27) yields

$$\sup_{t\in[0,T^n]} \|v^n(t) - v(t)\|_{L^p} \\ \leq C(\alpha, \beta, \gamma, \delta, m, \mu_p, T^*)\left\{\|u_0^n - u_0\|_{\mathrm{L}^p} + \sup_{s\in[0,T]} \|W_A^n(s) - W_A(s)\|_{L^p}\right\}, \tag{3.28}$$

for $p > \delta$. On passing $n \to \infty$, and using the continuity of supremum and of the processes $v^n(\cdot)$ and $v(\cdot)$, we obtain the required result.

Note that the existence of a mild solution ensures the existence of a weak solution (cf. [2]) also and hence the mapping $t \mapsto \|v^n(t)\|_{L^2}^2$ is absolutely continuous (for each fixed $\omega \in \Omega$). Thus, we know $v^n(\cdot)$ satisfies the following equation:

$$\frac{\partial v^n}{\partial t} = \nu\frac{\partial^2 v^n}{\partial x^2} - \alpha(v^n + W_A^n)^\delta \frac{\partial}{\partial x}(v^n + W_A^n) \\ + \beta(v^n + W_A^n)(1 - (v^n + W_A^n)^\delta)((v^n + W_A^n)^\delta - \gamma), \tag{3.29}$$

$\mathbb{P}$-a.s. in $\mathrm{H}^{-1}(\mathcal{O})$. Multiplying (3.29) by $|v^n|^{p-2}v^n$, integrating over $\mathcal{O}$ and then using Taylor's formula, we find

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|v^n(t)\|_{L^p}^p + \nu(p-1)\||v^n(t)|^{\frac{p-2}{2}}\partial_x v^n(t)\|_{\mathrm{L}^2}^2 \\
&= -\alpha((v^n(t) + W_A^n(t))\partial_x(v^n(t) + W_A^n(t))^\delta, |v^n(t)|^{p-2}v^n(t)) \\
&\quad + \beta(1+\gamma)((v^n(t) + W_A^n(t))^{\delta+1}, |v^n(t)|^{p-2}v^n(t)) \\
&\quad - \beta\gamma(v^n(t) + W_A^n(t), |v^n(t)|^{p-2}v^n(t)) - \beta((v^n(t) + W_A^n(t))^{2\delta+1}, |v^n(t)|^{p-2}v^n(t)) \\
&= -\frac{\alpha}{\delta+1}(\partial_x(v^n(t) + W_A^n(t))^{\delta+1}, |v^n(t)|^{p-2}v^n(t)) \\
&\quad + \beta(1+\gamma)((v^n(t) + W_A^n(t))^{\delta+1}, |v^n(t)|^{p-2}v^n(t)) \\
&\quad - \beta\gamma\|v^n(t)\|_{L^p}^p - \beta\gamma(W_A^n(t), |v^n(t)|^{p-2}v^n(t)) - \beta\|v^n(t)\|_{L^{2\delta+p}}^{2\delta+p} \\
&\quad - \beta(2\delta+1)(W_A^n(t)(\theta_2 v^n(t) + (1-\theta_2)W_A^n(t))^{2\delta}, |v^n(t)|^{p-2}v^n(t)),
\end{aligned} \tag{3.30}$$

for $0 < \theta_2 < 1$. It can be easily deduced from (3.30) that

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\|v^n(t)\|_{L^p}^p + \nu(p-1)\||v^n(t)|^{\frac{p-2}{2}}\partial_x v^n(t)\|_{\mathrm{L}^2}^2 + \beta\gamma\|v^n(t)\|_{\mathrm{L}^p}^p + \beta\|v^n(t)\|_{\mathrm{L}^{2\delta+p}}^{2\delta+p} \\
&= -\frac{\alpha}{\delta+1}(\partial_x(v^n(t) + W_A^n(t))^{\delta+1}, |v^n(t)|^{p-2}v^n(t)) \\
&\quad + \beta(1+\gamma)((v^n(t) + W_A^n(t))^{\delta+1}, |v^n(t)|^{p-2}v^n(t)) - \beta\gamma(W_A^n(t), |v^n(t)|^{p-2}v^n(t)) \\
&\quad - \beta(2\delta+1)(W_A^n(t)(\theta_2 v^n(t) + (1-\theta_2)W_A^n(t))^{2\delta}, |v^n(t)|^{p-2}v^n(t)) =: \sum_{j=1}^4 J_j,
\end{aligned} \tag{3.31}$$

where J_j's, $j = 1, \ldots, 4$ represents the terms appearing in the right-hand side of the equality (3.31). An integration by parts, Taylor's formula and (2.4) yield

$$\begin{aligned}
&\frac{\alpha}{\delta+1}(\partial_x(v^n + W_A^n)^{\delta+1}, |v^n|^{p-2}v^n) \\
&= -\frac{\alpha}{\delta+1}(p-1)((v^n + W_A^n)^{\delta+1}, |v^n|^{p-2}\partial_x v^n) \\
&= -\frac{\alpha}{\delta+1}(p-1)((v^n)^{\delta+1}, |v^n|^{p-2}\partial_x v^n) \\
&\quad - \frac{\alpha}{\delta+1}(p-1)(\delta+1)((\theta_1 v^n + (1-\theta_1)W_A^n)^{\delta} W_A^n, |v^n|^{p-2}\partial_x v^n) \\
&= -\alpha(p-1)((\theta_1 v^n + (1-\theta_1)W_A^n)^{\delta} W_A^n, |v^n|^{p-2}\partial_x v^n),
\end{aligned} \tag{3.32}$$

for $0 < \theta_1 < 1$. The term on the right-hand side of the equality (3.32) can be estimated using Hölder's, interpolation and Young's inequalities as

