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Higher-Order Derivative of Self-Intersection Local Time for Fractional Brownian Motion

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Abstract
We consider the existence and Hölder continuity conditions for the k-th-order derivatives of self-intersection local time for d-dimensional fractional Brownian motion, where $k = (k_1, k_2, \dots, k_d)$. Moreover, we show a limit theorem for the critical case with $H = \frac{2}{3}$ and $d = 1$, which was conjectured by Jung and Markowsky [7].

Keywords Self-intersection local time · Fractional Brownian motion · Hölder continuity

Mathematics Subject Classification (2020) 60G22 · 60J55

1 Introduction

Fractional Brownian motion (fBm) on $\mathbb{R}^d$ with Hurst parameter $H \in (0, 1)$ is a d-dimensional centered Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with component processes being independent copies of a 1-dimensional centered Gaussian process $B^{H,i}$, $i = 1, 2, \dots, d$ and the covariance function given by

$$\mathbb{E}[B_t^{H,i} B_s^{H,i}] = \frac{1}{2}\left[t^{2H} + s^{2H} - |t - s|^{2H}\right].$$

Note that $B_t^{\frac{1}{2}}$ is a classical standard Brownian motion. Let $D = \{(r, s) : 0 < r < s < t\}$. The self-intersection local time (SLT) of fBm was first investigated in Rosen [11]

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and formally defined as:

$$\alpha_t(y) = \int_D \delta(B_s^H - B_r^H - y)drds,$$

where B^H is a fBm and δ is the Dirac delta function. It was further investigated in Hu [3], Hu and Nualart [4]. In particular, Hu and Nualart [4] showed its existence whenever $Hd < 1$. Moreover, $\alpha_t(y)$ is Hölder continuous in time of any order strictly less than $1 - H$ with $d = 1$, which can be derived from Xiao [14].

The derivative of self-intersection local time (DSLT) for fBm was first considered in the works by Yan et al. [15] [16], where the ideas were based on Rosen [12]. The DSLT for fBm has two versions: One is extended by the Tanaka formula (see in Jung and Markowsky [7]):

$$\widetilde{\alpha}'_t(y) = -H \int_D \delta'(B_s^H - B_r^H - y)(s-r)^{2H-1}drds.$$

The other is from the occupation-time formula (see Jung and Markowsky [8]):

$$\widehat{\alpha}'_t(y) = -\int_D \delta'(B_s^H - B_r^H - y)drds.$$

Motivated by the first-order DSLT for fBm in Jung and Markowsky [8] and the k-th-order derivative of intersection local time (ILT) for fBm in Guo et al. [1], we will consider the following k-th-order DSLT for fBm in this paper

$$\begin{aligned}\widehat{\alpha}_t^{(k)}(y) &= \frac{\partial^k}{\partial y_1^{k_1}\cdots\partial y_d^{k_d}} \int_D \delta(B_s^H - B_r^H - y)drds \\ &= (-1)^{|k|}\int_D \delta^{(k)}(B_s^H - B_r^H - y)drds,\end{aligned}$$

where $k = (k_1, \ldots, k_d)$ is a multi-index with all k_i being nonnegative integers and $|k| = k_1 + k_2 + \cdots + k_d$, δ is the Dirac delta function of d variables and $\delta^{(k)}(y) = \frac{\partial^k}{\partial y_1^{k_1}\cdots\partial y_d^{k_d}}\delta(y)$ is the k-th-order partial derivative of δ.

Set

$$f_\varepsilon(x) = \frac{1}{(2\pi\varepsilon)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2\varepsilon}} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i\langle p,x\rangle} e^{-\varepsilon\frac{|p|^2}{2}}dp,$$

where $\langle p, x\rangle = \sum_{j=1}^d p_j x_j$ and $|p|^2 = \sum_{j=1}^d p_j^2$.

Since the Dirac delta function δ can be approximated by $f_\varepsilon(x)$, we approximate $\delta^{(k)}$ and $\widehat{\alpha}_t^{(k)}(y)$ by

$$f_\varepsilon^{(k)}(x) = \frac{i^{|k|}}{(2\pi)^d}\int_{\mathbb{R}^d} p_1^{k_1}\cdots p_d^{k_d} e^{i\langle p,x\rangle} e^{-\varepsilon\frac{|p|^2}{2}}dp$$

and

$$\widehat{\alpha}_{t,\varepsilon}^{(k)}(y) = (-1)^{|k|} \int_D f_\varepsilon^{(k)}(B_s^H - B_r^H - y)drds, \tag{1}$$

respectively.

If $\widehat{\alpha}_{t,\varepsilon}^{(k)}(y)$ converges to a random variable in L^p as $\varepsilon \to 0$, we denote the limit by $\widehat{\alpha}_t^{(k)}(y)$ and call it the k-th DSLT of B^H.

Theorem 1 *For* $0 < H < 1$ *and* $\widehat{\alpha}_{t,\varepsilon}^{(k)}(y)$ *defined in* (1), *let* $\# := \#\{k_i \text{ is odd}, \ i = 1, 2, \ldots d\}$ *denotes the odd number of* k_i, *for* $i = 1, 2, \ldots, d$. *If* $H < \min\{\frac{2}{2|k|+d}, \frac{1}{|k|+d-\#}, \frac{1}{d}\}$ *for* $|k| = \sum_{j=1}^d k_j$, *then* $\widehat{\alpha}_t^{(k)}(0)$ *exists in* L^2.

Theorem 2 *If* $H(|k| + d) < 1$, *then* $\widehat{\alpha}_t^{(k)}(0)$ *exists in* L^p, *for all* $p \in (0, \infty)$.

Note that if $d = 1$ and $|k| = 1$, the condition for the existence of $\widehat{\alpha}_t^{(k)}(y)$ in Theorems 1 and 2 is consistent with that in Jung and Markowsky [8]. If $d = 2$ and $|k| = 1$ in Theorems 1, we can see $H < \frac{1}{2}$ is the best possible, since a limit theorem for threshold $H = \frac{1}{2}$ studied in Markowsky [9].

Theorem 3 *Assume that* $H(|k|+d) < 1$ *and* $t, \tilde{t} \in [0, T]$. *Then,* $\widehat{\alpha}_t^{(k)}(y)$ *is Hölder continuous in* y *of any order strictly less than* $\min(1, \frac{1-Hd-H|k|}{H})$ *and Hölder continuous in* t *of any order strictly less than* $1 - H|k| - Hd$,

$$\left|\mathbb{E}\left[\left(\widehat{\alpha}_t^{(k)}(x) - \widehat{\alpha}_t^{(k)}(y)\right)^n\right]\right| \leq C|x-y|^{n\lambda}, \tag{2}$$

where $\lambda < \min(1, \frac{1-Hd-H|k|}{H})$ *and*

$$\left|\mathbb{E}\left[\left(\widehat{\alpha}_t^{(k)}(y) - \widehat{\alpha}_{\tilde{t}}^{(k)}(y)\right)^n\right]\right| \leq C|t-\tilde{t}|^{n\beta}, \tag{3}$$

where $\beta < 1 - H|k| - Hd$.

Note that if $d = 1$ and $k = 1$. The results of (2) and (3) in Theorem 3 are consistent with the results in Jung and Markowsky [8]. When $d = 1$ and $k = 0$, the corresponding Hölder continuous in time of any order less than $1-H$, is the condition obtained in Xiao [14]. Moreover, we believe that our methodology also works well for k-th-order DSLT of solution of stochastic differential equation (SDE) driven by fBm, if the solution of SDE driven by fBm satisfies the property of local nondeterminism. For example, the special linear SDE, the solution is fractional Ornstein–Uhlenbeck processes.

Jung and Markowsky [7] proved that $\widehat{\alpha}_t^{(k)}(0)$ exists in L^2 for $d = 1$ and $k = 1$ with $0 < H < 2/3$, and conjectured that for the case $H > 2/3$, $\varepsilon^{-\gamma(H)}\widehat{\alpha}_{t,\varepsilon}^{'}(0)$ converges in law to a Gaussian distribution for some suitable constant $\gamma(H) > 0$, and at the critical point $H = \frac{2}{3}$, the variable $\log(\frac{1}{\varepsilon})^{-\gamma}\widehat{\alpha}_{t,\varepsilon}^{'}(0)$ converges in law to a Gaussian distribution for some $\gamma > 0$. Later, Jaramillo and Nualart [5] proved the case of $H > 2/3$ as

$$\varepsilon^{\frac{3}{2}-\frac{1}{H}}\widehat{\alpha}_{t,\varepsilon}^{'}(0) \overset{law}{\to} N(0, \sigma_0^2), \quad \varepsilon \to 0.$$

By the proof of Lemma 1 in Sect. 2, we can see the multinomial terms $(p_{i1}-\frac{\mu p_{i2}}{\rho})^{k_i}$ for $i=1,2,...,d$, are taken into account. But we are not sure if k_i is odd or even, there are many difficulties in the integral of $\int(p_{i1}-\frac{\mu p_{i2}}{\rho})^{k_i}e^{-\frac{\rho p_{i1}^2}{2}}dp_{i1}$; thus, we only consider the limit theorem in case $d=1$ and $k=1$ below.

Inspired by the results conjectured in [7] and the functional limit theorem for SLT of fBm given in Jaramillo and Nualart [6]. We will show a limit theorem of the critical case $H=\frac{2}{3}$.

Theorem 4 *For $\widehat{\alpha}^{(k)}_{t,\varepsilon}(y)$ defined in (1) with $y=0$. Suppose that $H=\frac{2}{3}$, $d=1$ and $k=1$, then as $\varepsilon\to 0$, we have*

$$\left(\log\left(\frac{1}{\varepsilon}\right)\right)^{-1}\widehat{\alpha}^{\prime}_{t,\varepsilon}(0)\overset{law}{\to}N(0,\sigma^2),$$

where $\sigma^2=\frac{t^{\frac{4}{3}}}{8\pi}B(2,1/3)$ and $B(\cdot,\cdot)$ is a Beta function.

The study of DSLT for fBm has a strong degree of heat, see in [5–8,17] and references therein. However, the corresponding results for higher-order derivative have not been studied, except for the higher-order derivative of ILT for two independent fBms and some general Gaussian processes in [1] and [2]. As we all know, SLT and ILT have different integral structures in form. In particular, the independence of two fBms is required for ILT. So that the nondeterminism property which used for higher-order derivative of ILT cannot be used directly here.

