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Moderate Deviations for Drift Parameter Estimations in Reflected Ornstein–Uhlenbeck Process

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Abstract
In this paper, we study the asymptotic properties of drift parameter estimations in reflected Ornstein–Uhlenbeck process, and establish their moderate deviations in both cases with one-sided barrier and two-sided barriers. The main methods consist of regenerative process techniques and the strong Markov property, as well as moderate deviations for martingales.

Keywords Maximum likelihood estimator · Moderate deviation principle · Reflected Ornstein–Uhlenbeck process · Regenerative process

Mathematics Subject Classification (2020) 60F10 · 60G40 · 60G44 · 62M05

1 Introduction and Main Results

1.1 Introduction

In many situations, the stochastic processes involved are not allowed to cross a certain boundary, or are even supposed to remain within two boundaries. For instance, the

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reflected Ornstein–Uhlenbeck process behaves like the standard Ornstein–Uhlenbeck process in the interior of its domain. However, when it reaches the boundary, the sample path returns to the interior in a manner that the "pushing" force is minimal. This kind of process has wide range of applications in the field of queueing system, financial engineering, and mathematical biology.

Consider the following reflected Ornstein–Uhlenbeck process with one-sided barrier b_L:

$$\begin{cases} \mathrm{d}X_t = (-\theta X_t + \gamma)\mathrm{d}t + \mathrm{d}W_t + \mathrm{d}L_t, \\ X_t \geq b_L, \text{ for all } t \geq 0, \\ X_0 = x_0 \geq b_L, \end{cases} \tag{1.1}$$

where $\theta \in (0, +\infty)$ and γ are unknown, $W = \{W_t, t \in [0, \infty)\}$ is a standard Brownian motion. Here, the process $L = \{L_t, t \geq 0\}$ is the minimal continuous increasing process with $L_0 = 0$, which makes the process $X_t \geq b_{\mathrm{L}}$ for all $t \geq 0$. The process L increases only when X hits the boundary b_{L}, satisfying $\int_0^\infty I_{\{X_t > b_{\mathrm{L}}\}} \mathrm{d}L_t = 0$. Denote by P_{θ,γ,x_0} the probability distribution of the solution of (1.1) on $C(\mathbb{R}_+, \mathbb{R})$, the space of continuous functions from $\mathbb{R}^+$ to $\mathbb{R}$. Without specific instruction, we will suppress θ, γ and denote by P_{x_0}.

For $\theta \in (0, +\infty)$, the reflected Ornstein–Uhlenbeck process (1.1) is an ergodic Markov process [36], [37], and its properties have been extensively studied. To be explicit, we can refer to [30], [32], [36] for the transition density analysis; and [8], [10], [11] for the study of first passages time, [36], [37] for the formula of stationary distribution, as well as [26], [31] for the limit theorems of the processes $\{X_t, L_t, t \geq 0\}$.

The reflected Ornstein–Uhlenbeck process (1.1), as an extended Vasicek model, can successfully characterize mean reversion property of short interest rate. Actually, θ indicates the mean reversion rate, whereas γ, along with θ, determines the long run average. Then, to estimate them is a crucial step for practical applications. By Girsanov formula in Ward and Glynn [36], Bo et al. [9], the log-likelihood ratio process can be written as

$$\log\left(\frac{\mathrm{d}P_{\theta,\gamma,x_0}}{\mathrm{d}P_{0,0,x_0}}\bigg|_{\mathcal{F}_T}\right) = -\theta\int_0^T X_t \mathrm{d}(X_t - L_t) + \gamma\big(X_T - L_T - x_0\big) - \frac{\theta^2}{2}\int_0^T X_t^2 \mathrm{d}t + \theta\gamma\int_0^T X_t \mathrm{d}t - \frac{\gamma^2}{2}T,$$

where $\mathcal{F}_T = \sigma(W_s, s \leq T)$. The maximum likelihood estimators of θ and γ are given by

$$\widehat{\theta}_T = \frac{-T\int_0^T X_t \mathrm{d}(X_t - L_t) + (X_T - L_T - x_0)\int_0^T X_t \mathrm{d}t}{T\int_0^T X_t^2 \mathrm{d}t - \big(\int_0^T X_t \mathrm{d}t\big)^2} \tag{1.2}$$

and

$$\widehat{\gamma}_T = \frac{-\int_0^T X_t \mathrm{d}t \int_0^T X_t \mathrm{d}(X_t - L_t) + (X_T - L_T - x_0)\int_0^T X_t^2 \mathrm{d}t}{T\int_0^T X_t^2 \mathrm{d}t - \left(\int_0^T X_t \mathrm{d}t\right)^2}. \tag{1.3}$$

In the case of $\theta > 0, \gamma \equiv 0$, Bo et al. [9] studied the strong consistency and asymptotic normality for $\widehat{\theta}_T$, while Zang and Zhang [38] analyzed the Cramér–Rao lower bound. Moreover, Hu et al. [25] constructed another estimator via discrete observations, and considered its asymptotic normality. For more details, one can refer to Hu and Lee [24], Lee and Song [29], and the references therein. On the other hand, Zang and Zhang [39] considered asymptotic behavior of the trajectory fitting estimator for nonergodic reflected Ornstein–Uhlenbeck processes ($\theta < 0, \gamma \equiv 0$).

Compared with huge literature in classical Ornstein–Uhlenbeck type process [3–7,14–23,27], the large and moderate deviations for estimators in reflected Ornstein–Uhlenbeck process have been in the ascendant. In this paper, our goal is to fill this gap, refining the already known results in Bo et al. [9], Zang and Zhang [38]. Here, we will analyze the reflected Ornstein–Uhlenbeck process (1.1) in view of regenerative process, and this method is quite different from the techniques in the existed work.

Generally speaking, moderate deviation fulfills the gap between the limiting distribution and large deviation. More precisely, consider the estimation $\mathbb{P}\left(\frac{\sqrt{T}}{\lambda_T}\begin{pmatrix}\widehat{\theta}_T - \theta \\ \widehat{\gamma}_T - \gamma\end{pmatrix} \in A\right)$, where A is a given domain of deviations and λ_T denotes the scale of deviation. When $\lambda_T \equiv 1$, this is exactly the estimation of limiting distribution result. When $\lambda_T \equiv \sqrt{T}$, this corresponds to the large deviation. And when λ_T between 1 and $\sqrt{T}$, that is, $\lambda_T \to \infty$ and $\frac{\lambda_T}{\sqrt{T}} \to 0$ as $T \to \infty$, this is the so-called moderate deviation.

1.2 Main Results

Denote the stationary distribution of (1.1) by ([36])

$$\pi(\mathrm{d}x) = \frac{e^{-\theta\left(x-\gamma/\theta\right)^2}}{M} I_{[b_\mathrm{L},\infty)}\mathrm{d}x, \quad M = \int_{\mathrm{b}_L}^{\infty} e^{-\theta(x-\gamma/\theta)^2}\mathrm{d}x. \tag{1.4}$$

Now, we state our main results as follows:

Theorem 1.1 *Let λ_T be positive numbers, satisfying as $T \to \infty$*

$$\lambda_T \to \infty, \quad \frac{\lambda_T}{\sqrt{T}} \to 0. \tag{1.5}$$

Then, the family $\left\{\frac{\sqrt{T}}{\lambda_T}\begin{pmatrix}\widehat{\theta}_T-\theta\\ \widehat{\gamma}_T-\gamma\end{pmatrix}, T>0\right\}$ *satisfies the large deviations with speed* λ_T^2 *and rate function*

$$I(x)=\frac{1}{2}x^{\tau}\Sigma^{-1}x, \quad x\in\mathbb{R}^2,$$

where $\mu_1=\int_{b_{\mathrm{L}}}^{\infty}x\pi(\mathrm{d}x)$, $\mu_2=\int_{b_{\mathrm{L}}}^{\infty}x^2\pi(\mathrm{d}x)$ *and*

$$\Sigma=\left(\mu_2-\mu_1^2\right)^{-1}\begin{pmatrix}1 & \mu_1\\ \mu_1 & \mu_2\end{pmatrix}.$$

Explicitly, for any $A\in\mathcal{B}(\mathbb{R}^2)$

$$\begin{aligned}-\inf_{x\in A^o}I(x)&\le\liminf_{T\to\infty}\frac{1}{\lambda_T^2}\log P\left(\frac{\sqrt{T}}{\lambda_T}\begin{pmatrix}\widehat{\theta}_T-\theta\\ \widehat{\gamma}_T-\gamma\end{pmatrix}\in A\right)\\ &\le\limsup_{T\to\infty}\frac{1}{\lambda_T^2}\log P\left(\frac{\sqrt{T}}{\lambda_T}\begin{pmatrix}\widehat{\theta}_T-\theta\\ \widehat{\gamma}_T-\gamma\end{pmatrix}\in A\right)\le-\inf_{x\in\bar{A}}I(x).\end{aligned}$$

Then, we can obtain immediately that

Corollary 1.1 *Under condition (1.5), the families*

$$\left\{\frac{\sqrt{T}}{\lambda_T}\left(\widehat{\theta}_T-\theta\right), T>0\right\}, \quad \left\{\frac{\sqrt{T}}{\lambda_T}\left(\widehat{\gamma}_T-\gamma\right), T>0\right\}$$

satisfy the large deviations with speed λ_T^2 *and rate functions*

$$J_\theta(x)=\frac{1}{2}\left(\mu_2-\mu_1^2\right)x^2, \quad J_\gamma(x)=\frac{1}{2\mu_2}\left(\mu_2-\mu_1^2\right)x^2,$$

respectively.