$$\begin{aligned}
&\alpha(p-1)|(W_A^n(\theta_1 v^n + (1-\theta_1)W_A^n)^{\delta}, |v^n|^{p-2}\partial_x v^n)| \\
&\leq \alpha(p-1)\||v^n|^{\frac{p-2}{2}} W_A^n(\theta_1 v^n + (1-\theta_1)W_A^n)^{\delta}\|_{\mathrm{L}^2}\||v^n|^{\frac{p-2}{2}}\partial_x v^n\|_{\mathrm{L}^2} \\
&\leq \frac{\nu}{2}\||v^n|^{\frac{p-2}{2}}\partial_x v^n\|_{\mathrm{L}^2}^2 + \frac{(p-1)^2\alpha^2}{2\nu}\||v^n|^{\frac{p-2}{2}} W_A^n(\theta_1 v^n + (1-\theta_1)W_A^n)^{\delta}\|_{\mathrm{L}^2}^2 \\
&\leq \frac{\nu}{2}\||v^n|^{\frac{p-2}{2}}\partial_x v^n\|_{\mathrm{L}^2}^2 + \frac{(p-1)^2\alpha^2 2^{\delta-1}}{2\nu}\left(\|W_A^n\|_{\mathrm{L}^\infty}^2\|v^n\|_{\mathrm{L}^{p+2\delta-2}}^{p+2\delta-2} + \|W_A^n\|_{\mathrm{L}^\infty}^{2\delta+2}\|v^n\|_{\mathrm{L}^{p-2}}^{p-2}\right) \\
&\leq \frac{\nu}{2}\||v^n|^{\frac{p-2}{2}}\partial_x v^n\|_{\mathrm{L}^2}^2 + \frac{(p-1)^2\alpha^2 2^{\delta-1}}{2\nu}\|W_A^n\|_{\mathrm{L}^\infty}^2\|v^n\|_{\mathrm{L}^{p+2\delta}}^{\frac{(\delta-1)(p+2\delta)}{\delta}}\|v^n\|_{\mathrm{L}^p}^{\frac{p}{\delta}} \\
&\quad + \frac{(p-1)^2\alpha^2 2^{\delta-1}}{2\nu}\|W_A^n\|_{\mathrm{L}^\infty}^{2\delta+2}\|v^n\|_{\mathrm{L}^p}^{p-2} \\
&\leq \frac{\nu}{2}\||v^n|^{\frac{p-2}{2}}\partial_x v^n\|_{\mathrm{L}^2}^2 + \frac{\beta}{4}\|v^n\|_{\mathrm{L}^{p+2\delta}}^{p+2\delta} + \left(\frac{(p-1)^2\alpha^2 2^{(\delta-1)}}{2\nu}\right)^{\delta}\frac{1}{\delta}\left(\frac{4(\delta-1)}{\beta\delta}\right)^{\delta-1} \\
&\quad \|W_A^n\|_{\mathrm{L}^\infty}^{2\delta}\|v^n\|_{\mathrm{L}^p}^{p} \\
&\quad + \frac{(p-1)^2\alpha^2 2^{\delta-2}}{\nu}\|v^n\|_{\mathrm{L}^p}^{p} + \frac{(p-1)^2\alpha^2 2^{\delta-1}}{p\nu}\left(\frac{p-2}{p}\right)^{\frac{2}{p-2}}\|W_A^n\|_{\mathrm{L}^\infty}^{p(\delta+1)}.
\end{aligned} \tag{3.33}$$

We estimate J_2 using Hölder's, interpolation and Young's inequalities as

$$\begin{aligned}
|J_2| &\leq \beta(1+\gamma)2^{\delta}(|v^n|^{\delta+1} + |W_A^n|^{\delta+1}, |v^n|^{p-1}) \\
&\leq \beta(1+\gamma)2^{\delta}\|v^n\|_{\mathrm{L}^{\delta+p}}^{\delta+p} + \beta(1+\gamma)2^{\delta}\|W_A^n\|_{\mathrm{L}^\infty}^{\delta+1}\|v^n\|_{\mathrm{L}^{p-1}}^{p-1} \\
&\leq \beta(1+\gamma)2^{\delta}\|v^n\|_{\mathrm{L}^{2\delta+p}}^{\frac{2\delta+p}{2}}\|v^n\|_{\mathrm{L}^p}^{\frac{p}{2}} + \beta(1+\gamma)2^{\delta}\|W_A^n\|_{\mathrm{L}^\infty}^{\delta+1}\|v^n\|_{\mathrm{L}^p}^{p-1} \\
&\leq \frac{\beta}{4}\|v^n\|_{\mathrm{L}^{2\delta+p}}^{2\delta+p} + \beta(1+\gamma)^2 2^{2\delta}\|v^n\|_{\mathrm{L}^p}^{p} + \beta(1+\gamma)2^{\delta}\|v^n\|_{\mathrm{L}^p}^{p} \\
&\quad + \frac{\beta(1+\gamma)2^{\delta}}{p}\left(\frac{p-1}{p}\right)^{p-1}\|W_A^n\|_{\mathrm{L}^\infty}^{p(\delta+1)}.
\end{aligned} \tag{3.34}$$