To obtain the main results, we would use the methods of sample configuration given in Jung and Markowsky [8] and chaos decomposition provided in Jaramillo and Nualart [5]. Chaos decomposition is more and more mature for the asymptotic properties of SLT (see in Hu [4], Jaramillo and Nualart [5] and the references therein). The sample configuration method gives a way to apply nondeterminism property, and it is very powerful to prove the Hölder regularity. But the corresponding results of higher-order DSLT for d-dimensional fBm still have certain difficulty. The main difficulty lies in the computational complexity of multiple integrals. Moreover, the related results can be extended to the general cases. By the Theorem 4.1 in Jaramillo and Nualart [5] and Theorem 4 here, two limit theorems of the case $H>\frac{2}{2|k|+d}$ and critical case $H=\frac{2}{2|k|+d}$, with general $k=(k_1,\ldots,k_d)$ are left open. Extending these limit theorems to general cases will be worked in the future.

The paper has the following structure. We present some preliminary properties of d-dimensional fBm and some basic lemmas in Sect. 2. Section 3 is to prove the main results. To be exact, we will split this section into four subsections to prove the four theorems given in Sect. 1. Throughout this paper, if not mentioned otherwise, the letter C, with or without a subscript, denotes a generic positive finite constant and may change from line to line.

2 Preliminaries

In this section, we first give some properties of d-dimensional fBm B^H. It is well known that d-dimensional fBm has self-similarity, stationary increments and Hölder continuity. When Hurst parameter $H > 1/2$, B^H exhibits long memory. When $H < 1/2$, it has short memory. But in this paper, we need the following nondeterminism property.

By Nualart and Xu [10] (see also in Song, Xu and Yu [13]), we can see that for any $n \in \mathbb{N}$, there exists two constants κ_H and β_H depending only on n and H, such that for any $0 = s_0 < s_1 < \cdots < s_n$, $1 \le i \le n$, we have

$$\kappa_H \sum_{i=1}^n |x_i|^2 (s_i - s_{i-1})^{2H} \le \mathrm{Var}\Big(\sum_{i=1}^n x_i \cdot (B^H_{s_i} - B^H_{s_{i-1}})\Big)$$
$$\le \beta_H \sum_{i=1}^n |x_i|^2 (s_i - s_{i-1})^{2H}.$$

Next, we present two basic lemmas, which will be used in Sect. 3.

Lemma 1 *For any $\lambda,\ \mu,\ \rho \in \mathbb{R}$ with $\lambda > 0$, $\rho > 0$ and $\lambda\rho - \mu^2 > 0$. For $k \in \mathbb{Z}^+$, there exists a constant C only depending on k, such that*

(i) if k is odd,

$$\Big|\int_{\mathbb{R}^2} x^k y^k e^{-\frac{1}{2}(\lambda x^2 + \rho y^2 + 2\mu x y)} dx dy\Big| \le \begin{cases} \frac{C|\mu|^k}{(\lambda\rho - \mu^2)^{k+\frac{1}{2}}}, & \text{if } \frac{\mu^2}{\lambda\rho-\mu^2} \ge 1, \\ \frac{C|\mu|}{(\lambda\rho - \mu^2)^{\frac{k}{2}+1}}, & \text{if } \frac{\mu^2}{\lambda\rho-\mu^2} < 1, \end{cases} \tag{4}$$

(ii) if k is even,

$$\Big|\int_{\mathbb{R}^2} x^k y^k e^{-\frac{1}{2}(\lambda x^2 + \rho y^2 + 2\mu x y)} dx dy\Big| \le \begin{cases} \frac{C|\mu|^k}{(\lambda\rho - \mu^2)^{k+\frac{1}{2}}}, & \text{if } \frac{\mu^2}{\lambda\rho-\mu^2} \ge 1, \\ \frac{C}{(\lambda\rho - \mu^2)^{\frac{k+1}{2}}}, & \text{if } \frac{\mu^2}{\lambda\rho-\mu^2} < 1. \end{cases} \tag{5}$$

Proof First, we consider the integral with respect to y,

$$\int_{\mathbb{R}} y^k e^{-\frac{\rho}{2}y^2 - \mu x y} dy = e^{\frac{\mu^2 x^2}{2\rho}} \int_{\mathbb{R}} y^k e^{-\frac{\rho}{2}(y + \frac{\mu x}{\rho})^2} dy$$
$$= e^{\frac{\mu^2 x^2}{2\rho}} \int_{\mathbb{R}} (y - \frac{\mu x}{\rho})^k e^{-\frac{\rho}{2}y^2} dy.$$

If k is odd, since

$$(y-\frac{\mu x}{\rho})^k=\sum_{i=0}^{k}C_k^i y^i(-\frac{\mu x}{\rho})^{k-i},$$

we have

$$\begin{aligned}\int_{\mathbb{R}} y^k e^{-\frac{\rho}{2}y^2-\mu xy}dy &= e^{\frac{\mu^2x^2}{2\rho}}\int_{\mathbb{R}} A_{odd}e^{-\frac{\rho}{2}y^2}dy\\ &= C_1e^{\frac{\mu^2x^2}{2\rho}}\frac{(\mu x)^k}{\rho^{k+\frac{1}{2}}}+C_3e^{\frac{\mu^2x^2}{2\rho}}\frac{(\mu x)^{k-2}}{\rho^{k-\frac{1}{2}}}\\ &\quad+\cdots+C_ke^{\frac{\mu^2x^2}{2\rho}}\frac{(\mu x)}{\rho^{k/2+1}}\\ &=: \widetilde{A}_{odd},\end{aligned}$$

where $C_1,\ C_3,\ldots,\ C_k$ are all positive constants and

$$A_{odd}=C_k^0\left(-\frac{\mu x}{\rho}\right)^k+C_k^2y^2\left(-\frac{\mu x}{\rho}\right)^{k-2}+\cdots+C_k^{k-1}y^{k-1}\left(-\frac{\mu x}{\rho}\right).$$

For the dx integral,

$$\begin{aligned}&\int_{\mathbb{R}} x^k e^{-\frac{1}{2}\lambda x^2}(\widetilde{A}_{odd})dx\\ &\leq C\left[\frac{\mu^k}{(\lambda\rho-\mu^2)^{k+\frac{1}{2}}}+\frac{\mu^{k-2}}{(\lambda\rho-\mu^2)^{k-\frac{1}{2}}}+\cdots+\frac{\mu}{(\lambda\rho-\mu^2)^{\frac{k}{2}+1}}\right],\end{aligned}$$

where the right-hand side is the sum of equal ratio series with the common ratio $\frac{\mu^2}{\lambda\rho-\mu^2}>0$. Then, we get (4).

If k is even,

$$\begin{aligned}\int_{\mathbb{R}} y^k e^{-\frac{\rho}{2}y^2-\mu xy}dy &= e^{\frac{\mu^2x^2}{2\rho}}\int_{\mathbb{R}}(y-\frac{\mu x}{\rho})^k e^{-\frac{\rho}{2}y^2}dy\\ &\leq Ce^{\frac{\mu^2x^2}{2\rho}}\Big[\int_{\mathbb{R}} y^k e^{-\frac{\rho}{2}y^2}dy+\int_{\mathbb{R}}(\frac{\mu x}{\rho})^k e^{-\frac{\rho}{2}y^2}dy\Big]\\ &=: B_1+B_2.\end{aligned}$$

It is easy to see that

$$B_1\leq Ce^{\frac{\mu^2x^2}{2\rho}}\rho^{-\frac{k+1}{2}}$$

and

$$B_2 \le Ce^{\frac{\mu^2 x^2}{2\rho}} (\mu x)^k \rho^{-\frac{2k+1}{2}}.$$

For the integral with respect to x,

$$\begin{aligned}\frac{1}{\rho^{\frac{k+1}{2}}} \int_{\mathbb{R}} x^k e^{-\frac{x^2}{2\rho}(\lambda\rho-\mu^2)} dx &\le \frac{C}{\rho^{\frac{k+1}{2}}} \int_{\mathbb{R}} x^k e^{-\frac{x^2}{2}} \left(\frac{\lambda\rho-\mu^2}{\rho}\right)^{-\frac{1+k}{2}} dx \\ &\le \frac{C}{(\lambda\rho-\mu^2)^{\frac{k+1}{2}}}\end{aligned}$$

and

$$\begin{aligned}\frac{\mu^k}{\rho^{k+\frac{1}{2}}} \int_{\mathbb{R}} x^{2k} e^{-\frac{x^2}{2\rho}(\lambda\rho-\mu^2)} dx &\le C \frac{\mu^k}{\rho^{k+\frac{1}{2}}} \int_{\mathbb{R}} x^{2k} e^{-\frac{x^2}{2}} \left(\frac{\lambda\rho-\mu^2}{\rho}\right)^{-\frac{1+2k}{2}} dx \\ &\le C \frac{\mu^k}{(\lambda\rho-\mu^2)^{k+\frac{1}{2}}}.\end{aligned}$$

This gives (5). □

The next lemma gives the bounds on the quantity of $\lambda\rho - \mu^2$, which could be obtained from the Appendix B in [7] or the Lemma 3.1 in [3].

Lemma 2 *Let*

$$\lambda = |s-r|^{2H}, \quad \rho = |s'-r'|^{2H},$$

and

$$\mu = \frac{1}{2}\Big(|s'-r|^{2H} + |s-r'|^{2H} - |s'-s|^{2H} - |r-r'|^{2H}\Big).$$

Case (i) *Suppose that $D_1 = \{(r, r', s, s') \in [0, t]^4 \mid r < r' < s < s'\}$, let $r'-r = a$, $s-r' = b$, $s'-s = c$. Then, there exists a constant K_1 such that*

$$\lambda\rho - \mu^2 \ge K_1 \left((a+b)^{2H} c^{2H} + a^{2H}(b+c)^{2H}\right)$$

and

$$2\mu = (a+b+c)^{2H} + b^{2H} - a^{2H} - c^{2H}.$$

Case (ii) *Suppose that $D_2 = \{(r, r', s, s') \in [0, t]^4 \mid r < r' < s' < s\}$, let $r'-r = a$, $s'-r' = b$, $s-s' = c$. Then, there exists a constant K_2 such that*

$$\lambda\rho - \mu^2 \ge K_2 b^{2H} \left(a^{2H} + c^{2H}\right)$$

and

$$2\mu = (a+b)^{2H} + (b+c)^{2H} - a^{2H} - c^{2H}.$$

Case (iii) *Suppose that* $D_3 = \{(r, r', s, s') \in [0,t]^4 \mid r < s < r' < s'\}$, *let* $s - r = a$, $r' - s = b$, $s' - r' = c$. *Then, there exists a constant* K_3 *such that*

$$\lambda\rho - \mu^2 \geq K_3(ac)^{2H}$$

and

$$2\mu = (a+b+c)^{2H} + b^{2H} - (a+b)^{2H} - (c+b)^{2H}.$$

3 Proof of the Main Results

In this section, the proof of Theorems 1, 2, 3 and 4 is taken into account. We will divide this section into four parts and give the proof of the corresponding theorem in each part.