In particular, for any $x\ge 0$, *we have*

$$\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log P_{x_0}\left(\frac{\sqrt{T}}{\lambda_T}\left|\widehat{\theta}_T-\theta\right|\ge x\right)=-J_\theta(x)$$

and

$$\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log P_{x_0}\left(\frac{\sqrt{T}}{\lambda_T}\left|\widehat{\gamma}_T-\gamma\right|\ge x\right)=-J_\gamma(x).$$

The paper is organized as follows. In Sect. 2, by using the regenerative process techniques, we first state some properties of reflected Ornstein–Uhlenbeck process (1.1), and then give exponential equivalence for the functionals $\int_0^T X_t \mathrm{d}t$, $\int_0^T X_t^2 \mathrm{d}t$ to their asymptotic expectations, respectively. The proof of the main result Theorem 1.1 will be postponed to Sect. 3. In Sect. 4, we extend our results to two-sided barriers case. The main methods of this paper consist of regenerative process techniques and strong Markov property, as well as the moderate deviations for martingales. Throughout this paper, C_0, C_1, depending only on b_L, θ, γ and the initial point x_0, denote positive constants whose values can differ at different places.

2 Regenerative Process and Exponential Equivalence

To obtain the moderate deviations for $\left(\hat{\theta}_T, \hat{\gamma}_T\right)$, the key point is to show the functionals $\int_0^T X_t dt$ and $\int_0^T X_t^2 dt$ are exponential equivalent to their asymptotic expectations, respectively. Notice that the existing methods (Girsanov formula technique [3–7], [14], [15,19]; multiple Wiener-Itô integral [21,27]; log-Sobolev inequality method [13,18,20]) maybe not work. Here, regenerative process techniques will be employed, and we benefit a lot from Banerjee and Mukherjee [2].

2.1 Regenerative Process View of Functionals

We first briefly recall the definition of regenerative process [33], [34].

Definition 2.1 The process $X = \left\{X_t, t \geq 0\right\}$ is a regenerative process, if there exist random times $0 \leq \Theta_0 \leq \Theta_1 \leq \cdots$, such that for $k \geq 1$,

(1) $\left(X_{\Theta_k+t}, t \geq 0\right\}$ has the same distribution as $\left(X_{\Theta_0+t}, t \geq 0\right\}$.
(2) $\left(X_{\Theta_k+t}, t \geq 0\right\}$ is independent of $\left(X_t, 0 \leq t \leq \Theta_k\right\}$.

In particular, if $\Theta_0 = 0$, the process X is called a non-delayed regenerative process. Else, X is called a delayed regenerative process.

Loosely speaking, regenerative process starts anew at regeneration times $\left\{\Theta_k, k \geq 1\right\}$, independent of the past. Moreover, the regeneration times split the process into renewal cycles that are independent and identically distributed, possibly except the first cycle.

For the reflected Ornstein–Uhlenbeck process X (1.1), let

$$\tau_X(x) = \inf \left\{t \geq 0 : X_t = x\right\}. \tag{2.1}$$

Now, we can define regenerative times in terms of hitting times as follows:

$$\begin{aligned} \alpha_{2k+1} &= \inf \left\{t \geq \alpha_{2k} : X_t = b_L + 1\right\}, \ \alpha_{2k+2} \\ &= \inf \left\{t \geq \alpha_{2k+1} : X_t = b_L + 2\right\}, \ \alpha_0 = 0, \end{aligned} \tag{2.2}$$

$$\Theta_k = \alpha_{2k+2}, \quad N_T = \sup \left\{k \geq -1 : \Theta_k \leq T\right\}. \tag{2.3}$$

The strong Markov property implies that X is a regenerative process with regeneration times given by $\left\{\Theta_k, k \geq -1\right\}$. Then, under P_{x_0} with $x_0 \geq b_L$,

$$\left\{\int_{\Theta_{k-1}}^{\Theta_k} X_t \mathrm{d}t, \Theta_k - \Theta_{k-1} : k \geq 1\right\}, \quad \left\{\int_{\Theta_{k-1}}^{\Theta_k} X_t^2 \mathrm{d}t, \Theta_k - \Theta_{k-1} : k \geq 1\right\}$$

are both independent and identically distributed sequences. Moreover, we also have the following important results

$$\left|\int_0^T X_t \mathrm{d}t - \sum_{k=1}^{N_\mathrm{T}} \int_{\Theta_{k-1}}^{\Theta_k} X_t \mathrm{d}t\right| \leq \left|\int_0^{\Theta_0 \wedge T} X_t \mathrm{d}t\right| + \left|\int_{\Theta_{N_\mathrm{T}}}^T X_t \mathrm{d}t\right| \tag{2.4}$$

and

$$\left|\int_0^T X_t^2 dt - \sum_{k=1}^{N_T} \int_{\Theta_{k-1}}^{\Theta_k} X_t^2 \mathrm{d}t\right| \leq \int_0^{\Theta_0} X_t^2 \mathrm{d}t + \int_{\Theta_{N_T}}^{\Theta_{N_T+1}} X_t^2 \mathrm{d}t, \tag{2.5}$$

where the sum is 0 if the upper index is strictly less than the lower index.

The tail asymptotic of regenerative time Θ_0, renewal rewards $\int_0^{\Theta_0} X_t \mathrm{d}t$, $\int_0^{\Theta_0} X_t^2 \mathrm{d}t$, and $\int_0^{\Theta_0 \wedge T} X_t \mathrm{d}t$, $\int_{\Theta_{N_\mathrm{T}}}^T X_t \mathrm{d}t$ can be analyzed as follows:

Lemma 2.1 *For all $x_0 \geq b_L$ and Θ_0 defined by (2.3), there exists some positive constants C_0, C_1 depending only on x_0, b_L, θ and γ, such that for T large enough*

$$P_{x_0}\Big(\Theta_0 > T\Big) \leq C_0 e^{-C_1 T}. \tag{2.6}$$

Moreover, there exists some $\eta > 0$ such that $E_{x_0} e^{\eta \Theta_0} < \infty$.

Proof Firstly, if $b_L \leq x_0 \leq b_L + 1$, then $\Theta_0 = \tau_X(b_L + 2)$. Define the following Ornstein–Uhlenbeck process

$$\mathrm{d}Y_t = (-\theta Y_t + \gamma)\mathrm{d}t + \mathrm{d}W_t, \quad Y_0 = x_0. \tag{2.7}$$

Under $X_0 = Y_0 = x_0$, we have $X_t \geq Y_t$ for $t \geq 0$, and then $\tau_X(b_\mathrm{L}+2) \leq \tau_Y(b_\mathrm{L}+2)$, where

$$\tau_Y(x) = \inf\Big\{t \geq 0 : Y_t = x\Big\}. \tag{2.8}$$

By Corollary 3.1 in Alili et al. ([1]), we have for T large enough

$$P_{x_0}\Big(\Theta_0 > T\Big) = P_{x_0}\Big(\tau_X(b_\mathrm{L} + 2) > T\Big) \leq P_{x_0}\Big(\tau_Y(b_L + 2) > T\Big) \leq C_0 e^{-C_1 T}, \tag{2.9}$$

where C_0, C_1 are positive constants depending only on x_0, b_L, θ and γ.

On the other hand, if $x_0 > b_L + 1$, we have $\tau_X(b_L + 1) = \tau_Y(b_L + 1) = \alpha_1$, and $X_t = Y_t$ on the interval $[0, \tau(b_L)]$, where Y is defined by (2.7). It holds by strong Markov property

$$\begin{aligned} P_{x_0}\Big(\Theta_0 > T\Big) &\le P_{x_0}\Big(\alpha_1 > T/2\Big) + P_{x_0}\Big(\alpha_2 - \alpha_1 > T/2\Big) \\ &\le P_{x_0}\Big(\tau_Y(b_L+1) > T/2\Big) + P_{b_L+1}\Big(\tau_X(b_L+2) > T/2\Big) \\ &\le P_{x_0}\Big(\tau_Y(b_L+1) > T/2\Big) + P_{b_L+1}\Big(\tau_Y(b_L+2) > T/2\Big). \end{aligned}$$

Using Corollary 3.1 in Alili et al. [1] again, we have for T large enough

$$P_{x_0}\Big(\Theta_0 > T\Big) \le C_0 e^{-C_1 T}, \tag{2.10}$$

where C_0, C_1 are positive constants depending only on x_0, b_L, θ and γ.