Similarly, we estimate J_3 and J_4 as

$$\begin{aligned} |J_3| &\le \beta\gamma \|W_A^n\|_{L^\infty}\|v^n\|_{L^{p-1}}^{p-1} \le \beta\gamma\|W_A^n\|_{\mathrm{L}^\infty}\|v^n\|_{\mathrm{L}^p}^{p-1} \\ &\le \beta\gamma\|v^n\|_{\mathrm{L}^p}^p + \frac{\beta\gamma}{p}\left(\frac{p-1}{p}\right)^{p-1}\|W_A^n\|_{\mathrm{L}^\infty}^p, \qquad (3.35)\\ |J_4| &\le \beta(2\delta+1)2^{2\delta-1}(|W_A^n||v^n|^{2\delta} + |W_A^n|^{2\delta+1}, |v^n|^{p-1}) \\ &\le \beta(2\delta+1)2^{2\delta-1}\|W_A^n\|_{\mathrm{L}^\infty}\|v^n\|_{\mathrm{L}^{2\delta+p-1}}^{2\delta+p-1} \\ &\quad + \beta(2\delta+1)2^{2\delta-1}\|W_A^n\|_{\mathrm{L}^\infty}^{2\delta+1}\|v^n\|_{\mathrm{L}^{p-1}}^{p-1} \\ &\le \beta(2\delta+1)2^{2\delta-1}\|W_A^n\|_{\mathrm{L}^\infty}\|v^n\|_{\mathrm{L}^{2\delta+p}}^{\frac{(2\delta-1)(2\delta+p)}{2\delta}}\|v^n\|_{\mathrm{L}^p}^{\frac{p}{2\delta}} \\ &\quad + \beta(2\delta+1)2^{2\delta-1}\|W_A^n\|_{\mathrm{L}^\infty}^{2\delta+1}\|v^n\|_{\mathrm{L}^p}^{p-1} \\ &\le \frac{\beta}{4}\|v^n\|_{\mathrm{L}^{2\delta+p}}^{2\delta+p} + \beta(2\delta+1)^{2\delta}2^{2\delta(2\delta-1)}\|W_A^n\|_{\mathrm{L}^\infty}^{2\delta}\|v^n\|_{L^p}^p \qquad (3.36)\\ &\quad + \beta(2\delta+1)2^{2\delta-1}\|v^n\|_{\mathrm{L}^p}^p + \frac{\beta(2\delta+1)2^{2\delta-1}}{p}\left(\frac{p-1}{p}\right)^{p-1}\|W_A^n\|_{\mathrm{L}^\infty}^{p(2\delta+1)}. \end{aligned}$$

Combining (3.33)–(3.36) and substituting it in (3.31) yield

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\|v^n(t)\|_{L^p}^p + \frac{\nu(p-1)}{2}\||v^n(t)|^{\frac{p-2}{2}}\partial_x v^n(t)\|_{L^2}^2 + \beta\gamma\|v^n(t)\|_{L^p}^p) \\ &\quad + \frac{\beta}{4}\|v^n(t)\|_{L^{2\delta+p}}^{2\delta+p} \\ &\le \left\{\frac{(p-1)\alpha^2 2^{\delta-2}}{\nu} + \beta(1+\gamma)^2 2^{2\delta} + \beta(1+\gamma)2^\delta + \beta\gamma \right.\\ &\quad + \beta(2\delta+1)^{2\delta}2^{2\delta(2\delta-1)}\|W_A^n(t)\|_{L^\infty}^{2\delta} + \beta(2\delta+1)2^{2\delta-1} \\ &\quad \left. + \left(\frac{(p-1)\alpha^2 2^{(\delta-1)}}{2\nu}\right)^\delta \frac{1}{\delta}\left(\frac{4(\delta-1)}{\beta\delta}\right)^{\delta-1}\|W_A^n(t)\|_{L^\infty}^{2\delta}\right\}\|v^n(t)\|_{\mathrm{L}^p}^p \\ &\quad + \left\{\frac{(p-1)\alpha^2 2^{\delta-1}}{p\nu}\left(\frac{p-2}{p}\right)^{\frac{2}{p-2}} + \frac{\beta(1+\gamma)2^\delta}{p}\left(\frac{p-1}{p}\right)^{p-1}\right\}\|W_A^n(t)\|_{L^\infty}^{p(\delta+1)} \\ &\quad + \frac{\beta(1+\gamma)2^\delta}{p}\left(\frac{p-1}{p}\right)^{p-1}\|W_A^n(t)\|_{\mathrm{L}^\infty}^{p(\delta+1)} \\ &\quad + \frac{\beta(2\delta+1)2^{2\delta-1}}{p}\left(\frac{p-1}{p}\right)^{p-1}\|W_A^n(t)\|_{L^\infty}^{p(2\delta+1)} \\ &\le \frac{\beta}{8}\|v^n(t)\|_{\mathrm{L}^{2\delta+p}}^{2\delta+p} + \left(\frac{2\delta}{2\delta+p}\right)\left(\frac{8p}{(2\delta+p)\beta}\right)^{\frac{p}{2\delta}}\left\{\frac{(p-1)\alpha^2 2^{\delta-2}}{\nu}\right. \\ &\quad + \beta(1+\gamma)^2 2^{2\delta} + \beta(1+\gamma)2^\delta \\ &\quad + \beta\gamma + \beta(2\delta+1)^{2\delta}2^{2\delta(2\delta-1)}\|W_A^n(t)\|_{L^\infty}^{2\delta} + \beta(2\delta+1)2^{2\delta-1} \end{aligned}$$

$$+\left(\frac{(p-1)\alpha^2 2^{(\delta-1)}}{2\nu}\right)^{\delta}\frac{1}{\delta}\left(\frac{4(\delta-1)}{\beta\delta}\right)^{\delta-1}\|W_A^n(t)\|_{\mathrm{L}^\infty}^{2\delta}\Bigg\}^{\frac{2\delta+p}{2\delta}}$$

$$+\left\{\frac{(p-1)\alpha^2 2^{\delta-1}}{p\nu}\left(\frac{p-2}{p}\right)^{\frac{2}{p-2}}+\frac{\beta(1+\gamma)2^{\delta}}{p}\left(\frac{p-1}{p}\right)^{p-1}\right\}\|W_A^n(t)\|_{\mathrm{L}^\infty}^{p(\delta+1)}$$

$$+\frac{\beta(1+\gamma)2^{\delta}}{p}\left(\frac{p-1}{p}\right)^{p-1}\|W_A^n(t)\|_{\mathrm{L}^\infty}^{p(\delta+1)}$$

$$+\frac{\beta(2\delta+1)2^{2\delta-1}}{p}\left(\frac{p-1}{p}\right)^{p-1}\|W_A^n(t)\|_{\mathrm{L}^\infty}^{p(2\delta+1)}. \tag{3.37}$$