3.1 Proof of Theorem 1

By (1) and the proof of Lemma 1,

$$\begin{aligned}
&\mathbb{E}\Big[\widehat{\alpha}^{(k)}_{t,\varepsilon}(0)\widehat{\alpha}^{(k)}_{t,\eta}(0)\Big] \\
&\quad = \frac{1}{(2\pi)^{2d}} \int_{D^2}\int_{\mathbb{R}^{2d}} p_1^k p_2^k e^{\frac{-(\varepsilon|p_1|^2+\eta|p_2|^2)}{2}} \mathbb{E}\Big[\prod_{j=1}^{2} e^{i\langle p_j, B^H_{s_j} - B^H_{r_j}\rangle}\Big] dp_1 dp_2 dr_1 dr_2 ds_1 ds_2 \\
&\quad = C\int_{D^2}\int_{\mathbb{R}^{2d}} \prod_{i=1}^{d} p_{i1}^{k_i} \prod_{i=1}^{d} p_{i2}^{k_i} e^{-\frac{1}{2}(|p_1|^2(\lambda+\varepsilon)+|p_2|^2(\rho+\eta)+2\langle p_1,p_2\rangle\mu)} dp_1 dp_2 dr' dr ds' ds \\
&\quad = C\int_{D^2}\Big[\int_{\mathbb{R}^2} p_{11}^{k_1} p_{12}^{k_1} e^{-\frac{1}{2}(p_{11}^2(\lambda+\varepsilon)+p_{12}^2(\rho+\eta)+2p_{11}p_{12}\mu)} dp_{11} dp_{12}\Big] \\
&\qquad \times \cdots \times \Big[\int_{\mathbb{R}^2} p_{d1}^{k_d} p_{d2}^{k_d} e^{-\frac{1}{2}(p_{d1}^2(\lambda+\varepsilon)+p_{d2}^2(\rho+\eta)+2p_{d1}p_{d2}\mu)} dp_{d1} dp_{d2}\Big] dr' dr ds' ds \\
&\quad =: C\int_{D^2} \prod_{i=1}^{d} \Xi_{k_i} dr' dr ds' ds,
\end{aligned}$$

where $\lambda = |s-r|^{2H}$, $\rho = |s'-r'|^{2H}$,

$$\mu = \frac{1}{2}\Big(|s'-r|^{2H} + |s-r'|^{2H} - |s'-s|^{2H} - |r-r'|^{2H}\Big)$$

and

$$\Xi_{k_i} = \begin{cases} \frac{C_{k_i}\mu^{k_i}}{((\lambda+\varepsilon)(\rho+\eta)-\mu^2)^{k_i+\frac{1}{2}}} + \frac{C_{k_i-2}\mu^{k_i-2}}{((\lambda+\varepsilon)(\rho+\eta)-\mu^2)^{k_i-\frac{1}{2}}} + \cdots + \frac{C_1\mu}{((\lambda+\varepsilon)(\rho+\eta)-\mu^2)^{\frac{k_i}{2}+1}}, & \text{if } k_i \text{ is odd,} \\ \frac{C_{k_i}\mu^{k_i}}{((\lambda+\varepsilon)(\rho+\eta)-\mu^2)^{k_i+\frac{1}{2}}} + \frac{C_{k_i-2}\mu^{k_i-2}}{((\lambda+\varepsilon)(\rho+\eta)-\mu^2)^{k_i-\frac{1}{2}}} + \cdots + \frac{C_0}{((\lambda+\varepsilon)(\rho+\eta)-\mu^2)^{\frac{k_i}{2}+\frac{1}{2}}}, & \text{if } k_i \text{ is even,} \end{cases}$$

for $i = 1, 2, \ldots, d$.

Note that for any $\varepsilon_1,\ \varepsilon_2 > 0$,

$$\begin{aligned} &\mathbb{E}\Big[\big(\widehat{\alpha}^{(k)}_{t,\varepsilon_1}(0) - \widehat{\alpha}^{(k)}_{t,\varepsilon_2}(0)\big)^2\Big] \\ &\leq C\int_{D^2}\Big|\int_{\mathbb{R}^{2d}}\prod_{j=1}^{2}\Big(e^{-\frac{\varepsilon_1}{2}|p_j|^2} - e^{-\frac{\varepsilon_2}{2}|p_j|^2}\Big) \\ &\quad\times p_1^k p_2^k \mathbb{E}\Big[\prod_{j=1}^{2} e^{i\langle p_j, B^H_{s_j} - B^H_{r_j}\rangle}\Big]dp_1dp_2\Big|dr_1dr_2ds_1ds_2 \\ &\leq C\int_{D^2}\prod_{j=1}^{2}\max_{p_j}\Big|e^{-\frac{\varepsilon_1}{2}|p_j|^2} - e^{-\frac{\varepsilon_2}{2}|p_j|^2}\Big|\prod_{i=1}^{d}\big|\widetilde{\Xi}_{k_i}\big|\,dr'drds'ds, \end{aligned}$$

where

$$\widetilde{\Xi}_{k_i} = \begin{cases} \frac{C_{k_i}\mu^{k_i}}{(\lambda\rho-\mu^2)^{k_i+\frac{1}{2}}} + \frac{C_{k_i-2}\mu^{k_i-2}}{(\lambda\rho-\mu^2)^{k_i-\frac{1}{2}}} + \cdots + \frac{C_1\mu}{(\lambda\rho-\mu^2)^{\frac{k_i}{2}+1}}, & \text{if } k_i \text{ is odd,} \\ \frac{C_{k_i}\mu^{k_i}}{(\lambda\rho-\mu^2)^{k_i+\frac{1}{2}}} + \frac{C_{k_i-2}\mu^{k_i-2}}{(\lambda\rho-\mu^2)^{k_i-\frac{1}{2}}} + \cdots + \frac{C_0}{(\lambda\rho-\mu^2)^{\frac{k_i}{2}+\frac{1}{2}}}, & \text{if } k_i \text{ is even.} \end{cases}$$

Consequently, if

$$\int_{D^2}\prod_{i=1}^{d}\big|\widetilde{\Xi}_{k_i}\big|\,dr'drds'ds < \infty,$$

then $\widehat{\alpha}^{(k)}_{t,\varepsilon}(0)$ converges in L^2 as $\varepsilon \to 0$.

By Lemma 1, we can see that $\int_{D^2}\prod_{i=1}^{d}\big|\widetilde{\Xi}_{k_i}\big|\,dr'drds'ds$ is less than

$$C\int_{D^2}\left[\frac{|\mu|^{|k|}}{(\lambda\rho-\mu^2)^{|k|+\frac{d}{2}}} + \frac{|\mu|^{\#}}{(\lambda\rho-\mu^2)^{\frac{|k|+d+\#}{2}}}\right]dr'drds'ds,$$

where $\# = \#\{k_i \ is\ odd,\ i = 1, 2, \ldots, d\}$ denotes the odd number of k_i, for $i = 1, 2, \ldots, d$, and $\# \in \{0, 1, 2, \ldots, d\}$.

Thus, to prove the finiteness of $\int_{D^2}\prod_{i=1}^d \left|\widetilde{\Xi}_{k_i}\right| dr'drds'ds$, we only need to prove

$$\int_{D^2} \frac{|\mu|^Q}{(\lambda\rho-\mu^2)^{\frac{|k|+d+Q}{2}}} dr'drds'ds < \infty \tag{6}$$

with $Q=|k|$ and $Q=\#$.

By Lemma 2, we can see D^2 is the union of the sets $D_1,\ D_2,\ D_3$.

When $(r,r',s,s')\in D_1$, then the left-hand side of (6) is less than

$$\begin{aligned}
&C\int_{[0,t]^3} \frac{a^{2HQ}+b^{2HQ}+c^{2HQ}}{(a+b)^{\frac{H}{2}(|k|+d+Q)}(b+c)^{\frac{H}{2}(|k|+d+Q)}(ac)^{\frac{H}{2}(|k|+d+Q)}} dadbdc\\
&\le C\int_{[0,t]^3} \frac{a^{2HQ}}{a^{\frac{H}{2}(|k|+d+Q)}b^{\frac{H}{2}(|k|+d+Q)}(ac)^{\frac{H}{2}(|k|+d+Q)}} dadbdc\\
&\quad + C\int_{[0,t]^3} \frac{b^{2HQ}}{b^{\frac{H}{2}(|k|+d+Q)}b^{\frac{H}{2}(|k|+d+Q)}(ac)^{\frac{H}{2}(|k|+d+Q)}} dadbdc\\
&\quad + C\int_{[0,t]^3} \frac{c^{2HQ}}{b^{\frac{H}{2}(|k|+d+Q)}c^{\frac{H}{2}(|k|+d+Q)}(ac)^{\frac{H}{2}(|k|+d+Q)}} dadbdc\\
&\le C\int_{[0,t]^3} \frac{1}{x^{H(|k|+d-Q)}y^{\frac{H}{2}(|k|+d+Q)}z^{\frac{H}{2}(|k|+d+Q)}} dxdydz < \infty.
\end{aligned}$$

When $(r,r',s,s')\in D_2$. Note that for the condition $H<\min\{\frac{2}{2|k|+d},\frac{1}{|k|+d-\#},\frac{1}{d}\}$, only in the case $d=1$, H can get the value bigger than $\frac{1}{2}$, while these the case have been studied in [4] and [7], respectively. So, we only need to consider the case $H<\frac{1}{2}$.