Finally, by using Fubini theorem and (2.6), we can choose some $\eta > 0$ such that

$$E_{x_0} e^{\eta \Theta_0} \le e^{\eta T} + \eta \int_T^\infty e^{\eta x} P_{x_0}(\Theta > x)\mathrm{d}x < \infty,$$

which concludes the proof of this lemma. □

Lemma 2.2 *For all $x_0 \ge b_L$, there exists some positive constants C_0, C_1 depending only on x_0, b_L, θ and γ, such that for T large enough*

$$P_{x_0}\Big(\Big|\int_0^{\Theta_0} X_t \mathrm{d}t\Big| \vee \Big|\int_0^{\Theta_0\wedge T} X_t \mathrm{d}t\Big| > T\Big) \le C_0 e^{-C_1 T}, \quad P_{x_0}\Big(\int_0^{\Theta_0} X_t^2 dt > T\Big) \le C_0 e^{-C_1 T}. \tag{2.11}$$

In particular, there exists some $\eta > 0$ such that

$$E_{x_0} \exp\left\{\eta \left(\left|\int_0^{\Theta_0\wedge T} X_t \mathrm{d}t\right| \vee \left|\int_0^{\Theta_0} X_t \mathrm{d}t\right|\right)\right\} < \infty, \quad E_{x_0} \exp\left\{\eta \int_0^{\Theta_0} X_t^2 \mathrm{d}t\right\} < \infty. \tag{2.12}$$

Proof Firstly, if $b_L \le x_0 \le b_L + 1$, then $\Theta_0 = \tau_X(b_L + 2)$, and

$$\int_0^{\Theta_0} X_t^2 dt \le \Big((b_L+2)^2 \vee b_L^2\Big)\tau_X(b_L+2),$$

which implies by (2.9) that

$$P_{x_0}\Big(\int_0^{\Theta_0} X_t^2 dt > T\Big) \le P_{x_0}\Big(\Big((b_L+2)^2 \vee b_L^2\Big)\tau_X(b_L+2) > T\Big)$$
$$\le P_{x_0}\Big(\Big((b_L+2)^2 \vee b_L^2\Big)\tau_Y(b_L+2) > T\Big) \le C_0 e^{-C_1 T}.$$

On the other hand, suppose $x_0 > b_L + 1$. Then, for $t \in [0, \tau_X(b_L)]$, $X_t = Y_t$ and $\alpha_1 = \tau_X(b_L+1) = \tau_Y(b_L+1)$, where Y is defined by (2.7). Then, it holds that

$$\int_0^{\Theta_0} X_t^2 \mathrm{d}t = \int_0^{\tau_Y(b_\mathrm{L}+1)} Y_t^2 \mathrm{d}t + \int_{\tau_X(b_\mathrm{L}+1)}^{\Theta_0} X_t^2 dt$$
$$\le \int_0^{\tau_Y(b_\mathrm{L}+1)} Y_t^2 \mathrm{d}t + \Big((b_\mathrm{L}+2)^2 \vee b_\mathrm{L}^2\Big)\big(\Theta_0 - \tau_X(b_\mathrm{L}+1)\big).$$

By using the strong Markov property, we obtain

$$P_{x_0}\Big(\int_0^{\Theta_0} X_t^2 \mathrm{d}t \ge T\Big)$$
$$\le P_{x_0}\Big(\int_0^{\tau_Y(b_L+1)} Y_t^2 \mathrm{d}t \ge 2T/3\Big) + P_{x_0}\Big(\Big((b_L+2)^2 \vee b_L^2\Big)\big(\Theta_0 - \tau_X(b_L+1)\big) \ge T/3\Big)$$
$$\le P_{x_0}\Big(\tau_Y(b_L+1) \ge \frac{\theta^2 T}{\theta+2\gamma^2}\Big) + P_{x_0}\Big(\frac{\theta+2\gamma^2}{\theta^2 T}\int_0^{\frac{\theta^2 T}{\theta+2\gamma^2}} Y_t^2 \mathrm{d}t \ge \frac{2(\theta+2\gamma^2)}{3\theta^2}\Big)$$
$$+ P_{b_L+1}\Big(\Big((b_L+2)^2 \vee b_L^2\Big)\tau_X(b_L+2) \ge T/3\Big).$$

Since $\lim_{T\to\infty}\frac{1}{T}\int_0^T Y_t^2 \mathrm{d}t = \frac{\theta+2\gamma^2}{2\theta^2}$, by Lemma 2.3 in Gao and Jiang [20] and (2.9), we have for T large enough

$$P_{x_0}\Big(\int_0^{\Theta_0} X_t^2 \mathrm{d}t > T\Big) \le C_0 e^{-C_1 T}. \tag{2.13}$$

Finally, by Hölder inequality, Lemma 2.1 and (2.13), we have

$$P_{x_0}\Big(\Big|\int_0^{\Theta_0} X_t \mathrm{d}t\Big| > T\Big) \le P_{x_0}\Big(\Theta_0 \int_0^{\Theta_0} X_t^2 dt > T^2\Big)$$
$$\le P_{x_0}\Big(\Theta_0 > T\Big) + P_{x_0}\Big(\int_0^{\Theta_0} X_t^2 \mathrm{d}t > T\Big) \le C_0 e^{-C_1 T}$$

and

$$P_{x_0}\Big(\Big|\int_0^{\Theta_0\wedge T} X_t \mathrm{d}t\Big| > T\Big) \le P_{x_0}\Big(\Theta_0 > T\Big) + P_{x_0}\Big(\Big|\int_0^{\Theta_0} X_t \mathrm{d}t\Big| > T\Big) \le C_0 e^{-C_1 T},$$

which complete the proof of this lemma. □

2.2 Exponential Equivalence

In this subsection, we will show that the following functionals $\int_0^T X_t \mathrm{d}t$, $\int_0^T X_t^2 \mathrm{d}t$ are exponentially equivalent to their asymptotic expectations, respectively.

Since the stationary distribution of (1.1) is given by [36]

$$\pi(\mathrm{d}x) = \frac{e^{-\theta\left(x-\gamma/\theta\right)^2}}{M} I_{[b_\mathrm{L},\infty)} \mathrm{d}x, \quad M = \int_{b_\mathrm{L}}^{\infty} e^{-\theta(x-\gamma/\theta)^2} \mathrm{d}x,$$

using ergodic theorem (Theorem 1.16 in [28]), we have immediately that

Lemma 2.3 *As $T \to +\infty$, under P_{x_0} with $x_0 \geq b_\mathrm{L}$, for any $\beta \in \mathbb{R}$*

$$\frac{1}{T}\int_0^T X_t \mathrm{d}t \to \mu_1, \quad \int_0^T X_t^2 \mathrm{d}t \to \mu_2, \quad \frac{1}{T}\int_0^T (\beta - X_t)^2 \,\mathrm{d}t \to \beta^2 - 2\beta\mu_1 + \mu_2^2, \quad a.s.$$

where $\mu_1 = \int_{b_\mathrm{L}}^{\infty} x\pi(\mathrm{d}x)$, $\mu_2 = \int_{b_\mathrm{L}}^{\infty} x^2\pi(\mathrm{d}x)$.