Integrating the above inequality from 0 to t, we find

$$\|v^n(t)\|_{\mathrm{L}^p}^p+\frac{\nu p(p-1)}{2}\int_0^t\||v^n(s)|^{\frac{p-2}{2}}\partial_x v^n(s)\|_{\mathrm{L}^2}^2\mathrm{d}s+p\beta\gamma\int_0^t\|v^n(s)\|_{\mathrm{L}^p}^p ds$$
$$+\frac{p\beta}{8}\int_0^t\|v^n(s)\|_{\mathrm{L}^{2\delta+p}}^{2\delta+p}\mathrm{d}s$$
$$\leq\|u_0\|_{\mathrm{L}^p}^p+\left(\frac{2\delta t}{2\delta+p}\right)\left(\frac{8p}{(2\delta+p)\beta}\right)^{\frac{p}{2\delta}}\Bigg\{\frac{(p-1)\alpha^2 2^{\delta-2}}{\nu}+\beta(1+\gamma)^2 2^{2\delta}$$
$$+\beta(1+\gamma)2^{\delta}$$
$$+\beta\gamma+\beta(2\delta+1)^{2\delta}2^{2\delta(2\delta-1)}\sup_{s\in[0,t]}\|W_A^n(s)\|_{\mathrm{L}^\infty}^{2\delta}+\beta(2\delta+1)2^{2\delta-1}$$
$$+\left(\frac{(p-1)\alpha^2 2^{(\delta-1)}}{2\nu}\right)^{\delta}\frac{1}{\delta}\left(\frac{4(\delta-1)}{\beta\delta}\right)^{\delta-1}\sup_{s\in[0,t]}\|W_A^n(s)\|_{\mathrm{L}^\infty}^{2\delta}\Bigg\}^{\frac{2\delta+p}{2\delta}}$$
$$+t\left\{\frac{(p-1)\alpha^2 2^{\delta-1}}{p\nu}\left(\frac{p-2}{p}\right)^{\frac{2}{p-2}}+\frac{\beta(1+\gamma)2^{\delta}}{p}\left(\frac{p-1}{p}\right)^{p-1}\right\}$$
$$\times\sup_{s\in[0,t]}\|W_A^n(s)\|_{\mathrm{L}^\infty}^{p(\delta+1)}$$
$$+\frac{\beta(1+\gamma)2^{\delta}t}{p}\left(\frac{p-1}{p}\right)^{p-1}\sup_{s\in[0,t]}\|W_A^n(s)\|_{\mathrm{L}^\infty}^{p(\delta+1)}$$
$$+\frac{\beta(2\delta+1)2^{2\delta-1}t}{p}\left(\frac{p-1}{p}\right)^{p-1}\sup_{s\in[0,t]}\|W_A^n(t)\|_{\mathrm{L}^\infty}^{p(2\delta+1)}, \tag{3.38}$$

for all $t\in[0,T]$, and hence, the estimate (3.24) follows by taking $n\to\infty$ in (3.38) and applying the dominated convergence theorem. □

Remark 3.4 For $\delta=1$ case, the term containing $\delta-1$ will not appear in the estimate (3.24).

The following theorem can be immediately deduced from Theorem 3.2 and Lemma 3.3.

Theorem 3.5 *Let the $\mathscr{F}_0$-measurable initial data u_0 be given and $u_0 \in \mathrm{L}^p(\mathcal{O})$, $\mathbb{P}$-a.s. Then, there exists a unique mild solution of Eq. (3.6), which belongs to* $\mathrm{C}([0,T];\mathrm{L}^p(\mathcal{O}))$, *$\mathbb{P}$-a.s., for $p > \delta$.*

Proof The existence of a local mild solution up to a stopping time has been established in Theorem 3.2, and uniform bounds for the L^p-norm in $[0,T]$ have been obtained in Lemma 3.3. From Lemma 3.3, we have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}\|v(t)\|_{\mathrm{L}^p}^p\right]\\
&\leq \mathbb{E}\left[\|u_0\|_{\mathrm{L}^p}^p\right] + \left(\frac{2^{\frac{2\delta+p}{2\delta}}\delta T}{2\delta+p}\right)\left(\frac{8p}{(2\delta+p)\beta}\right)^{\frac{p}{2\delta}}\\
&\quad\times\Bigg\{\Bigg(\frac{(p-1)\alpha^2 2^{\delta-2}}{\nu} + \beta(1+\gamma)^2 2^{2\delta} + \beta(1+\gamma)2^{\delta}\\
&\qquad + \beta\gamma + \beta(2\delta+1)2^{2\delta-1}\Bigg)^{\frac{2\delta+p}{2\delta}} + \beta(2\delta+1)^{2\delta}2^{2\delta(2\delta-1)}\mathbb{E}\left[\mu_\infty^{2\delta}\right]\\
&\qquad + \left(\frac{(p-1)\alpha^2 2^{(\delta-1)}}{2\nu}\right)^{\delta}\frac{1}{\delta}\left(\frac{4(\delta-1)}{\beta\delta}\right)^{\delta-1}\mathbb{E}\left[\mu_\infty^{2\delta}\right]\Bigg\}\\
&\quad + T\left\{\frac{(p-1)\alpha^2 2^{\delta-1}}{p\nu}\left(\frac{p-2}{p}\right)^{\frac{2}{p-2}} + \frac{\beta(1+\gamma)2^{\delta}}{p}\left(\frac{p-1}{p}\right)^{p-1}\right\}\mathbb{E}\left[\mu_\infty^{p(\delta+1)}\right]\\
&\quad + \frac{\beta(1+\gamma)2^{\delta}T}{p}\left(\frac{p-1}{p}\right)^{p-1}\mu_\infty^{p(\delta+1)} + \frac{\beta(2\delta+1)2^{2\delta-1}T}{p}\left(\frac{p-1}{p}\right)^{p-1}\mathbb{E}\left[\mu_\infty^{p(2\delta+1)}\right]\\
&=: M_T.
\end{aligned}
\tag{3.39}
$$

Let us now define a sequence of stopping times by

$$
\tau_m := \inf_{t\geq 0}\left\{t : \|v(t)\|_{\mathrm{L}^p} > m\right\}, \tag{3.40}
$$

for $m \in \mathbb{N}$. Note that $\tau_m \leq \tau_k$, whenever $m \leq k$. Thus, τ_m is an increasing sequence and let us define $\tau_\infty := \lim_{m\to\infty}\tau_m$. We need to show that $\tau_\infty = T$, $\mathbb{P}$-a.s. Or in other words, one has to show that $\mathbb{P}\{\omega \in \Omega : \tau_\infty(\omega) < T\} = 0$.