Thus, the left-hand side of (6) is less than

$$\begin{aligned}
&C\int_{[0,t]^3} \frac{[(a+b)^{2H}+(b+c)^{2H}-a^{2H}-c^{2H}]^Q}{b^{H(|k|+d+Q)}(ac)^{\frac{H}{2}(|k|+d+Q)}} dadbdc\\
&\le C\int_{[0,t]^3} \frac{[(a+b)^{2H}-a^{2H}]^Q}{b^{H(|k|+d+Q)}(ac)^{\frac{H}{2}(|k|+d+Q)}} dadbdc\\
&\quad + C\int_{[0,t]^3} \frac{[(b+c)^{2H}-c^{2H}]^Q}{b^{H(|k|+d+Q)}(ac)^{\frac{H}{2}(|k|+d+Q)}} dadbdc\\
&\le 2C\int_{[0,t]^2} \frac{[(a+b)^{2H}-a^{2H}]^Q}{b^{H(|k|+d+Q)}a^{\frac{H}{2}(|k|+d+Q)}} dadb.
\end{aligned}$$

Since $H<\frac{1}{2}$, then

$$[(a+b)^{2H}-a^{2H}]^Q \le b^{2HQ}.$$

Thus,

$$\begin{aligned}&\int_{[0,t]^2}\frac{[(a+b)^{2H}-a^{2H}]^Q}{b^{H(|k|+d+Q)}a^{\frac{H}{2}(|k|+d+Q)}}dadb\\&\leq C\int_{[0,t]^2}\frac{b^{2HQ}}{b^{H(|k|+d+Q)}a^{\frac{H}{2}(|k|+d+Q)}}dadb\\&\leq C\int_{[0,t]^2}\frac{1}{b^{H(|k|+d-Q)}a^{\frac{H}{2}(|k|+d+Q)}}dadb<\infty.\end{aligned}$$

When $(r,r',s,s')\in D_3$. For $\alpha,\beta>0$ with $\alpha+\beta=1$, there exists a positive constant K such that

$$\begin{aligned}|\mu|&=\frac{1}{2}\Big|(a+b+c)^{2H}+b^{2H}-(a+b)^{2H}-(b+c)^{2H}\Big|\\&=\Big|H(2H-1)ac\int_0^1\int_0^1(b+au+cv)^{2H-2}dudv\Big|\\&\leq ac\int_0^1\int_0^1\Big[b^{\alpha}(au+cv)^{\beta}\Big]^{2H-2}dudv\\&\leq ac\int_0^1\int_0^1\Big[b^{\alpha}(au)^{\frac{\beta}{2}}(cv)^{\frac{\beta}{2}}\Big]^{2H-2}dudv\\&\leq K(ac)^{\beta(H-1)+1}b^{2\alpha(H-1)}.\end{aligned}$$

Thus, the left-hand side of (6) is less than

$$\begin{aligned}&C\int_{[0,t]^3}\frac{[(ac)^{\beta(H-1)+1}b^{2\alpha(H-1)}]^Q}{(ac)^{H(|k|+d+Q)}}dadbdc\\&\leq C\int_{[0,t]^3}\frac{1}{b^{2\alpha Q(1-H)}(ac)^{\beta(Q-HQ)+H|k|+Hd+HQ-Q}}dadbdc.\end{aligned}$$

Note that $|k|\geq 1$ (where $|k|=0$ with $H<\frac{1}{d}$ could deduced from [4]) and $H<\frac{1}{2}$. When $Q=0$ (all derivatives were of even order), the result of (6) is obvious by $H<\frac{1}{|k|+d}$. When $Q\geq 1$, we have $2Q(1-H)>1$. So, we first choose $\varepsilon_0>0$, such that

$$H(|k|+d)-\frac{1}{2}+\frac{\varepsilon_0}{2}\Big(\frac{2}{|k|+d+Q}-H\Big)<1.$$

Then, we can choose

$$\alpha\in\left(\frac{1-\varepsilon_0(\frac{2}{|k|+d+Q}-H)}{2Q(1-H)},\frac{1}{2Q(1-H)}\right).$$

Thus,

$$\begin{aligned}&\beta(Q-HQ)+H|k|+Hd+HQ-Q\\&\quad=(1-\alpha)(Q-HQ)+H|k|+Hd+HQ-Q\\&\quad<H(|k|+d)-\frac{1}{2}+\frac{\varepsilon_0}{2}\Big(\frac{2}{|k|+d+Q}-H\Big),\end{aligned}$$

which is less than one. This gives (6).

3.2 Proof of Theorem 2

By (1), we have

$$\Big|\mathbb{E}\Big[\big(\widehat{\alpha}^{(k)}_{t,\varepsilon}(0)\big)^n\Big]\Big|\le C\int_{D^n}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n|p_j^k|\mathbb{E}\Big[\prod_{j=1}^n e^{i\langle p_j,B^H_{s_j}-B^H_{r_j}\rangle}\Big]dpdrds,$$

where $k=(k_1,\ldots,k_d)$, $|p_j^k|=\prod_{i=1}^d|p_{ij}|^{k_i}$ for $j=1,...,n$, $drds=dr_1\cdots dr_nds_1\cdots ds_n$ and

$$dp=dp_1\cdots dp_n=dp_{11}dp_{12}\cdots dp_{1n}\cdots dp_{d1}dp_{d2}\cdots dp_{dn}.$$

We use the method of sample configuration as in Jung and Markowsky [8]. Fix an ordering of the set $\{r_1,s_1,r_2,s_2,\ldots,r_n,s_n\}$, and let $l_1\le l_2\le\cdots\le l_{2n}$ be a relabeling of the set $\{r_1,s_1,r_2,s_2,\ldots,r_n,s_n\}$. Let $u_1\ldots u_{2n-1}$ be the proper linear combinations of the p_j's so that

$$\mathbb{E}\Big[\prod_{j=1}^n e^{i\langle p_j,B^H_{s_j}-B^H_{r_j}\rangle}\Big]=\mathbb{E}\Big[\prod_{j=1}^{2n-1}e^{i\langle u_j,B^H_{l_{j+1}}-B^H_{l_j}\rangle}\Big].$$

A detailed description of how the u's are chosen can be found in [8]. Then, by the local nondeterminism of fBm,

$$\Big|\mathbb{E}\Big[\prod_{j=1}^n e^{i\langle p_j,B^H_{s_j}-B^H_{r_j}\rangle}\Big]\Big|\le e^{-c\sum_{j=1}^{2n-1}|u_j|^2(l_{j+1}-l_j)^{2H}}.$$

Fix j, and let j_1 to be the smallest value such that u_{j_1} contains p_j as a term and then choose j_2 to be the smallest value strictly larger than j_1 such that u_{j_2} does not contain p_j as a term. Then, $p_j=u_{j_1}-u_{j_1-1}=u_{j_2-1}-u_{j_2}$. Similar to Jung and Markowsky [8], we can see that with the convention that $u_0=u_{2n}=0$,

$$|p_j^k|=|(u_{j_1}-u_{j_1-1})^{\frac{k}{2}}||(u_{j_2-1}-u_{j_2})^{\frac{k}{2}}|$$

$$= \prod_{i=1}^{d} |u_{ij_1} - u_{i(j_1-1)}|^{\frac{k_i}{2}} |u_{i(j_2-1)} - u_{ij_2}|^{\frac{k_i}{2}}$$
$$\leq C \prod_{i=1}^{d} (|u_{ij_1}|^{\frac{k_i}{2}} + |u_{i(j_1-1)}|^{\frac{k_i}{2}})(|u_{i(j_2-1)}|^{\frac{k_i}{2}} + |u_{ij_2}|^{\frac{k_i}{2}})$$

Thus,

$$\prod_{j=1}^{n} |p_j^k| = \prod_{j=1}^{2n} |(u_j - u_{j-1})^{\frac{k}{2}}| \leq C \prod_{i=1}^{d} \prod_{j=1}^{2n} (|u_{ij}|^{\frac{k_i}{2}} + |u_{i(j-1)}|^{\frac{k_i}{2}}).$$

and

$$\begin{aligned} \prod_{i=1}^{d} \prod_{j=1}^{2n} (|u_{ij}|^{\frac{k_i}{2}} + |u_{i(j-1)}|^{\frac{k_i}{2}}) &= \sum_{S_1} \prod_{i=1}^{d} \prod_{j=1}^{2n} (|u_{ij}|^{\frac{k_i}{2}\gamma_{i,j}} |u_{i(j-1)}|^{\frac{k_i}{2}\overline{\gamma_{i,j}}}) \\ &\leq \sum_{S_2} \prod_{i=1}^{d} \prod_{j=1}^{2n-1} (|u_{ij}|^{\frac{k_i}{2}\alpha_{i,j}}), \end{aligned} \tag{7}$$

where

$$S_1 = \left\{\gamma_{i,j},\ \overline{\gamma_{i,j}} :\ \gamma_{i,j} \in \{0,1\},\ \gamma_{i,j} + \overline{\gamma_{i,j}} = 1,\ \ i = 1, \dots, d,\ \ j = 1, \dots, 2n\right\}$$

and

$$S_2 = \left\{\alpha_{i,j} :\ \alpha_{i,j} \in \{0,1,2\},\ i = 1, \dots, d,\ j = 1, \dots 2n-1\right\}.$$

Note that we have omitted the terms $j = 0, 2n$ in the final expression in (7) since $u_0 = u_{2n} = 0$. Then,

$$\begin{aligned} &\left|\mathbb{E}\left[(\widehat{\alpha}_{t,\varepsilon}^{(k)}(0))^n\right]\right| \\ &\leq C \int_{E^n} \int_{\mathbb{R}^{nd}} e^{-c\sum_{j=1}^{2n-1} |u_j|^2 (l_{j+1}-l_j)^{2H}} \prod_{i=1}^{d} \prod_{j=1}^{2n} (|u_{ij}|^{\frac{k_i}{2}} + |u_{i(j-1)}|^{\frac{k_i}{2}}) dp dl \\ &\leq C \sum_{S_2} \int_{E^n} \int_{\mathbb{R}^{nd}} e^{-c\sum_{j=1}^{2n-1} |u_j|^2 (l_{j+1}-l_j)^{2H}} \prod_{i=1}^{d} \prod_{j=1}^{2n-1} (|u_{ij}|^{\frac{k_i}{2}\alpha_{i,j}}) dp dl \\ &= C \sum_{S_2} \int_{E^n} \int_{\mathbb{R}^{nd}} e^{-c\sum_{j=1}^{2n-1} |u_j|^2 (l_{j+1}-l_j)^{2H}} \prod_{j=1}^{2n-1} (|u_j^{\frac{k}{2}\alpha_j}|) dp dl, \end{aligned}$$

where $E^n = \{0 < l_1 < \cdots < l_{2n} < t\}$, $|u_j^{\frac{k}{2}\alpha_j}| = \prod_{i=1}^{d} |u_{ij}|^{\frac{k_i}{2}\alpha_{i,j}}$ and $dl = dl_1 dl_2 \cdots dl_{2n}$.