Remark 2.1 By using Proposition 7.3 in Ross [33], Lemma 2.2, (2.4), (2.5) and strong Markov property, we have

$$\mu_1 = \frac{E_{b_\mathrm{L}+2}\int_{\Theta_0}^{\Theta_1} X_t \mathrm{d}t}{E_{b_\mathrm{L}+2}\Theta_0}, \quad \mu_2 = \frac{E_{b_\mathrm{L}+2}\int_{\Theta_0}^{\Theta_1} X_t^2 \mathrm{d}t}{E_{b_\mathrm{L}+2}\Theta_0}. \tag{2.14}$$

Now, we can state the exponential equivalence results as follows:

Proposition 2.1 *For λ_T defined by (1.5) and for all $\delta > 0$ and $x_0 \geq b_L$, we have*

$$\lim_{T\to\infty} \frac{1}{\lambda_T^2} \log P_{x_0}\left(\frac{1}{T}\left|\int_0^T X_t \mathrm{d}t - \mu_1 T\right| \geq \delta\right) = -\infty \tag{2.15}$$

and

$$\lim_{T\to\infty} \frac{1}{\lambda_T^2} \log P_{x_0}\left(\frac{1}{T}\left|\int_0^T X_t^2 \mathrm{d}t - \mu_2 T\right| \geq \delta\right) = -\infty. \tag{2.16}$$

In particular, for any $\beta \in \mathbb{R}$

$$\lim_{T\to\infty} \frac{1}{\lambda_T^2} \log P_{x_0}\left(\frac{1}{T}\left|\int_0^T (\beta - X_t)^2 \,\mathrm{d}t - \left(\beta^2 - 2\beta\mu_1 + \mu_2\right)T\right| \geq \delta\right) = -\infty. \tag{2.17}$$

Proof To prove (2.16), applying (2.5), we have by strong Markov property

$$
\begin{aligned}
&P_{x_0}\Big(\frac{1}{T}\Big|\int_0^T X_t^2\mathrm{d}t-\mu_2 T\Big|\geq\delta\Big)\\
&\leq P_{x_0}\Big(\Big|\sum_{k=1}^{N_{\mathrm{T}}}\int_{\Theta_{k-1}}^{\Theta_k}X_t^2\mathrm{d}t-\mu_2 T\Big|\geq\delta T/2\Big)+P_{x_0}\Big(\int_0^{\Theta_0}X_t^2\mathrm{d}t+\int_{\Theta_{N_{\mathrm{T}}}}^{\Theta_{N_{\mathrm{T}}+1}}X_t^2\mathrm{d}t\geq\delta T/2\Big)\\
&\leq P_{b_{\mathrm{L}}+2}\Big(\Big|\sum_{k=0}^{N_{\mathrm{T}}-1}\int_{\Theta_{k-1}}^{\Theta_k}X_t^2\mathrm{d}t-\mu_2 T\Big|\geq\delta T/2\Big)\\
&\quad+P_{x_0}\Big(\int_0^{\Theta_0}X_t^2\mathrm{d}t\geq\delta T/4\Big)+P_{b_{\mathrm{L}}+2}\Big(\int_0^{\Theta_0}X_t^2\mathrm{d}t\geq\delta T/4\Big).
\end{aligned}
$$

From (2.11), it follows that

$$
\begin{aligned}
&\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log\left(P_{x_0}\Big(\int_0^{\Theta_0}X_t^2\mathrm{d}t\geq\delta T/4\Big)\vee P_{b_{\mathrm{L}}+2}\Big(\int_0^{\Theta_0}X_t^2\mathrm{d}t\geq\delta T/4\Big)\right)\\
&\leq\lim_{T\to\infty}\frac{1}{\lambda_T^2}\left(\log C_0-C_1\delta T/4\right)=-\infty.
\end{aligned}
\tag{2.18}
$$

Now, it is sufficient to show

$$
\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log P_{b_{\mathrm{L}}+2}\Big(\Big|\sum_{k=0}^{N_{\mathrm{T}}-1}\int_{\Theta_{k-1}}^{\Theta_k}X_t^2\mathrm{d}t-\mu_2 T\Big|\geq\delta T/2\Big)=-\infty.
$$

Firstly, we give some estimations for N_{T}. In fact, we have for any $\delta'>0$

$$
\begin{aligned}
&P_{b_{\mathrm{L}}+2}\left(N_T\geq\left[T(1+\delta')/E_{b_{\mathrm{L}}+2}\Theta_0\right]\right)\\
&=P_{b_{\mathrm{L}}+2}\left(\sum_{i=0}^{\left[T(1+\delta')/E_{b_{\mathrm{L}}+2}\Theta_0\right]}(\Theta_i-\Theta_{i-1})\leq T\right)\\
&=P_{b_{\mathrm{L}}+2}\left(\frac{\sum_{i=0}^{\left[T(1+\delta')/E_{b_{\mathrm{L}}+2}\Theta_0\right]}(\Theta_i-\Theta_{i-1})}{\left[T(1+\delta')/E_{b_{\mathrm{L}}+2}\Theta_0\right]+1}\leq\frac{T}{\left[T(1+\delta')/E_{b_{\mathrm{L}}+2}\Theta_0\right]+1}\right).
\end{aligned}
$$

Take T large enough such that $\frac{T}{\left[T(1+\delta')/E_{b_{\mathrm{L}}+2}\Theta_0\right]+1}<\frac{E_{b_{\mathrm{L}}+2}\Theta_0}{1+\delta'/2}$. Then, Under $P_{b_{\mathrm{L}}+2}$, by using Lemma 2.1, $\{\Theta_i-\Theta_{i-1}:i\geq 0\}$ is a sequence of independent and identical distributed variables with some finite exponential moment. Then, we have by the large deviation results

$$
\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log P_{b_{\mathrm{L}}+2}\left(N_{\mathrm{T}}\geq\left[T(1+\delta')/E_{b_L+2}\Theta_0\right]\right)=-\infty.
\tag{2.19}
$$

Similarly, it holds

$$\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log P_{b_L+2}\left(N_T \le \left[T(1-\delta')/E_{b_L+2}\Theta_0\right]\right) = -\infty. \tag{2.20}$$

Secondly, take T large enough such that $\frac{T}{\left[T(1+\delta')/E_{b_L+2}\Theta_0\right]} > \frac{E_{b_L+2}\Theta_0}{1+2\delta'}$. By (2.14), we have

$$\begin{aligned}
&P_{b_L+2}\left(\sum_{k=0}^{N_T-1}\int_{\Theta_{k-1}}^{\Theta_k} X_t^2\mathrm{d}t - \mu_2 T \ge \delta T/4\right)\\
&\le P_{b_L+2}\left(\sum_{k=0}^{N_T-1}\int_{\Theta_{k-1}}^{\Theta_k} X_t^2\mathrm{d}t - \mu_2 T \ge \delta T/4,\, N_T \le \left[T(1+\delta')/E_{b_L+2}\Theta_0\right]\right)\\
&\quad + P_{b_L+2}\left(N_T > \left[T(1+\delta')/E_{b_L+2}\Theta_0\right]\right)\\
&\le P_{b_L+2}\left(\frac{\sum_{k=0}^{\left[T(1+\delta')/E_{b_L+2}\Theta_0\right]-1}\int_{\Theta_{k-1}}^{\Theta_k} X_t^2\mathrm{d}t}{\left[T(1+\delta')/E_{b_L+2}\Theta_0\right]} \ge \frac{1}{1+2\delta'}\left(\frac{\delta E_{b_L+2}\Theta_0}{4} + E_{b_L+2}\int_{\Theta_0}^{\Theta_1} X_t^2\mathrm{d}t\right)\right)\\
&\quad + P_{b_L+2}\left(N_T > \left[T(1+\delta')/E_{b_L+2}\Theta_0\right]\right)
\end{aligned}$$

Now, choose $\delta' < \frac{\delta E_{b_L+2}\Theta_0}{8E_{b_L+2}\int_{\Theta_0}^{\Theta_1} X_t^2\mathrm{d}t}$, and then

$$\frac{1}{1+2\delta'}\left(\frac{\delta E_{b_L+2}\Theta_0}{4} + E_{b_L+2}\int_{\Theta_0}^{\Theta_1} X_t^2\mathrm{d}t\right) > E_{b_L+2}\int_{\Theta_0}^{\Theta_1} X_t^2\mathrm{d}t. \tag{2.21}$$

Notice that, under P_{b_L+2}, by using Lemma 2.2, $\left\{\int_{\Theta_{k-1}}^{\Theta_k} X_t^2\mathrm{d}t : k \ge 0\right\}$ is a sequence of independent and identical distributed variables with some finite exponential moment. Together with (2.19), (2.21) and the large deviation results,

$$\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log P_{b_L+2}\left(\sum_{k=0}^{N_T-1}\int_{\Theta_{k-1}}^{\Theta_k} X_t^2\mathrm{d}t - \mu_2 T \ge \delta T/4\right) = -\infty.$$

Finally, using (2.20) and following above procedures, we also have

$$\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log P_{b_L+2}\left(\sum_{k=0}^{N_T-1}\int_{\Theta_{k-1}}^{\Theta_k} X_t^2\mathrm{d}t - \mu_2 T \le -\delta T/4\right) = -\infty.$$

Therefore,

$$\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log P_{b_{\mathrm{L}}+2}\left(\left|\sum_{k=0}^{N_{\mathrm{T}}-1}\int_{\Theta_{k-1}}^{\Theta_k}X_t^2\mathrm{d}t-\mu_2 T\right|\geq \delta T/4\right)=-\infty.$$