From Theorem 3.2, we know that a mild solution to Eq. (3.6) exists up to the stopping time $T \wedge \tau_m$. Let us consider the sets $A = \{\omega \in \Omega : \tau_\infty(\omega) < T\}$ and $B = \{\omega \in \Omega : \tau_m(\omega) < T\}$. Then, $\omega \in A$ implies $\tau_m \leq \tau_\infty < T$, and hence, $\omega \in B$, so that $A \subseteq B$. From the stopping time definition given in (3.40), we have

$$\begin{aligned}\mathbb{P}\left\{\omega\in\Omega:\tau_\infty(\omega)<T\right\}\leq\mathbb{P}\left\{\omega\in\Omega:\tau_m(\omega)<T\right\}\leq\mathbb{P}\left\{\sup_{t\in[0,T]}\|v(t)\|_{\mathrm{L}^p}\leq m\right\}\\ \leq\frac{1}{m^p}\mathbb{E}\left[\sup_{t\in[0,T]}\|v(t)\|_{\mathrm{L}^p}^p\right]=\frac{M_T}{m^p}\to 0\ \text{ as }\ m\to\infty,\end{aligned} \tag{3.41}$$

and hence, $\mathbb{P}\left\{\omega\in\Omega:\tau_\infty(\omega)<T\right\}=0$. Thus, we have $\tau_\infty=T$, $\mathbb{P}$-a.s. □

4 The Inviscid Limit

In this section, we take $\delta=1$ and discuss the inviscid limit of Eq. (3.3) as $\beta\to 0$. Let $u_\beta(\cdot)$ be the unique mild solution of Eq. (3.3). Equivalently, $v_\beta=u_\beta-W_A$ is a mild solution of Eq. (3.6) with $\delta=1$. One can easily show that $v_\beta(\cdot)$ satisfies:

$$\begin{aligned}&\sup_{0\leq t\leq T}\|v_\beta(t)\|_{\mathrm{L}^2}^2+\nu\int_0^T\|\partial_x v_\beta(t)\|_{\mathrm{L}^2}^2\mathrm{d}t+\frac{\beta}{4}\int_0^T\|v_\beta(t)\|_{\mathrm{L}^4}^4\mathrm{d}t\\ &\leq C(\alpha,\beta,\gamma,\nu,T)\left\{1+\|u_0\|_{\mathrm{L}^2}^2+\mu_\infty^4\right\}=:L_T,\ \mathbb{P}\text{-a.s.}\end{aligned} \tag{4.1}$$

We consider the following stochastic Burgers equation:

$$\begin{cases}\mathrm{d}u(t)=[-\nu Au(t)-\alpha B(u(t))]\mathrm{d}t+\mathrm{d}W(t),\ t\in(0,T),\\ u(0)=u_0\in\mathrm{L}^2(\mathcal{O}).\end{cases} \tag{4.2}$$

The existence and uniqueness of mild solution

$$u(t)=u_0-\alpha\int_0^t R(t-s)B(u(s))\mathrm{d}s+\int_0^t R(t-s)\mathrm{d}W(s),$$

of the above equation can be established in a similar way as in Sect. 3 (see [11] also). Equivalently, $v=u-W_A$ is the unique mild solution of the equation

$$\begin{cases}\dfrac{dv(t)}{dt}=[-\nu Av(t)-\alpha B(v(t)+W_A(t))],\ t\in(0,T),\\ u(0)=u_0\in\mathrm{L}^2(\mathcal{O}).\end{cases} \tag{4.3}$$

The existence of a mild solution to Eq. (4.3) ensures the existence of weak solution also. For $u_0\in\mathrm{L}^2(\mathcal{O})$, $\mathbb{P}$-a.s., the unique mild solution of Eq. (4.3) satisfies the following energy inequality:

$$\sup_{0\leq t\leq T}\|v(t)\|_{\mathrm{L}^2}^2+\nu\int_0^T\|\partial_x v(t)\|_{\mathrm{L}^2}^2\mathrm{d}t\leq\left(\|u_0\|_{\mathrm{L}^2}^2+\frac{\alpha^2T\mu_\infty^4}{2\nu}\right)e^{\frac{2\alpha^2T\mu_\infty^2}{\nu}}=:K_T,\ \mathbb{P}\text{-a.s.,} \tag{4.4}$$

where μ_∞ is defined in (3.25). Also, $u(\cdot)$ has the regularity $u \in \mathrm{C}([0,T];\mathrm{L}^2(\mathcal{O}))$, $\mathbb{P}$-a.s. Making use of Gagliardo–Nirenberg's inequality (Theorem 1, [28]), we also have

$$
\begin{aligned}
\int_0^T \|v(t)\|_{\mathrm{L}^4}^p dt &\leq C\int_0^T \|\partial_x v(t)\|_{\mathrm{L}^2}^{\frac{p}{4}}\|v(t)\|_{\mathrm{L}^2}^{\frac{3p}{4}}\mathrm{d}t\\
&\leq CT^{\frac{8-p}{8}}\sup_{t\in[0,T]}\|v(t)\|_{\mathrm{L}^2}^{\frac{3p}{4}}\left(\int_0^T \|\partial_x v(t)\|_{\mathrm{L}^2}^2\mathrm{d}t\right)^{\frac{p}{8}}\\
&\leq CT^{\frac{8-p}{8}}K_T<\infty,
\end{aligned}
$$

for $1\leq p\leq 8$.