It is easy to observe that $\{u_1, u_2, \ldots, u_{2n-1}\}$ is contained in the span of $\{p_1, p_2, \ldots, p_n\}$ and conversely, so we can let $\mathcal{A}$ be a subset of $\{1, \ldots, 2n-1\}$ such that the set $\{u_j\}_{j\in\mathcal{A}}$ spans $\{p_1, p_2, \ldots, p_n\}$. We let $\mathcal{A}^c$ denote the complement of $\mathcal{A}$ in $\{1, \ldots, 2n-1\}$. Note that

$$\begin{aligned}
&e^{-c\sum_{j\in\mathcal{A}^c}|u_j|^2(l_{j+1}-l_j)^{2H}}\prod_{j\in\mathcal{A}^c}(|u_j^{\frac{k}{2}\alpha_j}|)\\
&=e^{-c\sum_{j\in\mathcal{A}^c}|u_j|^2(l_{j+1}-l_j)^{2H}}\prod_{j\in\mathcal{A}^c}\left(|u_j^{\frac{k}{2}\alpha_j}|(l_{j+1}-l_j)^{\frac{H|k\alpha_j|}{2}}\right)\prod_{j\in\mathcal{A}^c}(l_{j+1}-l_j)^{-\frac{H|k\alpha_j|}{2}}\\
&\leq C\prod_{j\in\mathcal{A}^c}(l_{j+1}-l_j)^{-\frac{H|k\alpha_j|}{2}},
\end{aligned}$$

where $|k\alpha_j| = k_1\alpha_{1,j} + \cdots + k_d\alpha_{d,j}$. Then, we perform a linear transformation changing $(p_1,\ p_2, \ldots, p_n)$ into an integral with respect to variables $\{u_j\}_{j\in\mathcal{A}}$,

$$\begin{aligned}
&\int_{\mathbb{R}^{nd}} e^{-c\sum_{j=1}^{2n-1}|u_j|^2(l_{j+1}-l_j)^{2H}}\prod_{j=1}^{2n-1}(|u_j^{\frac{k}{2}\alpha_j}|)dp\\
&\quad\leq C\prod_{j\in\mathcal{A}^c}(l_{j+1}-l_j)^{-\frac{H|k\alpha_j|}{2}}\int_{\mathbb{R}^{nd}} e^{-c\sum_{j\in\mathcal{A}}|u_j|^2(l_{j+1}-l_j)^{2H}}\prod_{j\in\mathcal{A}}(|u_j^{\frac{k}{2}\alpha_j}|)dp\\
&\quad= C|J|\prod_{j\in\mathcal{A}^c}(l_{j+1}-l_j)^{-\frac{H|k\alpha_j|}{2}}\int_{\mathbb{R}^{nd}} e^{-c\sum_{j\in\mathcal{A}}|u_j|^2(l_{j+1}-l_j)^{2H}}\prod_{j\in\mathcal{A}}(|u_j^{\frac{k}{2}\alpha_j}|)du,
\end{aligned}$$

where $|J|$ is the Jacobian determinant of changing variables $(p_1,\ p_2, \ldots, p_n)$ to $(u_j, j\in\mathcal{A})$.

Therefore, we may reduce the convergence of $\left|\mathbb{E}[(\widehat{\alpha}_{t,\varepsilon}^{(k)}(0))^n]\right|$ to show the finiteness of

$$\int_{E^n}\int_{\mathbb{R}^{nd}} e^{-c\sum_{j\in\mathcal{A}}|u_j|^2(l_{j+1}-l_j)^{2H}}\prod_{j\in\mathcal{A}}(|u_j^{\frac{k}{2}\alpha_j}|)\prod_{j\in\mathcal{A}^c}(l_{j+1}-l_j)^{-\frac{H|k\alpha_j|}{2}}dudl =: \Lambda.$$

Since

$$\int_{\mathbb{R}} e^{-c\,u_{ij}^2(l_{j+1}-l_j)^{2H}}|u_{ij}|^{\frac{k_i}{2}\alpha_{i,j}}du_{ij}\leq C(l_{j+1}-l_j)^{-H-\frac{H}{2}k_i\alpha_{i,j}},$$

we have

$$\begin{aligned}
\Lambda &\leq C\int_{E^n}\prod_{j\in\mathcal{A}}(l_{j+1}-l_j)^{-Hd-\frac{H}{2}|k\alpha_j|}\prod_{j\in\mathcal{A}^c}(l_{j+1}-l_j)^{-\frac{H|k\alpha_j|}{2}}dl\\
&\leq C_{n,H,t}\int_{E^n}\prod_{j\in\mathcal{A}}(l_{j+1}-l_j)^{-Hd-H|k|}\prod_{j\in\mathcal{A}^c}(l_{j+1}-l_j)^{-H|k|}dl
\end{aligned}$$

$$\leq C_{n,H,t} \frac{\Gamma^n(1-Hd-H|k|)\Gamma^{n-1}(1-H|k|)}{\Gamma\Big(n(1-Hd-H|k|)+(n-1)(1-H|k|)+1\Big)},$$

where $C_{n,H,t}$ is a constant dependent on $n,\ H$ and t.

Thus, we can see that $\Big|\mathbb{E}[(\widehat{\alpha}_{t,\varepsilon}^{(k)}(0))^n]\Big|$ is finite for all $\varepsilon > 0$ under condition $H(d+|k|) < 1$. Then, we need to prove $\{\widehat{\alpha}_{t,\varepsilon}^{(k)}(0)\}_{\varepsilon>0}$ is a Cauchy sequence.

Notice that for any $\varepsilon_1,\ \varepsilon_2 > 0$,

$$\Big|\mathbb{E}\Big[(\widehat{\alpha}_{t,\varepsilon_1}^{(k)}(0)-\widehat{\alpha}_{t,\varepsilon_2}^{(k)}(0))^n\Big]\Big| \leq C\int_{D^n}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n |e^{-\frac{\varepsilon_1}{2}|p_j|^2}-e^{-\frac{\varepsilon_2}{2}|p_j|^2}| \\ \times\prod_{j=1}^n |p_j^k|\mathbb{E}\Big[\prod_{j=1}^n e^{i\langle p_j, B_{s_j}^H-B_{r_j}^H\rangle}\Big]dpdrds.$$

By the dominated convergence theorem and

$$\int_{D^n}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n |p_j^k|\mathbb{E}\Big[\prod_{j=1}^n e^{i\langle p_j, B_{s_j}^H-B_{r_j}^H\rangle}\Big]dpdrds < \infty,$$

we can obtain the desired result. This completes the proof.

3.3 Proof of Theorem 3

Let us first prove (2). For any $\lambda \in [0, 1]$, we have the following inequalities:

$$|e^{-i\langle p_j,x\rangle}-e^{-i\langle p_j,y\rangle}| \leq C|p_j|^\lambda|x-y|^\lambda$$

and

$$|p_j|^\lambda = (p_{1j}^2+\cdots+p_{dj}^2)^{\frac{\lambda}{2}} \leq C\Big(|p_{1j}|^\lambda+\cdots+|p_{dj}|^\lambda\Big).$$

Using the similar methods as in the proof of Theorem 2, we find that

$$\Big|\mathbb{E}\Big[\Big(\widehat{\alpha}_t^{(k)}(x)-\widehat{\alpha}_t^{(k)}(y)\Big)^n\Big]\Big| \\ \leq C|x-y|^{n\lambda}\int_{D^n}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n |p_j^k|\Big(|p_{1j}|^\lambda+\cdots+|p_{dj}|^\lambda\Big) \\ \times\mathbb{E}\Big[\prod_{j=1}^n e^{i\langle p_j, B_{s_j}^H-B_{r_j}^H\rangle}\Big]dpdrds \\ \leq C|x-y|^{n\lambda}\sum_{l=1}^d\int_{D^n}\int_{\mathbb{R}^{nd}}\prod_{j=1}^n |p_j^{k+\tilde{\lambda}_l}|\mathbb{E}\Big[\prod_{j=1}^n e^{i\langle p_j, B_{s_j}^H-B_{r_j}^H\rangle}\Big]dpdrds$$

$$=: C|x-y|^{n\lambda}\Lambda_1,$$

where $\tilde{\lambda}_l = (\lambda_1, \ldots, \lambda_d)$ with $\lambda_l = \lambda$ and all other $\lambda_j = 0$. So $|k+\tilde{\lambda}| = |k| + \lambda$ and Λ_1 is less than (with $\mathcal{A}$ defined as before)

$$\begin{aligned}
C \int_{E^n} \prod_{j\in\mathcal{A}} (l_{j+1}-l_j)^{-Hd-\frac{H}{2}|\alpha_j(k+\tilde{\lambda})|} \prod_{j\in\mathcal{A}^c} (l_{j+1}-l_j)^{-\frac{H}{2}|\alpha_j(k+\tilde{\lambda})|} dl \\
\le C_{n,H,t} \frac{\Gamma^n(1-Hd-H|k+\tilde{\lambda}|)\Gamma^{n-1}(1-H|k+\tilde{\lambda}|)}{\Gamma\Big(n(1-Hd-H|k+\tilde{\lambda}|)+(n-1)(1-H|k+\tilde{\lambda}|)+1\Big)},
\end{aligned}$$

which is finite if $1-Hd-H|k+\tilde{\lambda}| > 0$.