Now, we turn to proving (2.15). Indeed, by (2.4), we can write

$$\begin{aligned}
&\left|\int_0^T X_t\mathrm{d}t-\sum_{k=1}^{N_{\mathrm{T}}}\int_{\Theta_{k-1}}^{\Theta_k}X_t\mathrm{d}t\right|\\
&\leq \Theta_0^{1/2}\left|\int_0^{\Theta_0}X_t^2\mathrm{d}t\right|^{1/2}+\left(\Theta_{N_{\mathrm{T}}+1}-\Theta_{N_{\mathrm{T}}}\right)^{1/2}\left|\int_{\Theta_{N_{\mathrm{T}}}}^{\Theta_{N_{\mathrm{T}}+1}}X_t^2\mathrm{d}t\right|^{1/2}\\
&\leq \frac{1}{2}\left(\Theta_0+\int_0^{\Theta_0}X_t^2\mathrm{d}t+\left(\Theta_{N_{\mathrm{T}}+1}-\Theta_{N_{\mathrm{T}}}\right)+\int_{\Theta_{N_{\mathrm{T}}}}^{\Theta_{N_{\mathrm{T}}+1}}X_t^2\mathrm{d}t\right).
\end{aligned}$$

Applying Lemma 2.1, we have

$$\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log\mathbb{P}_{x_0}\left(\Theta_0>T\delta\right)\leq\lim_{T\to\infty}\frac{1}{\lambda_T^2}\left(\log C_0-C_1\delta T\right)=-\infty. \tag{2.22}$$

Together with (2.18) and strong Markov property, we obtain

$$\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log P_{x_0}\left(\Theta_{N_{\mathrm{T}}+1}-\Theta_{N_{\mathrm{T}}}>T\delta\right)=\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log P_{b_{\mathrm{L}}+2}\left(\Theta_0>T\delta\right)=-\infty, \tag{2.23}$$

and

$$\begin{aligned}
&\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log P_{x_0}\left(\int_{\Theta_{N_{\mathrm{T}}}}^{\Theta_{N_{\mathrm{T}}+1}}X_t^2\mathrm{d}t>T\delta\right)\\
&=\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log P_{b_{\mathrm{L}}+2}\left(\int_0^{\Theta_0}X_t^2\mathrm{d}t>T\delta\right)=-\infty.
\end{aligned} \tag{2.24}$$

Together with (2.18, 2.22, 2.23) 2.24, following the similar line in the proof of (2.16), we have for any $\delta>0$,

$$\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log\mathbb{P}_{x_0}\left(\frac{1}{T}\left|\int_0^T X_t\mathrm{d}t-\sum_{k=1}^{N_{\mathrm{T}}}\int_{\Theta_{k-1}}^{\Theta_k}X_t\mathrm{d}t\right|>\delta\right)=-\infty,$$

$$\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log\mathbb{P}_{x_0}\left(\frac{1}{T}\left|\sum_{k=1}^{N_{\mathrm{T}}}\int_{\Theta_{k-1}}^{\Theta_k}X_t\mathrm{d}t-\mu_1 T\right|>\delta\right)=-\infty,$$

and thus

$$\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log\mathbb{P}_{x_0}\left(\frac{1}{T}\left|\int_0^T X_t\mathrm{d}t-\mu_1 T\right|>\delta\right)=-\infty,$$

which completes the proof of this proposition. □

3 Moderate Deviations for $\left(\widehat{\theta}_T, \widehat{\gamma}_T\right)$

Set

$$\widehat{\mu}_T=\frac{1}{T}\int_0^T X_t\mathrm{d}t,\quad \widehat{\sigma}_T^2=\frac{1}{T}\int_0^T X_t^2\mathrm{d}t-\widehat{\mu}_T^2. \tag{3.1}$$

For $\widehat{\theta}_T$ and $\widehat{\gamma}_T$ defined by (1.2) and (1.3), we have the following key martingale decomposition by straightforward calculations

$$\sqrt{T}\begin{pmatrix}\widehat{\theta}_T-\theta\\ \widehat{\gamma}_T-\gamma\end{pmatrix}=\frac{M_\mathrm{T}}{\sqrt{T}}+R_T, \tag{3.2}$$

with the martingale

$$M_\mathrm{T}=(\mu_2-\mu_1^2)^{-1}\begin{pmatrix}\int_0^T(\mu_1-X_t)\,\mathrm{d}W_t\\ \int_0^T(\mu_2-\mu_1X_t)\,\mathrm{d}W_t\end{pmatrix} \tag{3.3}$$

and the remainder term

$$R_\mathrm{T}=\frac{1}{\sqrt{T}\widehat{\sigma}_T^2}\begin{pmatrix}W_\mathrm{T}\big(\widehat{\mu}_T-\mu_1\big)+\big(1-(\mu_2-\mu_1^2)^{-1}\widehat{\sigma}_T^2\big)\int_0^T(\mu_1-X_t)\,\mathrm{d}W_t\\ \widehat{\mu}_T W_\mathrm{T}\big(\widehat{\mu}_T-\mu_1\big)+\big(\widehat{\mu}_T-\mu_1(\mu_2-\mu_1^2)^{-1}\widehat{\sigma}_T^2\big)\int_0^T(\mu_1-X_t)\,\mathrm{d}W_t\end{pmatrix}. \tag{3.4}$$

As a martingale, $\left\{M_T, T>0\right\}$ is the main term in our moderate deviation analysis, while R_T will be negligible.

Lemma 3.1 *For λ_T defined by (1.5) and M_T defined by (3.3), the families*

$$\left\{\frac{M_T}{\sqrt{T}\lambda_T}, T>0\right\},\quad \left\{\frac{1}{\sqrt{T}\lambda_T}\int_0^T(\mu_1-X_t)\,\mathrm{d}W_t, T>0\right\}$$

satisfy the large deviations with speed λ_T^2 and rate function

$$I(x)=\frac{1}{2}x^\tau\Sigma^{-1}x,\quad J(y)=\frac{y^2}{2\big(\mu_2-\mu_1^2\big)},\quad x\in\mathbb{R}^2, y\in\mathbb{R},$$

respectively, where $\mu_1 = \int_{b_L}^{\infty} x\pi(\mathrm{d}x)$, $\mu_2 = \int_{b_L}^{\infty} x^2\pi(\mathrm{d}x)$ *and*

$$\Sigma = \left(\mu_2 - \mu_1^2\right)^{-1} \begin{pmatrix} 1 & \mu_1 \\ \mu_1 & \mu_2 \end{pmatrix}.$$

Proof Note that $\left\{M_{\mathrm{T}}, T > 0\right\}$ and $\left\{ \int_0^T (\mu_1 - X_t)\,\mathrm{d}W_t, T > 0\right\}$ are martingales with predictable quadratic variations

$$< M >_T = (\mu_2 - \mu_1^2)^{-2} \begin{pmatrix} \int_0^T (\mu_1 - X_t)^2\,\mathrm{d}t & \int_0^T (\mu_1 - X_t)(\mu_2 - \mu_1 X_t)\,\mathrm{d}t \\ \int_0^T (\mu_1 - X_t)(\mu_2 - \mu_1 X_t)\,\mathrm{d}t & \int_0^T (\mu_2 - \mu_1 X_t)^2\,dt \end{pmatrix}$$

and $\left\langle \int_0^{\cdot} (\mu_1 - X_t)\,\mathrm{d}W_t \right\rangle_T = \int_0^T (\mu_1 - X_t)^2\,\mathrm{d}t$. By Proposition 2.1, we can get for any $\delta > 0$,

$$\lim_{T\to\infty} \frac{1}{\lambda_T^2} \log P_{x_0}\left(\frac{1}{T}\left\| < M >_T - \Sigma \cdot T \right\| \geq \delta\right) = -\infty,$$
$$\lim_{T\to\infty} \frac{1}{\lambda_T^2} \log P_{x_0}\left(\frac{1}{T}\left|\left\langle \int_0^{\cdot} (\mu_1 - X_t)\,\mathrm{d}W_t \right\rangle_T - \left(\mu_2 - \mu_1^2\right)T\right| \geq \delta\right) = -\infty.$$

Therefore, Proposition 1 in Dembo ([12]) yields the conclusion of this lemma. □

Lemma 3.2 *For the remainder term* R_{T} *defined by (3.4), we have the following results.*

(1) For any $\delta > 0$,

$$\lim_{T\to\infty} \frac{1}{\lambda_T^2} \log P_{x_0}\left(\left|\widehat{\sigma}_T^2 - \left(\mu_2 - \mu_1^2\right)\right| \geq \delta\right) = -\infty. \tag{3.5}$$

(2) For any $\delta > 0$,

$$\lim_{T\to\infty} \frac{1}{\lambda_T^2} \log P_{x_0}\left(\frac{1}{\lambda_T}\left|R_T\right| \geq \delta\right) = -\infty. \tag{3.6}$$