Proposition 4.1 *Let $u_\beta(\cdot)$ be the unique mild solution of the stochastic Burgers–Huxley equation (see (3.3) with $\delta=1$), for $u_0\in\mathrm{L}^2(\mathcal{O})$, $\mathbb{P}$-a.s. As $\beta\to 0$, the mild solution $u_\beta(\cdot)$ of Eq. (3.3) tends to the mild solution of the stochastic Burgers equation (4.2), that is,*

$$
u_\beta \to u \ \ in\ \ \mathrm{C}([0,T];\mathrm{L}^2(\mathcal{O}))\cap\mathrm{L}^2(0,T;\mathrm{C}(\overline{\mathcal{O}})),\ \ \mathbb{P}\text{-}a.s.,\ \ as\ \ \beta\to 0.
$$

Proof Let us define $w_\beta = u_\beta - u = (u_\beta - W_A) - (u - W_A) = v_\beta - v$, then w_β satisfies:

$$
\begin{cases}
\dfrac{dw_\beta(t)}{\mathrm{d}t} = [-\nu Aw_\beta(t)+\alpha[B(v_\beta(t)+W_A(t))-B(v(t)+W_A(t))]+\beta c(v_\beta(t)+W_A(t))],\\
w_\beta(0)=0,
\end{cases}
\tag{4.5}
$$

in $\mathrm{H}^{-1}(\mathcal{O})$ for a.e. $t\in[0,T]$. Taking the inner product with $w_\beta(\cdot)$ to the first equation in (4.5) and then applying integration by parts, we find

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w_\beta(t)\|_{\mathrm{L}^2}^2+\nu\|\partial_x w_\beta(t)\|_{\mathrm{L}^2}^2\\
&=-\alpha[b(v_\beta(t)+W_A(t),v_\beta(t)+W_A(t),w_\beta(t))-b(v(t)+W_A(t),v(t)+W_A(t),w_\beta(t))]\\
&\quad+\beta(c(v_\beta(t)+W_A(t)),w_\beta(t))\\
&=-\alpha[b(w_\beta(t),v(t),w_\beta(t))+b(v(t),w_\beta(t),w_\beta(t))+b(w_\beta(t),W_A(t),w_\beta(t))\\
&\quad+b(w_\beta(t),W_A(t),w_\beta(t))+b(W_A(t),w_\beta(t),w_\beta(t))]\\
&\quad+\beta(c(v_\beta(t)+W_A(t))-c(v(t)+W_A(t)),w_\beta(t))+\beta(c(v(t)+W_A(t)),w_\beta(t))\\
&=\alpha[b(v(t),w_\beta(t),w_\beta(t))+b(W_A(t),w_\beta(t),w_\beta(t))]\\
&\quad+\beta(c(v_\beta(t)+W_A(t))-c(v(t)+W_A(t)),w_\beta(t))+\beta(c(v(t)+W_A(t)),w_\beta(t)),
\end{aligned}
\tag{4.6}
$$

for a.e. $t\in[0,T]$, where we have used the fact that $b(u,v,u)=-2b(v,u,u)$. It can be easily seen that $\|u\|_{\mathrm{L}^4}\leq\sqrt{2}\|u\|_{\mathrm{L}^4}^{1/2}\|\partial_x u\|_{\mathrm{L}^2}^{1/2}$, for all $u\in\mathrm{H}_0^1(\mathcal{O})$. The first two terms from the right-hand side of the equality (4.6) can be estimated using Hölder's and Young's inequalities as

$$\begin{aligned}\alpha|b(v,w_\beta,w_\beta)| &\leq \|v\|_{\mathrm{L}^4}\|\partial_x w_\beta\|_{\mathrm{L}^2}\|w_\beta\|_{\mathrm{L}^4} \leq \sqrt{2}\alpha\|v\|_{\mathrm{L}^4}\|\partial_x w_\beta\|_{\mathrm{L}^2}^{3/2}\|w_\beta\|_{\mathrm{L}^2}^{1/2}\\ &\leq \frac{\nu}{8}\|\partial_x w_\beta\|_{\mathrm{L}^2}^2 + \frac{216\alpha^4}{\nu^3}\|v\|_{\mathrm{L}^4}^4\|w_\beta\|_{\mathrm{L}^2}^2,\\ \alpha|b(W_A,w_\beta,w_\beta)| &\leq \alpha\|W_A\|_{\mathrm{L}^4}\|\partial_x w_\beta\|_{\mathrm{L}^2}\|w_\beta\|_{\mathrm{L}^4} \leq \sqrt{2}\alpha\|W_A\|_{\mathrm{L}^4}\|\partial_x w_\beta\|_{\mathrm{L}^2}^{3/2}\|w_\beta\|_{\mathrm{L}^2}^{1/2}\\ &\leq \frac{\nu}{8}\|\partial_x w_\beta\|_{\mathrm{L}^2}^2 + \frac{216\alpha^4}{\nu^3}\|W_A\|_{\mathrm{L}^4}^4\|w_\beta\|_{\mathrm{L}^2}^2.\end{aligned}$$

It can be easily seen that (cf. Theorem 2.2., [25])

$$\beta|(c(v_\beta(t)+W_A(t)) - c(v(t)+W_A(t)), w_\beta(t))| \leq \beta(1+\gamma+\gamma^2)\|w_\beta\|_{\mathrm{L}^2}^2.$$

Using Hölder's, Poincaré's and Young's inequalities, we estimate $\beta|(c(v+W_A), w_\beta)|$ as

$$\begin{aligned}&\beta|(c(v+W_A), w_\beta)|\\ &\leq \beta(\gamma\|v+W_A\|_{\mathrm{L}^2} + (1+\gamma)\|v+W_A\|_{\mathrm{L}^4}^2)\|w_\beta\|_{\mathrm{L}^2} + \beta\|v+W_A\|_{\mathrm{L}^4}^3\|w_\beta\|_{\mathrm{L}^4}\\ &\leq \frac{\nu}{4}\|\partial_x w\|_{\mathrm{L}^2}^2 + \frac{2\beta^2\gamma^2}{\pi^2\nu}\|v+W_A\|_{\mathrm{L}^2}^2 + \frac{2}{\pi^2\nu}\beta^2(1+\gamma)^2\|v+W_A\|_{\mathrm{L}^4}^4 + \frac{2\beta^2}{\pi\nu}\|v+W_A\|_{\mathrm{L}^4}^6.\end{aligned}$$