For the proof of (3), let $\tilde{D} = \{(r,s) : 0 < r < s < \tilde{t}\}$ and without loss of generality, we assume that $t < \tilde{t}$. Then,

$$\begin{aligned}
&\Big|\mathbb{E}\Big[\Big(\widehat{\alpha}_t^{(k)}(y) - \widehat{\alpha}_{\tilde{t}}^{(k)}(y)\Big)^n\Big]\Big| \\
&\le C \int_{(\tilde{D}\setminus D)^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n |p_j^k| \mathbb{E}\Big[\prod_{j=1}^n e^{i\langle p_j, B_{s_j}^H - B_{r_j}^H\rangle}\Big] dp dr ds \\
&\le C \int_{[t,\tilde{t}]^n} \int_{[0,s_1]\times\cdots\times[0,s_n]} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n |p_j^k| \mathbb{E}\Big[\prod_{j=1}^n e^{i\langle p_j, B_{s_j}^H - B_{r_j}^H\rangle}\Big] dp dr ds \\
&\le C \int_{\tilde{D}^n} \prod_{j=1}^n 1_{[t,\tilde{t}]}(s_j) \int_{\mathbb{R}^{nd}} \prod_{j=1}^n |p_j^k| \mathbb{E}\Big[\prod_{j=1}^n e^{i\langle p_j, B_{s_j}^H - B_{r_j}^H\rangle}\Big] dp dr ds \\
&\le C|t-\tilde{t}|^{n\beta} \Big(\int_{\tilde{D}^n} \Big(\int_{\mathbb{R}^{nd}} \prod_{j=1}^n |p_j^k| \mathbb{E}\Big[\prod_{j=1}^n e^{i\langle p_j, B_{s_j}^H - B_{r_j}^H\rangle}\Big] dp\Big)^{\frac{1}{1-\beta}} dr ds\Big)^{1-\beta} \\
&=: C|t-\tilde{t}|^{n\beta}\Lambda_2,
\end{aligned}$$

where we use the Hölder's inequality in the last inequality with $\beta < 1 - H|k| - Hd$.

Using the similar methods as in the proof of Theorem 2, Λ_2 is bounded by

$$\Big(\int_{E^n} \prod_{j\in\mathcal{A}} (l_{j+1}-l_j)^{-\frac{Hd}{1-\beta}-\frac{H}{2(1-\beta)}|k\alpha_j|} \prod_{\mathcal{A}^c} (l_{j+1}-l_j)^{-\frac{H}{2(1-\beta)}|k\alpha_j|} dl\Big)^{1-\beta}.$$

Since $1-\beta > H(|k|+d)$, there exists a constant $C > 0$, such that

$$\Big|\mathbb{E}\Big[\Big(\widehat{\alpha}_t^{(k)}(y) - \widehat{\alpha}_{\tilde{t}}^{(k)}(y)\Big)^n\Big]\Big| \le C|t-\tilde{t}|^{n\beta}.$$

3.4 Proof of Theorem 4

In this section, we mainly use the method given in Jaramillo and Nualart [5]. We first give the chaos decomposition of the random variable $\widehat{\alpha}^{(k)}_{t,\varepsilon}(0)$ defined in (1) with $d = 1$ and $k = 1$. We write

$$\widehat{\alpha}'_{t,\varepsilon}(0) = \int_0^t \int_0^s \alpha_{\varepsilon,s,r} dr ds,$$

where

$$\alpha_{\varepsilon,s,r} = f'_\varepsilon(B^H_s - B^H_r) = \sum_{q=1}^{\infty} I_{2q-1}(f_{2q-1,\varepsilon,s,r})$$

with

$$f_{2q-1,\varepsilon,s,r}(x_1,\ldots,x_{2q-1}) = (-1)^q \beta_q \Big(\varepsilon + (s-r)^{2H}\Big)^{-q-\frac{1}{2}} \prod_{j=1}^{2q-1} \mathbb{1}_{[r,s]}(x_j)$$

and

$$\beta_q = \frac{1}{2^{q-\frac{1}{2}}(q-1)!\sqrt{\pi}}.$$

Then, $\widehat{\alpha}'_{t,\varepsilon}(0)$ has the following chaos decomposition:

$$\widehat{\alpha}'_{t,\varepsilon}(0) = \sum_{q=1}^{\infty} I_{2q-1}(f_{2q-1,\varepsilon}),$$

where

$$f_{2q-1,\varepsilon}(x_1,\ldots,x_{2q-1}) = \int_D f_{2q-1,\varepsilon,s,r}(x_1,\ldots,x_{2q-1}) dr ds$$

with $D = \{(r,s) : 0 < r < s < t\}$.

For $q = 1$,

$$\mathbb{E}\Big[\Big|I_1(f_{1,\varepsilon})\Big|^2\Big] = \int_{D^2} \langle f_{1,\varepsilon,s_1,r_1}, f_{1,\varepsilon,s_2,r_2}\rangle_{\mathfrak{H}} dr_1 dr_2 ds_1 ds_2, \tag{8}$$

where $\mathfrak{H}$ is the Hilbert space obtained by taking the completion of the step functions endowed with the inner product

$$\langle \mathbb{1}_{[a,b]}, \mathbb{1}_{[c,d]}\rangle_{\mathfrak{H}} := \mathbb{E}[(B^H_b - B^H_a)(B^H_d - B^H_c)].$$

For $q > 1$, we have to describe the terms $\langle f_{2q-1,\varepsilon,s_1,r_1}, f_{2q-1,\varepsilon,s_2,r_2}\rangle_{\mathfrak{H}^{\otimes(2q-1)}}$, where $\mathfrak{H}^{\otimes(2q-1)}$ is the $(2q-1)$-th tensor product of $\mathfrak{H}$. For every $x,\ u_1,\ u_2 > 0$, we define

$$\mu(x, u_1, u_2) = \mathbb{E}[B^H_{u_1}(B^H_{x+u_2} - B^H_x)].$$

Then, from Eq. (2.19) in Jaramillo and Nualart [5],

$$\langle f_{2q-1,\varepsilon,s_1,r_1}, f_{2q-1,\varepsilon,s_2,r_2}\rangle_{\mathfrak{H}^{\otimes(2q-1)}} = \beta_q^2 G^{(q)}_{\varepsilon,r_2-r_1}(s_1 - r_1, s_2 - r_2),$$

where

$$G^{(q)}_{\varepsilon,x}(u_1, u_2) = \left(\varepsilon + u_1^{2H}\right)^{-\frac{1}{2}-q}\left(\varepsilon + u_2^{2H}\right)^{-\frac{1}{2}-q}\mu(x, u_1, u_2)^{2q-1}.$$

Before we give the proof of the main result, we give some useful lemmas below. In the sequel, we just consider the case $H = \frac{2}{3}$.

Lemma 3

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\Big|\frac{1}{\log\frac{1}{\varepsilon}}\widehat{\alpha}'_{t,\varepsilon}(0)\Big|^2\Big] = \sigma^2.$$

Proof From Lemma 5.1 in Jaramillo and Nualart [5], we can see

$$\mathbb{E}\Big[\Big|\frac{1}{\log\frac{1}{\varepsilon}}\widehat{\alpha}'_{t,\varepsilon}(0)\Big|^2\Big] = \frac{1}{(\log\frac{1}{\varepsilon})^2}\Big(V_1(\varepsilon) + V_2(\varepsilon) + V_3(\varepsilon)\Big)$$

and

$$V_i(\varepsilon) = \frac{1}{\pi}\int_{D_i} |\varepsilon I + \Sigma|^{-\frac{3}{2}}\mu dr ds dr' ds'$$

where D_i (i=1, 2, 3) defined in Lemma 2 and Σ is the covariance matrix of $(B^H_s - B^H_r, B^H_{s'} - B^H_{r'})$ with $\Sigma_{1,1} = \lambda$, $\Sigma_{2,2} = \rho$, $\Sigma_{1,2} = \mu$ given in Lemma 2.

Next, we will split the proof into three parts to consider $V_1(\varepsilon)$, $V_2(\varepsilon)$ and $V_3(\varepsilon)$, respectively.

For the $V_1(\varepsilon)$ term, changing the coordinates (r, r', s, s') by $(r, a = r' - r, b = s - r', c = s' - s)$ and integrating the r variable, we get

$$\begin{aligned} V_1(\varepsilon) &\le \frac{1}{\pi}\int_{[0,t]^4} |\varepsilon I + \Sigma|^{-\frac{3}{2}}\mu dr da db dc \\ &= \frac{t}{\pi}\int_{[0,t]^3} |\varepsilon I + \Sigma|^{-\frac{3}{2}}\mu da db dc \\ &=: \widetilde{V}_1(\varepsilon). \end{aligned}$$

Since

$$\mu = \frac{1}{2}((a+b+c)^{\frac{4}{3}} + b^{\frac{4}{3}} - a^{\frac{4}{3}} - c^{\frac{4}{3}}) \le \sqrt{\lambda\rho} = (a+b)^{\frac{2}{3}}(b+c)^{\frac{2}{3}}$$

and

$$\begin{aligned} |\varepsilon I + \Sigma| &= (\varepsilon + \Sigma_{1,1})(\varepsilon + \Sigma_{2,2}) - \Sigma_{1,2}^2 \\ &\ge C\Big[\varepsilon^2 + \varepsilon((a+b)^{\frac{4}{3}} + (b+c)^{\frac{4}{3}}) + a^{\frac{4}{3}}(c+b)^{\frac{4}{3}} + c^{\frac{4}{3}}(a+b)^{\frac{4}{3}}\Big] \\ &\ge C\Big[\varepsilon^2 + (a+b)^{\frac{2}{3}}(b+c)^{\frac{2}{3}}(\varepsilon + (ac)^{\frac{2}{3}})\Big] \\ &\ge C(a+b)^{\frac{2}{3}}(b+c)^{\frac{2}{3}}(\varepsilon + (ac)^{\frac{2}{3}}), \end{aligned}$$

where we use the Young's inequality in the second to last inequality.