Proof (1) Since $\widehat{\sigma}_T^2 = \frac{1}{T}\int_0^T X_t^2 dt - \frac{1}{T^2}\left(\int_0^T X_t dt\right)^2$, we have

$$\begin{aligned} &P_{x_0}\left(\left|\widehat{\sigma}_T^2 - \left(\mu_2 - \mu_1^2\right)\right| \geq \delta\right) \\ &\leq P_{x_0}\left(\frac{1}{T}\left|\int_0^T X_t^2\mathrm{d}t - \mu_2 T\right| \geq \delta/2\right) \\ &\quad + P_{x_0}\left(\frac{1}{T}\left|\int_0^T X_t\mathrm{d}t + \mu_1 T\right| \cdot \frac{1}{T}\left|\int_0^T X_t dt - \mu_1 T\right| \geq \delta/2\right) \\ &\leq P_{x_0}\left(\frac{1}{T}\left|\int_0^T X_t^2\mathrm{d}t - \mu_2 T\right| \geq \delta/2\right) + P_{x_0}\left(\frac{1}{T}\left|\int_0^T X_t\mathrm{d}t - \mu_1 T\right| \geq |\mu_1| + 1\right) \end{aligned}$$

$$+ P_{x_0}\Big(\frac{1}{T}\Big|\int_0^T X_t \mathrm{d}t + \mu_1 T\Big| \cdot \frac{1}{T}\Big|\int_0^T X_t \mathrm{d}t - \mu_1 T\Big|$$
$$\geq \delta/2, \frac{1}{T}\Big|\int_0^T X_t dt - \mu_1 T\Big| < |\mu_1| + 1\Big)$$
$$\leq P_{x_0}\Big(\frac{1}{T}\Big|\int_0^T X_t^2 \mathrm{d}t - \mu_2 T\Big| \geq \delta/2\Big) + P_{x_0}\Big(\frac{1}{T}\Big|\int_0^T X_t \mathrm{d}t - \mu_1 T\Big| \geq |\mu_1| + 1\Big)$$
$$+ P_{x_0}\Big(\frac{1}{T}\Big|\int_0^T X_t \mathrm{d}t - \mu_1 T\Big| \geq \frac{\delta}{2(3|\mu_1| + 1)}\Big).$$

Now, we can complete the proof of (3.5) by using Proposition 2.1.

(2) For any $L > 0$,

$$P_{x_0}\left(\frac{1}{\lambda_T\sqrt{T}}\Big|W_{\mathrm{T}}\big(\widehat{\mu}_T - \mu_1\big) + \big(1 - (\mu_2 - \mu_1^2)^{-1}\widehat{\sigma}_T^2\big)\int_0^T (\mu_1 - X_t)\,\mathrm{d}W_t\Big| \geq \delta\right)$$
$$\leq P_{x_0}\left(\frac{1}{\lambda_T\sqrt{T}}\Big|W_{\mathrm{T}}\big(\widehat{\mu}_T - \mu_1\big)\Big| \geq \delta/2\right)$$
$$+ P_{x_0}\left(\frac{1}{\lambda_T\sqrt{T}}\Big|\big(1 - (\mu_2 - \mu_1^2)^{-1}\widehat{\sigma}_T^2\big)\int_0^T (\mu_1 - X_t)\,\mathrm{d}W_t\Big| \geq \delta/2\right)$$
$$\leq P_{x_0}\left(\frac{1}{\lambda_T\sqrt{T}}\Big|W_{\mathrm{T}}\Big| \geq L\right) + P_{x_0}\left(\Big|\widehat{\mu}_T - \mu_1\Big| \geq \frac{\delta}{2L}\right)$$
$$+ P_{x_0}\left(\frac{1}{\lambda_T\sqrt{T}}\Big|\int_0^T (\mu_1 - X_t)\,\mathrm{d}W_t\Big| \geq L\right)$$
$$+ P_{x_0}\left(\Big|1 - (\mu_2 - \mu_1^2)^{-1}\widehat{\sigma}_T^2\Big| \geq \frac{\delta}{2L}\right).$$

Applying Proposition 2.1, Lemma 3.1, (3.5) and classical moderate deviations for the Brownian motion, we can obtain that

$$\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log P_{x_0}\left(\frac{1}{\lambda_T\sqrt{T}}\Big|W_{\mathrm{T}}\big(\widehat{\mu}_T - \mu_1\big)\right.$$
$$\left. +\big(1 - (\mu_2 - \mu_1^2)^{-1}\widehat{\sigma}_T^2\big)\int_0^T (\mu_1 - X_t)\,\mathrm{d}W_t\Big| \geq \delta\right)$$
$$\leq -\frac{L^2}{2}\Big(1 \vee \big(\mu_2 - \mu_1^2\big)^{-1}\Big),$$

which implies immediately by letting $L \to \infty$ that

$$\lim_{T\to\infty} \frac{1}{\lambda_T^2} \log P_{x_0} \left(\frac{1}{\lambda_T \sqrt{T}} \Big| W_T(\widehat{\mu}_T - \mu_1) + \big(1 - (\mu_2 - \mu_1^2)^{-1} \widehat{\sigma}_T^2\big) \int_0^T (\mu_1 - X_t)\, \mathrm{d}W_t \Big| \geq \delta \right) = -\infty. \tag{3.7}$$

Similarly, we can also have that

$$\lim_{T\to\infty} \frac{1}{\lambda_T^2} \log P_{x_0} \left(\frac{1}{\lambda_T \sqrt{T}} \Big| \widehat{\mu}_T W_T(\widehat{\mu}_T - \mu_1) + \big(\widehat{\mu}_T - \mu_1 (\mu_2 - \mu_1^2)^{-1} \widehat{\sigma}_T^2\big) \int_0^T (\mu_1 - X_t)\, \mathrm{d}W_t \Big| \geq \delta \right) = -\infty. \tag{3.8}$$

Then, together with (3.5), (3.7) and (3.8), we can complete the proof of (3.6). ◻

Proof of Theorem 1.1 By (3.2) and Lemma 3.2, $\left\{ \frac{\sqrt{T}}{\lambda_T} \begin{pmatrix} \widehat{\theta}_T - \theta \\ \widehat{\gamma}_T - \gamma \end{pmatrix}, T > 0 \right\}$ is exponential equivalent to $\left\{ \frac{1}{\lambda_T \sqrt{T}} M_{\mathrm{T}}, T > 0 \right\}$ with speed λ_T^2. Theorem 1.1 follows from Lemma 3.1. ◻

4 The Case of Two-Sided Barriers

In this section, we focus on the drift parameter estimations for the reflected Ornstein–Uhlenbeck process with two-sided barriers b_L and b_U ($b_U > b_L$):

$$\begin{cases} \mathrm{d}X_t = (-\theta X_t + \gamma)\mathrm{d}t + \mathrm{d}W_t + \mathrm{d}L_t - \mathrm{d}U_t, \\ X_t \in [b_L, b_U], \text{ for all } t \geq 0, \\ X_0 = x_0 \in [b_{\mathrm{L}}, b_{\mathrm{U}}], \end{cases} \tag{4.1}$$

where $\theta \in (0, +\infty)$ and γ are unknown, the processes $L = \{L_t, t \geq 0\}$ and $U = \{U_t, t \geq 0\}$ are the minimal continuous increasing processes with $L_0 = U_0 = 0$, which make the process $X_t \in [b_L, b_U]$ for all $t \geq 0$ and satisfy

$$\int_0^\infty I_{\{X_t > b_L\}} \mathrm{d}L_t = 0, \quad \int_0^\infty I_{\{X_t < b_U\}} \mathrm{d}U_t = 0.$$

The stationary distribution of (4.1) is given by

$$\widetilde{\pi}(dx) = \frac{e^{-\theta(x-\gamma/\theta)^2}}{\widetilde{M}} I_{[b_L, b_U]} \mathrm{d}x, \quad \widetilde{M} = \int_{b_L}^{b_U} e^{-\theta(x-\gamma/\theta)^2} \mathrm{d}x.$$

4.1 Maximum Likelihood Estimators of θ and γ

By Girsanov formula in Ward and Glynn [36], Bo et al. [9], the log-likelihood ratio process can be written as

$$\log\left(\frac{dP_{\theta,\gamma,x_0}}{\mathrm{d}P_{0,0,x_0}}\bigg|_{\mathcal{F}_T}\right) = -\theta\int_0^T X_t\mathrm{d}(X_t - L_t + U_t) + \gamma\big(X_T - L_T + U_T - x_0\big) - \frac{\theta^2}{2}\int_0^T X_t^2\mathrm{d}t + \theta\gamma\int_0^T X_t\mathrm{d}t - \frac{\gamma^2}{2}T,$$

where $\mathcal{F}_T = \sigma(W_s, s \le T)$. Therefore, the maximum likelihood estimators of θ and γ are given by

$$\widetilde{\theta}_T = \frac{-T\int_0^T X_t\mathrm{d}(X_t - L_t + U_t) + (X_T - L_T + U_T - x_0)\int_0^T X_t\mathrm{d}t}{T\int_0^T X_t^2\mathrm{d}t - \big(\int_0^T X_t\mathrm{d}t\big)^2}$$

and

$$\widetilde{\gamma}_T = \frac{-\int_0^T X_t\mathrm{d}t\int_0^T X_t\mathrm{d}(X_t - L_t + U_t) + (X_T - L_T + U_T - x_0)\int_0^T X_t^2\mathrm{d}t}{T\int_0^T X_t^2\mathrm{d}t - \big(\int_0^T X_t\mathrm{d}t\big)^2}.$$