Combining the above estimates and then substituting it in (4.6), we deduce that

$$\begin{aligned}&\|w_\beta(t)\|_{\mathrm{L}^2}^2 + \nu\int_0^t \|\partial_x w_\beta(s)\|_{\mathrm{L}^2}^2\mathrm{d}s\\ &\leq \frac{512\alpha^4}{\nu^3}\int_0^t \left(\|v(s)\|_{\mathrm{L}^4}^4 + \|W_A(s)\|_{\mathrm{L}^4}^4\right)\|w_\beta(s)\|_{\mathrm{L}^2}^2\mathrm{d}s\\ &\quad + 2\beta(1+\gamma+\gamma^2)\int_0^t \|w_\beta(s)\|_{\mathrm{L}^2}^2\mathrm{d}s\\ &\quad + \frac{8\beta^2\gamma^2}{\pi^2\nu}\int_0^t \left(\|v(s)\|_{\mathrm{L}^2}^2 + \|W_A(s)\|_{\mathrm{L}^2}^2\right)\mathrm{d}s\\ &\quad + \frac{32\beta^2}{\pi^2\nu}(1+\gamma)^2\int_0^t \left(\|v(s)\|_{\mathrm{L}^4}^4 + \|W_A(s)\|_{\mathrm{L}^4}^4\right)\mathrm{d}s\\ &\quad + \frac{64\beta^2}{\pi\nu}\int_0^t \left(\|v(s)\|_{\mathrm{L}^4}^6 + \|W_A(s)\|_{\mathrm{L}^4}^6\right)\mathrm{d}s,\end{aligned} \tag{4.7}$$

for all $t \in [0,T]$. An application of Gronwall's inequality in (4.7) yields

$$\begin{aligned}&\sup_{t\in[0,T]}\|w_\beta(t)\|_{\mathrm{L}^2}^2 + \int_0^T \|\partial_x w_\beta(t)\|_{\mathrm{L}^2}^2\mathrm{d}t\\ &\leq \frac{8\beta^2}{\pi^2\nu}\left\{\gamma^2\int_0^T \left(\|v(t)\|_{\mathrm{L}^2}^2 + \|W_A(t)\|_{\mathrm{L}^2}^2\right)\mathrm{d}t\right.\end{aligned}$$

$$
\begin{aligned}
&+4(1+\gamma)^2\int_0^T\left(\|v(t)\|_{\mathrm{L}^4}^4+\|W_A(t)\|_{\mathrm{L}^4}^4\right)\mathrm{d}t\\
&+8\pi\int_0^T\left(\|v(t)\|_{\mathrm{L}^4}^6+\|W_A(t)\|_{\mathrm{L}^4}^6\right)\mathrm{d}t\Bigg\}e^{2\beta(1+\gamma+\gamma^2)T}\\
&\times\exp\left\{\frac{512\alpha^4}{\nu^3}\int_0^T\left(\|v(t)\|_{\mathrm{L}^4}^4+\|W_A(t)\|_{\mathrm{L}^4}^4\right)\mathrm{d}t\right\},
\end{aligned}
\tag{4.8}
$$

and the required result follows by taking $\beta \to 0$ in (4.8) and using the fact that $\mathrm{H}_0^1(\mathcal{O}) \subset \mathrm{C}(\overline{\mathcal{O}})$. □

For $\delta = 1$, let us now discuss the inviscid limit of Eq. (3.3) as $\alpha \to 0$. We consider the following Huxley equation for $(x, t) \in \mathcal{O} \times (0, T)$:

$$
\begin{cases}
dz(t) = -\nu Az(t) + \beta c(z(t)) + \mathrm{d}W(t),\\
z(0) = u_0 \in \mathrm{L}^2(\mathcal{O}).
\end{cases}
\tag{4.9}
$$

The existence and uniqueness of a mild solution $z \in \mathrm{C}([0, T]; \mathrm{L}^2(\mathcal{O}))$, $\mathbb{P}$-a.s. to Eq. (4.9) can be proved in a similar way as in Theorem 3.5. Equivalently, $y = z - W_A$ is the unique mild solution of the equation:

$$
\begin{cases}
dy(t) = -\nu Ay(t) + \beta c(y(t) + W_A(t)),\\
y(0) = u_0 \in \mathrm{L}^2(\mathcal{O}).
\end{cases}
\tag{4.10}
$$

It can be easily seen that $y(\cdot)$ satisfies:

$$
\begin{aligned}
&\sup_{t\in[0,T]}\|y(t)\|_{\mathrm{L}^2}^2+2\nu\int_0^T\|\partial_x y(t)\|_{\mathrm{L}^2}^2\mathrm{d}t+\frac{\beta}{4}\int_0^T\|y(t)\|_{\mathrm{L}^4}^4\mathrm{d}t\\
&\leq C(\beta,\gamma,\nu,T)\left\{1+\|u_0\|_{\mathrm{L}^2}^2+\mu_\infty^4\right\}:=H_T.
\end{aligned}
\tag{4.11}
$$

Then, we have the following result:

Proposition 4.2 *Let $u_\alpha(\cdot)$ be the unique mild solution of the Burgers–Huxley equation (see (3.3) with $\delta = 1$), for $u_0 \in \mathrm{L}^2(\mathcal{O})$, $\mathbb{P}$-a.s.. As $\alpha \to 0$, the mild solution $u_\alpha(\cdot)$ of Eq. (3.3) tends to the mild solution of the Huxley equation (4.9), that is,*

$$
u_\alpha \to z \ \text{ in } \ \mathrm{C}([0,T];\mathrm{L}^2(\mathcal{O}))\cap\mathrm{L}^2(0,T;\mathrm{C}(\overline{\mathcal{O}})),\ \mathbb{P}\text{-a.s.},\ \text{ as } \ \alpha\to 0.
$$