Then, we have

$$\widetilde{V}_1(\varepsilon) \le C \int_{[0,t]^3} (a+b)^{-\frac{1}{3}}(b+c)^{-\frac{1}{3}}\Big(\varepsilon + (ac)^{\frac{2}{3}}\Big)^{-\frac{3}{2}} dadbdc.$$

We will estimate this integral over the regions $\{b \le (a \vee c)\}$ and $\{b > (a \vee c)\}$ separately, and we will denote these two integrals by $\widetilde{V}_{1,1}$ and $\widetilde{V}_{1,2}$, respectively. If $b \le (a \vee c)$, without loss of generality, we can assume $a \ge c$ and thus $b \le a$. For a given small enough constant $\varepsilon_1 > 0$,

$$\begin{aligned} \widetilde{V}_{1,1}(\varepsilon) &\le C \int_{[0,t]^3} (a+b)^{-\frac{1}{3}-\varepsilon_1}(b+c)^{-\frac{1}{3}} \frac{(a+b)^{\varepsilon_1}}{a^{\varepsilon_1}} a^{\varepsilon_1}\Big(\varepsilon + (ac)^{\frac{2}{3}}\Big)^{-\frac{3}{2}} dadbdc \\ &\le C \int_{[0,t]^3} b^{-\frac{2}{3}-\varepsilon_1} a^{\varepsilon_1}\Big(\varepsilon + (ac)^{\frac{2}{3}}\Big)^{-\frac{3}{2}} dadbdc \\ &\le C \int_0^t \int_0^{t\varepsilon^{-\frac{3}{2}}} a^{\varepsilon_1}\Big(1 + (ac)^{\frac{2}{3}}\Big)^{-\frac{3}{2}} dcda, \end{aligned}$$

where we make the change of variable $c = c\,\varepsilon^{-\frac{3}{2}}$ in the last inequality.

By L'Hôspital's rule, we have

$$\limsup_{\varepsilon \to 0} \frac{\widetilde{V}_{1,1}(\varepsilon)}{\log \frac{1}{\varepsilon}} \le C < \infty.$$

If $b > (a \vee c)$, we can see that

$$\mu = \frac{1}{2}((a+b+c)^{\frac{4}{3}} + b^{\frac{4}{3}} - a^{\frac{4}{3}} - c^{\frac{4}{3}}) \le C\, b^{\frac{4}{3}}$$

and

$$
\begin{aligned}
|\varepsilon I + \Sigma| &\geq C\Big[\varepsilon^2 + \varepsilon((a+b)^{\frac{4}{3}} + (b+c)^{\frac{4}{3}}) + a^{\frac{4}{3}}(c+b)^{\frac{4}{3}} + c^{\frac{4}{3}}(a+b)^{\frac{4}{3}}\Big] \\
&\geq C\, b^{\frac{4}{3}}(\varepsilon + (a \vee c)^{\frac{4}{3}}).
\end{aligned}
$$

Then,

$$
\begin{aligned}
\limsup_{\varepsilon\to 0} \frac{\widetilde{V}_{1,2}(\varepsilon)}{\log\frac{1}{\varepsilon}} &\leq \limsup_{\varepsilon\to 0} \frac{C}{\log\frac{1}{\varepsilon}} \int_{[0,t]^3} b^{-\frac{2}{3}}\Big(\varepsilon + (a\vee c)^{\frac{4}{3}}\Big)^{-\frac{3}{2}} dadbdc \\
&\leq \limsup_{\varepsilon\to 0} \frac{C}{\log\frac{1}{\varepsilon}} \int_0^t \int_0^a (\varepsilon + a^{\frac{4}{3}})^{-\frac{3}{2}} dcda \\
&= \limsup_{\varepsilon\to 0} \frac{C}{\log\frac{1}{\varepsilon}} \int_0^t a(\varepsilon + a^{\frac{4}{3}})^{-\frac{3}{2}} da < \infty.
\end{aligned}
$$

So, by the above result, we can obtain

$$
\lim_{\varepsilon\to 0} \frac{V_1(\varepsilon)}{(\log\frac{1}{\varepsilon})^2} = 0. \tag{9}
$$

For the $V_2(\varepsilon)$ **term**, changing the coordinates (r, r', s, s') by $(r, a = r' - r, b = s' - r', c = s - s')$ and integrating the r variable, we get

$$
V_2(\varepsilon) \leq \frac{t}{\pi} \int_{[0,t]^3} |\varepsilon I + \Sigma|^{-\frac{3}{2}} \mu dadbdc =: \widetilde{V}_2(\varepsilon).
$$

By

$$
\begin{aligned}
\mu &= \frac{1}{2}\Big((a+b)^{\frac{4}{3}} + (b+c)^{\frac{4}{3}} - a^{\frac{4}{3}} - c^{\frac{4}{3}}\Big) \\
&= \frac{2b}{3} \int_0^1 \Big((a+bv)^{\frac{1}{3}} + (c+bv)^{\frac{1}{3}}\Big) dv \\
&\leq \frac{4}{3} b(a+b+c)^{\frac{1}{3}}
\end{aligned}
$$

and

$$
\begin{aligned}
|\varepsilon I + \Sigma| &= (\varepsilon + \Sigma_{1,1})(\varepsilon + \Sigma_{2,2}) - \Sigma_{1,2}^2 \\
&\geq \varepsilon^2 + \varepsilon((a+b+c)^{\frac{4}{3}} + b^{\frac{4}{3}}) + C\, b^{\frac{4}{3}}(a^{\frac{4}{3}} + c^{\frac{4}{3}}),
\end{aligned}
$$

we have

$$
\widetilde{V}_2(\varepsilon) \leq C \int_{[0,t]^3} b(a+b+c)^{\frac{1}{3}} \Big(\varepsilon((a+b+c)^{\frac{4}{3}} + b^{\frac{4}{3}}) + b^{\frac{4}{3}}(a^{\frac{4}{3}} + c^{\frac{4}{3}})\Big)^{-\frac{3}{2}} dadbdc.
$$

We again estimate this integral over the regions $\{b \le (a \vee c)\}$ and $\{b > (a \vee c)\}$ separately, and denote these two integrals by $\widetilde{V}_{2,1}$ and $\widetilde{V}_{2,2}$, respectively. If $b \le (a \vee c)$,

$$\begin{aligned}\widetilde{V}_{2,1}(\varepsilon) &\le C \int_{[0,t]^3} b(a \vee c)^{\frac{1}{3}} \Big(\varepsilon (a \vee c)^{\frac{4}{3}} + b^{\frac{4}{3}} (a \vee c)^{\frac{4}{3}}\Big)^{-\frac{3}{2}} dadbdc \\ &\le C \int_{[0,t]^3} b(a \vee c)^{-\frac{5}{2}} \Big(\varepsilon + b^{\frac{4}{3}}\Big)^{-\frac{3}{2}} dadbdc \\ &\le C \int_0^t b\Big(\varepsilon + b^{\frac{4}{3}}\Big)^{-\frac{3}{2}} db \\ &\le C \int_0^{t\varepsilon^{-\frac{3}{4}}} b(1 + b^{\frac{4}{3}})^{-\frac{3}{2}} db.\end{aligned}$$

Thus,

$$\limsup_{\varepsilon \to 0} \frac{\widetilde{V}_{2,1}(\varepsilon)}{\log \frac{1}{\varepsilon}} \le C \limsup_{\varepsilon \to 0} \varepsilon^{-\frac{3}{2}} (1 + t^{\frac{4}{3}} \varepsilon^{-1})^{-\frac{3}{2}} < \infty. \tag{10}$$

If $b > (a \vee c)$, similarly, we have

$$\begin{aligned}\limsup_{\varepsilon \to 0} \frac{\widetilde{V}_{2,2}(\varepsilon)}{\log \frac{1}{\varepsilon}} &\le \limsup_{\varepsilon \to 0} \frac{C}{\log \frac{1}{\varepsilon}} \int_0^t b^{-\frac{2}{3}} db \int_{[0,t]^2} \Big(\varepsilon + (a \vee c)^{\frac{4}{3}}\Big)^{-\frac{3}{2}} dadc \\ &\le \limsup_{\varepsilon \to 0} \frac{C}{\log \frac{1}{\varepsilon}} \int_0^t \int_0^a (\varepsilon + a^{\frac{4}{3}})^{-\frac{3}{2}} dcda \\ &= \limsup_{\varepsilon \to 0} \frac{C}{\log \frac{1}{\varepsilon}} \int_0^t a(\varepsilon + a^{\frac{4}{3}})^{-\frac{3}{2}} da < \infty.\end{aligned} \tag{11}$$

So, by the above result, we can obtain

$$\lim_{\varepsilon \to 0} \frac{V_2(\varepsilon)}{(\log \frac{1}{\varepsilon})^2} = 0.$$

For the $V_3(\varepsilon)$ term. We first change the coordinates (r, r', s, s') by $(r, a = s - r, b = r' - s, c = s' - r')$ and then by

$$\begin{aligned}\mu &= \frac{1}{2}\Big((a + b + c)^{\frac{4}{3}} + b^{\frac{4}{3}} - (b + c)^{\frac{4}{3}} - (a + b)^{\frac{4}{3}}\Big) \\ &= \frac{2}{9} ac \int_0^1 \int_0^1 (b + ax + cy)^{-\frac{2}{3}} dxdy \\ &=: \mu(a + b, a, c),\end{aligned}$$

and $|\varepsilon I + \Sigma| = \varepsilon^2 + \varepsilon(a^{\frac{4}{3}} + c^{\frac{4}{3}}) + (ac)^{\frac{4}{3}} - \mu(a+b,a,c)^2$, we can find

$$\begin{aligned} V_3(\varepsilon) &= \frac{1}{\pi}\int_{[0,t]^3} \mathbb{1}_{(0,t)}(a+b+c)(t-a-b-c)\mu|\varepsilon I + \Sigma|^{-\frac{3}{2}} dadbdc \\ &= \frac{1}{\pi}\int_{[0,t\varepsilon^{-\frac{3}{4}}]^2\times[0,t]} \mathbb{1}_{(0,t)}(b+\varepsilon^{\frac{3}{4}}(a+c)) \\ &\quad\times \frac{(t-b-\varepsilon^{\frac{3}{4}}(a+c))\varepsilon^{-\frac{3}{2}}\mu(\varepsilon^{\frac{3}{4}}a+b,\varepsilon^{\frac{3}{4}}a,\varepsilon^{\frac{3}{4}}c)}{\left[(1+a^{\frac{4}{3}})(1+c^{\frac{4}{3}}) - \varepsilon^{-2}\mu(\varepsilon^{\frac{3}{4}}a+b,\varepsilon^{\frac{3}{4}}a,\varepsilon^{\frac{3}{4}}c)^2\right]^{\frac{3}{2}}} dbdadc, \end{aligned}$$

where we change the coordinates (a,b,c) by $(\varepsilon^{-\frac{3}{4}}a, b, \varepsilon^{-\frac{3}{4}}c)$ in the last equality.