Similar to the one-sided barrier case in Sect. 3, we have the following key martingale decomposition

$$\sqrt{T}\begin{pmatrix}\widetilde{\theta}_T - \theta\\ \widetilde{\gamma}_T - \gamma\end{pmatrix} = \frac{\widetilde{M}_T}{\sqrt{T}} + \widetilde{R}_T, \tag{4.2}$$

where

$$\widetilde{M}_T = (\widetilde{\mu}_2 - \widetilde{\mu}_1^2)^{-1}\begin{pmatrix}\int_0^T(\widetilde{\mu}_1 - X_t)\,\mathrm{d}W_t\\ \int_0^T(\widetilde{\mu}_2 - \widetilde{\mu}_1X_t)\,\mathrm{d}W_t\end{pmatrix}, \tag{4.3}$$

$$\widetilde{R}_T = \frac{1}{\sqrt{T}\widehat{\sigma}_T^2}\begin{pmatrix}W_T\big(\widehat{\mu}_T - \widetilde{\mu}_1\big) + \big(1 - (\widetilde{\mu}_2 - \widetilde{\mu}_1^2)^{-1}\widehat{\sigma}_T^2\big)\int_0^T(\widetilde{\mu}_1 - X_t)\,\mathrm{d}W_t\\ \widehat{\mu}_T W_T\big(\widehat{\mu}_T - \widetilde{\mu}_1\big) + \big(\widehat{\mu}_T - \widetilde{\mu}_1(\widetilde{\mu}_2 - \widetilde{\mu}_1^2)^{-1}\widehat{\sigma}_T^2\big)\int_0^T(\widetilde{\mu}_1 - X_t)\,\mathrm{d}W_t\end{pmatrix} \tag{4.4}$$

and

$$\widetilde{\mu}_1 = \int_{b_{\mathrm{L}}}^{b_{\mathrm{U}}} x\widetilde{\pi}(\mathrm{d}x),\ \widetilde{\mu}_2 = \int_{b_L}^{b_U} x^2\widetilde{\pi}(\mathrm{d}x),\ \widehat{\mu}_T = \frac{1}{T}\int_0^T X_t\mathrm{d}t,\ \widehat{\sigma}_T^2 = \frac{1}{T}\int_0^T X_t^2\mathrm{d}t - \widehat{\mu}_T^2. \tag{4.5}$$

4.2 Regenerative Process View of $\int_0^T X_t dt$ and $\int_0^T X_t^2 dt$

To analyze the deviation properties of $\int_0^T X_t dt$ and $\int_0^T X_t^2 dt$, we will also employ regenerative process techniques. Let $\tau_X(x) = \inf\left\{t \geq 0 : X_t = x\right\}$. We can define regenerative times in terms of hitting times, which are slightly different from the one-sided barrier case.

$$\begin{aligned}
\widetilde{\alpha}_0 &= 0,\ \widetilde{\alpha}_{2k+1} = \inf\left\{t \geq \widetilde{\alpha}_{2k} : X_t = b_{\mathrm{L}} + \frac{b_{\mathrm{U}} - b_{\mathrm{L}}}{4} \text{ or } X_t = b_L + \frac{3(b_U - b_L)}{4}\right\},\\
\widetilde{\alpha}_{2k+2} &= \inf\left\{t \geq \widetilde{\alpha}_{2k+1} : X_t = b_L + \frac{b_U - b_L}{2}\right\},\ \widetilde{\Theta}_k = \widetilde{\alpha}_{2k+2},\\
\widetilde{N}_T &= \sup\left\{k \geq -1 : \widetilde{\Theta}_k \leq T\right\}.
\end{aligned}$$

From the strong Markov property of reflected Ornstein–Uhlenbeck process, X is a regenerative process with regeneration times given by $\left\{\widetilde{\Theta}_k, k \geq 0\right\}$. Then, under P_{x_0} with $x_0 \in [b_{\mathrm{L}}, b_{\mathrm{U}}]$,

$$\left\{\int_{\widetilde{\Theta}_{k-1}}^{\widetilde{\Theta}_k} X_t \mathrm{d}t, \widetilde{\Theta}_k - \widetilde{\Theta}_{k-1} : k \geq 1\right\}, \quad \left\{\int_{\widetilde{\Theta}_{k-1}}^{\widetilde{\Theta}_k} X_t^2 \mathrm{d}t, \widetilde{\Theta}_k - \widetilde{\Theta}_{k-1} : k \geq 1\right\}$$

are both independent and identically distributed sequences. Moreover, we also have the following crucial formulas

$$\left|\int_0^T X_t \mathrm{d}t - \sum_{k=1}^{\widetilde{N}_T} \int_{\widetilde{\Theta}_{k-1}}^{\widetilde{\Theta}_k} X_t \mathrm{d}t\right| \leq \left|\int_0^{\widetilde{\Theta}_0 \wedge T} X_t \mathrm{d}t\right| + \left|\int_{\widetilde{\Theta}_{\widetilde{N}_T}}^T X_t \mathrm{d}t\right| \tag{4.6}$$

and

$$\left|\int_0^T X_t^2 \mathrm{d}t - \sum_{k=1}^{\widetilde{N}_T} \int_{\widetilde{\Theta}_{k-1}}^{\widetilde{\Theta}_k} X_t^2 \mathrm{d}t\right| \leq \int_0^{\widetilde{\Theta}_0} X_t^2 \mathrm{d}t + \int_{\widetilde{\Theta}_{\widetilde{N}_T}}^{\widetilde{\Theta}_{\widetilde{N}_T+1}} X_t^2 \mathrm{d}t. \tag{4.7}$$

Parallel to Lemmas 2.1 and 2.2, we have the following decay of tail probabilities.

Lemma 4.1 *For all $x_0 \in [b_{\mathrm{L}}, b_{\mathrm{U}}]$, there exists some positive constants C_0, C_1 depending only on $x_0, b_{\mathrm{L}}, b_{\mathrm{U}}, \theta$ and γ, such that for T large enough*

$$P_{x_0}\left(\widetilde{\Theta}_0 > T\right) \leq C_0 e^{-C_1 T} \tag{4.8}$$

and

$$P_{x_0}\Big(\Big|\int_0^{\widetilde{\Theta}_0} X_t \mathrm{d}t\Big| \vee \Big|\int_0^{\widetilde{\Theta}_0\wedge T} X_t \mathrm{d}t\Big| > T\Big) \le C_0 e^{-C_1 T}, \quad P_{x_0}\Big(\int_0^{\widetilde{\Theta}_0} X_t^2 dt > T\Big) \le C_0 e^{-C_1 T}. \tag{4.9}$$

In particular, there exists some $\eta > 0$ *such that* $E_{x_0} e^{\eta \widetilde{\Theta}_0} < \infty$, *and*

$$E_{x_0} \exp\left\{\eta\left(\left|\int_0^{\widetilde{\Theta}_0\wedge T} X_t \mathrm{d}t\right| \vee \left|\int_0^{\widetilde{\Theta}_0} X_t \mathrm{d}t\right|\right)\right\} < \infty, \quad E_{x_0} \exp\left\{\eta \int_0^{\widetilde{\Theta}_0} X_t^2 \mathrm{d}t\right\} < \infty.$$

Proof Firstly, if $b_L \le x_0 \le b_{\mathrm{L}} + \frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{4}$, then $\widetilde{\Theta}_0 = \tau_X(b_{\mathrm{L}} + \frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{2})$. Under $X_0 = Y_0 = x_0$, we have $X_t \ge Y_t$ for $t \le \tau_X(b_{\mathrm{U}})$, and then $\tau_X(b_{\mathrm{L}} + \frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{2}) \le \tau_Y(b_{\mathrm{L}} + \frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{2})$, where Y and τ_Y are defined by (2.7) and (2.8). By Corollary 3.1 in Alili et al. [1], we have for T large enough

$$\begin{aligned} P_{x_0}\Big(\widetilde{\Theta}_0 > T\Big) &= P_{x_0}\Big(\tau_X(b_{\mathrm{L}} + \frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{2}) > T\Big) \le P_{x_0}\Big(\tau_Y(b_{\mathrm{L}} + \frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{2}) > T\Big) \\ &\le C_0 e^{-C_1 T}, \end{aligned} \tag{4.10}$$

where C_0, C_1 are positive constants depending only on x_0, b_L, b_U, θ and γ.