Proof Let us define $w_\alpha = u_\alpha - z = (u_\alpha - W_A) - (z - W_A) = v_\alpha - y$, where v_α is the unique mild solution of Eq. (3.6) with $\delta = 1$. Then, $w_\alpha(\cdot)$ satisfies:

$$
\begin{cases}
\dfrac{dw_\alpha(t)}{\mathrm{d}t} = -\nu Aw_\alpha(t) + \beta[c(v_\alpha(t) + W_A(t)) - c(y(t) + W_A(t))] + \alpha B(v_\alpha(t) + W_A(t)),\\
w(0) = 0,
\end{cases}
\tag{4.12}
$$

in $\mathrm{H}^{-1}(\mathcal{O})$ for a.e. $t \in [0, T]$. Taking the inner product with $w_\alpha(\cdot)$ to the first equation in (4.5) and then applying integration by parts, we find

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|w_\alpha(t)\|_{\mathrm{L}^2}^2 + \nu\|\partial_x w_\alpha(t)\|_{\mathrm{L}^2}^2 \\
&= -\alpha b(v_\alpha + W_A(t), v_\alpha + W_A(t), w_\alpha(t)) \\
&\quad + \beta(c(v_\alpha(t) + W_A(t)) - c(y(t) + W_A(t)), w_\alpha(t)),
\end{aligned} \tag{4.13}$$

for a.e. $t \in [0, T]$. It is an immediate consequence of the property of $c(\cdot)$ that (cf. Theorem 2.2, [25])

$$\beta(c(v_\alpha + W_A) - c(y + W_A), w_\alpha) \leq \beta(1 + \gamma + \gamma^2)\|w_\alpha\|_{\mathrm{L}^2}^2.$$

Using the property of $b(\cdot, \cdot, \cdot)$, we find

$$\begin{aligned}
b(v_\alpha + W_A, v_\alpha + W_A, w_\alpha) &= b(w_\alpha, w_\alpha, w_\alpha) + b(w_\alpha, y + W_A, w_\alpha) \\
&\quad + b(y + W_A, w_\alpha, w_\alpha) + b(y + W_A, y + W_A, w_\alpha) \\
&= -b(y + W_A, w_\alpha, w_\alpha) - \frac{1}{2}b(y + W_A, w_\alpha, y + W_A).
\end{aligned}$$

Using Hölder's, Poincaré's and Young's inequalities, we estimate the final two terms from the right-hand side of the above equality as

$$\begin{aligned}
\alpha|b(y + W_A, w_\alpha, w_\alpha)| &\leq \alpha\|y + W_A\|_{\mathrm{L}^4}\|\partial_x w_\alpha\|_{\mathrm{L}^2}\|w_\alpha\|_{\mathrm{L}^4} \\
&\leq \sqrt{2}\alpha\|y + W_A\|_{\mathrm{L}^4}\|\partial_x w_\alpha\|_{\mathrm{L}^2}^{3/2}\|w_\alpha\|_{\mathrm{L}^2}^{1/2} \\
&\leq \frac{\nu}{4}\|\partial_x w_\alpha\|_{\mathrm{L}^2}^2 + \frac{27\alpha^4}{\nu^3}\|y + W_A\|_{\mathrm{L}^4}^4\|w_\alpha\|_{\mathrm{L}^2}^2, \\
\frac{\alpha}{2}|b(y + W_A, w_\alpha, y + W_A)| &\leq \frac{\alpha}{2}\|y + W_A\|_{\mathrm{L}^4}^2\|w_\alpha\|_{\mathrm{L}^2} \\
&\leq \frac{\nu}{4}\|\partial_x w_\alpha\|_{\mathrm{L}^2}^2 + \frac{\alpha^2}{4\pi^2\nu}\|y + W_A\|_{\mathrm{L}^4}^4.
\end{aligned}$$

Thus, from (4.13), it is immediate that

$$\begin{aligned}
&\|w_\alpha(t)\|_{\mathrm{L}^2}^2 + \nu\int_0^t \|\partial_x w_\alpha(s)\|_{\mathrm{L}^2}^2 \mathrm{d}s \\
&\leq 2\beta(1 + \gamma + \gamma^2)\int_0^t \|w_\alpha(s)\|_{\mathrm{L}^2}^2 \mathrm{d}s + \frac{54\alpha^4}{\nu^3}\int_0^t \|y(s) + W_A(s)\|_{\mathrm{L}^4}^4\|w_\alpha(s)\|_{\mathrm{L}^2}^2 \mathrm{d}s \\
&\quad + \frac{\alpha^2}{4\pi^2\nu}\int_0^t \|y(s) + W_A(s)\|_{\mathrm{L}^4}^4 \mathrm{d}s,
\end{aligned} \tag{4.14}$$

for all $t \in [0, T]$. An application of Gronwall's inequality in (4.14) gives

$$\begin{aligned}
&\sup_{t\in[0,T]} \|w_\alpha(t)\|_{\mathrm{L}^2}^2 + \int_0^T \|\partial_x w_\alpha(t)\|_{\mathrm{L}^2}^2 \mathrm{d}t \\
&\quad \leq \left\{ \frac{2\alpha^2}{\pi^2\nu} \int_0^T \left(\|y(t)\|_{\mathrm{L}^4}^4 + \|W_A(t)\|_{\mathrm{L}^4}^4 \right) \mathrm{d}t \right\} \\
&\qquad \times e^{2\beta(1+\gamma+\gamma^2)T} \exp\left\{ \frac{512\alpha^4}{\nu^3} \int_0^t \left(\|y(t)\|_{\mathrm{L}^4}^4 + \|W_A(t)\|_{\mathrm{L}^4}^4 \right) \mathrm{d}t \right\}. \qquad (4.15)
\end{aligned}$$

On passing $\alpha \to 0$ in the above inequality (see (4.1) also) provides the required result. □

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