By the definition of $\mu(a+b,a,c)$, it is easy to find

$$\mu(\varepsilon^{\frac{3}{4}}a+b,\varepsilon^{\frac{3}{4}}a,\varepsilon^{\frac{3}{4}}c) = \frac{2}{9}\varepsilon^{\frac{3}{2}}ac\int_{[0,1]^2}(b+\varepsilon^{\frac{3}{4}}av_1+\varepsilon^{\frac{3}{4}}cv_2)^{-\frac{2}{3}}dv_1dv_2$$

and

$$\varepsilon^{-\frac{3}{2}}\mu(\varepsilon^{\frac{3}{4}}a+b,\varepsilon^{\frac{3}{4}}a,\varepsilon^{\frac{3}{4}}c) = \frac{2}{9}acb^{-\frac{2}{3}} + O(\varepsilon^{\frac{3}{4}}ac(a+c)).$$

The other part of the integrand in $V_3(\varepsilon)$ is

$$\begin{aligned} &\left[(1+a^{\frac{4}{3}})(1+c^{\frac{4}{3}}) - \varepsilon^{-2}\mu(\varepsilon^{\frac{3}{4}}a+b,\varepsilon^{\frac{3}{4}}a,\varepsilon^{\frac{3}{4}}c)^2\right]^{-\frac{3}{2}} \\ &\quad= \left[(1+a^{\frac{4}{3}})(1+c^{\frac{4}{3}})\right]^{-\frac{3}{2}} + O\Big(\varepsilon a^2c^2[(1+a^{\frac{4}{3}})(1+c^{\frac{4}{3}})]^{-\frac{5}{2}}\Big). \end{aligned}$$

Since

$$\begin{aligned} &\frac{1}{(\log\frac{1}{\varepsilon})^2}\int_{[0,t\varepsilon^{-\frac{3}{4}}]^2} \varepsilon^{\frac{3}{4}}ac(a+c)\left[(1+a^{\frac{4}{3}})(1+c^{\frac{4}{3}})\right]^{-\frac{3}{2}}dadc \\ &\quad+ \frac{1}{(\log\frac{1}{\varepsilon})^2}\int_{[0,t\varepsilon^{-\frac{3}{4}}]^2} \varepsilon a^3c^3\left[(1+a^{\frac{4}{3}})(1+c^{\frac{4}{3}})\right]^{-\frac{5}{2}}dadc \\ &\to 0, \end{aligned} \tag{12}$$

as $\varepsilon \to 0$. Then, by L'Hôspital's rule, we have

$$\begin{aligned} \lim_{\varepsilon\to 0}\frac{V_3(\varepsilon)}{(\log\frac{1}{\varepsilon})^2} &= \frac{2}{9\pi}\int_0^t (t-b)b^{-\frac{2}{3}}db \\ &\qquad\times \lim_{\varepsilon\to 0}\frac{1}{(\log\frac{1}{\varepsilon})^2}\int_{[0,t\varepsilon^{-\frac{3}{4}}]^2} ac\left[(1+a^{\frac{4}{3}})(1+c^{\frac{4}{3}})\right]^{-\frac{3}{2}}dadc \\ &= \frac{t^{\frac{4}{3}}}{8\pi}B\left(2,\frac{1}{3}\right). \end{aligned} \tag{13}$$

Together (9)–(13), we can see

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\Big|\frac{1}{\log\frac{1}{\varepsilon}}\widehat{\alpha}'_{t,\varepsilon}(0)\Big|^2\Big]=\frac{t^{\frac{4}{3}}}{8\pi}B\left(2,\frac{1}{3}\right)=:\sigma^2.$$

□

Lemma 4 *For $I_1(f_{1,\varepsilon})$ given in (8), then*

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\Big|\frac{1}{\log\frac{1}{\varepsilon}}I_1(f_{1,\varepsilon})\Big|^2\Big]=\sigma^2.$$

Proof Form (8), we can find

$$\mathbb{E}\Big[\Big|\frac{1}{\log\frac{1}{\varepsilon}}I_1(f_{1,\varepsilon})\Big|^2\Big]=\frac{1}{(\log\frac{1}{\varepsilon})^2}\Big(V_1^{(1)}(\varepsilon)+V_2^{(1)}(\varepsilon)+V_3^{(1)}(\varepsilon)\Big),$$

where $V_i^{(1)}(\varepsilon)=2\int_{D_i}\langle f_{1,\varepsilon,s_1,r_1},f_{1,\varepsilon,s_2,r_2}\rangle_{\mathfrak{H}}dr_1dr_2ds_1ds_2$ for $i=1,2,3$. Then, we have

$$0\le V_i^{(1)}(\varepsilon)\le V_i(\varepsilon), \tag{14}$$

since $H>\frac{1}{2}$ and μ can only take positive values.

Combining (14) with (9)–(11), we can see

$$\lim_{\varepsilon\to 0}\frac{1}{(\log\frac{1}{\varepsilon})^2}\Big(V_1^{(1)}(\varepsilon)+V_2^{(1)}(\varepsilon)\Big)=0.$$

Thus, we only need to consider $\frac{1}{(\log\frac{1}{\varepsilon})^2}V_3^{(1)}(\varepsilon)$ as $\varepsilon\to 0$.

By the proof of Lemma 5.7 in Jaramillo and Nualart [5], we have

$$\begin{aligned}V_3^{(1)}(\varepsilon)&=\frac{1}{\pi}\int_{S_3}G^{(1)}_{\varepsilon,r'-r}(s-r,s'-r')\\&=\frac{1}{\pi}\int_{[0,t]^3}\int_0^{t-(a+b+c)}\mathbb{1}_{(0,t)}(a+b+c)(\varepsilon+a^{\frac{4}{3}})^{-\frac{3}{2}}\\&\quad(\varepsilon+c^{\frac{4}{3}})^{-\frac{3}{2}}\mu(a+b,a,c)ds_1dadbdc\\&=\frac{2}{9\pi}\int_0^t\int_{[0,t\varepsilon^{-\frac{3}{4}}]^2}\int_{[0,1]^2}\mathbb{1}_{(0,t)}\Big((b+\varepsilon^{\frac{3}{4}}(a+c)\Big)\Big(t-b-\varepsilon^{\frac{3}{4}}(a+c)\Big)\\&\quad\times\Big[(1+a^{\frac{4}{3}})(1+c^{\frac{4}{3}})\Big]^{-\frac{3}{2}}ac\Big(b+\varepsilon^{\frac{3}{4}}(av_1+cv_2)\Big)^{-\frac{2}{3}}dv_1dv_2dadcdb.\end{aligned}$$

Note that

$$\int_{[0,1]^2} \Big(b+\varepsilon^{\frac{3}{4}}(av_1+cv_2)\Big)^{-\frac{2}{3}} dv_1 dv_2 = b^{-\frac{2}{3}} + O(\varepsilon^{\frac{3}{4}}(a+c))$$

and

$$\begin{aligned}
&\int_{[0,1]^2} \Big(t-b-\varepsilon^{\frac{3}{4}}(a+c)\Big)\Big[(1+a^{\frac{4}{3}})(1+c^{\frac{4}{3}})\Big]^{-\frac{3}{2}} ac\Big(b+\varepsilon^{\frac{3}{4}}(av_1+cv_2)\Big)^{-\frac{2}{3}} dv_1 dv_2 \\
&= (t-b)b^{-\frac{2}{3}} ac\Big[(1+a^{\frac{4}{3}})(1+c^{\frac{4}{3}})\Big]^{-\frac{3}{2}} \\
&+ O\left(\varepsilon^{\frac{3}{4}}(a+c)ac\Big[(1+a^{\frac{4}{3}})(1+c^{\frac{4}{3}})\Big]^{-\frac{3}{2}}\right).
\end{aligned}$$

Similar to (12) and (13), we can find that

$$\lim_{\varepsilon\to 0} \frac{1}{(\log\frac{1}{\varepsilon})^2} \int_{[0,t\varepsilon^{-\frac{3}{4}}]^2} \varepsilon^{\frac{3}{4}}(a+c)ac\Big[(1+a^{\frac{4}{3}})(1+c^{\frac{4}{3}})\Big]^{-\frac{3}{2}} da dc = 0$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{(\log\frac{1}{\varepsilon})^2} \int_{[0,t\varepsilon^{-\frac{3}{4}}]^2} ac\Big[(1+a^{\frac{4}{3}})(1+c^{\frac{4}{3}})\Big]^{-\frac{3}{2}} da dc = \frac{9}{16}.$$

Thus,

$$\begin{aligned}
\lim_{\varepsilon\to 0} \frac{V_3^{(1)}(\varepsilon)}{(\log\frac{1}{\varepsilon})^2} &= \frac{2}{9\pi}\int_0^t \frac{9}{16}(t-b)b^{-\frac{2}{3}} db \\
&= \lim_{\varepsilon\to 0} \frac{V_3(\varepsilon)}{(\log\frac{1}{\varepsilon})^2} \\
&= \sigma^2.
\end{aligned}$$

□

Proof of Theorem 4 By Lemmas 3–4 and

$$\widehat{\alpha}'_{t,\varepsilon}(0) = I_1(f_{1,\varepsilon}) + \sum_{q=2}^{\infty} I_{2q-1}(f_{2q-1,\varepsilon}),$$

we can see

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big[\Big|\frac{1}{\log\frac{1}{\varepsilon}} \sum_{q=2}^{\infty} I_{2q-1}(f_{2q-1,\varepsilon})\Big|^2\Big] = 0.$$

Since $I_1(f_{1,\varepsilon})$ is Gaussian, then we have, as $\varepsilon \to 0$,

$$\Big(\log \frac{1}{\varepsilon}\Big)^{-1} I_1(f_{1,\varepsilon}) \overset{law}{\to} N(0,\sigma^2).$$

Thus,

$$\Big(\log \frac{1}{\varepsilon}\Big)^{-1} \widehat{\alpha}'_{t,\varepsilon}(0) \overset{law}{\to} N(0,\sigma^2)$$

as $\varepsilon \to 0$. This completes the proof.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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