Secondly, if $b_L + \frac{3(b_{\mathrm{U}}-b_{\mathrm{L}})}{4} \le x_0 \le b_{\mathrm{U}}$, then $\widetilde{\Theta}_0 = \tau_X(b_{\mathrm{L}} + \frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{2})$. Under $X_0 = Y_0 = x_0$, we have $X_t \le Y_t$ for $t \le \tau_X(b_{\mathrm{L}})$, and then $\tau_X(b_{\mathrm{L}} + \frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{2}) \le \tau_Y(b_{\mathrm{L}} + \frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{2})$. By Corollary 3.1 in Alili et al. [1], we have for T large enough

$$\begin{aligned} P_{x_0}\Big(\widetilde{\Theta}_0 > T\Big) &= P_{x_0}\Big(\tau_X(b_{\mathrm{L}} + \frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{2}) \ge T\Big) \le P_{x_0}\Big(\tau_Y(b_{\mathrm{L}} + \frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{2}) > T\Big) \\ &\le C_0 e^{-C_1 T}. \end{aligned} \tag{4.11}$$

Thirdly, if $b_{\mathrm{L}} + \frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{4} \le x_0 \le b_L + \frac{3(b_{\mathrm{U}}-b_{\mathrm{L}})}{4}$, we have $X_t = Y_t$ on the interval $[0, \tau_X(b_{\mathrm{L}}) \wedge \tau_X(b_{\mathrm{U}})]$. Under $X_0 = Y_0 = x_0$, we have

$$\begin{aligned} &\tau_X\left(b_{\mathrm{L}} + \frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{4}\right) \wedge \tau_X\left(b_{\mathrm{L}} + \frac{3(b_{\mathrm{U}}-b_{\mathrm{L}})}{4}\right) \\ &\quad = \tau_Y\left(b_L + \frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{4}\right) \wedge \tau_Y\left(b_{\mathrm{L}} + \frac{3(b_{\mathrm{U}}-b_{\mathrm{L}})}{4}\right) = \widetilde{\alpha}_1. \end{aligned}$$

Consequently, it holds by strong Markov property

$$\begin{aligned} &P_{x_0}\Big(\widetilde{\Theta}_0 > T\Big) \\ &\quad \le P_{x_0}\Big(\widetilde{\alpha}_1 > T/2\Big) + P_{x_0}\Big(\widetilde{\alpha}_2 - \widetilde{\alpha}_1 > T/2\Big) \\ &\quad \le P_{x_0}\Big(\tau_X(b_{\mathrm{L}} + \frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{4}) \wedge \tau_X(b_{\mathrm{L}} + \frac{3(b_{\mathrm{U}}-b_{\mathrm{L}})}{4}) > T/2\Big) \end{aligned}$$

$$
\begin{aligned}
&+ P_{b_{\mathrm{L}}+\frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{4}}\Big(\tau_X(b_{\mathrm{L}}+\frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{2}) > T/2\Big)\\
&\cdot P_{x_0}\Big(\tau_X(b_{\mathrm{L}}+\frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{4}) < \tau_X(b_{\mathrm{L}}+\frac{3(b_{\mathrm{U}}-b_{\mathrm{L}})}{4})\Big)\\
&+ P_{b_{\mathrm{L}}+\frac{3(b_{\mathrm{U}}-b_{\mathrm{L}})}{4}}\Big(\tau_X(b_{\mathrm{L}}+\frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{2}) > T/2\Big)\\
&\cdot P_{x_0}\Big(\tau_X(b_{\mathrm{L}}+\frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{4}) \geq \tau_X(b_{\mathrm{L}}+\frac{3(b_{\mathrm{U}}-b_{\mathrm{L}})}{4})\Big)\\
\leq\, & P_{x_0}\Big(\tau_Y(b_{\mathrm{L}}+\frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{4}) \wedge \tau_Y(b_{\mathrm{L}}+\frac{3(b_{\mathrm{U}}-b_{\mathrm{L}})}{4}) > T/2\Big)\\
&+ P_{b_{\mathrm{L}}+\frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{4}}\Big(\tau_Y(b_{\mathrm{L}}+\frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{2}) > T/2\Big) + P_{b_{\mathrm{L}}+\frac{3(b_{\mathrm{U}}-b_{\mathrm{L}})}{4}}\Big(\tau_Y(b_{\mathrm{L}}+\frac{b_{\mathrm{U}}-b_{\mathrm{L}}}{2}) > T/2\Big).
\end{aligned}
$$

Using Corollary 3.1 in Alili et al. [1] again, we have for T large enough

$$
P_{x_0}\Big(\widetilde{\Theta}_0 > T\Big) \leq C_0 e^{-C_1 T}. \tag{4.12}
$$

Therefore, together with (4.10,4.11, 4.12), we can complete the proof of (4.8).

Finally, since $\sup_{t\in[0,\infty)} |X_t| \leq |b_L| \vee |b_U|$, then (4.9) can be achieved by (4.8).

□

4.3 Exponential Equivalence and Moderate Deviations

By (4.6, 4.7) and Lemma 4.1, and using the same procedure as in the proof of Proposition 2.1, we can state the following exponential equivalence results, while the proofs are omitted.

Proposition 4.1 *Let λ_T be defined by (1.5). For all $\delta > 0$ and $b_L \leq x_0 \leq b_U$, we have*

$$
\lim_{T\to\infty} \frac{1}{\lambda_T^2} \log P_{x_0}\Big(\frac{1}{T}\Big|\int_0^T X_t \mathrm{d}t - \widetilde{\mu}_1 T\Big| \geq \delta\Big) = -\infty
$$

and

$$
\lim_{T\to\infty} \frac{1}{\lambda_T^2} \log P_{x_0}\Big(\frac{1}{T}\Big|\int_0^T X_t^2 \mathrm{d}t - \widetilde{\mu}_2 T\Big| \geq \delta\Big) = -\infty.
$$

In particular, for any $\beta \in \mathbb{R}$

$$
\lim_{T\to\infty} \frac{1}{\lambda_T^2} \log P_{x_0}\Big(\frac{1}{T}\Big|\int_0^T (\beta - X_t)^2\, \mathrm{d}t - \big(\beta^2 - 2\beta\widetilde{\mu}_1 + \widetilde{\mu}_2\big)T\Big| \geq \delta\Big) = -\infty.
$$

Now, following the similar line as in the proof of Lemma 3.2, we have

Lemma 4.2 *Let λ_T, $\widetilde{M}_T$, $\widetilde{R}_T$, $\widehat{\mu}_T$, $\widetilde{\sigma}_T^2$, $\widetilde{\mu}_1$, $\widetilde{\mu}_2$ be defined by (1.5,4.3, 4.4, 4.5).*

(1) The family $\left\{\frac{\widetilde{M}_T}{\sqrt{T}\lambda_T}, T > 0\right\}$ *satisfy the large deviations with speed* λ_T^2 *and rate function*

$$\widetilde{I}(x) = \frac{1}{2}x^\tau \widetilde{\Sigma}^{-1}x, \quad x \in \mathbb{R}^2,$$

where

$$\widetilde{\Sigma} = \left(\widetilde{\mu}_2 - \widetilde{\mu}_1^2\right)^{-1}\begin{pmatrix} 1 & \widetilde{\mu}_1 \\ \widetilde{\mu}_1 & \widetilde{\mu}_2 \end{pmatrix}.$$

(2) For any $\delta > 0$,

$$\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log P_{x_0}\left(\left|\widehat{\sigma}_T^2 - \left(\widetilde{\mu}_2 - \widetilde{\mu}_1^2\right)\right| \geq \delta\right) = -\infty,$$
$$\lim_{T\to\infty}\frac{1}{\lambda_T^2}\log P_{x_0}\left(\frac{1}{\lambda_T}\left|\widetilde{R}_T\right| \geq \delta\right) = -\infty.$$

By (4.2), Lemma 4.2, and using the same way as in the proof of Theorem 1.1, we can establish the moderate deviations for $\left(\widetilde{\theta}_T - \theta, \widetilde{\gamma}_T - \gamma\right)$.

Theorem 4.1 *For* λ_T *defined by (1.5), the family* $\left\{\frac{\sqrt{T}}{\lambda_T}\begin{pmatrix}\widetilde{\theta}_T - \theta \\ \widetilde{\gamma}_T - \gamma\end{pmatrix}, T > 0\right\}$ *satisfies the large deviations with speed* λ_T^2 *and rate function* $\widetilde{I}(x)$.

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